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ON CERTAIN HOMOTOPY ACTIONS OF GENERAL LINEAR GROUPS ON ITERATED PRODUCTS

by R. LEVI and S. PRIDDY

1. Introduction.

In this paper we consider splitting iterated products of H-spaces after one suspension. Several important advances in homotopy theory [3], [6], [5] have used wedge decompositions or splittings to construct spaces with desirable properties. For example one often seeks spaces whose mod-p cohomology is free over a subalgebra of the Steenrod algebra. Previous work [9], [7], [8] along these lines has considered iterated products of the abelian H-spaces $X = B(\mathbb{Z}/p^r\mathbb{Z})$ and $B(\mathbb{Z}_p)$. Here we show that the same sort of splitting occurs when X satisfies much weaker requirements. As applications we give decompositions for iterated products of $\Omega^2 S^3$, SO(4), and G_2. We note the latter two are not even homotopy commutative.

Throughout this note by an H-space we mean a homotopy associative H-space with homotopy inverses and a two-sided homotopy unit. Let $\mathbb{F}$ be a commutative ring with a unit and let E_* be a homology theory taking values in the category of cocommutative coalgebras over $\mathbb{F}$. Then, restricted to the category of H-spaces, the theory E_* takes values in the category $\mathcal{HA}$ of cocommutative Hopf algebras over $\mathbb{F}$. Let R denote an associative ring with a unit. An H-space X is said to be an E_*-R-module H-space if the following conditions are satisfied:

a) $E_*(X)$ is a commutative Hopf algebra with respect to the Pontryagin product and

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b) there exists a function $\alpha : \mathrm{R} \to \mathrm{Map}(X, X)$, which induces a ring homomorphism $\alpha_* : \mathrm{R} \to \mathrm{End}_{\mathcal{HA}}(E_*(X))$.

Since $E_*(X)$ is assumed to be a commutative and cocommutative Hopf algebra, the set of all Hopf algebra endomorphisms $\mathrm{End}_{\mathcal{HA}}(E_*(X))$ is an associative ring with a unit, where multiplication is given by composition of maps and addition of two endomorphisms f and g is given by the composite

$$E_*(X) \xrightarrow{\Delta} E_*(X) \otimes E_*(X) \xrightarrow{f \otimes g} E_*(X) \otimes E_*(X) \xrightarrow{\mu} E_*(X).$$

On a more intuitive level, an E_*-R-module H-space is an H-space whose E_* homology admits a natural structure of an R-module as detailed above.

As we observe below examples of E_*-R-module H-spaces exist in abundance. In particular, if X is an H-space and E_* a homology theory such that $E_*(X)$ is a commutative Hopf algebra then X is an E_*-$\mathbb{Z}$ module H-space. An analogue of this observation for more general rings will be pointed out later in the paper.

If X is an E_*-R-module H-space, then one can use the structure map α to define a pairing $\phi : M_n(\mathrm{R}) \times X^n \to X^n$ with k-th component given by

$$\phi\big((a_{i,j}), (x_1, x_2, \dots, x_n)\big)_k = a_{k,1}x_1 * a_{k,2}x_2 * \cdots * a_{k,n}x_n,$$

where $M_n(\mathrm{R})$ is the ring of all $n \times n$ matrices over R with the usual addition and multiplication. In the definition of ϕ, the symbol $*$ denotes multiplication in the H-structure of X and

$$a_{k,i}x_i = \alpha(a_{k,i})(x_i).$$

This pairing may sometimes be regarded as turning X^n up to homotopy into an $M_n(\mathrm{R})$-module, but generally of course this is not the case. The main observation in this paper is that the pairing ϕ, for an E_*-R-module H-space X, induces an action of $M_n(\mathrm{R})$ on $E_*(X^n)$, thus turning X^n into an E_*-$M_n(\mathrm{R})$-module H-space. This idea is then applied to obtain new functorial splittings of iterated products of E_*-R-module H-spaces after a single suspension and localization with respect to the homology theory E_*.

Before we state our results, one more definition is needed. As we observe below, if X is a homotopy commutative H-space then the subset

$[X,X]_H$ of all homotopy classes of self H-maps of X forms a subgroup of the abelian group $[X,X]$. Furthermore, composition of maps induces a multiplicative structure on this set, which satisfies the appropriate distributivity laws and thus endows $[X,X]_H$ with the structure of an associative ring with a unit. If X is an H-space, which is not necessarily homotopy commutative, then one can still define an associative ring with a unit $\langle X,X\rangle_H$, which is the analogue of $[X,X]_H$ but does not require homotopy commutativity. If X is homotopy commutative then $\langle X,X\rangle_H = [X,X]_H$. The construction of $\langle X,X\rangle_H$ is given in Section 2.

DEFINITION 1.1. — *Let* R *be an associative ring with a unit and let* X *be an* H*-space. We say that* X *is an* R*-module* H*-space if there exists a ring homomorphism*

$$\alpha : \mathrm{R} \longrightarrow \langle X,X\rangle_H.$$

In particular, we observe below that every H-space is a $\mathbb{Z}$-module H-space. The main result of this paper may be interpreted as saying that whenever X is an R-module H-space, the question whether for a given homology theory E_*, the iterated product X^n is an E_*-$M_n(\mathrm{R})$ module H-space depends only on whether or not $E_*(X)$ is a commutative Hopf algebra.

A statement of our main results comes next. The main result of the paper is the following

THEOREM 1.2. — *Let* X *be an* R*-module* H*-space and let* E_* *be a homology theory such that* $E_*(X)$ *is a commutative Hopf algebra. Then the pairing*

$$\phi_n : M_n(\mathrm{R}) \times X^n \longrightarrow X^n$$

with k-th component given by

$$\phi\big((a_{ij}),(x_1,x_2,\cdots,x_n)\big)_k = a_{k,1}x_1 * a_{k,2}x_2 * \cdots * a_{k,n}x_n$$

makes X^n *into an* E_**-*$M_n(\mathrm{R})$*-module* H*-space.*

If X satisfies the conditions of the theorem and E_* is a homology theory such that $E_*(X)$ is a commutative Hopf algebra, which as a module is a free module in each degree over the ring $\mathbb{F} = \mathbb{Z}/r\mathbb{Z}$ for some $r \geq 0$, then $E_*(X^n)$ becomes a module over the semigroup algebra $\mathbb{F}[M_n(\mathrm{R})]$.

