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QUANTITATIVE ESTIMATES FOR THE GREEN FUNCTION AND AN APPLICATION TO THE BERGMAN METRIC

by K. DIEDERICH and G. HERBORT

1. Introduction.

The pluricomplex Green function $G_D(z, w)$ with a single pole w for bounded pseudoconvex domains $D \subset\subset \mathbb{C}^n$ was introduced in [15]. Its connection with the complex Monge-Ampère operator and other properties were extensively studied in [5]. By a recent result of M. Carlehed, U. Cegrell, F. Wikström [4] one has the following important fact for the case of hyperconvex domains:

THEOREM 1.1. — *Let $D \subset\subset \mathbb{C}^n$ be a bounded hyperconvex domain and let $(w_k) \subset D$ be a sequence with $\lim_{k\to\infty} w_k = w_0 \in \partial D$. Then there is a pluripolar set $E \subset D$ such that for every $z \in D \setminus E$*

$$\overline{\lim_{k\to\infty}}\, G_D(z, w_k) = 0.$$

Since $G_D(\cdot, w)$ has a logarithmic pole at w, it can be used as a weight in Hörmander's $\overline{\partial}$-theory in order to construct holomorphic L^2-functions with certain prescribed properties. This has turned out to be a useful tool in studying the Bergman theory of such domains D, the most recent result

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in this direction being the Bergman completeness of hyperconvex domains which was proved in [2] and [12] using Theorem 1.1.

Considering Theorem 1.1 one might want to ask, whether the exceptional pluripolar set E (which might even depend on the choice of the sequence (w_k) converging to a fixed boundary point $w_0 \in \partial D$) necessarily appears. This seems to be unknown for general bounded hyperconvex domains $D \subset\subset \mathbb{C}^n$. However, under a slightly stronger hypothesis on D, the following statement was proved in [13]:

THEOREM 1.2. — *Let $D \subset\subset \mathbb{C}^n$ be a bounded domain admitting a bounded plurisubharmonic exhaustion function which is Hölder continuous on $\overline{D}$. Then one has for any compact set $K \subset D$ and any point $w_0 \in \partial D$*

$$\lim_{w\to w_0} \sup_K |G_D(\cdot, w)| = 0. \tag{1.1}$$

Remark 1.3. — Notice, that, according to [6], all bounded pseudoconvex domains in $\mathbb{C}^n$ with $\mathcal{C}^2$-smooth boundary satisfy the hypothesis of Theorem 1.2.

It might be useful for future applications to go beyond the qualitative result of Theorem 1.2, by asking under which circumstances (1.1) can be put into a more quantitative form. An indication, that this might be possible can be seen in the result of Carlehed who showed in [3] that one has on any strongly pseudoconvex domain D with smooth $\mathcal{C}^2$-boundary the inequality

$$|G_D(z,w)| \leqslant C\frac{\delta_D(z)\delta_D(w)}{|z-w|^4}$$

for all $z, w \in D$, with a constant $C > 0$. Here, as always in this article, $\delta_D(\cdot)$ denotes the boundary distance in D.

The first goal of this article now is to show

THEOREM 1.4. — *Assume that on the bounded pseudoconvex domain $D \subset\subset \mathbb{C}^n$ there is a bounded plurisubharmonic exhaustion function $\rho : D \to [-1,0)$, such that for some number $0 < \alpha < 1$ and constants $C_1, C_2 > 0$ the inequality*

$$C_2\delta_D^\alpha(z) \leqslant |\rho(z)| \leqslant C_1\delta_D^\alpha(z) \tag{1.2}$$

holds everywhere on D. Then for any fixed number $0 < t \ll 1$ there exists a constant $\tilde{C}_t \gg 1$, such that for all compact subsets $K \subset D$ and all $w \in D$

satisfying

$$\delta_D(w) < \tilde{C}_t^{-1}\delta_D(K)$$

the pluricomplex Green function G_D of D can be estimated by

$$\sup_K |G_D(\cdot, w)| \leqslant \tilde{C}_t\left(\left(\frac{\delta_D^{1-2nt}(w)}{\delta_D(K)}\right)^{\alpha/n}\left(\log\frac{2R_D}{\delta_D(w)}\right)^{1/n} + \delta_D^{t\alpha}(w)|\log\delta_D(w)|\right)$$

where R_D denotes the diameter of D.

Remark 1.5. — Notice, that the second term on the right hand side of this inequality is clearly dominating for w very close to ∂D. However, the first term becomes relevant near the other extreme, namely near $\delta_D(w) = \tilde{C}_t\delta_D(K)$.

