ANNALES

DE

L'INSTITUT FOURIER

Mauricio GARAY

On vanishing inflection points of plane curves

Tome 52, nº 3 (2002), p. 849-880.

<http://aif.cedram.org/item?id=AIF_2002__52_3_849_0>

Ann. Inst. Fourier, Grenoble
52, 3 (2002), 849–880

ON VANISHING INFLECTION POINTS OF PLANE CURVES

by Mauricio D. GARAY

Dédié à V.I. Arnold pour son 65^e anniversaire

Introduction.

The study of inflection points of plane curves has a long history going back to Descartes, Newton and Plücker. Nevertheless, it seems that a study "à la Poincaré" *i.e.* a study of local normal forms is lacking. In this paper, we try to fill this gap. More precisely, we consider the following problem.

Let U be a neighbourhood of a point p in the projective plane $\mathbb{C}P^2$. Let $f: U \to \mathbb{C}$ be a holomorphic function with an isolated critical point at p. The fibres of f may have inflection points. We wish to find a local normal form of f in a neighbourhood of p which "takes into account" the inflection points of the curves $f^{-1}(\varepsilon)$ when $\varepsilon \to f(p)$.

The basic technique for finding normal forms is to consider the map f up to (biholomorphic) change of variables. This is of course inadequate for our problem because such a change of variables does not send a line to a line. Consequently an inflection point of a curve is, as a general rule, not sent to an inflection of the image of the curve. In fact, to define a local normal form (and a versal deformation) with respect to inflections of the germ of f at a point $p \in U$ is one of the first goals of this paper. Once this is done, we shall give an analog of Arnold's A, D, E classification [1] for the

Keywords : Plücker formulas – Normal forms – Inflection points – Bifurcation diagrams – Projective geometry.
Math. classification : 37G25 – 14N15.

case of inflections. Afterwards, we shall compute the versal deformations with respect to inflections of Morse function-germs. We hope that the normal form method adopted in this paper will be useful for solving other geometrical problems.

This paper is divided as follows.

In the first section, we recall basic facts on the geometry of plane projective curves.

In the second section, we introduce, following V.V. Goryunov, the equivalence of functions on plane curves.

In the third section, we define local normal forms and versal deformation with respect to inflections. Then, we give the classification of simple function-germs with respect to inflections and the versal deformation with respect to inflections of Morse functions.

In the fourth section, we prove the classification theorem stated in Section 3.

In the fifth section, we prove the versal deformation theorem stated in Section 3.

We have chosen for our exposition the complex holomorphic situation for simplicity but the results can be easily adapted for the case of real C^∞ curves in $\mathbb{R}P^2$.

1. Inflection points of plane curves.

We recall basic facts on the geometry of curves in the projective plane, standard references are [14], [10].

1.1. Basic definitions.

By *plane curve*, we mean a complex holomorphic submanifold of the projective plane $\mathbb{C}P^2$ of dimension 1, possibly singular. Let $V \subset \mathbb{C}P^2$ be a plane curve and L the tangent line to V at a point $p \in V$.

DEFINITION 1.1. — The *anomaly* a_p of the plane curve V at the point p is equal to the intersection multiplicity at p of V with L minus 2:

$$a_p = (V \cdot L)_p - 2.$$

Example. — Fix an integer number k. The anomaly of the plane curve

$$V = \{(x, y) \in \mathbb{C}^2 : y - x^{k+2} = 0\}$$

at the origin is equal to k.

DEFINITION 1.2. — A point $p \in V$ of a plane curve $V \subset \mathbb{C}P^2$ is called an *inflection point* of V if p is a smooth point of V and if the anomaly of V at p is not equal to zero.

DEFINITION 1.3. — An inflection point $p \in V$ of a plane curve $V \subset \mathbb{C}P^2$ is called a *degenerate inflection point* if p is an inflection point of V such that the anomaly of V at p is at least equal to 2.

Example. — The holomorphic curve

$$V = \{(x, y) \in \mathbb{C}^2 : y - x^{k+2} = 0\}$$

has a degenerate inflection point at the origin provided that $k > 1$.

1.2. The Hessian determinant.

Let U be an affine open subset of the projective plane $\mathbb{C}P^2$. Consider a holomorphic function $f : U \to \mathbb{C}$ and fix affine coordinates x, y in U.

Following Plücker [17], we give an equation for the inflection points of the fibres $f^{-1}(\varepsilon)$ of f in terms of the derivatives of f.

DEFINITION 1.4. — The *Hessian determinant* of f, denoted Δ_f, is the determinant of the bordered Hessian matrix:

$$\begin{pmatrix} f_{xx} & f_{xy} & f_x \\ f_{xy} & f_{yy} & f_y \\ f_x & f_y & 0 \end{pmatrix}.$$

Remark. — The Hessian determinant is not a projective invariant: it depends on the choice of the affine coordinate system (x, y).

We summarize well known properties due respectively to Hesse and Plücker (see [14] for details). Denote by $\mathcal{O}(U)$ the ring of holomorphic functions in U and fix $\varepsilon \in f(U)$. For simplicity assume that $f - \varepsilon$ and Δ_f are reduced equations.

PROPOSITION 1.1. — 1) *The ideal of $\mathcal{O}(U)$ generated by Δ_f is a projective invariant i.e., it does not depend on the choice of the affine coordinate system x, y.*

2) *The curves $f^{-1}(\varepsilon)$ and $\{p \in U : \Delta_f(p) = 0\}$ intersect at a point p with multiplicity k if an only if the anomaly of $f^{-1}(\varepsilon)$ at p is equal to k.*

Example 1. — Consider the function f defined by the polynomial $f(x, y) = y - x^3$. The Hessian determinant of f is equal to $6x$ (see figure 1, left part).

Example 2. — Consider the function f defined by the polynomial $f(x, y) = y - x^4$. The Hessian determinant of f is equal to $12x^2$ which is not a reduced equation. In order to extend part 2 of the proposition to the case where Δ_f is not necessarily reduced, we do as follows (part 1 holds even if the equation Δ_f is not reduced).

Let $\mathcal{O}$ be the ring of germs at p of holomorphic functions. Let I be the ideal of $\mathcal{O}$ generated by the germs of f and of Δ_f at the point p. Then the complex dimension of $\mathcal{O}/I$ is equal to the anomaly at p of $f^{-1}(\varepsilon)$.

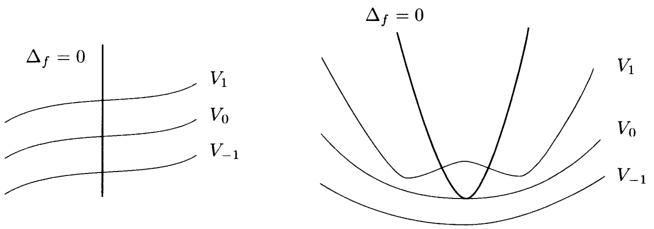

Figure 1. *The Hessian curve $X_f = \{(x, y) : \Delta_f(x, y) = 0\}$ intersects the curve $V_\varepsilon = \{(x, y) : f(x, y) = \varepsilon\}$ at an inflection point. If the inflection point is non-degenerate then the intersection is transversal (left-hand side).*

2. A brief review of the theory of functions on plane curves.

Given a holomorphic function-germ $f : U \to \mathbb{C}$ defined on an affine neighbourhood U of $\mathbb{C}P^2$, the ideal generated by a Hessian determinant Δ_f of f is well defined. Consequently, it makes sense to study the restriction of f to the curve of equation $\Delta_f = 0$. In this section, we recall basic facts of the study of functions on plane curves, due to Goryunov [9].

2.1. Notations.

Throughout the paper, the following notations will be used:

1) $\mathcal{O}$ is the ring of holomorphic function-germs of the type $g:(\mathbb{C}^2,0) \to (\mathbb{C},0)$ where $\mathbb{C}^2$ denotes the ordered pairs of complex numbers.

2) $\mathcal{O}^*$ is the group of units of $\mathcal{O}$, that is, the holomorphic function-germs of $\mathcal{O}$ that do not vanish at the origin.

3) $\mathcal{M}$ is the maximal ideal of $\mathcal{O}$, that is, the holomorphic function-germs of $\mathcal{O}$ that vanish at the origin.

4) $\mathcal{M}^k$ is the k-th power of the maximal ideal $\mathcal{M}$ of $\mathcal{O}$.

5) $\mathrm{Diff}(k)$ is the group of biholomorphic map-germs $\varphi:(\mathbb{C}^k,0) \to (\mathbb{C}^k,0)$.

6) $\mathrm{diff}(k)$ is the set of vector-field germs $v:(\mathbb{C}^k,0) \to (\mathbb{C}^k,0)$ that vanish at the origin.

7) $\mathrm{GL}(2,\mathcal{O})$ is the group of 2×2 invertible matrices with coefficients in $\mathcal{O}$.

8) T is the subgroup of $\mathrm{GL}(2,\mathcal{O})$ consisting of upper triangular matrices of the type

$$\begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix}, \quad \alpha \in \mathcal{O},\ \beta \in \mathcal{O}^*.$$

9) If $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ denotes an element of the group $\mathrm{GL}(2,\mathcal{O})$ then, for $g = (g_1, g_2) \in (\mathcal{O}\times\mathcal{O})$, the notations $A\times g$ stands for $(\alpha g_1 + \beta g_2, \gamma g_1 + \delta g_2)$.

