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A NAKAI-MOISHEZON CRITERION FOR NON-KÄHLER SURFACES

by Nicholas BUCHDAHL

0. Introduction.

In Corollary 15 of [B], the classical Nakai-Moishezon criterion for a compact complex surface X was generalised to yield a characterization of the set of classes in $H^{1,1}_{\mathbb{R}}(X)$ which can be represented by a Kähler form, a result obtained independently by Lamari [L]. Under the assumption that $b_1(X)$ is even, this result was further generalised in Theorem 16 of [B] to the case of $\bar{\partial}\partial$-closed modulo $\bar{\partial}\partial$-exact $(1,1)$-forms. The purpose of this paper is to demonstrate that the assumption on $b_1(X)$ can be dropped entirely. Namely, the following will be proved:

THEOREM. — *Let X be a compact complex surface equipped with a positive $\bar{\partial}\partial$-closed $(1,1)$-form ω and let φ be a smooth real $\bar{\partial}\partial$-closed $(1,1)$-form satisfying $\int_X \varphi \wedge \varphi > 0$, $\int_X \varphi \wedge \omega > 0$ and $\int_D \varphi > 0$ for every irreducible effective divisor $D \subset X$ with $D \cdot D < 0$. Then there is a smooth function g on X such that $\varphi + i\bar{\partial}\partial g$ is positive.*

Theorem 16 of [B] differs from this only in that it assumes $b_1(X)$ is even and that $\int_D \varphi > 0$ for *every* effective divisor $D \subset X$; however, this inequality must hold for any effective divisor D with $D \cdot D \geq 0$ by Proposition 5 of that paper.

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1. Proof of the theorem.

Let X be a compact complex surface. Since the theorem has already been proved in the case of even first Betti number, it will be assumed henceforth that $b_1(X)$ is odd. The same notation as in [B] is employed throughout, so $\Lambda^{p,q}$ denotes the sheaf of germs of smooth (p,q)-forms on X, with $\Lambda^{p,q}(X)$ denoting the global sections. A $\bar\partial\partial$-closed positive $(1,1)$-form $\omega \in \Lambda^{1,1}_{\mathbb{R}}(X)$ is chosen once and for all, its existence guaranteed by Gauduchon's theorem [G].

For any $f \in \Lambda^{1,1}(X)$ there is a function $g \in \Lambda^{0,0}(X)$, unique up to the addition of a constant, such that $\omega \wedge (f + g'')$ is a constant multiple of ω^2 where $g'' := i\bar\partial\partial g$. Since $b_1(X)$ is odd, the proof of Lemma 8 in [B] implies that there is a unique form $\sigma_0 \in \Lambda^{1,1}_{\mathbb{R}}(X)$ with the properties that it is d-exact and satisfies $\omega \wedge \sigma_0 = \omega^2$. The harmonic representative of a closed $(1,1)$-form f on X satisfying $\omega \wedge f = c\omega^2$ for some constant c is then $f - c\,\sigma_0$. This form is anti-self-dual with respect to ω, a manifestation of the fact that the intersection form on $H^2(X,\mathbb{R})$ restricted to $H^{1,1}_{\mathbb{R}}(X)$ is negative definite ([BPV], IV 2.13).

For a holomorphic line bundle L on X, there is a unique hermitian metric on L such that the corresponding hermitian connection has curvature f_L satisfying $\omega \wedge f_L = \mathrm{Const}\cdot\omega^2$. If $s \in \Gamma(X, \mathcal{O}(L))$ is non-zero and E is the associated effective divisor $s^{-1}(0)$, the equation of currents $2\pi\, E = i\, f - i\bar\partial\partial \log|s|^2$ holds by the Poincaré-Lelong theorem ([GH]). Therefore $\int_E \varphi = \frac{i}{2\pi}\int_X f_L \wedge \varphi$ for any smooth $\bar\partial\partial$-closed $(1,1)$-form φ. When the divisor E is given without reference to L, the notation f_E will be used to denote f_L for $L = \mathcal{O}(E)$.