This theorem has a useful consequence for the sublevel sets of the Green function, showing that they concentrate near the boundary with a certain speed. Namely, one has

COROLLARY 1.6. — *Let D be as in Theorem (1.4). Then there is for any fixed number $0 < t \ll 1$ a number $0 < \delta_t \ll 1$, such that for all $w \in D$ with $\delta_D(w) < \delta_t$ the sublevel set $A(w, D) := \{z \in D : G_D(z, w) < -1\}$ is contained in the collar $\{z \in D : \delta_D(z) < \delta_D^{1-3nt}(w)\}$.*

In addition, for strongly pseudoconvex domains, we can state a slight improvement of Theorem 1.4, namely:

THEOREM 1.7. — *Suppose $D \subset\subset \mathbb{C}^n$ is a strongly pseudoconvex domain with smooth $\mathcal{C}^2$-boundary. Then there is a constant $C > 0$, such that one has for any compact set $K \subset D$ and any point $w \in D$ with $\delta_D(w) < C^{-1}\delta_D(K)$ the inequality*

$$\sup_K |G_D(\cdot, w)| \leqslant C\frac{\delta_D(w)}{\delta_D(K)}.$$

In particular there exists a number $C_1 > 0$ such that for all $w \in D$ with $\delta_D(w) < C_1^{-1}$ the relation

$$\{z \in D : G_D(z, w) < -1\} \subset \{z \in D : \delta_D(z) < C_1\delta_D(w)\}$$

holds.

In [10] the Bergman distance function d_B on pseudoconvex domains $D \subset \mathbb{C}^n$ was studied that admit a continuous plurisubharmonic exhaustion function ρ which satisfies an estimate of the form

$$C\delta_D^m \leqslant |\rho| \leqslant C^*\delta_D^{1/m}$$

with certain constants $C, C^*, m > 0$. For these domains an effective lower bound was obtained for $d_B(\cdot, z^0)$ (with a fixed $z^0 \in D$). Namely, there exist constants $C, C' > 0$ and $\delta_0 > 0$ such that $d_B(z, z^0) \geqslant C|\log|\log C'\delta_D(z)||$, whenever $\delta_D(z) < \delta_0$. This result applies for example to all those domains, which can be represented as intersections of finitely many C^2-smooth bounded pseudoconvex domains.

We will show in this article two applications of Theorems 1.2 and Theorem 1.4 to questions about the boundary behavior of the Bergman differential metric B_D of D. We have at first:

THEOREM 1.8. — *Let $D \subset\subset \mathbb{C}^n$ be a pseudoconvex domain admitting a bounded uniformly Hölder continuous plurisubharmonic exhaustion function and having the property, that any point $q \in \partial D$ is a peak point for the family $\mathcal{P}(D) := \mathcal{C}^0(\overline{D}) \cap PSH(D)$ of continuous functions on $\overline{D}$ plurisubharmonic on D. Then we have for any $q \in \partial D$ and any vector $X \in \mathbb{C}^n \setminus \{0\}$*

$$\lim_{w \to q} B_D(w; X) = +\infty.$$

Remark 1.9. — Note, that no finite type assumption is made in this theorem. For instance, all regular domains (in the sense of [7]) satisfy the hypothesis of the theorem.

Before stating the second result on the boundary behavior of B_D we remind the reader of the definition of two invariants associated to boundary points of D:

DEFINITION 1.10. — *Let $D \subset\subset \mathbb{C}^n$ be a pseudoconvex domain and $w^0 \in \partial D$ a boundary point such that ∂D is $\mathcal{C}^\infty$-smooth near w^0.*

a) *The boundary ∂D is called pseudoconvexly extendable of an order $m \geqslant 2$ at w^0, if there is an open neighborhood U of w^0 and a C^3-function ρ on U, such that $\rho(w^0) = 0$, the surface $S := \{\rho = 0\}$ is smooth and pseudoconvex from the side $\{\rho < 0\}$ and for some constant $C > 0$ the estimate*

$$-C\Big(\delta_D(z) + |z - w^0|^2\Big) \leqslant \rho(z) < -|z - w^0|^m$$

holds on $D \cap U$.

b) *The number*

$$N(\partial D; w^0) := \inf\{m \geqslant 2 : \partial D \text{ is pseudoconv. ext. of order } m \text{ at } w^0\}$$

is called the order of pseudoconvex extendability of ∂D *at* w^0.

An important analytic local invariant of ∂D at w^0 is

DEFINITION 1.11. — *Let* D, w^0 *be as in Definition 1.10. Then the number*

$$h_D(w^0)$$
$$:= \sup\left\{ s : \liminf_{z \to w^0} \delta_D^s(z)\, B_D(z; X) = \infty \text{ nontangentially}, \forall X \in \mathbb{C}^n \setminus \{0\} \right\}$$

is called the growth exponent of the Bergman metric at w^0.

Remark 1.12. — For further details and properties concerning the order of pseudoconvex extendability and the growth exponent of the Bergman metric see [8] and [9].