10) The product of two elements $A, B \in \mathrm{GL}(2,\mathcal{O})$ is denoted by $A\times B$.

2.2. The $\mathcal{G}$-equivalence relation.

The action of the groups $\mathrm{Diff}(2)$ and $\mathrm{GL}(2,\mathcal{O})$ on $\mathcal{O}\times\mathcal{O}$ induces a semi-direct product group structure on the product $\mathrm{Diff}(2)\times \mathrm{GL}(2,\mathcal{O})$. This semi-direct product is given by the formula

$$(1) \qquad (\varphi, A)\cdot(\varphi', A') = \big(\varphi\circ\varphi', A\times(A'\circ\varphi)\big).$$

This group, denoted $\mathcal{K}$, is sometimes called the contact-group but we shall not use this terminology. We call it the *V-equivalence group.*

DEFINITION 2.1. — Two map-germs $\tilde{f}, \tilde{g} : (\mathbb{C}^2,0) \to (\mathbb{C}\times\mathbb{C},0)$ are called *V-equivalent* provided that they are in the same orbit under the action of the V-equivalence group $\mathcal{K}$.

Similarly, the set $\mathrm{Diff}(2) \times T$ is endowed with a semi-direct product group structure.

DEFINITION 2.2. — The *group* $\mathcal{G}$ is the set $\mathrm{Diff}(2) \times T$ with the group structure defined by formula (1) above.

DEFINITION 2.3. — Two holomorphic map-germs $\tilde{f} = (f, E_f)$ and $\tilde{g} = (g, E_g) : (\mathbb{C}^2, 0) \to (\mathbb{C}\times\mathbb{C}, 0)$ are called *$\mathcal{G}$-equivalent* provided that there exists a biholomorphic map germ $\psi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $(\psi \circ f, E_f)$ and $\tilde{g}$ are in the same orbit under the action of the group $\mathcal{G}$.

2.3. The finite determinacy theorem for $\mathcal{G}$-equivalence.

In this subsection $\tilde{f} = (f, E_f) : (\mathbb{C}^2, 0) \to (\mathbb{C} \times \mathbb{C}, 0)$ denotes a holomorphic map-germ.

DEFINITION. — The *extended tangent space* to the map germ $\tilde{f} = (f, E_f)$, denoted $T\tilde{f}$, is the $\mathcal{O}$-submodule of $(\mathcal{O} \times \mathcal{O})$ generated by the 4 map-germs

$$\partial_x \tilde{f},\ \partial_y \tilde{f},\ (E_f, 0),\ (0, E_f).$$

Remark. — The extended tangent space to $\tilde{f}$ is the tangent space to the $\mathcal{G}$-orbit of $\tilde{f}$ at the point $\tilde{f} \in (\mathcal{O} \times \mathcal{O})$ to which we have added (for convenience only) the $\mathbb{C}$ vector space generated by $\partial_x \tilde{f}$ and $\partial_y \tilde{f}$.

DEFINITION 2.5. — The *$\mathcal{G}$-Milnor number* of the holomorphic map-germ $\tilde{f}$, denoted $\mu_{\mathcal{G}}(\tilde{f})$, is the complex codimension of the vector-space $T\tilde{f}$ in $(\mathcal{O} \times \mathcal{O})$:

$$\mu_{\mathcal{G}}(\tilde{f}) = \dim_{\mathbb{C}}\big[(\mathcal{O} \times \mathcal{O})/T\tilde{f}\,\big].$$

Remark. — When no confusion is possible we simply write $\mu_{\mathcal{G}}$ instead of $\mu_{\mathcal{G}}(\tilde{f})$.

The following theorem is the *finite determinacy theorem* for $\mathcal{G}$-equivalence. It can be proved along the same lines as the standard finite determinacy theorem (see [7] and [16], [19], [15] for the standard theorem) or using Damon's general theory [6].

THEOREM 2.1. — *Assume that $\mu_{\mathcal{G}}(\tilde{f}) < +\infty$. Then for any map-germ $\psi = (\psi_1, \psi_2)$ such that $\psi_1, \psi_2 \in \mathcal{M}^{\mu_{\mathcal{G}}+2}$, $\tilde{f} + \psi$ is $\mathcal{G}$-equivalent to $\tilde{f}$.*

2.4. Versal deformation theory for $\mathcal{G}$-equivalence.

In this subsection $\tilde{f} = (f, E_f) : (\mathbb{C}^2, 0) \to (\mathbb{C} \times \mathbb{C}, 0)$ denotes a holomorphic map-germ.

DEFINITION 2.6. — A holomorphic map-germ $\widetilde{F} : (\mathbb{C}^k \times \mathbb{C}^2, 0) \to (\mathbb{C} \times \mathbb{C}, 0)$ such that $\widetilde{F}(0, .) = \tilde{f}$ is called a *deformation of* $\tilde{f}$.

DEFINITION 2.7. — A deformation $\widetilde{G} : (\mathbb{C}^r \times \mathbb{C}^2, 0) \to (\mathbb{C} \times \mathbb{C}, 0)$ of $\tilde{f}$ is *induced* from a deformation $\widetilde{F} : (\mathbb{C}^k \times \mathbb{C}^2, 0) \to (\mathbb{C} \times \mathbb{C}, 0)$ of $\tilde{f}$ by a holomorphic map-germ $h : (\mathbb{C}^r, 0) \to (\mathbb{C}^k, 0)$, denoted $\widetilde{G} = h^* \widetilde{F}$, provided that the equality $\widetilde{G}(\lambda, .) = \widetilde{F}(h(\lambda), .)$ holds.

Given a deformation $\widetilde{F} : (\mathbb{C}^k \times \mathbb{C}^2, 0) \to (\mathbb{C} \times \mathbb{C}, 0)$ of $\tilde{f}$ and a holomorphic map-germ $\psi : (\mathbb{C}^k \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$, we denote by $\psi_* F$ the deformation

$$(\lambda, p) \longrightarrow \big(\psi(\lambda, F_1(\lambda, p)), F_2(\lambda, p)\big).$$

DEFINITION 2.8. — Two deformations $\widetilde{F}, \widetilde{G} : (\mathbb{C}^k \times \mathbb{C}^2, 0) \to (\mathbb{C} \times \mathbb{C}, 0)$ are called *$\mathcal{G}$-equivalent* provided that there exist holomorphic map-germs $\varphi : (\mathbb{C}^k \times \mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, $A : (\mathbb{C}^k, 0) \to T$, $\psi : (\mathbb{C}^k \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that the following three conditions hold:

1) $\psi^* F = A \times (G \circ \varphi)$,

2) $\psi(0, .)$ is a biholomorphic map-germ.

3) $\varphi(0, .)$ is a biholomorphic map-germ.

DEFINITION 2.9. — A deformation $\widetilde{F}$ is called *$\mathcal{G}$-versal* if any other deformation of the same germ is $\mathcal{G}$-equivalent to a deformation induced from $\widetilde{F}$.

Denote respectively by $(\lambda_1, \dots, \lambda_k)$ and (x, y) the coordinates in $\mathbb{C}^k$ and $\mathbb{C}^2$.

Let $\widetilde{F} : (\mathbb{C}^k \times \mathbb{C}^2, 0) \to (\mathbb{C} \times \mathbb{C}, 0)$ be a deformation of $\tilde{f} = (f, E_f)$.

DEFINITION 2.10. — *The tangent space* to the deformation $\widetilde{F}$, denoted $T\widetilde{F}$, is the sum of the following $\mathbb{C}$-vector spaces:

- the $\mathcal{O}$-module generated by the four map-germs $\partial_x \tilde{f}$, $\partial_y \tilde{f}$, $(E_f, 0)$, $(0, E_f)$,

- the $\mathbb{C}$-vector space generated by the restriction to $\lambda = 0$ of the $\partial_{\lambda_i}\widetilde{F}$'s,
- the $\mathbb{C}$-vector space generated by $(1,0)$.

The following theorem is the *versal deformation theorem for $\mathcal{G}$-equivalence.* The proof is given in [7] using standard arguments (see [16], [19], [15]). Damon's general theory [6] of geometrical subgroups can also be applied.

THEOREM 2.2. — *A deformation $\widetilde{F} : (\mathbb{C}^k \times \mathbb{C}^2, 0) \to (\mathbb{C} \times \mathbb{C}, 0)$ of the map-germ $\tilde{f} = \widetilde{F}(0,.)$ is $\mathcal{G}$-versal provided that $T\widetilde{F} = (\mathcal{O} \times \mathcal{O})$.*

3. Normal forms and versal deformations with respect to inflections.

3.1. Normal forms with respect to inflections.

We keep the notations than those of Subsection 1.2.

To the holomorphic function $f : U \to \mathbb{C}$ is associated the Hessian curve

$$X = \{p \in U : \Delta_f(p) = 0\}$$

and a map

$$\xi[f] : X \longrightarrow \mathbb{C}, \quad p \longmapsto f(p)$$

whose fibre at $\varepsilon \in f(U)$ consists of the inflection points and the singular points of $f^{-1}(\varepsilon)$.