A *real divisor* on X is by definition a finite formal sum of the form $D = \sum_i \nu_i D_i$ where $D_i \subset X$ is an irreducible effective divisor on X and ν_i is a real number; D is *effective* if $\nu_i \geq 0$ for all i, in which case the usual notation $D \geq 0$ is employed; similarly, $D \geq E$ iff $D - E \geq 0$. As for integral divisors, the notation f_D is used to denote $\sum_i \nu_i f_{D_i}$.

The intersection form on $H^2(X,\mathbb{R})$ is denoted by the dot product symbol. Thus $E \cdot E$ is the self-intersection number of an effective divisor

E in X, realised by the integral $-\frac{1}{4\pi^2}\int_X f_E \wedge f_E$. The notation extends by $\mathbb{R}$-linearity to all real divisors, and is further extended to denote the pairing between $\bar\partial\partial$-closed $(1,1)$-forms: $\varphi \cdot \psi := \int_X \varphi \wedge \psi$ for $\bar\partial\partial$-closed $\varphi, \psi \in \Lambda^{1,1}_{\mathbb{R}}(X)$. If $\psi = if_D$ for some real divisor D, the notation $\varphi \cdot D$ may also be used in place of $\frac{1}{2\pi}\varphi \cdot if_D$.

LEMMA 1. — *Let $E \subset X$ be an effective integral divisor such that $E \cdot E = 0$. Then for any $\varepsilon > 0$ there is a smooth function g such that $if_E + g'' \geq -\varepsilon\omega$.*

Proof. — If there is no smooth function g on X such that $if_E + g'' + \varepsilon\omega$ is positive *in a neighbourhood of* E, the Hahn-Banach Theorem implies the existence of a current T and a constant c such that $T(if_E + \varepsilon\omega + g'') \leq c$ for every smooth function g and $T(\psi) > c$ for every smooth 2-form ψ whose $(1,1)$-component is positive in a neighbourhood of E.

It follows immediately that T is a $(1,1)$-current, that $\bar\partial\partial T = 0$, that c must be non-positive, that $T(if_E + \varepsilon\omega) \leq c$, that $T(\psi) \geq 0$ for any smooth $(1,1)$-form ψ which is positive in a neighbourhood of E and finally that the support of T must be contained in E. By Lemma 32 of [HL], it follows that $T = \sum_i h_i E_i$ where h_i is a non-negative constant and $E_1, E_2, \ldots$ are the irreducible components of E. Since $E \cdot E = 0$ and $b_1(X)$ is odd, $[E] = 0$ in $H^2(X, \mathbb{R})$. Hence $E_i \cdot E = 0$ for all i, and this gives a contradiction since then $c \geq T(if_E + \varepsilon\omega) = T(\varepsilon\omega) > c$.

It can therefore be supposed that E is the zero set of a section s of a holomorphic line bundle L which has a hermitian connection whose curvature form f satisfies $if > -\varepsilon\omega$ in an open neighbourhood U of E. After rescaling s if necessary, it can be assumed that $\{x \in X \mid |s(x)| \leq 1\} \subset U$.

Let χ be a smooth convex increasing function on $\mathbb{R}$ such that $0 \leq \chi'(t) \leq 1$ for all t, with $\chi(t) = t$ for $t \geq 0$ and with $\chi(t) = -1$ for $t \leq -1$. Then $i\bar\partial\partial(\chi(\log|s|^2)) = \chi'(\log|s|^2)\, i\bar\partial\partial \log|s|^2 + \chi''(\log|s|^2)\, i\bar\partial(\log|s|^2) \wedge \partial(\log|s|^2) \leq \chi'(\log|s|^2)\, if$, so $if - i\bar\partial\partial(\chi(\log|s|^2)) \geq (1 - \chi'(\log|s|^2))\, if \geq -\varepsilon\omega$, as required. □

Remark. — The above proof also works in some cases when $b_1(X)$ is even. For example, if E is irreducible (with $E \cdot E = 0$), or if every effective divisor on X has non-negative self-intersection.