Using Theorem 1.4 we can show

THEOREM 1.13. — *Assume that the pseudoconvex domain* $D \subset\subset \mathbb{C}^n$ *has* $\mathcal{C}^\infty$*-smooth boundary near* $w^0 \in \partial D$. *If the type of* ∂D *at* w^0 *is finite, then we have the inequality*

$$h_D(w^0) \geqslant \frac{1}{N(\partial D; w^0)}.$$

The structure of this article is as follows: Section 2 contains the proof of Theorem 1.4, which, in fact, is completely parallel to the proof of Theorem 1.1 from [13]. The short Section 3 then contains what is needed to get Corollary 1.6 and the estimate in the strictly pseudoconvex case as stated in Theorem 1.7. Starting with Section 4 we turn to the boundary behavior of the Bergman metric showing at first the basic comparison lemma (Proposition 4.1) between the Bergman metrics on D and on the sublevel set $A(w, D)$ of the pluricomplex Green function of D. The boundary behavior of such sublevel sets is studied in Section 5. The proof of Theorem 1.8 in Section 6 is then very short. Section 7, finally, contains the proof of Theorem 1.13.

2. Proof of Theorem 1.4.

As said already we can completely follow the pattern of the proof of Theorem 1.1 from [13]. For this we always assume in this section, that the domain D and the exhaustion function ρ with the exponent $0 < \alpha < 1$ have been chosen as in Theorem 1.4. We fix a compact subset $K \subset D$ and introduce two auxiliary functions depending on the choice of points $z \in K$ and $w \in D \setminus K$, a number $\eta_w > 0$, which will be chosen later and an integer $k > 0$. We put

$$U_{z,w}(\zeta) := \max\{G_D(\zeta, z), -\eta_w\} \tag{2.1}$$

and

$$V_{w,k}(\zeta) := \max\{G_D(\zeta, w), -k\}. \tag{2.2}$$

We will study the following three integrals:

$$\begin{aligned} \mathcal{I}_k(z,w) &:= \int_D |V_{w,k}|(dd^c U_{z,w})^n \\ \mathcal{I}'_k(z,w) &:= \int_D |V_{w,k}|^n (dd^c U_{z,w})^n \\ \mathcal{I}(z,w) &:= \int_D |G_D(\cdot, w)|(dd^c U_{z,w})^n. \end{aligned} \tag{2.3}$$

The main load in proving Theorem 1.4 consists in showing suitable upper and lower estimates for the integral $\mathcal{I}(z,w)$ as formulated precisely in the following two lemmas

LEMMA 2.1. — *There is a constant $C^* > 0$ (depending on n and ρ) such that for all $z \in K$ and all $w \in D$ with $\delta_D(w) < \frac{\delta_D(K)}{2}$ the estimate*

$$\mathcal{I}(z,w) \leqslant C^* \eta_w^{1-\frac{1}{n}} \left(\delta_D^{-\alpha}(K) \log \frac{2R_D}{\delta_D(K)} \right)^{1/n} \delta_D^{\alpha/n}(w) \tag{2.4}$$

holds.

In formulating the decisive lower estimate for $\mathcal{I}(z,w)$ we specify the free constant $\eta_w > 0$ from (2.1). We get

LEMMA 2.2. — *If, for $0 < t \ll 1$, the number η_w is defined to be*

$$\eta_w := |\rho(w)|^{-2t} \tag{2.5}$$

then we have for all $z \in K$ and $w \in D$ with $\delta_D(w) < \delta_0(t, K)$ the estimate

$$\mathcal{I}(z,w) \geqslant |G_D(z,w)| - C'_t \delta_D^{t\alpha}(w) |\log \delta_D(w)| \tag{2.6}$$

with some unimportant constant $C_t' > 0$. *Furthermore, the constant* $\delta_0(t,K)$ *is given by*

$$\delta_0(t,K) = \min\left\{ \delta_1(t,\alpha), \frac{C_2}{C_1}\delta_D(K) \right\}$$

with a constant $\delta_1(t,\alpha)$ *not depending on* K *and the constants* C_1, C_2 *from (1.2).*

Before showing these lemmas we explain at first, how Theorem 1.4 follows immediately from them:

Proof of Theorem 1.4. — Notice that one has according to (2.5) and (1.2) the inequality $\eta_w \leqslant C_2^{-2t}\delta_D^{-2t\alpha}(w)$. Hence we get from (2.4) and (2.6)

$$\begin{aligned}
|G_D(z,w)| &\leqslant C_t'\delta_D^{t\alpha}(w)|\log\delta_D(w)| + \mathcal{I}(z,w) \\
&\leqslant C_t'\delta_D^{t\alpha}(w)\left|\log_D\delta(w)\right| + C^*\left(\delta_D^{-\alpha}(K)\log\frac{2R_D}{\delta_D(K)}\right)^{1/n} \eta_w^{1-\frac{1}{n}}\delta_D^{\alpha/n}(w) \\
&\leqslant C_t'\delta_D^{t\alpha}(w)\left|\log_D\delta(w)\right| + C^*\left(\delta_D^{-\alpha}(K)\log\frac{2R_D}{\delta_D(K)}\right)^{1/n} \delta_D^{(1-2nt)\alpha/n}(w).
\end{aligned}$$