In order to find a local normal form of the function f with respect to inflections, we can forget the local projective structure on U and study the map $\xi[f]$ up to analytic equivalence. Of course, the problem of defining analytical equivalence for the map $\xi[f]$ is not straightforward because X is not necessarily smooth. This problem is solved (at least for germs) by applying $\mathcal{G}$-equivalence to the map-germ $(f, \Delta_f) : (\mathbb{C}^2, 0) \to \mathbb{C} \times \mathbb{C}$.

DEFINITION 3.1. — Two holomorphic function-germs $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ are called *$\mathcal{P}$-equivalent*(1) if the map-germs $(f, \Delta_f) : (\mathbb{C}^2, 0) \to (\mathbb{C} \times \mathbb{C}, 0)$ and $(g, \Delta_g) : (\mathbb{C}^2, 0) \to (\mathbb{C} \times \mathbb{C}, 0)$ are $\mathcal{G}$-equivalent.

Remark 1. — The definition of $\mathcal{G}$-equivalence and the first part of Proposition 1.1 imply that the $\mathcal{P}$-equivalence class of a holomorphic

(1) The letter $\mathcal{P}$ stands for Plücker.

function-germ is a projective invariant *i.e.*, it is invariant under projective transformations that preserve the origin.

Remark 2. — The finite determinacy theorem for $\mathcal{G}$-equivalence implies that any holomorphic function-germ $f:(\mathbb{C}^2,0)\to(\mathbb{C},0)$ such that (f,Δ_f) has a finite $\mathcal{G}$-Milnor number is $\mathcal{P}$-equivalent to a polynomial.

3.2. $\mathcal{P}$-versal deformations.

Let $f:(\mathbb{C}^2,0)\to(\mathbb{C},0)$ be a holomorphic function-germ.

Definition 3.2. — A *deformation* of the function-germ f is a function-germ $F:(\mathbb{C}^k\times\mathbb{C}^2,0)\to(\mathbb{C},0)$ such that $F(0,.)=f$.

Given a deformation F of f, we denote by Δ_F the Hessian determinant with respect to the variables $(x,y)\in\mathbb{C}^2$.

Definition 3.3. — A deformation $F:(\mathbb{C}^k\times\mathbb{C}^2,0)\to(\mathbb{C},0)$ is called *$\mathcal{P}$-versal* if for any other deformation $G:(\mathbb{C}^r\times\mathbb{C}^2,0)\to(\mathbb{C},0)$ of f, the deformation (G,Δ_G) of (f,Δ_f) is induced by (F,Δ_F).

Remark. — It is readily verified that this definition is projectively invariant (see Remark 1 following Definition 3.1).

3.3 A local projective Morse lemma.

One of the most basic results of singularity theory is the Morse lemma asserting that for any non-degenerate function-germ $f:(\mathbb{C}^2,0)\to(\mathbb{C},0)$ with a critical point at 0 there exist local coordinates (x,y) in $\mathbb{C}^2$ such that f is given by $f(x,y)=xy$. The non-degeneracy condition means that the second differential of f is a non-degenerate quadratic form.

In the local projective case, we have an analogous result but the non-degeneracy condition is stronger.

Definition 3.4. — A critical point p of a function $f:U\to\mathbb{C}$ is called a *Plücker critical point* if

1) the quadratic differential at p is non-degenerate,

2) the anomalies at p of the branches at p of the curve $\{f=f(p)\}$ are both equal to 0.

The following proposition is a local projective analog of the Morse lemma.

PROPOSITION 3.1. — *Let p be a Plücker critical point of $f : U \to \mathbb{C}$ then the germ of $f - f(p)$ at p is $\mathcal{P}$-equivalent to the function-germ defined by*

$$(\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0), \quad (x, y) \longmapsto xy + x^3 + y^3.$$

3.4. The $\mathcal{P}$-simple function-germs.

Following Arnold, who introduced the modality and the simplicity notions in [1] for the case of critical points of functions, we introduce the notion of modality in Plücker space.

Let $f \in \mathcal{M}^2$ be a holomorphic function-germ such that (f, Δ_f) has finite $\mathcal{G}$-Milnor number, that is, $\mu_{\mathcal{G}}(f, \Delta_f) < +\infty$.

Denote by $j^k f \in J^k\mathcal{M}^2$, the k-jet at the origin of f.

DEFINITION 3.5. — The holomorphic function-germ f has *$\mathcal{P}$-modality* m provided that m is the least number satisfying the following property. For any $k \geq (\mu_{\mathcal{G}}(f, \Delta_f) + 1)$, a sufficiently small neighbourhood of $j^k f \in J^k\mathcal{M}^2$ is the union of the $\mathcal{P}$-equivalence classes of a finite number of m-parameter families of k-jets. If the modality of the function-germ f is equal to 0 then f is called *$\mathcal{P}$-simple.*

Remark. — The $\mathcal{G}$-finite determinacy theorem implies that m does not depend on the choice of k.

THEOREM 3.1. — *For any $k > 3$, the variety of the non $\mathcal{P}$-simple function-germs is of codimension 2 in the space $J^k\mathcal{M}^2$.*

THEOREM 3.2. — *Any $\mathcal{P}$-simple function-germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ with a critical point at 0 is $\mathcal{P}$-equivalent to one of the following function-germs:*

$\mathcal{P}A_1$	$\mathcal{P}A_1^p$	$\mathcal{P}A_1^{p,q}$	$\mathcal{P}A_2$
$xy + x^3 + y^3$	$xy + x^{p+3} + y^3$	$xy + x^{p+3} + y^{q+3}$	$y^2 + x^3$
$p\tau = 0$	$p\tau = p$	$p\tau = p + q$	$p\tau = 1$

where p, q are strictly positive integers. Here $p\tau$ denotes the Plücker-Tyurina number of f (see Section 3.5 below).

This theorem is proved in Section 4.

3.5. Adjacencies of the $\mathcal{P}$-singularity classes.

We keep the notations of the preceding subsection.

DEFINITION 3.6. — A *$\mathcal{P}$-singularity class* is a subset of the space $\mathcal{M}$ which invariant under $\mathcal{P}$-equivalence.

If X is a $\mathcal{P}$-singularity class, we write $f \in X$ if $f:(\mathbb{C}^2,0) \to (\mathbb{C},0)$ is a function-germ belonging to X. Denote by

$$\Sigma[X] = \{f \in \mathcal{M}^2 : f \in X\}$$

the set of function-germs belonging to the $\mathcal{P}$-singularity class X.

DEFINITION 3.7. — The *Plücker-Tyurina number* of a function-germ $f \in X$, denoted $p\tau(f)$, is the codimension of $\Sigma[X]$ in $\mathcal{M}^2$.

The Plucker-Tyurina numbers of the $\mathcal{P}$-simple function-germs are given in the table of Theorem 3.2.

DEFINITION 3.8. — A $\mathcal{P}$-singularity class L is called *adjacent* to a $\mathcal{P}$-singularity class K, denoted $L \to K$, if every function-germ $f \in L$ can be deformed to a function-germ of class K by an arbitrary small perturbation.

If L is adjacent to K and K is adjacent to J then L is adjacent to J. We simply write

$$L \longrightarrow K \longrightarrow J,$$

omitting the arrow between L and J.

Here is the list of all the adjacencies for the $\mathcal{P}$-simple singularities in the space $\mathcal{M}$.

1) The $\mathcal{P}$-singularity class A_3 denotes the function-germs $f:(\mathbb{C}^2,0) \to (\mathbb{C},0)$ equal to y^2+x^4 up to a biholomorphic change of variables in $\mathbb{C}^2$.

2) The $\mathcal{P}$-singularity class[2] K_j denotes the set of function-germs $g:(\mathbb{C}^2,0) \to (\mathbb{C},0)$ such that the origin is an inflection point of $\{g=0\}$ with anomaly $j+1$.

[2] The letter K refers to Kazarian.

The parenthesis (X) means that the $\mathcal{P}$-singularity class X is not $\mathcal{P}$-simple.

$$\begin{tikzcd}
K_0 & K_1 \arrow[l] & K_2 \arrow[l] & K_3 \arrow[l] & K_4 \arrow[l] & K_5 \arrow[l] & \cdots \arrow[l] \arrow[ul] \\
 & \mathcal{P}A_1 \arrow[ul] & \mathcal{P}A_1^1 \arrow[l] \arrow[ul] & \mathcal{P}A_1^2 \arrow[l] \arrow[ul] & \mathcal{P}A_1^3 \arrow[l] \arrow[ul] & \mathcal{P}A_1^4 \arrow[l] \arrow[ul] & \cdots \arrow[l] \\
 & & \mathcal{P}A_2 \arrow[ul] & \mathcal{P}A_1^{1,1} \arrow[ul] & \mathcal{P}A_1^{1,2} \arrow[l] \arrow[ul] & \mathcal{P}A_1^{1,3} \arrow[l] \arrow[ul] & \cdots \arrow[l] \arrow[ul] \\
 & & & (A_3) \arrow[ul] \arrow[uul] & & \mathcal{P}A_1^{2,2} \arrow[ul] & \cdots \arrow[l] \arrow[ul]
\end{tikzcd}$$

This table should be read as follows:

1) The $\mathcal{P}$-singularity class $\mathcal{P}A_1^{p,q}$, $p \leq q$, is adjacent only to the $\mathcal{P}A_1^{j,k}$'s such that $j \leq p$, $k \leq q$ and to the K_j for which $j \leq q$.