Here $M_n(\mathrm{R})$ is considered as a semigroup (a unital associative monoid) with respect to matrix multiplication. Then a decomposition of the identity element as a sum of orthogonal idempotents in this algebra gives rise to a corresponding splitting of a suitable localization of X^n. More precisely, we have

THEOREM 1.3. — *Let $\mathbb{F}$ denote $\mathbb{Z}/r\mathbb{Z}$ for some $r \geq 0$. Let X be an E_*-R-module H-space and assume that $E_*(X)$ is a free $\mathbb{F}$-module in each degree. Then any decomposition of the identity element in $\mathbb{F}[M_n(\mathrm{R})]$ by orthogonal idempotents, $1 = \sum_{i=1}^k e_i$ gives rise to a splitting*

$$L_E\big(\Sigma(X^n)\big) \simeq E_1(X) \vee E_2(X) \vee \ldots \vee E_k(X)$$

where L_E denotes the Bousfield localization functor with respect to the homology theory E_*. Moreover, this splitting is natural with respect to H-maps defined on X.

Finally we consider a specific example, namely when E_* is given by ordinary singular homology with coefficients in $\mathbb{F}_p$. Restrict to $\mathbb{F}_p[\mathrm{GL}_n(\mathbb{F}_p)]$. For the class of spaces under consideration, homology localization coincides with Bousfield-Kan p-completion and we obtain the following

THEOREM 1.4. — *Let H_* denote ordinary mod-p homology theory and let X be an H_*-$\mathbb{Z}_p$-module H-space such that for some $r > 0$ the p^r power map induces the zero map on $H_*(X^n)$. Then any decomposition of the identity element in $\mathbb{F}_p[\mathrm{GL}_n(\mathbb{F}_p)]$ by orthogonal idempotents, $1 = \sum_{i=1}^k e_i$, gives rise to a splitting*

$$\Sigma(X^n)^\wedge_p \simeq E_1(X) \vee E_2(X) \vee \ldots \vee E_k(X).$$

Moreover, this splitting is natural with respect to H-maps defined on X.

The paper is organized as follows. Section 2 is devoted to a general discussion of E_*-R-module H-spaces. We show that if X is an R module H-space and E_* is a homology theory such that $E_*(X)$ is a commutative Hopf algebra, then X is an E_*-R-module H-space. In Section 3 we prove Theorem 1.2. In Section 4 we apply these observations to prove Theorems 1.4 and 1.3. In Section 5 we describe examples which arise from considering an orthogonal idempotent decomposition of the identity in the group algebra $\mathbb{F}_2[\mathrm{GL}_2(\mathbb{F}_2)]$.

2. E_*-R-module H-space.

If X is an H-space then the set $[X, X]$ of pointed homotopy classes of self maps of X is a group, which is generally non-commutative. Furthermore, in this group the product (induced by the H-structure) of two H-maps is not generally an H-map. However, if X is homotopy commutative then the product of two H-maps is again an H-map. Let $[X, X]_H \subseteq [X, X]$ denote the subset of all classes of self maps of X, which are H-maps up to homotopy. The following proposition shows that, in the case where X is homotopy commutative, this subset is in fact a subgroup of $[X, X]$ and furthermore, that composition of maps induces a multiplicative structure, turning it into an associative ring with a unit.

PROPOSITION 2.1. — *Let X be a homotopy commutative H-space. Then the set $[X, X]_H$ has the structure of an associative ring with a unit, where multiplication is given by composition and addition is induced by the H-space structure on X.*

Proof. — Since composition of H-maps is an H-map, one has a unital multiplication operation on $[X, X]_H$, induced by composition. Let f and g be self H-maps of X and consider the composite

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f\times g} X \times X \xrightarrow{\text{mult}} X,$$

which we denote by $f + g$. The claim that $f + g$ is an H-map amounts to showing that the external rectangle in the diagram

$$\begin{tikzcd}
X\times X \arrow[r, "\Delta\times\Delta"] \arrow[dd, "\mu"'] & (X\times X)\times(X\times X) \arrow[r, "(f\times g)\times(f\times g)"] \arrow[d, "1\times T\times 1"'] & (X\times X)\times(X\times X) \arrow[r, "\mu\times\mu"] \arrow[d, "1\times T\times 1"'] & X\times X \arrow[dd, "\mu"'] \\
& (X\times X)\times(X\times X) \arrow[r, "(f\times f)\times(g\times g)"] \arrow[d, "\mu\times\mu"'] & (X\times X)\times(X\times X) \arrow[d, "\mu\times\mu"'] & \\
X \arrow[r, "\Delta"] & X\times X \arrow[r, "f\times g"] & X\times X \arrow[r, "\mu"] & X,
\end{tikzcd}$$

where μ denotes the H-space multiplication map, Δ is the diagonal map and T denotes the twist map, commutes up to homotopy. But, the left and top center squares in this diagram commute strictly for obvious reasons. The bottom center square commutes up to homotopy because both f and g are assumed to be H-maps up to homotopy. Finally the right square commutes since the assumption that X is homotopy commutative can be rephrased as saying that the multiplication map μ is itself an H-map up to homotopy, which is equivalent to saying that this square homotopy commutes. This shows that $[X,X]_H$ is closed under addition and multiplication.

To prove that it is in fact a ring, we must prove the two distributivity laws hold. That is, if f,g,h are self H-maps of X, then we must show that the diagram

$$\begin{tikzcd} X \arrow[r, "\Delta"] & X \times X \arrow[r, "f\times g"] & X \times X \arrow[r, "\mu"] \arrow[d, "h\times h"] & X \arrow[d, "h"] \\ X \arrow[u, "h"] \arrow[r, "\Delta"] & X \times X \arrow[u, "h\times h"'] & X \times X \arrow[r, "\mu"] & X \end{tikzcd}$$

commutes up to homotopy, which is obvious. In particular the right square homotopy commutes because h is assumed an H-map. □

The above proposition motivates the following definition. A more general version of it will appear later.