If we put, for instance,

$$\tilde{C}_t := \max\left\{ C_t', \frac{C_2}{C_1}, 2\frac{1+R_D}{\delta_1(t,\alpha)} \right\}$$

Theorem 1.4 follows immediately, since $z \in K$ was chosen arbitrarily. □

We now come to the proof of the upper bound for $\mathcal{I}(z,w)$ from Lemma 2.1. The main tool for it is the following lemma by Z. Blocki ([1]):

LEMMA 2.3. — *Assume that* $u, v_1, \ldots, v_n : D \to \mathbb{R}^-$ *are bounded plurisubharmonic functions such that* $u(\zeta) \to 0$ *for* $\zeta \to \partial D$. *Then one has*

$$\int_D |u|^n (dd^c v_1) \wedge \ldots \wedge (dd^c v_n) \leqslant n!\|v_1\|_\infty \cdots \|v_{n-1}\|_\infty \int_D |v_n|(dd^c u)^n.$$

Applying this to $u := V_{w,k}$ and $v_1 = \cdots = v_n = U_{z,w}$ and observing that $\|v_j\|_\infty \leqslant \eta_w$ for all $j = 1, \ldots, n$, we get the inequality

$$\mathcal{I}_k'(z,w) \leqslant n!\eta_w^{n-1}\int_D |U_{z,w}|(dd^c V_{w,k})^n. \tag{2.7}$$

With this we can estimate

$$\begin{aligned}\mathcal{I}_k(z,w) &= \int_D |V_{w,k}|(dd^c U_{z,w})^n \\ &\leqslant \left(\int_D (dd^c U_{z,w})^n\right)^{\frac{n-1}{n}} \left(\int_D |V_{w,k}|^n (dd^c U_{z,w})^n\right)^{\frac{1}{n}} \\ &= (2\pi)^{n-1}(\mathcal{I}'_k(z,w))^{1/n} \\ &\leqslant C_n \eta_w^{1-\frac{1}{n}} \left(\int_D |U_{z,w}|(dd^c V_{w,k})^n\right)^{1/n}.\end{aligned}$$

(Here we used the fact that all the positive measures $(dd^c U_{z,w})^n$ have total mass $(2\pi)^n$, see e.g. Lemma 2.1 in [13].)

So, we have shown, that for all k

$$\mathcal{I}_k(z,w) \leqslant C_n \eta_w^{1-\frac{1}{n}} \left(\int_D |U_{z,w}|(dd^c V_{w,k})^n\right)^{1/n}. \tag{2.8}$$

Now notice, that for $k \to \infty$ one has $|V_{w,k}| \nearrow |G_D(\cdot, w)|$ and, hence, by Beppo Levi,

$$\mathcal{I}_k(z,w) \underset{k\to\infty}{\to} \mathcal{I}(z,w). \tag{2.9}$$

Furthermore, since all $(dd^c V_{w,k})^n$ have the same mass as the measure $(2\pi)^n \delta_w$, where δ_w denotes the Dirac measure at w, they tend for $k \to \infty$ in the weak-star topology to $(2\pi)^n \delta_w$. Hence, we even have

$$\int_D f(dd^c V_{w,k})^n \underset{k\to\infty}{\to} (2\pi)^n f(w)$$

for any bounded continuous function f on D. Applying this to $f := |U_{z,w}|$, we get for $k \to \infty$

$$\int_D |U_{z,w}|(dd^c V_{w,k})^n \to (2\pi)^n |U_{z,w}(w)| \leqslant (2\pi)^n |G_D(w,z)|.$$

Combining this with (2.8) and (2.9), we get the estimate

$$\mathcal{I}(z,w) \leqslant C_n \eta_w^{1-\frac{1}{n}} |G_D(w,z)|^{1/n}. \tag{2.10}$$

This inequality implies Lemma 2.1, since one also has

LEMMA 2.4. — *Let D, ρ, α be as in Theorem 1.4. Then there exists a constant $C_\rho > 0$, such that for any compact set $K \subset D$ the estimate*

$$\sup_{z\in K} |G_D(\zeta, z)| \leqslant C_\rho \left(\delta_D^{-\alpha}(K) \log \frac{2R_D}{\delta_D(K)}\right) \delta_D^\alpha(\zeta) \tag{2.11}$$

holds for all points $\zeta \in D$ with $\delta_D(\zeta) \leqslant \frac{1}{2}\delta_D(K)$.