2) The $\mathcal{P}$-singularity class $\mathcal{P}A_2$ is adjacent only to $\mathcal{P}A_1$ (and not to K_1 for instance).

3) The only $\mathcal{P}$-singularity class which is not $\mathcal{P}$-simple and which bounds the list of $\mathcal{P}$-simple singularities is A_3.

The first statement follows from Theorem 3.3. The second statement is elementary (see [7]). The third statement will be proved in Section 4.

3.6. $\mathcal{P}$-versal deformations of germs belonging to $\mathcal{P}A_1^{p,q}$.

THEOREM 3.3. — 1) *The deformation* $F : (\mathbb{C}^{p+q} \times \mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ *defined by the polynomial*

$$F(\alpha,\beta,x,y) = xy + x^{p+3} + y^{q+3} + \sum_{j=1}^{p} \alpha_j x^{j+2} + \sum_{k=1}^{q} \beta_k y^{k+2}$$

is $\mathcal{P}$*-versal.*

2) *The deformation* (F, Δ_F) *is* $\mathcal{G}$*-equivalent to the deformation* $(\alpha,\beta,x,y) \mapsto (P(\alpha,\beta,x,y), xy)$ *where* P *is the polynomial defined by the formula*

$$P(\alpha,\beta,x,y) = x^{p+3} + y^{q+3} + \sum_{j=1}^{p} \alpha_j x^{j+2} + \sum_{k=1}^{q} \beta_k y^{k+2}.$$

The proof of this theorem is given in Section 5.

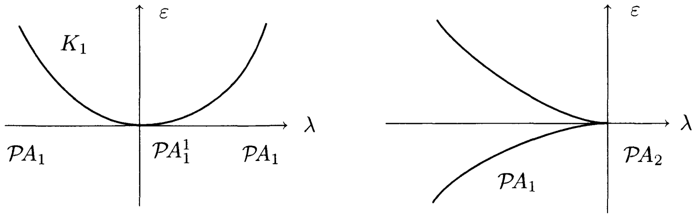

Figure 2. The Plücker discriminant of the families of curves $\mathcal{P}A_1^1 : xy + x^4 + \lambda x^3 + y^3 = \varepsilon$ (left-hand side) and $\mathcal{P}A_2 :\ y^2 + x^3 + \lambda_1 x = \varepsilon$ (right-hand side) for real values of the parameters λ, ε.

In [7], we remarked that the multiplicity of intersection of the Hessian curve $\{\Delta_F = 0\}$ with the plane projective curve $\{F(\lambda, .) = \varepsilon\}$ gives rise to a stratification of the space of the parameters $(\lambda, \varepsilon) \in \mathbb{C}^k \times \mathbb{C}$. The closure of the strata of codimension one is called the *Plücker discriminant.*

The figures labelled from 2 to 4 show the list of the typical singularities of Plücker discriminants for two and three parameter families of functions with $\mathcal{P}$-simple critical points (of course transverse intersections of these Plücker discriminants are allowed). This result is obtained using Theorem 5 and the list of adjacencies of the $\mathcal{P}$-simple singularities.

3.7. Further results.

The results of this paper can be used to compute local topological projective invariants for the $\mathcal{P}$-simple function-germs. Such invariants were defined and computed in [7]. The $\mathcal{P}$-simple function-germs have a $K(\pi, 1)$ property analogous to the $K(\pi, 1)$ property of the complement of the bifurcation diagram of the simple function-germs (see [7]).

The methods of this paper can be carried out for the case of flattening points of curves in $\mathbb{C}P^n$ (*i.e.*, the points of a curve for which the osculating hyperplane has a higher order of tangency than usual). A positive lower bound for the modality was found in [7] for $n > 2$. In particular, there are no $\mathcal{P}$-simple objects in dimension higher than 2. As E. Ghys remarked this lower bound is defined in a way analogous to that of Dirichlet series.

The two starting points of this research where the Plücker formula and the Kazarian theory of flattening point of rational curves (see [11], [13], [12]).

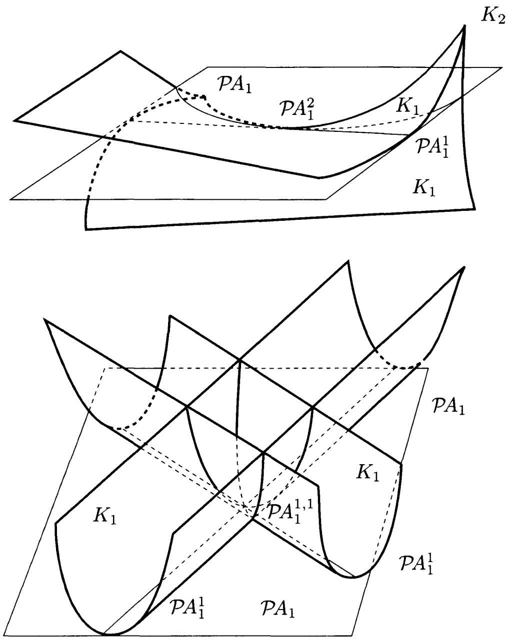

Figure 3. *The Plücker discriminants of the families of curves* $\mathcal{P}A_1^2:\ xy + x^5 + \lambda_1 x^4 + \lambda_2 x^3 + y^3 = \varepsilon$ *and* $\mathcal{P}A_1^{1,1}:\ xy + x^4 + y^4 + \lambda_1 x^3 + \lambda_2 y^3 = \varepsilon$. *A transversal slice of the surface of the left hand side gives two curves intersecting with multiplicity* 4 *at the origin.*

The Plücker formulas were generalized by several authors among which Arnold, Griffiths, Kleimann, Piene, Pohl (see [18] and references therein, [4]).

The Kazarian theory was to some extent treated as a particular of the Legendre mapping deformation theory introduced in [7] and pursued in [8].

This Legendre deformation theory is just an extension of the theory of Legendre mappings developed by Arnold, Zakalyukin and others (see [2], [3], [21]). It enables us to give, for instance, versal deformations of surfaces in $\mathbb{R}P^3$ with respect to parabolic curves and special parabolic points.

The Ph. D. thesis of R. Uribe [20] contains also much interesting material on the geometry of curves and surfaces and their relation to symplectic and contact geometries.

E. Ghys raised the interesting question of studying the case where the lines do not come from local projective structure but from a more

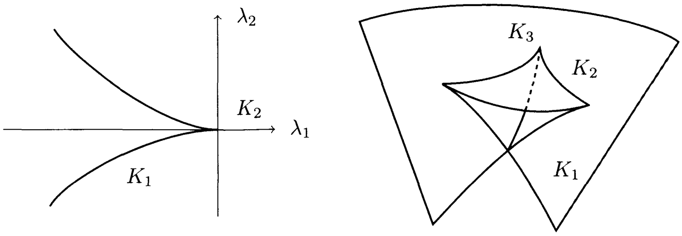

Figure 4. The Plücker discriminants of the families of curves $K_3: y+x^5+\lambda_1 x^3+\lambda_2 x^2$ and $K_4: y+x^6+\lambda_1 x^4+\lambda_2 x^3+\lambda_3 x^2$.

general second order differential equation. In such a case, one might wonder whether there is a relation between the vanishing inflections and the violation of Desargues theorem or not.

4. Proof of Theorem 3.2.

4.1. A preliminary remark.

Let $f:\mathbb{C}^2 \to (\mathbb{C},0)$ be a holomorphic function-germ. Fix a linear coordinate-system x,y in $\mathbb{C}^2$. Denote the Hamilton vector-field of f by

$$X_f = \partial_y f \partial_x - \partial_x f \partial_y.$$

Remark that this vector field depends on the choice of the coordinate system. By a straightforward computation, we get the following proposition.

PROPOSITION 4.1. — *The Hessian determinant of f is equal to the determinant of the 2×2 matrix whose columns are the first and second derivatives of the affine coordinates (x,y) along X_f.*

4.2. Normal form $\mathcal{P}A_1^{p,q}$.

Let $f:(\mathbb{C}^2,0) \to (\mathbb{C},0)$ be a Morse function-germ such that the anomalies at the origin of the branches of $f^{-1}(0)$ are finite. Then there exists a linear coordinate system (x,y) such that f admits the preliminary normal form

$$f(x,y) = xyr(x,y) + x^{3+p}s(x) + y^{3+q}t(y) \tag{2}$$

with $r(0)=s(0)=t(0)=1$ and $p \leq q$. Remark that the numbers p and q are the anomalies at the origin of the branches of $f^{-1}(0)$.

The proof of the following proposition is given in the next three lemmas.

PROPOSITION 4.2. — *The following $\mathcal{G}$-equivalence relation holds:*

$$(f, \Delta_f) \sim (x^{3+p} + y^{3+q}, xy). \tag{3}$$

LEMMA 4.1. — *There exist holomorphic function-germs $a, b : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that the following $\mathcal{G}$-equivalence holds:*

$$(f, \Delta_f) \sim \big(a(x) + b(y), xy\big).$$

Proof. — The Hamilton vector-field of f is of the following form, with $r_1, r_2 \in \mathcal{M}^2$:

$$X_f = (x + r_1)\partial_x - (y + r_2)\partial_y.$$

Using Proposition 4.1, we get by a straightforward computation that the Hessian determinant Δ_f of f is of the type

$$\Delta_f(x, y) = 2xy + r_3$$

where $r_3 \in \mathcal{M}^3$.