LEMMA 2. — *Suppose $\psi \in \Lambda^{1,1}_{\mathbb{R}}(X)$ satisfies $\bar\partial\partial\psi = 0$, $\psi \cdot \psi = 0$, $\psi \cdot \omega \geq 0$ and $\psi \cdot D \geq 0$ for every effective divisor $D \subset X$. Then for any*

$\varepsilon > 0$ *there is a smooth function* g *such that* $\psi + g'' \geq -\varepsilon\omega$.

Proof. — By Lemma 7 of [B], ψ can be approximated arbitrarily closely in L^2 norm by forms of the kind $p - g''$ where p is smooth and positive and g is smooth. Following exactly the same argument as used in the proof of Theorem 11 of [B], a sequence of smooth functions g_n and smooth positive $(1,1)$-forms p_n can be found such that $\|\psi + g_n'' - p_n\|_{L^2(\omega)}$ is converging to 0 and g_n is converging in L^1 to define an almost-positive closed $(1,1)$-current $P = g_\infty'' \geq -\psi$. Applying the same arguments as in the proofs of Theorems 11 and 16 in [B] shows that for any given $\varepsilon > 0$ there is a real effective divisor D_ε and a smooth function g_ε such that $-if_{D_\varepsilon} + g_\varepsilon'' \geq -\psi - \varepsilon\omega$. The construction of D_ε is such that it can be assumed that $D_{\varepsilon'} \geq D_\varepsilon$ for $\varepsilon' < \varepsilon$ and the coefficient of an irreducible component common to both D_ε and $D_{\varepsilon'}$ is the same in both.

Now take a sequence of positive numbers ε converging monotonically to 0. Since $\chi_\varepsilon := \varepsilon\omega + \psi - if_{D_\varepsilon} + g_\varepsilon''$ is positive, $0 \leq \chi_\varepsilon \cdot \chi_\varepsilon = \varepsilon^2\,\omega\cdot\omega + 4\pi^2\,D_\varepsilon\cdot D_\varepsilon + 2\varepsilon\,\omega\cdot\psi - 4\pi\varepsilon\,\omega\cdot D_\varepsilon - 2\pi\,\psi\cdot D_\varepsilon$. The hypotheses on ψ and negativity of the intersection form restricted to $H^{1,1}_{\mathbb{R}}(X)$ therefore imply that the cohomology classes $[D_\varepsilon] \in H^2(X,\mathbb{R})$ are uniformly bounded. After passing to a subsequence if necessary, the corresponding sequence of harmonic representatives can be assumed to converge smoothly. Moreover, the inequality $0 \leq \omega\cdot\chi_\varepsilon = \varepsilon\,\omega\cdot\omega + \omega\cdot\psi - 2\pi\,\omega\cdot D_\varepsilon$ implies that the increasing sequence of non-negative numbers $\{\omega \cdot D_\varepsilon\}$ is bounded above and hence converges. Therefore the sequence of forms $\{f_{D_\varepsilon}\}$ converges smoothly to a closed $(1,1)$-form $f_{\mathcal{D}}$ satisfying $f_{\mathcal{D}} \cdot f_{\mathcal{D}} = 0 = \psi \cdot f_{\mathcal{D}}$ and $\omega \wedge i f_{\mathcal{D}} = c\,\omega^2$ for some constant $c \geq 0$. Since $[if_{\mathcal{D}}] = 0$ in $H^2(X,\mathbb{R})$ it follows $if_{\mathcal{D}} = c\sigma_0$.

If $c = 0$, it follows from the fact that $\{\omega \cdot D_\varepsilon\}$ is non-negative and increasing that $\omega\cdot D_\varepsilon = 0$ for all ε; in this case $D_\varepsilon = 0$ for all ε and therefore $\psi + g_\varepsilon'' \geq -\varepsilon\omega$ as required.