Proof of Lemma 2.4. — We proceed analogously to [5]. First we note that we have (1.2). Let, now, $x \in K$ be arbitrary and $B_{x,K} := B(x; \frac{\delta_D(K)}{2})$. We put

$$\phi_x(\zeta) := \begin{cases} \max\left\{ C\rho(\zeta), \log \dfrac{|\zeta - x|}{R_D} \right\} & \text{if} \quad \zeta \in D \setminus B_{x,K} \\ \log \dfrac{|\zeta - x|}{R_D} & \text{if} \quad \zeta \in B_{x,K} \end{cases}$$

where

$$C = 2^\alpha C_2^{-1} \delta_D^{-\alpha}(K) \log \frac{2R_D}{\delta_D(K)}.$$

Note that

$$\inf_{\partial B_{x,K}} |\rho| \geqslant C_2 \inf_{\partial B_{x,K}} \delta_D^\alpha \geqslant C_2 2^{-\alpha} \delta_D^\alpha(K).$$

With this choice of C we, therefore, have

$$C\rho = -C|\rho| \leqslant -C \inf_{\partial B_{x,K}} |\rho| \leqslant \log \frac{\delta_D(K)}{2R_D}$$

on $\partial B_{x,K}$.

The above definition of ϕ_x, hence, is consistent, and we obtain a negative plurisubharmonic function ϕ_x on D serving as a candidate for $G_D(\cdot, x)$.

Let now $\zeta \in D$ be a point with $\delta_D(\zeta) \leqslant \frac{\delta_D(K)}{2}$. Then $\zeta \notin B_{x,K}$ and

$$\begin{aligned} |G_D(\zeta, x)| &\leqslant |\phi_x(\zeta)| \leqslant C|\rho(\zeta)| \\ &\leqslant 2^\alpha C_2^{-1} \left(\delta_D^{-\alpha}(K) \log \frac{2R_D}{\delta_D(K)} \right) |\rho(\zeta)| \\ &\leqslant C_\rho \left(\delta_D^{-\alpha}(K) \log \frac{2R_D}{\delta_D(K)} \right) \delta_D^\alpha(\zeta) \end{aligned}$$

with $C_\rho = 2^\alpha \frac{C_1}{C_2}$. This proves Lemma 2.4. □

In order to show the lower bound for $\mathcal{I}(z, w)$ as stated in Lemma 2.2, we use the following result shown in [13]:

LEMMA 2.5. — *Let $D \subset\subset \mathbb{C}^n$ be a hyperconvex domain admitting a continuous plurisubharmonic exhaustion function $\sigma : D \to [-1, 0)$ which is uniformly Hölder continuous on D with exponent $\beta > 0$. Then there is a constant $C > 0$ (depending on β) and for any compact subset $K \subset D$ a* *constant $\delta_0 > 0$ (only depending on K and β), such that one has for any $z \in K$ and $w \in D \setminus K$ with $\delta_D(w) < \delta_0$ the estimate*

$$\sup\left\{ G_D(\zeta, w) : |\zeta - z| < R_D \exp\left(-\sigma^{-2}(w)\right)\right\} \leqslant G_D(z,w) + C|\sigma(w)|\left|\log|\sigma(w)|\right|.$$

As can be seen from [13], the proof of the lemma yields some more precise information on the number $\delta_0(K,\beta)$, namely, δ_0 is of the form

$$\delta_0(K,\beta) = \min\{\delta_1(\beta), \delta_2(K)\} \tag{2.12}$$

where $\delta_1(\beta)$ is independent of K and $\delta_2(K) > 0$ is a number such that

$$|\sigma(x)| < \inf_K |\sigma| \ \forall x \in D \text{ with } \delta_D(x) < \delta_2(K). \tag{2.13}$$

In the situation of Theorem 1.4 we apply Lemma 2.5 for getting Lemma 2.2 with

$$\sigma = \sigma_t := -(-\rho)^t. \tag{2.14}$$

Then we get with $\beta = t\alpha$ the estimate

$$C_2^t \delta_D^{t\alpha} \leqslant |\sigma| \leqslant C_1^t \delta_D^{t\alpha}.$$

This implies that the number

$$\delta_2(K) := \frac{C_2}{C_1}\delta_D(K)$$

satisfies (2.13) such that the number δ_0 in Lemma 2.5 becomes in this case

$$\delta_0 = \delta_0(t,K) = \min\left\{\delta_1(t\alpha), \frac{C_2}{C_1}\delta_D(K)\right\}. \tag{2.15}$$

As we will see in the following, this is the number $\delta_0(t,K)$ needed in Lemma 2.2, such that the number $\delta_1(t,\alpha)$ used there is, in fact, $\delta_1(t,\alpha) = \delta_1(t\alpha)$. With this we can now give the