The Morse lemma implies that there exists a biholomorphic map-germ

$$\varphi : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$$

such that we have the equality

$$(\Delta_f \circ \varphi)(x, y) = xy. \tag{4}$$

By definition of $\mathcal{G}$-equivalence, we have the $\mathcal{G}$-equivalence relation

$$(f, \Delta_f) \sim (f \circ \varphi, \Delta_f \circ \varphi).$$

The division theorem implies that one can represent $f \circ \varphi$ in the form

$$(f \circ \varphi)(x, y) = a(x) + b(y) + c(x, y)xy,$$

where a, b, c are holomorphic function-germs.

The definition of $\mathcal{G}$-equivalence implies the $\mathcal{G}$-equivalence relation

$$\big(a(x) + b(y) + c(x, y)xy, xy\big) \sim \big(a(x) + b(y), xy\big).$$

The lemma is proved. □

LEMMA 4.2. — *The orders of the holomorphic function-germs a, b of the preceding lemma are respectively equal to $3+p$ and $3+q$, i.e.,*

$$a(x) = a_0 x^{3+p} + o(x^{3+p}), \quad b(y) = b_0 y^{3+q} + o(y^{3+q})$$

with $a_0 b_0 \neq 0$.

Proof. — Put $E(x,y) = xy$. Denote by j (resp. k) the highest number such that $a \in \mathcal{M}^j$ (resp. $b \in \mathcal{M}^k$). That is the first term in the Taylor series of a (resp. b) appearing with a non-zero coefficient is of degree j (resp. k). A priori j or k can be infinite but we shall see that this is not the case.

Denote by

- C_1 (resp. C_2) the branch of the plane curve-germ of equation $f = 0$ tangent to the x-axis (resp. to the y-axis),
- Δ_1 (resp. Δ_2) the branch of the plane curve-germ of equation $\Delta_f = 0$ tangent to the x-axis (resp. to the y-axis).
- $(C_\ell \cdot \Delta_m)$ the multiplicity of intersection at the origin of the curve germs C_ℓ and Δ_m.

We have

$$j = (C_1 \cdot \Delta_1) + (C_2 \cdot \Delta_1), \quad k = (C_1 \cdot \Delta_2) + (C_2 \cdot \Delta_2).$$

The curve-germ C_1 is tangent to the x-axis while Δ_2 is tangent to the y-axis. Hence, their intersection number is equal to

$$(C_1 \cdot \Delta_2) = 1.$$

Similarly, we have the equality

$$(C_2 \cdot \Delta_1) = 1.$$

Consequently the numbers j, k are given by the following system of equations:

$$\begin{cases} j = (C_1 \cdot \Delta_1) + (C_1 \cdot \Delta_2), \\ k = (C_2 \cdot \Delta_1) + (C_2 \cdot \Delta_2). \end{cases} \tag{5}$$

This means that $j = (C_1 \cdot \Delta)$ and $k = (C_2 \cdot \Delta)$.

The preliminary normal form given by formula (2) above implies that the curve-germ C_1 is parameterized by a holomorphic map-germ of the type

$$\begin{cases} x(t) = t, \\ y(t) = -t^{2+p} + o(t^{2+p}). \end{cases} \tag{6}$$

Denote by $\delta \in \mathcal{O}_t$ the holomorphic function-germ of the parameter t obtained by restricting Δ_f to the curve-germ C_1.

The number $(C_1 \cdot \Delta)$ is equal to the degree of the first term in the Taylor series of δ appearing with a non-zero coefficient.

Using the old-fashioned notations, the Hamilton vector field X_f of f is defined by the Hamilton differential equations

$$\begin{cases} \dot{x} = x + m_1(x,y), \\ \dot{y} = -y + m_2(x,y), \end{cases} \tag{7}$$

with $m_1, m_2 \in \mathcal{M}^2$.

The parameterization given in (6) allows us to identify the ring of holomorphic function-germs on C_1 with a subring of $\mathcal{O}_t$. Via this identification, the restriction D of the derivation along the Hamilton vector-field X_f to C_1 is a (holomorphic) derivation of $\mathcal{O}_t$. The first equality of the system of equations (7) implies that

$$Dt = t + o(t). \tag{8}$$

Using (6) and (8), we get the Taylor expansion

$$\delta(t) = \begin{vmatrix} Dt & D(-t^{2+p}) \\ D^2 t & D^2(-t^{2+p}) \end{vmatrix} + o(t^{3+p}). \tag{9}$$

Thus $\delta(t) = (2+p)(1+p)t^{3+p} + o(t^{3+p})$ and consequently

$$(C_1 \cdot \Delta) = 3 + p.$$

The proof that $(C_2 . \Delta) = 3 + q$ differs only in notations. This concludes the proof of the lemma. $\square$

We keep the same notations.

LEMMA 4.3. — *We have the $\mathcal{G}$-equivalence relation*

$$\big(a(x)+b(y),xy\big)\sim\big(x^{3+p}+y^{3+q},xy\big).$$

Proof. — Since $a_0b_0\neq 0$, there exist $\alpha,\beta\in\mathcal{O}^*$ such that the equalities

$$a(x)=\big(\alpha(x)x\big)^{3+p},\quad b(y)=\big(\beta(y)y\big)^{3+q}$$

hold. The biholomorphic map-germ

$$\begin{aligned}\varphi:(\mathbb{C}^2,0)&\longrightarrow(\mathbb{C}^2,0),\\(x,y)&\longmapsto\big(\alpha(x)x,\beta(y)y\big)\end{aligned}$$

induces the $\mathcal{G}$-equivalence relation

$$(x^{3+p}+y^{3+q},xy)\sim\big((\alpha(x)x)^{3+p}+(\beta(y)y)^{3+q},\alpha(x)\beta(y)xy\big).$$

By definition of $\mathcal{G}$-equivalence, we have the $\mathcal{G}$-equivalence relation

$$\big((\alpha(x)x)^{3+p}+(\beta(y)y)^{3+q},\alpha(x)\beta(y)xy\big)\sim\big((\alpha(x)x)^{3+p}+(\beta(y)y)^{3+q},xy\big).$$

This concludes the proof of the lemma and of Proposition 4.2. □

4.3. Quasi-homogeneous filtration and notations.

1) In Arnold's R-classification [1], the notation $f\in A_k$ means that $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ is a holomorphic function-germ such that there exists a coordinate system for which f admits the representation

$$f(x_1,\dots,x_n)=x_1^{k+1}+x_2^2+x_3^2+\cdots+x_n^2.$$

2) In the next subsections, we assume that the reader is acquainted with the notion of quasi-homogeneous filtration (for details see [5]). Given a quasi-homogeneous filtration in $\mathcal{O}$, in order to avoid confusions with the singularity class A_k, we shall denote by $F_d\subset\mathcal{O}$ the $\mathbb{C}$-vector subspace of holomorphic function-germs of order d instead of the notations A_d which is commonly used.

3) We write $f=f_0+\tilde{o}(d)$ if $f,f_0\in\mathcal{O}$ are holomorphic function-germs such that $f-f_0\in F_{d+1}$.

4.4. $\mathcal{P}A_2$ normal form.

PROPOSITION 4.3. — *For any holomorphic function-germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ belonging to the R-singularity class A_2, the two following statements hold:*

1) *The $\mathcal{G}$-Milnor number of (f, Δ_f) is finite.*

2) *We have the $\mathcal{G}$-equivalence relation $(f, \Delta_f) \sim (y^2 + x^3, 3x^4 + 4xy^2)$.*

Proof. — We fix a linear coordinate system (x, y) in $(\mathbb{C}^2, 0)$.

In $\mathcal{O}$, we introduce the quasi-homogeneous filtration for which the weight of the monomial $x^i y^j$ is $2i + 3j$.

A non-degenerate linear map $\alpha : \mathbb{C}^2 \to \mathbb{C}^2$ sends the inflection points of a curve to the inflection points of its image, thus the maps f and $f \circ \alpha$ are $\mathcal{P}$-equivalent. Therefore, without loss of generality, we can assume that $f \in A_2$ admits an expansion of the type

$$f(x, y) = y^2 + x^3 + \tilde{o}(6).$$

LEMMA 4.4. — *The holomorphic function-germ f is $\mathcal{P}$-equivalent to a holomorphic function-germ of the type $y^2 + x^3 + \tilde{o}(7)$.*

Proof. — We have

$$f(x, y) = y^2 + x^3 + cx^2 y + \tilde{o}(7).$$

Moreover

$$y^2 + x^3 + cx^2 y + \tilde{o}(7) = y^2 + \left(x + \frac{c}{3} y\right)^3 + \tilde{o}(7).$$

Put $\alpha(x, y) = (x - \frac{1}{3} cy, y)$. The function-germ $f \circ \alpha$ is of the type

$$(f \circ \alpha)(x, y) = y^2 + x^3 + \tilde{o}(7).$$

This proves the lemma. □

The previous lemma implies that without loss of generality, we can assume that our holomorphic function-germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is of the type

$$f(x, y) = y^2 + x^3 + \tilde{o}(7). \tag{10}$$

LEMMA 4.5. — *The holomorphic function-germ Δ_f admits the expansion $\Delta_f = -18x^4 - 24xy^2 + \tilde{o}(9)$.*

Proof. — The preliminary normal form (10) implies that the Hamilton vector-field X_f of f admits an expansion of the type

$$X_f = \big(2y + \tilde{o}(4)\big)\partial_x - \big(3x^2 + \tilde{o}(5)\big)\partial_y.$$

Using this expansion, we get the lemma by a direct computation. □

Define the holomorphic function-germs $H, E \in \mathcal{O}$ by

$$H(x,y) = y^2 + x^3, \quad E(x,y) = 3x^4 + 4xy^2.$$

We have proved the existence of a $\mathcal{G}$-equivalence relation of the type

$$(f, \Delta_f) \sim \big(H + \tilde{o}(7), E + \tilde{o}(9)\big)$$

with $r \in F_{10}$.