Definition 2.2. — *Let X be a connected homotopy commutative H-space. Let* R *be an associative ring with a unit. We say that X is an* R*-module H-space if there exists a ring homomorphism*

$$\alpha : \mathrm{R} \longrightarrow [X,X]_H.$$

Notice that X is an R-module H-space if an only if there exists a function from R to the subset of all self H-maps of X, such that the obvious requirements with respect to addition, multiplication and distributivity hold up to homotopy.

To motivate the discussion, we consider some obvious examples. Restrict attention first to $\mathrm{R} = \mathbb{Z}$ and assume that X is a homotopy commutative H-space. Then X is an $\mathbb{Z}$-module H-space, since $[X,X]$ in that case has the structure of an abelian group and so $[X,X]_H$ is an associative ring with a unit by Proposition 2.1. The map α is given by sending $n \in \mathbb{Z}$ to the class of the n-th power map on X.

Similarly, assume $\mathrm{R} = \mathbb{Z}_p$, the p-adic integers. Then it is well-known that under appropriate conditions $\mathbb{Z}_p$ acts on p-complete H-spaces up to homotopy. The following version of this is due to A. Bousfield [2].

PROPOSITION 2.3. — *Let X be a connected homotopy commutative H-space. Let W be a pointed connected, CW-complex. If X is p-complete then the abelian group $[W,X]$ is* Ext-*p-complete and hence has a canonical $\mathbb{Z}_p$-module structure.*

Proof. — The proof is analogous to that of [2], Proposition 7.2, and uses induction on the skeleta of W together with a $\lim^1$ argument. It uses the fact that Ext-*p-complete* abelian groups are closed under extensions, cokernels, and arbitrary inverse limits. An Ext-*p-complete* abelian group has a canonical $\mathbb{Z}_p$-module structure by [2], Proposition 4.3. □

As before, if X satisfies the assumptions of the proposition then it is a $\mathbb{Z}_p$ module H-space.

If X is a homotopy commutative H-space and E_* is an a homology theory such that $E_*(X)$ is a commutative and cocommutative Hopf algebra, then there is an obvious ring homomorphism

$$\beta : [X,X]_H \longrightarrow \mathrm{End}_{\mathcal{HA}}\big(E_*(X)\big).$$

Hence if X is in addition an R-module H-space, then the composite

$$\mathrm{R} \xrightarrow{\alpha} [X,X]_H \xrightarrow{\beta} \mathrm{End}_{\mathcal{HA}}\big(E_*(X)\big)$$

gives X the structure of an E_*-R-module H-space.

Our next observation is that X may qualify as an E_*-R-module H-space without being homotopy commutative. Thus assume X is an H-space and E_* is a homology theory such that $E_*(X)$ is a commutative Hopf algebra over a ground ring $\mathbb{F}$. To avoid complications, which may arise if $E_*(X)$ contains $\mathbb{F}$-torsion, we assume throughout that $E_*(X)$ is a free $\mathbb{F}$-module in each dimension. There is a map of sets

$$\beta : [X,X]_H \xrightarrow{\beta} \mathrm{End}_{\mathcal{HA}}\big(E_*(X)\big),$$

which

1) sends composites to composites and

2) sends products of maps to the sum of the induced maps.

Let $[X,X]^{\mathrm{ab}}$ denote the abelianization of the group $[X,X]$, where the group structure in the later is induced by the H-structure on X. Consider the composite

$$[X,X]_H \longrightarrow [X,X] \longrightarrow [X,X]^{\mathrm{ab}}$$

and let $[X,X]_H^{\mathrm{ab}}$ denote the subgroup of $[X,X]^{\mathrm{ab}}$ generated by the image of $[X,X]_H$ under this composite. Then $[X,X]_H^{\mathrm{ab}}$ is an abelian group by construction, where addition is induced by the H-space structure on X and an associative monoid with a unit, where the product operation is induced by composition of maps. This set fails to be a ring only because one of the distributivity laws is not satisfied. Specifically, if f, g and k represent classes of elements in $[X,X]_H^{\mathrm{ab}}$, then since k may be a sum of H maps rather than an H-map itself the rule

$$k(f+g) \simeq kf + kg$$

may not hold. Thus let $\langle X,X\rangle_H$ denote the quotient group of $[X,X]_H^{\mathrm{ab}}$ by the subgroup generated by all elements of the form

$$k(f+g) - kf - kg.$$

Then $\langle X,X\rangle_H$ is an associative ring with a unit. Notice that if X is homotopy commutative then the discussion above implies that $\langle X,X\rangle_H = [X,X]_H$. This motivates the following

DEFINITION 2.4. — *Let X be an H-space and let* R *be an associative ring with a unit. We say that X is an* R*-module H-space if there is a ring homomorphism*

$$\alpha : \mathrm{R} \longrightarrow \langle X,X\rangle_H.$$

From the discussion above the following proposition is straight forward.

PROPOSITION 2.5. — *Let X be an* R*-module H-space and let E_* be a homology theory such that $E_*(X)$ is a commutative Hopf algebra. Then X is an E_*-*R*-module H-space.*

As a particular case we consider the case where E_* is given by ordinary homology with coefficients in a ring.

PROPOSITION 2.6. — *Let* R *be an associative ring with a unit and let X be an* R*-module H-space. Let $\mathbb{F}$ be a commutative ring with a unit such that $H_*(X,\mathbb{F})$ is a commutative Hopf algebra. Then X is an E_*-*R*-module H-space, where $E_*(X) = H_*(X,\mathbb{F})$.*

3. X^n as an $hM_n(\mathrm{R})$ module H-space.

This section is devoted to the proof of Theorem 1.2. Namely, we show that if X is an R-module H-space and E_* is a homology theory such that $E_*(X)$ is a commutative Hopf algebra, then the map ϕ defined in the Introduction turns X^n into an E_*-$M_n(\mathrm{R})$-module H-space. For the convenience of the reader we restate the theorem as

THEOREM 3.1. — *Let X be an* R *module H-space and let E_* be a homology theory such that $E_*(X)$ is a commutative Hopf algebra. Then the pairing*

$$\phi_n : M_n(\mathrm{R}) \times X^n \longrightarrow X^n$$

with k-th component given by

$$\phi\big((a_{ij}),(x_1,x_2,\cdots,x_n)\big)_k = a_{k,1}x_1 * a_{k,2}x_2 * \cdots * a_{k,n}x_n$$

makes X^n into an E_*-M_n(R)*-module H-space.*

Let $\alpha : \mathrm{R} \to \langle X, X\rangle_H$ be the ring homomorphism defining X as an R-module H-space. If $r \in \mathrm{R}$ is any element, denote $\alpha(r)$ by $\hat{r}$ to simplify the notation. Let $M_n(\mathrm{R})$ be the ring of $n \times n$ matrices over R, where product is given by matrix multiplication.