End of the proof of Lemma 2.2.— The measure $(dd^cU_{z,w})^n$ is supported in the ball $B_{z,w} := B(z, R_D\exp(-\eta_w)) = B(z, R_D\exp(-\sigma_t^{-2}(w)))$. We, therefore, obtain for all $z \in K$ and w with $\delta_D(w) < \delta_0(t,K)$ the inequality

$$\begin{aligned}\mathcal{I}(z,w) &\geqslant (2\pi)^n \inf_{B_{z,w}} |G_D(\cdot,w)| \\ &= -(2\pi)^n \sup_{B_{z,w}} G_D(\cdot,w) \\ &\geqslant |G_D(z,w)| - (2\pi)^n C_t|\sigma_t(w)|\left|\log|\sigma_t(w)|\right|\end{aligned}$$

where the inequality and the constant C_t come from Lemma 2.5. This finishes the proof of Lemma 2.2 (and, hence, also of Theorem 1.4). $\square$

3. Proofs of Corollary 1.6 and Theorem 1.7.

We start with the proof of the corollary. Let for this ρ, t and $\tilde{C}_t$ be as in Theorem 1.4. We take $w \in D$ such that

$$\delta_D(w) < \tilde{C}_t^{-1/3nt}$$

and put

$$K := \left\{ x \in D : \delta_D(x) \geqslant \delta_D^{1-3nt}(w) \right\}.$$

Then we have

$$\delta_D(w) = \delta_D^{3nt}(w)\delta_D(K) < \tilde{C}_t^{-1}\delta_D(K).$$

Hence we get

$$\begin{aligned}\sup_K |G_D(\cdot, w)| &\leqslant \tilde{C}_t \left(\left(\frac{\delta_D^{1-2nt}(w)}{\delta_D(K)} \right)^{\alpha/n} \left(\log \frac{2R_D}{\delta_D^{1-3nt}(w)} \right)^{1/n} \right. \\ &\qquad\qquad \left. + \delta_D^{t\alpha}(w) \log \tfrac{1}{\delta_D(w)} \right) \\ &\leqslant C_t' \delta_D^{t\alpha}(w) \log \frac{1}{\delta_D(w)} \\ &< C_t' \delta_t^{t\alpha} \log \frac{1}{\delta_t} \text{ for } \delta_D(w) < \delta_t \\ &< 1\end{aligned}$$

provided $\delta_t \ll 1$. This means that $A(w, D) \cap K = \emptyset$ as claimed. $\square$

In showing Theorem 1.7, it suffices to prove the first claim, since the second follows from it as in the proof of the corollary. First we choose a $\delta_1 > 0$ such that on $\{w \in D : \delta_D(w) < \delta_1\}$ the orthogonal projection to ∂D is well-defined. We denote the image of w by w^*. Let now $F : \overline{D} \times \partial D \to \Delta$, where Δ denotes the unit disc in $\mathbb{C}$, be a continuous function with the properties:

i) $F(\cdot, q)$ is a holomorphic peak function at $q \in \partial D$ defined on a fixed neighborhood of $\overline{D}$;

ii) with some constant $C_2 > 0$ one has for all $w \in D$ with $\delta_D(w) < \delta_1$

$$|1 - F(w, w^*)| \leqslant C_2 \delta_D(w).$$

(Such a function is constructed for instance in [11].) Furthermore, we notice that one has for any two points $a, b \in \Delta$ the estimate

$$\left|\frac{a-b}{1-\overline{a}b}\right|^2 \geqslant 1 - 4\frac{1-|b|}{1-|a|}. \tag{3.1}$$

This yields for all points $z \in K \subset D$ and $w \in D$ with $\delta_D(w) < \delta_1$

$$\begin{aligned} |G_D(z,w)| &\leqslant |G_\Delta(F(z,w^*), F(w,w^*))| \\ &= -\frac{1}{2}\log\left|\frac{F(z,w^*) - F(w,w^*)}{1-\overline{F(z,w^*)}F(w,w^*)}\right|^2 \\ &\leqslant -\frac{1}{2}\log\left(1 - 4\frac{1-|F(w,w^*)|}{1-|F(z,w^*)|}\right) \\ &\leqslant -\frac{1}{2}\log\left(1 - 4\frac{C_2\delta_D(w)}{1-|F(z,w^*)|}\right). \end{aligned} \tag{3.2}$$

However, from the Hopf Lemma applied to the plurisubharmonic function $|F(\cdot,w^*)| - 1$, we obtain with a constant $C_3 > 0$ which can be chosen to be independent of w^*

$$|F(\cdot,w^*)| - 1 \leqslant -C_3\delta_D(\cdot)\,.$$

In particular, we obtain (since $z \in K$)

$$-\frac{1}{2}\log\left(1 - 4\frac{C_2\delta_D(w)}{1-|F(z,w^*)|}\right) \leqslant -\frac{1}{2}\log\left(1 - 4\frac{C_2\delta_D(w)}{C_3\delta_D(K)}\right).$$