We now prove that the $\mathcal{G}$-equivalence

$$(H + r_1, E + r_2) \sim (H, E). \tag{11}$$

holds for any $r_1 \in F_8,\ r_2 \in F_{10}$. This will conclude the proof of Proposition 4.3.

In order to prove the $\mathcal{G}$-equivalence relation (11), we introduce the $\mathcal{O}$-submodule $\mathcal{T}$ of the extended tangent space to (H, E) defined as follows.

The $\mathcal{O}$-submodule $\mathcal{T}$ of $\mathcal{O} \times \mathcal{O}$ is generated by

- $\{(a\partial_x + b\partial_y) \cdot (H, E) : a, b \in \mathcal{M}^2\}$,
- $(E, 0)$, $(0, xE)$, $(0, yE)$.

Here $v \cdot (H, E)$ denotes the Lie derivative of the map-germ (H, E) along the vector-field v.

Next, consider the $\mathcal{O}$-module M defined by

$$M = \big\{(g, h) \in \mathcal{O} \times \mathcal{O} :\ g \in F_8,\ h \in F_{10}\big\}.$$

LEMMA 4.6. — *The $\mathcal{G}$-equivalence $(H + r_1, E + r_2) \sim (H, E)$ holds for any $r_1 \in F_8,\ r_2 \in F_{10}$ provided that M is contained in $\mathcal{T}$.*

Proof. — The proof is standard. First we prove the assertion for formal power series and then use the $\mathcal{G}$-finite determinacy theorem.

The induction step is as follows.

Assume that we have proved the $\mathcal{G}$-equivalence relation

$$(H + r_1, E + r_2) \sim (H + m_1, E + m_2) \tag{12}$$

with $m_1 \in F_{d_1}$ and $m_2 \in F_{d_2}$, for some pair $d_1 \geq 8$, $d_2 \geq 10$.

Assertion. — The $\mathcal{G}$-equivalence relation (12) implies that

$$(H + r_1, E + r_2) \sim \big(H + \tilde{o}(d_1), E + \tilde{o}(d_2)\big).$$

We prove this assertion.

The inclusion $M \subset \mathcal{T}$ implies that (m_1, m_2) admits a representation of the type

$$(m_1, m_2) = a(E, 0) + b(0, E) + v \cdot (H, E) \tag{13}$$

with $a, b \in \mathcal{O}$, $v \in \operatorname{diff}(2)$.

Put $v(x,y) = v_1(x,y)\partial_x + v_2(x,y)\partial_y$.

Define the biholomorphic map-germ $\varphi\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ by

$$\varphi(x,y) = \big(x - v_1(x,y), y - v_2(x,y)\big)$$

and the matrix $A \in \mathrm{GL}(2, \mathcal{O})$ by

$$A = \begin{pmatrix} 1 & -a \\ 0 & 1-b \end{pmatrix}.$$

With the notations of Subsection 2.2, we get the equality

$$\begin{aligned} &A \times (H + m_1, E + m_2) \circ \varphi \\ &\quad = (H, E) - v.(H, E) - (aE, bE) + (m_1, m_2) + \big(\tilde{o}(d_1), \tilde{o}(d_2)\big). \end{aligned}$$

Using the equality (13), we get that

$$A \times (H + m_1, E + m_2) \circ \varphi = (H, E) + \big(\tilde{o}(d_1), \tilde{o}(d_2)\big).$$

This proves the assertion formulated above.

This assertion implies that for any k, there exists a map-germ $\psi = (\psi_1, \psi_2) \in (\mathcal{O} \times \mathcal{O})$ with $\psi_1 \in \mathcal{M}^k$, $\psi_2 \in \mathcal{M}^k$ such that the following $\mathcal{G}$-equivalence holds:

$$(H + r_1, E + r_2) \sim ((H, E) + \psi).$$

The inclusion $M \subset \mathcal{T}$ implies that the $\mathcal{G}$-Milnor number of (H, E) is finite. Thus, the $\mathcal{G}$-finite determinacy theorem (Theorem 2.2) implies that

$$\big((H, E) + \psi\big) \sim (H, E)$$

provided that k is chosen big enough. This concludes the proof of the lemma. □

The following lemma concludes the proof of Proposition 4.3.

LEMMA 4.7. — *The $\mathcal{O}$-module M is contained in $\mathcal{T}$.*

Proof. — Denote by $\mathcal{M}M$ the $\mathcal{O}$-submodule of M defined by

$$\mathcal{M}M = \big\{am : a \in \mathcal{M},\ m \in M\big\}.$$

Let $\pi : M \to (M/\mathcal{M}M)$ be the canonical projection. The Nakayama lemma (see [15]) implies that the following two equalities are equivalent:

1) $M = M \cap \mathcal{T}$,

2) $\pi(M) = \pi(M \cap \mathcal{T})$.

The $\mathbb{C}$-vector space $\pi(M)$ is generated by the images under π of the eight following map-germs

$$\begin{array}{llll} (x^4, 0), & (xy^2, 0), & (0, x^5), & (0, x^2y^2), \\ (x^3y, 0), & (y^3, 0), & (0, x^4y), & (0, x^3y). \end{array}$$

Direct computations show that the images under π of the eight following map-germs of $M \cap \mathcal{T}$ are linearly independent:

$$\begin{array}{llll} (E, 0), & (0, xE), & x^2\partial_x(H, E), & xy\partial_y(H, E), \\ (0, yE), & xy\partial_x(H, E), & y^3\partial_y(H, E), & x^3y\partial_y(H, E). \end{array}$$

Thus, the equality $\pi(M) = \pi(M \cap \mathcal{T})$ holds. The lemma is proved and so is Proposition 4.3. □

4.5. A_3 is not $\mathcal{P}$-simple.

Arnold's classification implies that any holomorphic function-germ with a critical point at the origin of critical value equal to zero which is neither in A_1 nor in A_2 is adjacent to A_3 (see [1], [5]).

Next, the function-germs in A_1 which do not belong to a $\mathcal{P}A_1^{p,q}$ for some value of p, q form a set of infinite codimension in $\mathcal{M}^2$. Consequently, in order to prove that there are no other $\mathcal{P}$-simple singularity classes than the $\mathcal{P}A_1^{p,q}$'s and $\mathcal{P}A_2$ it is sufficient to prove the following proposition.

PROPOSITION 4.4. — *Any holomorphic function-germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ such that $f \in A_3$ is not $\mathcal{P}$-simple.*

Proof. — That $f \in A_3$ means that there exists a biholomorphic map-germ $\varphi\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that

$$H = f \circ \varphi, \quad \text{with} \quad H(x, y) = y^2 + x^4.$$

A non-degenerate linear map sends the inflection points of a curve to the inflection points of its image. Hence, without loss of generality we can assume that

$$(D\varphi)(0) = \mathrm{Id}.$$

Consider the one-parameter family f_λ of function-germs f defined as follows. Let $\varphi_\lambda(x, y) = \varphi(x, y) + (0, \lambda x^2)$ and put $f_\lambda = H \circ \varphi_\lambda^{-1}$.

We denote by $\mathcal{D}[\varphi_\lambda]$ the determinant of the 2×2 matrix whose lines are the first and second derivative of φ_λ along the Hamilton vector-field of H.

The following lemma will not be proved. It can be obtained by a direct computation.

LEMMA 4.8. — *The holomorphic map-germ $(H, \mathcal{D}[\varphi_\lambda])$ is $\mathcal{G}$-equivalent to $(f_\lambda, \Delta_{f_\lambda})$.*

The proof of Proposition 4.4 is based on the three following assertions:

- *Assertion* 1: if f_a is $\mathcal{P}$-equivalent to f_b then $a^2 = b^2$.
- *Assertion* 2: if $(H, \mathcal{D}[\varphi_a])$ is V-equivalent to $(H, \mathcal{D}[\varphi_b])$ then $a^2 = b^2$.

• *Assertion* 3: if f_a is $\mathcal{P}$-equivalent to f_b then $(H, \mathcal{D}[\varphi_a])$ is V-equivalent to $(H, \mathcal{D}[\varphi_b])$.

Assertion 1 implies that λ is a modulus. Hence the modality of f is at least 1 provided that assertion 1 is proved.