As before, we start by considering the homotopy commutative case. If X is homotopy commutative, then the same holds for X^n for every integer $n > 0$. Hence to prove the theorem in this case it suffices show that X^n is an $M_n(\mathrm{R})$-module H-space, *i.e.* to produce a ring homomorphism

$$\alpha_n : M_n(\mathrm{R}) \longrightarrow [X^n, X^n]_H = \langle X, X\rangle_H.$$

The map ϕ_n defined above certainly induces a map of sets

$$\bar{\phi}_n : M_n(\mathrm{R}) \longrightarrow [X^n, X^n].$$

Thus we must show that $\bar{\phi}_n$ in fact takes values in $[X^n, X^n]_H$ and that it preserves the ring structure on $M_n(\mathrm{R})$.

First notice that $[X^n, X^n]$ is canonically isomorphic as an abelian group to $[X^n, X]^n$ and that a self map of X^n is an H-map if and only if its projection to each factor is an H-map. In the group $[X^n, X]$ there are the homotopy classes of the projections $p_j : X^n \to X$ to the j-th coordinate, which we shall denote by $\hat{p}_j$. It is then easy to observe that for $A = (a_{i,j}) \in M_n(\mathrm{R})$, the class the self map of X^n given by $\bar{\phi}_n(A)$ is represented on the k-th coordinate by

$$\bar{\phi}_n(A)_k = \sum_{j=1}^{k} \hat{a}_{k,j}\,\hat{p}_j,$$

where the sum is induced by the H-space structure on X. Since X is homotopy commutative, a sum of H-maps is again an H-map. This shows that $\bar{\phi}_n$ takes values in $[X^n, X^n]_H$.

To see that $\bar{\phi}_n$ preserves the product structure, let $A, B \in M_n(\mathrm{R})$ be given by $(a_{i,j})$ and $(b_{i,j})$ respectively and let $C = AB$. Then the entries $c_{k,j}$ of C satisfy the equation

$$c_{k,j} = \sum_{m=1}^{n} a_{k,m} b_{m,j}.$$

Thus

$$\bar{\phi}_n(C)_k = \sum_{j=1}^{n} \hat{c}_{k,j}\hat{p}_j = \sum_{j=1}^{n} \Big(\sum_{m=1}^{n} \hat{a}_{k,m}\hat{b}_{m,j} \Big) \hat{p}_j = \sum_{j=1}^{n} \sum_{m=1}^{n} \hat{a}_{k,m}\hat{b}_{m,j}\hat{p}_j.$$

On the other hand, one easily verifies that

$$\big(\bar{\phi}_n(A) \circ \bar{\phi}_n(B)\big)_k = \sum_{m=1}^{n} \hat{a}_{k,m} \sum_{j=1}^{n} \hat{b}_{m,j}\hat{p}_j.$$

Since X is homotopy commutative, the maps $\hat{r}$ can be represented by genuine H-maps. Since composition with H-maps is distributive over addition in the H-space structure up to homotopy, and any two permutations on the order of addition give homotopic maps, the two maps above are homotopic, proving that $\bar{\phi}_n$ preserves the product structure in $M_n(\mathrm{R})$.

The argument that $\mathrm{ad}\,\phi_*$ preserves the additive structure is analogous and will be omitted. This completes the proof of the theorem in the homotopy commutative case.

Now assume that Y an H-space, which is not necessarily homotopy commutative. Let E_* be a homology theory such that $E_*(Y)$ is a commutative Hopf algebra. Then the obvious multiplicative map $[Y,Y]_H \to \mathrm{End}_{\mathcal{HA}}(E_*(Y))$ factors uniquely through a ring homomorphism

$$\psi : \langle Y,Y\rangle_H \longrightarrow \mathrm{End}_{\mathcal{HA}}\big(E_*(Y)\big).$$

Thus in order to prove the theorem in this case, we only need to observe that if X is an R-module H-space then X^n is an $M_n(\mathrm{R})$-module H-space, namely, that there is a ring homomorphism from $M_n(\mathrm{R})$ to $\langle X,X\rangle_H$.

Again, the structure map $\alpha : \mathrm{R} \to \langle X,X\rangle_H$ can be lifted to $\mathrm{Map}(X,X)$. For each $r \in \mathrm{R}$ we fix some self map φ_r of X representing $\alpha(r)$. These maps can be used to define a pairing

$$\phi_n : M_n(\mathrm{R}) \times X^n \longrightarrow X^n.$$

This pairing in turn defines a function

$$\bar{\phi}_n : M_n(\mathrm{R}) \longrightarrow [X^n, X^n]$$

and we must show that it induces the required ring homomorphism.

First consider the elementary matrices $E_{i,j}(r)$, with $r \in \mathrm{R}$ in the (i,j)-th entry and 0 everywhere else. By construction, for each $r \in \mathrm{R}$ the map φ_r can be thought of as a sum of self H-maps of X. Thus for each $r \in \mathrm{R}$, the class $\bar{\phi}_n(E_{i,j}(r))$ can be represented by the composite

$$X^n \xrightarrow{p_j} X \xrightarrow{\varphi_r} X \xrightarrow{\mathrm{inc}_i} X^n$$

where p_j denotes the projection to the j-th coordinate and inc_i inclusion to the i-th coordinate. Since each map in this composite is either an H-map or a sum of H-maps, $\bar{\phi}_n(E_{i,j}(r))$ is an element in the subgroup $[X^n,X^n]_H^{\mathrm{ab}}$. Since every $A \in M_n(\mathrm{R})$ is a sum of such matrices, one has that $\bar{\phi}_n(A)$ is an element in $[X^n,X^n]_H^{\mathrm{ab}}$. This shows that

$$\bar{\phi}_n : M_n(\mathrm{R}) \longrightarrow \langle X^n, X^n\rangle_H$$

is well defined as a map of sets.