If now

$$\delta_D(w) < \frac{C_3}{8C_2}\delta_D(K)$$

we obtain from (3.2)

$$|G_D(z,w)| \leqslant -\frac{1}{2}\log\left(1 - \frac{C_2\delta_D(w)}{C_3\delta_D(K)}\right) \leqslant \frac{8C_2}{C_3}\frac{\delta_D(w)}{\delta_D(K)}.$$

This implies the claim. □

4. A comparison lemma for the Bergman metric.

The basic tool allowing to deduce estimates on the Bergman metric of a bounded domain $D \subset\subset \mathbb{C}^n$ from estimates on the Green function $G_D(\cdot,\cdot)$ of D is the following fundamental comparison lemma:

PROPOSITION 4.1. — *For any bounded pseudoconvex domain $D \subset\subset \mathbb{C}^n$ there is a constant $C_1 > 0$, such that for all $(w, X) \in D \times (\mathbb{C}^n \setminus \{0\})$ the following estimate holds:*

$$\frac{1}{C_1} B_{A(w,D)}(w; X) \leqslant B_D(w; X) \leqslant C_1 B_{A(w,D)}(w; X) . \tag{4.1}$$

The proof of this proposition is based on the well-known fact, that for arbitrary bounded domains $\Omega \subset\subset \mathbb{C}^n$ the Bergman kernel $K_\Omega(z) := K_\Omega(z, z)$ can be characterized by the extremal property

$$K_\Omega(z) = \max\left\{ |f(z)|^2 : f \in H^2(\Omega), \|f\|_\Omega := \|f\|_{L^2(\Omega)} \leqslant 1 \right\} \tag{4.2}$$

and the quantity $\Omega \times \mathbb{C}^n \ni (z, X) \longmapsto b^2_\Omega(z; X) := K_\Omega(z) B^2_\Omega(z; X)$ is given by the variational problem

$$b^2_\Omega(z; X) = \max\left\{ |\langle \partial f(z), X\rangle|^2 : f \in H^2(\Omega), f(z) = 0, \|f\|_\Omega \leqslant 1 \right\} \tag{4.3}$$

where $H^2(\Omega) := L^2(\Omega) \cap \mathcal{O}(\Omega)$. Therefore, the following interpolation result can be used for proving Proposition 4.1.

LEMMA 4.2. — *Let $D \subset\subset \mathbb{C}^n$ be a bounded pseudoconvex domain and fix a point $w \in D$ and a number $S > 0$. Denote by $A_S(w, D)$ the sublevel set*

$$A_S(w, D) := \{G_D(\cdot; w) < -S\} .$$

Then we can find for every function $f \in H^2(A_S(w, D))$ a function $\tilde{f} \in H^2(D)$ such that

i) $\tilde{f}(w) = f(w)$

ii) $\partial\tilde{f}(w) = \partial f(w)$

iii) *the estimate*

$$\left\|\tilde{f}\right\|^2_D \leqslant \left(1 + \frac{e^{(4n+6)S+1+R^2}}{S^2}\right) \int_{A_S(w,D)} |f|^2 \, d^{2n} z$$

holds with $R := \max_D |z|$.

We will show at first, how this lemma implies Proposition 4.1:

Proof of Proposition 4.1. — Choosing $S = 1$ in Lemma 4.2 we get for any nonzero $f \in H^2(A(w,D))$ because of (4.2) resp. (4.3)

$$K_D(w) \geqslant \frac{\left|\tilde{f}(w)\right|^2}{\left\|\tilde{f}\right\|^2} \geqslant \frac{1}{C_D(1+4e^{4n+6})} \frac{|f(w)|^2}{\|f\|^2_{A(w,D)}}$$

and

$$b_D^2(w,X) \geqslant \frac{\left|\langle \partial \tilde{f}(w), X\rangle\right|^2}{\left\|\tilde{f}\right\|^2} \geqslant \frac{1}{C_D(1+4e^{4n+6})} \frac{|\langle \partial f(w), X\rangle|^2}{\|f\|^2_{A(w,D)}}$$

for all $X \in \mathbb{C}^n$. Here the constant C_D can be defined as

$$C_D = 1 + 4e^{1+R^2}.$$

Taking the supremum over all $f \in H^2(A(w,D))$ with $\|f\|_{A(w,D)} = 1$ we get at first

$$K_D(w) \geqslant \frac{1}{C_D(1+4e^{4n+6})} K_{A(w,D)}(w) \tag{4.4}$$

and taking the supremum over all $f \in H^2(A(w,D))$ with $\|f\|_{A(w,D)} = 1$ and $f(w) = 0$ we also get

$$b_D^2(w,X) \geqslant \frac{1}{C_D(1+4e^{4n+6})} b^2_{A(w,D)}(w,X) \tag{4.5}$$

for any $X \in \mathbb{C}^n$.