Assertion 3 follows from Lemma 4.8 and from the fact that the $\mathcal{G}$-equivalence group is a subgroup of the V-equivalence group $\mathcal{K}$.

It remains to prove assertion 2.

Define the quasi-homogeneous weight of $x^i y^j$ to be $i + 2j$.

LEMMA 4.9. — *The map-germ $(H, \mathcal{D}[\varphi_\lambda])$ is V-equivalent to a holomorphic map-germ of the form $(H, \frac{7}{6}(c+\lambda)y^3 + x^6 + \tilde{o}(6))$ with $c \in \mathbb{C}$.*

Proof. — The holomorphic map-germ $\varphi\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ admits an expansion of the form

$$\varphi = \big(x + \tilde{o}(1), y + cx^2 + \tilde{o}(2)\big).$$

Hence $\varphi_\lambda : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is of the form

$$\varphi_\lambda(x, y) = \big(x + \tilde{o}(1), y + (\lambda + c)x^2 + \tilde{o}(2)\big).$$

Consequently, we get the expansion

$$\mathcal{D}[\varphi_\lambda] = (\lambda + c)[x, x^2] + [x, y] + \tilde{o}(6),$$

where the brackets are equal to the following 2×2 determinants:

$$[x, y] = \begin{vmatrix} 2y & -4x^3 \\ -12x^3 & -24x^2 y \end{vmatrix}, \quad [x, x^2] = \begin{vmatrix} 2y & 4xy \\ -7x^3 & 8y^2 - 16x^4 \end{vmatrix}.$$

Denote by $\equiv$ the V-equivalence relation.

We substitute y^2 by $-x^4$ in $[x, y]$ and x^4 by $-y^2$ in $[x, x^2]$. We get the V-equivalence

$$[x, y] + (\lambda + c)[x, x^2] \equiv 24x^6 + 28(\lambda + c)y^3.$$

This proves the lemma. □

Define the family of function-germs E_α depending on the parameter $\alpha \in \mathbb{C}$ by $E_\alpha(x, y) = \alpha y^3 + x^6$.

The next lemma concludes the proof of Proposition 4.4.

LEMMA 4.10. — *If a holomorphic map-germ of the type $(H, E_a + \tilde{o}(6))$ is V-equivalent to a holomorphic map germ of the type $(H, E_b + \tilde{o}(6))$ then $a^2 = b^2$.*

Proof. — Denote the two holomorphic map-germs by $(H, E_a + r_1)$ and $(H, E_b + r_2)$ with $r_1, r_2 \in F_7$.

Assume that there exists an invertible 2×2 matrix A with elements in $\mathcal{O}$ and biholomorphic map-germs $g : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, $\psi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that

$$A \times (H, E_a + r_1) \circ g = (\psi \circ H, E_b + r_2). \tag{14}$$

Remark that this equation is in fact a system of two equations. We shall call them the first and second equation of the system (14).

The matrix A is of the type

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with $\alpha, \beta, \gamma, \delta \in \mathcal{O}$.

Write

$$g(x, y) = \big(mx + \tilde{o}(1), nx + py + qx^2 + \tilde{o}(2)\big), \quad m, p \in \mathbb{C} \setminus \{0\}.$$

Equating the terms of quasi-homogeneous weight 2 in the first equation of the system (14), we get that $n = 0$.

Equating the terms of quasi-homogeneous weight 4 in the first equation of the system (14), we get the equality

$$p^2y^2 + m^4x^4 + 2pqx^2y = c(y^2 + x^4)$$

where c denotes a non-zero constant.

Thus $p = \pm m^2$, $q = 0$. Consequently, the map-germ g admits an expansion of the type

$$g(x, y) = (mx + \tilde{o}(1), \pm m^2y + \tilde{o}(2)).$$

The second equation of the system (14) is of the form

$$\gamma H + \delta(E_a + r_1) = E_b + r_2.$$

Equating successively the terms of weight 4, 5 and 6, we get that $\gamma \in F_3$. Consequently the following equality holds:

$$\delta(0)(\pm a m^6 y^3 + m^6 x^6) = b y^3 + x^6.$$

By identification of the coefficients of x^6 and y^3 in this equality, we get the equalities

$$\delta(0) = m^{-6}, \quad a = \pm b.$$

The lemma is proved. □

This lemma concludes the proof of Proposition 4.4 and that of Theorem 3.2.

5. Proof of Theorem 3.3.

For notational reasons, we assume that p and q are strictly positive integers.

Let $P : (\mathbb{C}^{p+q} \times \mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be the holomorphic function-germ defined by the polynomial

$$P(\alpha, \beta, x, y) = x^{3+p} + y^{3+q} + \sum_{j=1}^{p} \alpha_j x^{2+j} + \sum_{k=1}^{q} \beta_k y^{2+k},$$

with $\alpha = (\alpha_1, \dots, \alpha_p)$, $\beta = (\beta_1, \dots, \beta_q)$.

Denote by

$$G : (\mathbb{C}^k \times \mathbb{C}^2, 0) \to (\mathbb{C}, 0)$$

an arbitrary deformation of the holomorphic function-germ

$$\begin{aligned} f : (\mathbb{C}^2, 0) &\longrightarrow (\mathbb{C}, 0), \\ (x, y) &\longmapsto xy + x^{3+p} + y^{3+q} \end{aligned}$$

and by $\overline{G}$ a representative of G.

The function $\overline{G}(\gamma, .)$ has a Morse critical point in a neighbourhood the origin provided that γ is small enough.

A translation sends an inflection point of a curve to an inflection point of the translated curve. Consequently, we can assume without loss of generality that the Morse critical point of $\overline{G}(\gamma, .)$ is the origin.

LEMMA 5.1. — *The deformation* (G, Δ_G) *is* $\mathcal{G}$*-equivalent to a deformation induced from* $(P(\alpha, \beta, x, y), xy)$.

Proof. — The holomorphic function-germ $f = G(0, .)$ belongs to the $\mathcal{P}$-singularity class $\mathcal{P}A_1^{p,q}$. Thus Proposition 4.2 implies that the following $\mathcal{G}$-equivalence holds:

$$(f, \Delta_f) \sim (x^{3+p} + y^{3+q}, xy). \tag{15}$$

Hence (G, Δ_G) is $\mathcal{G}$-equivalent to a deformation of $(x^{3+p} + y^{3+q}, xy)$.

Define the deformation $A : (\mathbb{C}^{p+q+5} \times \mathbb{C}^2, 0) \to (\mathbb{C} \times \mathbb{C}, 0)$ of (f, Δ_f) by the polynomial mapping

$$A(\mu, \alpha, \beta, x, y) = (P(\alpha, \beta, x, y) + \mu_1 x^2 + \mu_2 y^2 + \mu_3 x + \mu_4 y, xy + \mu_5).$$

Using the $\mathcal{G}$-equivalence (15), we get by direct computations that the tangent space to the deformation A is $TA = (\mathcal{O} \times \mathcal{O})$. Thus, the $\mathcal{G}$-versal deformation theorem (Theorem 2.2) implies that the deformation A is $\mathcal{G}$-versal. Consequently (G, Δ_G) is $\mathcal{G}$-equivalent to a deformation induced from A.

Denote respectively by γ, λ the parameters of the deformations G and P. We have $\lambda = (\alpha, \beta) \in \mathbb{C}^{p+q}$. Put $\mu = (\mu_1, \dots, \mu_5)$. We use the old-fashioned notation

$$\lambda = \lambda(\gamma), \quad \mu = \mu(\gamma)$$

for a map inducing the deformation equivalent to (G, Δ_G) from A. Lemma 5.1 is a consequence of the following lemma.

LEMMA 5.2. — *The map-germ* μ *vanishes identically.*

Proof. — Recall that $\overline{G}$ denotes a representative of the germ G.

For γ small enough, the function $\overline{G}(\gamma, .)$ has a Morse critical point at the origin. Consequently, Proposition 4.2, implies that for any γ small enough:

1) $\Delta_{\overline{G}}(\gamma, .)$ has a Morse critical point at the origin of critical value 0;

2) the restriction of $\overline{G}(\gamma, .)$ to each branch of the plane curve

$$\{(x, y) \in \mathbb{C}^2 : \Delta_{\overline{G}}(\gamma, x, y) = 0\}$$

has a critical point of the type A_k with $k \geq 2$.

Condition 1) implies that μ_5 vanishes identically. Then, condition 2) implies that $\mu_1, \dots, \mu_4$ also vanish identically. The lemma is proved. □

LEMMA 5.3. — *Assume that* $\forall j \in \{1,\dots,p\}$, $\forall k \in \{1,\dots,q\}$, *the vectors* $(x^{2+j},0)$, $(y^{2+k},0)$ *are contained in the tangent space to the deformation* (G,Δ_G). *Then the* $\mathcal{G}$*-equivalence* $(G,\Delta_G) \sim (P(\alpha,\beta,x,y),xy)$ *holds.*

Proof. — Consider the deformation $B:(\mathbb{C}^{k+5}\times\mathbb{C}^2,0)\to(\mathbb{C}\times\mathbb{C},0)$ of (f,Δ_f) defined by the formula

$$B(\mu,\gamma,x,y) = \big(G(\gamma,x,y)+\mu_1x^2+\mu_2y^2+\mu_3x+\mu_4y, \Delta_G(\gamma,x,y)+\mu_5\big).$$

By direct computations, we get that under the assumptions of the lemma the tangent space to the deformation B is $TB = (\mathcal{O}\times\mathcal{O})$. Thus, the $\mathcal{G}$-versal deformation theorem implies that B is $\mathcal{G}$-versal.