Finally, notice that the ring operations are clearly preserved by $\bar{\phi}_n$. This is obvious for composition and follows at once from the definitions for addition. This completes the proof of the theorem in the general case.

4. Splitting X^n after one suspension.

Let X be an E_*-R-module H-space. Then for every n, the Hopf algebra $E_*(X^n)$ is a module over the ring $M_n(\mathrm{R})$. If $\mathbb{F}$ is commutative ring with a unit and $E_*(X)$ is a free $\mathbb{F}$ module in each degree (*e.g.* if $\mathbb{F}$ is a field), then the $M_n(\mathrm{R})$ action can be extended to an action of the semigroup ring $\mathbb{F}[M_n(\mathrm{R})]$ on $E_*(X^n)$, now considered as an $\mathbb{F}$ module rather than a Hopf algebra. Furthermore, if $\mathbb{F} = \mathbb{Z}/r\mathbb{Z}$, $r \geq 0$, and a single suspension is allowed, then the action of $\mathbb{F}[M_n(\mathrm{R})]$ on $E_*(X^n)$ can be realized by maps of spaces, where the suspension coordinate is used to add maps. The following is a restatement of Theorem 1.3.

THEOREM 4.1. — *Let X be an E_*-R-module H-space and assume that $E_*(X)$ is a free $\mathbb{F} = \mathbb{Z}/r\mathbb{Z}$ in each degree for some $r \geq 0$. Then any decomposition of the identity element in $\mathbb{F}[M_n(\mathrm{R})]$ by orthogonal idempotents, $1 = \sum_{i=1}^k e_i$ gives rise to a splitting*

$$L_E\big(\Sigma(X^n)\big) \simeq e_1(X^n) \vee e_2(X^n) \vee \ldots \vee e_k(X^n)$$

where L_E denotes the Bousfield localization functor with respect to the homology theory E_*. Moreover, this splitting is natural with respect to maps of H-spaces.

Proof. — The technique of the proof is completely standard and has been used in the literature. Let $1 = \sum_{i=1}^k e_i$ be an orthogonal idempotent decomposition of 1 in $\mathbb{F}[M_n(\mathrm{R})]$. Each element e_i induces a self map ϵ_i of $\Sigma(X^n)$ using the suspension coordinate for addition. Define $e_i(X^n)$ to be the mapping telescope of ϵ_i. Then one has

$$E_*\big(e_i(X)\big) \cong \operatorname{colim}_{\epsilon_i} E_*(X^n) \cong e_i E_*(X^n).$$

Since the idempotents e_i are orthogonal, the map

$$\Sigma(X^n) \longrightarrow \bigvee_i e_i(X^n)$$

induces an isomorphism

$$E_*\big(\Sigma(X^n)\big) \xrightarrow{\simeq} E_*\Big(\bigvee_i e_i(X^n)\Big)$$

and hence a homotopy equivalence after localization with respect to E_*.

Let X and Y be R-module H-spaces. If $f : X \to Y$ is a map of R-module H-spaces and $A \in M_n(\mathrm{R})$, then showing that $f^n \circ \phi_n(A) \simeq \phi_n(A) \circ f^n$ involves a simple diagram chasing using only the assumption that f is an H-map. Here $f^n : X^n \to Y^n$ means the n-fold product of f. Thus it follows at once that the splitting described in the theorem is natural with respect to maps of R-module H-spaces. □

4.1. *Example: The splitting in $\mathbb{F}_p[\mathrm{GL}_n(\mathbb{F}_p)]$ corresponding to idempotents.* — Fix a prime p and restrict attention to $\mathrm{GL}_n(\mathbb{F}_p) \subseteq M_n(\mathbb{F}_p)$. Recall that the kernel of the reduction $\mathrm{GL}_n(\mathbb{Z}/p^r\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{F}_p)$ is a p-group and hence the kernel of the induced (surjective) map of group rings $\mathbb{F}_p[\mathrm{GL}_n(\mathbb{Z}/p^r\mathbb{Z})] \to \mathbb{F}_p[\mathrm{GL}_n(\mathbb{F}_p)]$ is a nilpotent ideal [4]. Thus one can lift an orthogonal idempotent decomposition

$$e_1 + \cdots + e_N = 1 \in \mathbb{F}_p\big[\mathrm{GL}_n(\mathbb{F}_p)\big]$$

to a corresponding orthogonal decomposition

$$\tilde{e}_1 + \cdots + \tilde{e}_N = 1 \in \mathbb{F}_p\big[\mathrm{GL}_n(\mathbb{Z}/p^r\mathbb{Z})\big].$$

Let $H_*(-)$ denote ordinary mod-p homology theory. Let X be an H_*-$\mathbb{Z}_p$ module H-space. Then $H_*(X^n)$ is a $\mathbb{F}_p[\mathrm{GL}_n(\mathbb{Z}_p)]$ module. In particular if for some $r > 0$ the p^r power map on X induces the zero map on mod-p homology then the action of $\mathrm{GL}_n(\mathbb{Z}_p)$ on $H_*(X^n)$ factors through an action of $\mathrm{GL}_n(\mathbb{Z}/p^r\mathbb{Z})$. Hence $H_*(X^n)$ is a module over the group ring $\mathbb{F}_p[\mathrm{GL}_n(\mathbb{Z}/p^r\mathbb{Z})]$.