If $c > 0$, the identity $\psi \cdot \sigma_0 = 0$ and Proposition 5 of [B] imply that $\psi + g''$ is a non-negative multiple of σ_0 for some smooth function g. If there is a non-zero integral effective divisor E on X such that $E \cdot E = 0$, since $[\sigma_0] = 0$ in $H^2(X,\mathbb{R})$ it follows that $\sigma_0 \cdot E = 0$ and by Proposition 5 of [B] again, that σ_0 is a positive multiple of if_E; in this case, the desired result follows from Lemma 1. If X has algebraic dimension 1, it is well-known that X is an elliptic surface ([BPV], VI 4.1) and therefore such a divisor E exists.

If X has algebraic dimension 0, then by [BPV], IV 6.2, there are only finitely many irreducible curves on X so that for ε sufficiently small, the real divisors D_ε are independent of ε. Hence $f_{\mathcal{D}} = f_D$ for some genuine real effective divisor D on X satisfying $D \cdot D = 0$. By Lemma 4 in §3.5 of Ch. V of [Bou], the symmetric negative semi-definite intersection matrix M associated with the irreducible components of a connected component of D has a 1-dimensional kernel, and the entries in a generating vector $\mathbf{v}$ all have the same sign. Since $\mathbf{v}$ must be a multiple of a column of the cofactor matrix of M, after multiplying by a real constant it has positive integer entries. This implies that there is an effective non-zero integral divisor E on X with $E \cdot E = 0$, so the desired result follows from the previous paragraph. □

The proof of the main theorem can now be completed. Let $\varphi \in \Lambda^{1,1}_{\mathbb{R}}(X)$ be a $\bar\partial\partial$-closed form satisfying the hypotheses of the theorem. By the proof of Theorem 14 of [B], there is a form $u \in \Lambda^{0,1}(X)$ such that $\tilde\varphi := \varphi + \partial u + \bar\partial \bar u$ is positive; (the hypothesis that $b_1(X)$ be even in that theorem is used only in the final sentence of the proof).

By Proposition 5 of [B], $\tilde\varphi \cdot \varphi$ is strictly positive. Let t_0 be the smaller solution of the equation $(\varphi - t_0\tilde\varphi) \cdot (\varphi - t_0\tilde\varphi) = 0$, and set $\psi := \varphi - t_0\tilde\varphi$. Since $(\varphi - t\tilde\varphi) \cdot (\varphi - t\tilde\varphi) > 0$ for t satisfying $0 \leq t < t_0$, the sign of $\omega \cdot (\varphi - t\tilde\varphi)$ cannot change for such t so $\omega \cdot \psi \geq 0$. Since $(\varphi - \tilde\varphi) \cdot (\varphi - \tilde\varphi) = -2\|\bar\partial u\|^2 \leq 0$, it follows that $t_0 \leq 1$ and therefore for any effective divisor $E \subset X$, $\psi \cdot E = (1 - t_0)\, \varphi \cdot E \geq 0$.

The form ψ therefore satisfies the hypotheses of Lemma 2. Applying that lemma, given $\varepsilon > 0$ there is a smooth function g_ε such that $\psi + g_\varepsilon'' \geq -\varepsilon\omega$, so if ε is chosen so small that $t_0\tilde\varphi - \varepsilon\,\omega > 0$, it follows that $\varphi + g_\varepsilon'' > 0$, as required. □

Remark. — The methods of this paper show that if $\varphi \in \Lambda^{1,1}_{\mathbb{R}}(X)$ satisfies the hypotheses of the theorem except for the condition that $\int_E \varphi$ be positive for every effective $E \subset X$ with negative self-intersection, there is an effective real divisor D on X such that $\varphi - i f_D$ is $i\bar\partial\partial$-homologous to a positive form.

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