Consequently the deformation

$$\begin{aligned}(\mathbb{C}^2,0) &\longrightarrow (\mathbb{C}\times\mathbb{C},0),\\ (\alpha,\beta,x,y) &\longmapsto (P(\alpha,\beta,x,y),xy)\end{aligned}$$

is induced from a deformation equivalent to B. The same argument as the one given in Lemma 5.2 implies that $(P(\alpha,\beta,x,y),xy)$ is induced from a deformation equivalent to (G,Δ_G).

On the other hand, Lemma 5.2 implies that (G,Δ_G) is induced from a deformation equivalent to $(P(\alpha,\beta,x,y),xy)$. We have shown that:

- (G,Δ_G) is induced from a deformation equivalent to $(P(\alpha,\beta,x,y),xy)$;
- $(P(\alpha,\beta,x,y),xy)$ is induced from a deformation equivalent to (G,Δ_G).

Consequently (G,Δ_G) and $(P(\alpha,\beta,x,y),xy)$ are $\mathcal{G}$-equivalent. The lemma is proved. □

By definition of $\mathcal{P}$-versality, these two lemmas imply that a function-germ G satisfying the conditions of Lemma 5.3 is a $\mathcal{P}$-versal deformation. It remains to find such a deformation of f.

We assert that the deformation $F:(\mathbb{C}^{p+q}\times\mathbb{C}^2,0)\to(\mathbb{C},0)$ defined by the formula

$$F(\alpha,\beta,x,y) = xy + x^{3+p} + y^{3+q} + \sum_{j=1}^{p}\alpha_jx^{2+j} + \sum_{k=1}^{q}\beta_ky^{2+k},$$

satisfies the assumptions of Lemma 5.3.

We show that the tangent space to the deformation (F, Δ_F) contains the $(x^{2+j}, 0)$'s, for $j \leq p$. The proof that the $(y^{2+k}, 0)$'s for $k \leq q$ are contained in the tangent space to the deformation (F, Δ_F) differs only in notations.

We fix an integer $j \leq p$ and in order to avoid too many indices we put $\alpha_j = \tau$. We call $\lambda = (\alpha, \beta)$ the parameter of the deformation F.

Final assertion: the vector $(x^{2+j}, 0)$ is contained in the tangent space to the deformation (F, Δ_F).

Denote by M the $\mathcal{O}$-module

$$M = \{(0, m) : m \in \mathcal{M}^2\}.$$

LEMMA 5.4. — *The $\mathcal{O}$-module M is contained in the tangent space to the deformation (F, Δ_F).*

Proof. — The function-germ $f = F(0, .)$ belongs to the $\mathcal{P}$-singularity class $\mathcal{P}A_1^{p,q}$. Thus Proposition 4.2 implies that the $\mathcal{G}$-equivalence relation

$$(f, \Delta_f) \sim (x^{3+p} + y^{3+q}, xy)$$

holds.

Hence the tangent space to the deformation (F, Δ_F) contains the extended tangent space to the map-germ

$$(x, y) \longmapsto (x^{3+p} + y^{3+q}, xy).$$

It is readily verified that the extended tangent space to this map-germ contains the $\mathcal{O}$-module M. This proves the lemma. □

LEMMA 5.5. — *The restriction to $\lambda = 0$ of the function-germ $\partial_\tau \Delta_F$ belongs to $\mathcal{M}^2$.*

Proof. — The Hamilton vector-field of $F(\lambda, .)$ is

$$h_\lambda = \big(x + r_1(\lambda, x, y)\big)\partial_x - \big(y + r_2(\lambda, x, y)\big)\partial_y$$

with $r_1(\lambda, .), r_2(\lambda, .) \in \mathcal{M}^2$.

A direct computation shows that this equality implies that for any value of λ, we have $\Delta_F \in \mathcal{M}^2$. This proves the lemma. □

We conclude the proof of the final assertion stated above.

By definition of the tangent space to a deformation, the restriction to $\tau = 0$ of the map-germ

$$\partial_\tau(F, \Delta_F)$$

belongs to $T(F, \Delta_F)$. Thus Lemma 5.4 and Lemma 5.5 imply that the restriction of the map-germ

$$(\partial_\tau F, 0) = \partial_\tau(F, \Delta_F) - (0, \partial_\tau \Delta_F)$$

to $\tau = 0$ belongs to $T(F, \Delta_F)$. Since $(\partial_\tau F, 0) = (x^{2+j}, 0)$ this concludes the proof of the final assertion and the proof of Theorem 3.3.

Acknowledgements. — This work owes his existence to V.I. Arnold who taught to me singularity theory and posed the initial problem from which originated this research. The author is also deeply grateful to M.E. Kazarian for his constant help and interest.

Finally, the author acknowledges Yu. Barishnikov, R. Uribe, V.M. Zakaliukine, V.V. Goryunov, E. Ferrand and E. Ghys for their help.

BIBLIOGRAPHY

[1] V.I. ARNOLD, Normal forms for functions near degenerate critical points, the Weyl groups A_k, D_k, E_k and Lagrangian singularities, Funct. Anal. Appl., 6-4 (1972), 254–272.

[2] V.I. ARNOLD, Mathematical methods of classical mechanics, Nauka, 1974; English transl. Springer, 1978, 462 p.

[3] V.I. ARNOLD, Wave front evolution and the equivariant Morse lemma, Comm. Pure Appl. Math., 29-6 (1976), 557–582.

[4] V.I. ARNOLD, Vanishing inflexions, Funct. Anal. Appl., 18-2 (1984), 51–52.

[5] V.I. ARNOLD, V.I. VASSILIEV, V.V. GORYUNOV & O.V. LYASHKO, Singularity theory I, dynamical systems VI, VINITI, Moscow; English transl. Springer-Verlag, 1993, 245 p.

[6] J. DAMON, The unfolding and determinacy theorems for subgroups of $\mathcal{A}$ and $\mathcal{K}$, Memoirs of the AMS, vol. 50, 1984.

[7] M. GARAY, Théorie des points d'aplatissement évanescents des courbes planes et spatiales, Thèse, Université de Paris 7, février 2001 (in English).

[8] M. GARAY, On simple families of functions, Preprint Mainz Universität, 2002, 25 pp.

[9] V.V. GORYUNOV, Vector fields and functions on discriminants of complete intersections and bifurcation diagrams of projections, Funct. Anal. Appl., 15 (1981), 77–82.

[10] P. GRIFFITHS & J. HARRIS, Principles of algebraic geometry, Wiley Interscience, 1978.

[11] M.E. KAZARIAN, Singularities of the boundary of fundamental systems, flattening of projective curves, and Schubert cells, Itogi Nauli Tekh., Ser. Sovrem Probl. Math., Noviejshie dostizh., 33 (1988), 215–234; English transl. J. Soviet Math., 52 (1990), 3338–3349.

[12] M.E. KAZARIAN, Bifurcations of flattening and Schubert cells, Advances Soviet Math. (AMS), I (1990), 145–156.

[13] M.E. KAZARIAN, Flattening of projective curves, singularities of Schubert stratifications of Grassmann and flag varieties, bifurcations of Weierstrass points of algebraic curves, Usp. Mat. Nauk, 46-5 (1961), 79–119; English transl. Russ. Math. Surveys, 46-5 (1992), 91–136.

[14] F. KLEIN, Development of mathematics in the XIXth century, Lie groups history, frontiers and applications, vol. 9, Math. Sci. Press, Massachussets, 1979, 630 p.

[15] J. MARTINET, Singularities of smooth functions and maps, London Math. Soc. Lecture Notes Series, vol. 58, Cambridge University Press, 1982.

[16] J. MATHER, Stability of C^∞ mappings, I–VI, Ann. of Math., 87 (1968), Ann. of Math., 89 (1969), Pub. Sci. IHES, 35 (1969), 37 (1970), Adv. in Math., 4 (1970), Lecture Notes in Math. 192 (1971).

[17] J. PLÜCKER, System der Analytischen Geometrie, Gessam. Wissen. Abhand., vol. Band 1, vol. 2, B.G. Teubner, 1834, 1898.

[18] R. PIENE, Numerical characters of a curve in projective n-space, Real and complex singularities, 1977, 475–495.

[19] G.N. TYURINA, Locally semi-universal plane deformations of isolated singularities in complex space, Math. USSR Izv., 32-3 (1968), 967–999.

[20] R. URIBE, Singularités symplectiques et de contact en géométrie différentielle des courbes et des surfaces, Thèse, Université de Paris 7, 2001 (in English).

[21] V.M. ZAKALYUKIN, Lagrangian and Legendrian singularities, Funct. Anal. Appl., 10-1 (1976).

Manuscrit reçu le 22 janvier 2001,
révisé le 17 septembre 2001,
accepté le 15 novembre 2001.

Mauricio D. GARAY,
Université Paris 7
UFR de Mathématiques
Case 7012 – 2 place Jussieu
75005 Paris (France).
garay@math.jussieu.fr