Now, let $1 = \sum e_i$ in $\mathbb{F}_p[\mathrm{GL}_n(\mathbb{F}_p)]$ be an orthogonal idempotent decomposition of the identity. By the discussion above, this decomposition lifts to a decomposition of the identity in $\mathbb{F}_p[\mathrm{GL}_n(\mathbb{Z}/p^r\mathbb{Z})]$ by orthogonal idempotents $\tilde{1} = \sum \tilde{e}_i$. Each idempotent $\tilde{e}_i$ can be thought of as an endomorphism of $H_*(X^n)$. Then for each i, choosing an arbitrary lift of each $\tilde{e}_i$ to $\mathbb{Z}[\mathrm{GL}_n(\mathbb{Z}_p)]$ defines an element of the group $[\Sigma X^n, \Sigma X^n]$ which induces the same map on $H_*(X^n)$ as $\tilde{e}_i$. Proceeding as in Theorem 4.1, we obtain Theorem 1.4, which we restate here as

THEOREM 4.2. — *Let X be an H_*-$\mathbb{Z}_p$ module H-space such that for some $r > 0$ the p^r power map induces the zero map on $H_*(X^n)$. Then any decomposition of the identity element in $\mathbb{F}_p[\mathrm{GL}_n(\mathbb{F}_p)]$ by orthogonal idempotents, $1 = \sum_{i=1}^k e_i$, gives rise to a splitting*

$$\Sigma(X^n)^\wedge_p \approx e_1(X) \vee e_2(X) \vee \ldots \vee e_k(X).$$

Moreover this splitting is natural with respect to maps of H-spaces.

5. Examples.

This section is dedicated to a few examples arising from the general linear group $\mathrm{GL}_2(\mathbb{F}_2)$. We do not attempt to analyze the resulting spaces in any depth, but rather to demonstrate some of their homological properties. An important and unusual feature of this example is that the group epimorphism $\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{F}_2)$ induced by the natural projection $\mathbb{Z} \to \mathbb{F}_2$ admits a section. Thus the corresponding splittings which arise exist for any E_*-R-module H-space.

One of our aims in this section will be to compute Poincaré series for the homology of the factors obtained in our splittings. We start by an easy generalization of a well known theorem, due to Molien, on invariants of finite groups on symmetric algebras. This will be useful in the calculations we present later. The examples we look at arise by taking $X = \Omega^2 S^3$, $\mathrm{SO}(4)$ and the Lie group G_2. One feature of all these examples is that the Hopf algebras under consideration are primitively generated, which makes calculations much easier than in the general case.

5.1. *Molien's theorem.* — Let G be a finite group. Let V be a graded $k[G]$-module, where k is a field of characteristic 0. The G-action is of course required to preserve degrees. Let $S[V]$ denote the symmetric algebra of V. Thus $S[V]$ is a tensor product of a polynomial algebra on even dimensional generators and an exterior algebra on odd dimensional generators. Then $S[V]$ inherits a natural G-action. The classical Molien theorem gives the Poincaré series for the ring of invariants $S[V]^G$ in the case V is concentrated in a single even degree. A more general version of the theorem appears in [1]. The case discussed there is where $V = W \oplus dW$, with W in a single even degree and dW in degree $\deg(W) - 1$. Here we observe that Benson's generalization in fact applies to the case of a general symmetric algebra $S[V]$ on a G-module V. The proof is included for the convenience of the reader but we make no claim for originality here.

THEOREM 5.1. — *Let G be a finite group and let k be a field of characteristic* 0. *Let V_i, $i = 1 \ldots n$ be $k[G]$-modules with V_i concentrated in degree d_i. Let V denote the direct sum of the V_i. Then*

$$P_{S[V]^G}(t) = \frac{1}{|G|} \sum_{g \in G} \frac{\prod_{d_i \text{ odd}} \det(1 + gt^{d_i}, V_i)}{\prod_{d_i \text{ even}} \det(1 - gt^{d_i}, V_i)}.$$

Proof. — Let $\pi = |G|^{-1}\sum_{g\in G} g$. Then π projects $S[V]$ onto $S[V]^G$. Thus the dimension of $(S[V]^G)_j$ is equal to the trace of the matrix representing π on $S[V]_j$. As the trace is an additive function we have

$$P_{S[V]^G}(t) = \frac{1}{|G|}\sum_{g\in G}\sum_{j=0}^{\infty} \mathrm{Tr}\big(g, S[V]_j\big)t^j.$$

The rest of the proof consists of evaluation the sum on the right hand side of this equation. Notice that

$$S[V] \cong S[V_1]\otimes S[V_2]\otimes\cdots\otimes S[V_n].$$

For an element $g\in G$, the trace of the matrix of g on the tensor product is the product of the traces on the factors. Thus

$$\mathrm{Tr}\big(g, S[V_1]_{j_1}\otimes S[V_2]_{j_2}\otimes\cdots\otimes S[V_n]_{j_n}\big) = \prod_{i=1}^{n}\mathrm{Tr}\big(g, S[V_i]_{j_i}\big)$$

and we have

$$\begin{aligned}\sum_{j=0}^{\infty}\mathrm{Tr}(g, S[V]_j)t^j &= \sum_{j=0}^{\infty}\sum_{j_1+\cdots+j_n=j}\Big(\prod_{i=1}^{n}\mathrm{Tr}\big(g, S[V_i]_{j_i}\big)\Big)t^j\\ &= \prod_{i=1}^{n}\sum_{j=0}^{\infty}\big(\mathrm{Tr}\big(g, S[V_i]_j\big)t^j\big).\end{aligned}$$

The theorem now follows from Lemma 5.2 below. □

The proof of the following lemma is contained in [1] in sufficient detail and we omit it.

LEMMA 5.2. — *Suppose V is concentrated in degree d. Then*

$$\sum_{j=0}^{\infty}\mathrm{Tr}\big(g, S[V]_j\big)t^j = \begin{cases} 1/\det(1-gt^d, V) & d \text{ even},\\ \det(1+gt^d, V) & d \text{ odd}.\end{cases}$$

5.2. *Orthogonal idempotent splitting of* $\mathbb{F}_2[\mathrm{GL}_2(\mathbb{F}_2)]$. — We proceed by considering examples arising from an orthogonal idempotent decomposition of the identity in $\mathbb{F}_2[\mathrm{GL}_2(\mathbb{F}_2)]$. Such a decomposition is easy to obtain in this case, and we start by recording it.

Consider the elements $\tau = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, $\sigma = \left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$ and $u = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Then the elements

$$e_0 = 1 + \sigma + \sigma^2, \quad e_1 = 1 + u + \sigma + \tau, \quad e_1' = 1 + u + \sigma^2 + \tau$$

are easily verified to be primitive orthogonal idempotents in $\mathbb{F}_2[\mathrm{GL}_2(\mathbb{F}_2)]$.

If X is an H-space and E_* a homology theory such that $E_*(X)$ is a commutative Hopf algebra, then X is an E_*-$\mathbb{Z}$ module H space. Hence X^2 is an E_*-$M_2(\mathbb{Z})$-module H-space, and so $E_*(X^2)$ is a module over the group ring $\mathbb{Z}[\mathrm{GL}_2(\mathbb{Z})]$. Using the right inverse of the projection $\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{F}_2)$ one has that $E_*(X^2)$ is in fact a module over the group ring $\mathbb{Z}[\mathrm{GL}_2(\mathbb{F}_2)]$. Now if $E_*(X)$ takes values in mod-2 vector spaces then $E_*(X^2)$ is a module over $\mathbb{F}_2[\mathrm{GL}_2(\mathbb{F}_2)]$, and the above decomposition of the identity in $\mathbb{F}_2[\mathrm{GL}_2(\mathbb{F}_2)]$ gives rise to a decomposition of $L_E(\Sigma(X \times X))$ into three wedge summands, which we shall denote $e_0(X^2)$, $e_1(X^2)$ and $e_1'(X^2)$ respectively.

Notice that up to the obvious dimension shift $E_*(e_0(X^2))$ is given precisely by the invariants of the action of the unique cyclic subgroup of order 3 in $\mathrm{GL}_2(\mathbb{F}_2)$ on $E_*(X \times X)$. Notice also that conjugation by $\tau\sigma^2$ carries e_1 to e_1'. Thus the eigenspaces of e_1 and e_1' in $E_*(X \times X)$ are isomorphic as vector spaces.

In the calculations below E_* will always be taken to be mod-2 homology. Thus let $H_*(-)$ denote $H_*(-, \mathbb{F}_2)$.

5.3. *Example* 1: $X = \Omega^2\Sigma^3 X$. — A large class of examples propagates from looking at products of spaces of the form $\Omega^2\Sigma^3 X$ for any space X (not necessarily connected). Fix $p = 2$. Then $H_*((\Omega^2\Sigma^3 X)^2)$ is a primitively generated Hopf algebra, which as an algebra is a polynomial algebra on infinitely many pairs of generators corresponding to Dyer-Lashof operations on classes coming from the homology of X. The simplest example of this kind arises by considering $X = S^0$. Thus

$$H_*(\Omega^2 S^3) \cong \mathbb{F}_2\big[x_{2^k-1} \mid k = 1, 2, \ldots\big],$$

where degrees are given by subscripts (x_{2^k-1} is of course the element $Q^{2^{k-1}, 2^{k-2}, \ldots, 2}\iota_1$).

Let $\mathbf{A} = \mathbb{F}_2[x_{2^k-1}, y_{2^k-1} \mid k = 1, 2, \ldots]$. Thus

$$H_*(\Omega^2 S^3 \times \Omega^2 S^3) \cong \mathbf{A}.$$

Let $\mathbf{A}_n = \mathbb{F}_2[x_{2^k-1}, y_{2^k-1} \mid k = 1, 2, \ldots, n]$. Then $\mathbf{A}$ is the colimit of the $\mathbf{A}_n$'s. In order to be able to use the generalized Molien formula to compute the Poincaré series of the splitting, we need to lift σ to an integral (or 2-adic) matrix. Indeed, observe that $\widehat{\sigma} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$ reduces to σ and is an element of order 3. Let $P_n(t)$ denote the Poincaré series for the invariants in $\mathbf{A}_n$ under the action of $\widehat{\sigma}$. By Theorem 5.1

$$\begin{aligned} P_n &= \frac{1}{3} \sum_{i=0}^{2} \frac{1}{\prod_{k=1}^{n} \det(1 - \widehat{\sigma}^i t^{2^k-1})} \\ &= \frac{1}{3} \Big(\frac{1}{\prod_{k=1}^{n} (1 - t^{2^k-1})} + \frac{2}{\prod_{k=1}^{n} (1 + t^{2^k-1} + t^{2^{k+1}-2})} \Big). \end{aligned}$$

Notice that if x is in the submodule of $\mathbf{A}$ given by the homology of $\Omega^2 S^3 \vee \Omega^2 S^3$, then x is not an invariant. Thus the submodule of invariants is in fact a submodule of $H_*(\Omega^2 S^3 \wedge \Omega^2 S^3)$. Obviously the Poincaré series for $e_0((\Omega^2 S^3)^2)$ is the limit of the functions P_n as n goes to infinity. The Poincaré series for the remaining pieces can now easily be computed from the formula above and the known Poincaré series for the homology of $\Omega^2 S^3 \times \Omega^2 S^3$ and $\Omega^2 S^3 \vee \Omega^2 S^3$. The expression is rather complicated and we omit it.

5.4. *Example* 2: $X = \mathrm{SO}(4)$, *the special orthogonal group.* — Slightly more delicate is the following example.

Let $X = \mathrm{SO}(4)$. Then X is not a homotopy commutative H-space, however its mod-2 homology is a commutative Hopf algebra. Thus by the discussion above $H_*(X^2)$ has the structure of an $\mathbb{F}_2[\mathrm{GL}_2(\mathbb{F}_2)]$ module and the corresponding splitting is obtained after a single suspension.

We recall that

$$\mathbf{B} := H^*\big(\mathrm{SO}(4)^{\times 2}\big) = \mathbb{F}_2[s, u, t, v]/(s^4, u^4, t^2, v^2),$$

where $|s| = |u| = 1$, $|t| = |v| = 3$ and $\mathrm{Sq}^i(t) = \mathrm{Sq}^i(v) = 0$ for $i > 0$. Notice that working with cohomology rather than homology here does not make a difference, since the action on homology can be obtained by taking the Hom dual.

The action of $\sigma \in \mathrm{GL}_2(\mathbb{F}_2)$ is given by $s \to u \to s + u$. Similarly $t \to v \to t + v$ since $\mathbf{B}$ is primitively generated. Now by direct calculation (checked with Magma) we find the invariants $e_0\mathbf{B} = \mathbf{B}^{\langle\sigma\rangle}$, the invariants are generated by

$$\boxed{\begin{array}{lll} a_2 = s^2 + su + u^2, & a_3 = s^2u + su^2, & b_3 = s^3 + su^2 + u^3, \\ c_4 = st + tu + uv, & d_4 = sv + tu, & c_5 = s^2t + tu^2 + u^2v, \\ d_5 = s^2v + tu^2, & e_6 = tv, & \end{array}}$$

where the subscripts on generators correspond to the dimensions. Algebra relations are determined by those of $\mathbf{B}$ and are straightforward. A vector space basis for $\mathbf{B}^{\langle\sigma\rangle}$ with Steenrod operations is given by Diagram (1) below. All higher Steenrod operations are zero. Similarly e_1R is given by Diagram (2), where

$$f_6 = (s^3 + u^3)(t + v) + s^2ut + su^2v, \quad g_6 = (s^2u + su^2)(t + v).$$

5.5. *Example* 3: $X = G_2$, *the exceptional Lie group.* — This example is algebraically similar to that of SO(4). Topologically however, they are quite different. With mod-2 coefficients we have

$$\mathbf{C} =: H^*(G_2 \times G_2) = \mathbb{F}_2[s, u, t, v]/(s^4, u^4, t^2, v^2)$$

where $|s| = |u| = 3$, $|t| = |v| = 5$, and $\mathrm{Sq}^2(s) = t$, $\mathrm{Sq}^2(u) = v$, $\mathrm{Sq}^1(t) = s^2$, $\mathrm{Sq}^1(v) = u^2$.

Since $H^*(X)$ is primitively generated, Theorem 4.1 again applies. We have chosen the same symbols for generators as in Example 2. As ungraded rings the $\mathbf{B}$ and $\mathbf{C}$ are isomorphic and the $\mathbb{Z}/3$ action is identical. Thus we find generators for $e_0\mathbf{C} = \mathbf{C}^{\langle\sigma\rangle}$ given by

$$\boxed{\begin{array}{lll} a_6 = s^2 + su + u^2, & a_9 = s^2u + su^2, & b_9 = s^3 + su^2 + u^3, \\ c_8 = st + tu + uv, & d_8 = sv + tu, & c_{11} = s^2t + tu^2 + u^2v, \\ d_{11} = s^2v + tu^2, & e_{10} = tv. & \end{array}}$$

A basis for $e_0\mathbf{C}$ has the same form as that in Example 2 with the elements in different degrees; it is given by Diagram (3) below. We do not list the Steenrod operations as they are much more elaborate than those in Example 2.

Similarly, $e_1\mathbf{C}$ has exactly the same form as that of Example 2.

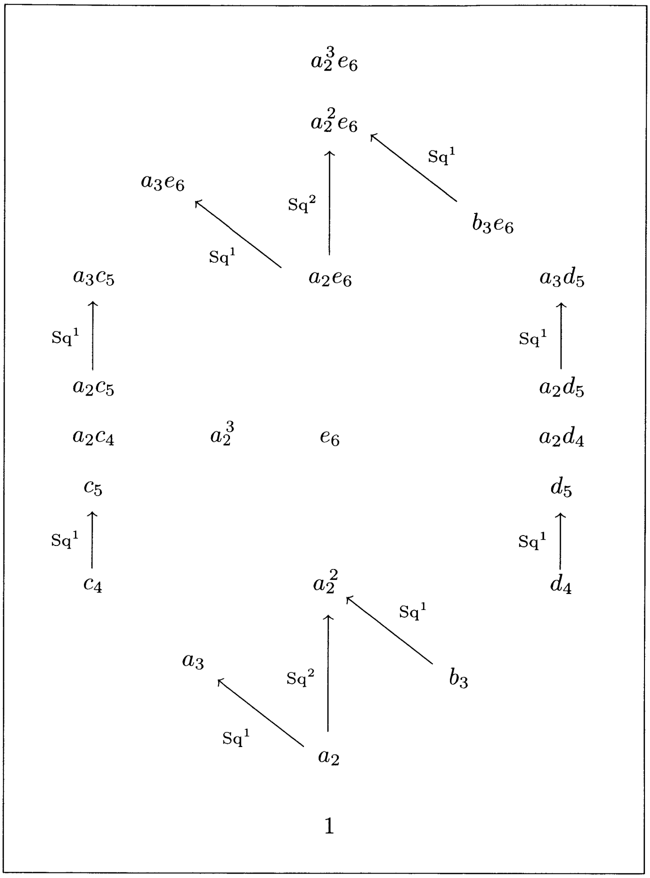

Diagram (1)

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$$
\begin{tikzcd}
 & & s^2tu^2v(s+u) & & \\
 & & stuv(s^2+u^2) \arrow[u, "\mathrm{Sq}^1"] & & \\
(s^3+u^3)tv & & & & s^3u^3(t+v) \\
(s^2+u^2)tv & & & & s^3u^2t+s^2u^3v \\
(s+u)tv \arrow[u, "\mathrm{Sq}^1"] & & s^2u^2(t+v) & & s^3ut+u^3sv \arrow[u, "\mathrm{Sq}^1"] \\
s^3t+u^3v & f_6 \arrow[ur, "\mathrm{Sq}^1"] & & g_6 & \\
s^2t+u^2v & & (s^2+su+u^2)(t+v) \arrow[uu, "\mathrm{Sq}^2"] \arrow[ur, "\mathrm{Sq}^1"] & & s^3u^2+s^2u^3 \\
st+uv \arrow[u, "\mathrm{Sq}^1"] & & & & s^3u+su^3 \arrow[u, "\mathrm{Sq}^1"] \\
s^3+u^3 & & s^2+u^2 & & t+v \\
 & & s+u \arrow[u, "\mathrm{Sq}^1"] & &
\end{tikzcd}
$$

Diagram (2)

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	$a_6^3 e_{10}$	
	$a_6^2 e_{10}$	
a_9c_{11}		a_9d_{11}
a_9e_{10}		b_9e_{10}
	a_9b_9	
a_6c_{11}		a_6d_{11}
	a_6e_{10}	
a_6c_8		a_6d_8
	a_6^2	
c_{11}		d_{11}
	e_{10}	
b_9		a_9
c_8		d_8
	a_6	
	1	

Diagram (3)

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