Putting together (4.3) and (4.5) yields

$$B_D^2(w,X) \geqslant \frac{1}{C_D(1+4e^{4n+6})} B^2_{A(w,D)}(w,X)$$

the lower estimate of Proposition 4.1. The upper estimate follows similarly using (4.4). □

It remains to show Lemma 4.2:

Proof of Lemma 4.2. — The proof is a refinement of the argument showing Proposition 3.6 of [12]. We denote for small $\tau > 0$ by D_τ the set $D_\tau := \{z \in D : \operatorname{dist}(z, \partial D) > \tau\}$. Furthermore, let $\psi_1 \in C_0^\infty(\mathbb{C}^n)$ be a non-negative polyradially symmetric function with $\int_{\mathbb{C}^n} \psi_1 \, d^{2n}z = 1$. We put

$$\psi_\tau(z) := \tau^{-2n} \psi_1\left(\tau^{-1} z\right) \text{ for } z \in \mathbb{C}^n.$$

With it we define on D_τ the weight functions

$$\Phi_\tau(z) := (2n+2)V_\tau(z) + e^{V_\tau(z)} + \tau|z|^2$$

with $V_\tau := G_D(\cdot, w) * \psi_\tau$ and

$$\Phi := (2n+2)G_D(\cdot, w) + e^{G_D(\cdot,w)}.$$

Finally, we choose a cut-off function $\chi \in \mathcal{C}^\infty(\mathbb{R})$ with $|\chi'| \leqslant 2S^{-1}$ and such that

$$\chi(x) = \begin{cases} 1 \text{ for } x \leqslant -2S \\ 0 \text{ for } x \geqslant -S. \end{cases}$$

Let us now take an arbitrary $f \in H^2(A_S(w,D))$ and define the $(0,1)$-form

$$\alpha_\tau = \sum_{j=1}^n \alpha_{\tau,j}\, d\bar{z}_j := \bar\partial(\chi \circ V_\tau \cdot f) = \chi' \circ V_\tau \cdot f \bar\partial V_\tau.$$

It is smooth and its support is contained in the set $\{-2S \leqslant V_\tau \leqslant -S\} \subset A_S(w,D)$. We also observe that the weight function Φ_τ satisfies on D_τ the estimate

$$\Phi_\tau \leqslant 1 + R^2 \tag{4.6}$$

and on $\operatorname{supp}(\alpha_\tau)$ we have

$$\Phi_\tau \geqslant -(4n+4)S. \tag{4.7}$$

Next we want to convince ourselves that

$$\int_{D_\tau} |\alpha_\tau|^2 e^{-\Phi_\tau}\, d^{2n}z < \infty. \tag{4.8}$$

Namely, $\|G_D(\cdot;w)\|_{L^1(D)} \leqslant C$ for all $w \in D$, $C > 0$ a constant, hence $|\bar\partial V_\tau| \leqslant C\tau^{-2n-1}$ on D_τ. In conjunction with (4.6) and (4.7) the inequality (4.8) therefore follows.

As a next step the Levi form of Φ_τ can be estimated from below by

$$\mathcal{L}_{\Phi_\tau} \geqslant e^{V_\tau} \partial V_\tau \otimes \overline{\partial V_\tau} \geqslant e^{-2S} \partial V_\tau \otimes \overline{\partial V_\tau} \text{ on } \operatorname{supp}\alpha_\tau. \tag{4.9}$$

If we write $Q = (Q_{jk})$ for the inverse of the coefficient matrix of $\mathcal{L}_{\Phi_\tau}$, we obtain using at first (4.9) and then (4.7) the crucial estimate

$$\begin{aligned}\int_D \sum_{j,k=1}^n Q_{jk}(z)\alpha_{\tau,j}(z)\overline{\alpha_{\tau,k}(z)} e^{-\Phi_\tau(z)}\, d^{2n}z \\ \leqslant e^{2S} \int_{\operatorname{supp}\alpha_\tau} |\chi' \circ V_\tau|^2 |f(z)|^2 e^{-\Phi_\tau(z)}\, d^{2n}z \\ \leqslant 4S^{-2} e^{(4n+6)S} \|f\|^2_{A_S(w,D)}.\end{aligned}$$

This shows, that the methods of the proof of Lemma 4.4.1 from [14] apply to the pseudoconvex open set D_τ for small $\tau > 0$. We obtain a solution $u_\tau \in \mathcal{C}^\infty(D_\tau)$ of the equation $\overline{\partial} u_\tau = \alpha_\tau$ satisfying the estimate

$$(4.10) \qquad \int_{D_\tau} |u_\tau|^2 e^{-\Phi_\tau}\, d^{2n}z \leqslant 4S^{-2} e^{(4n+6)S} \int_{\mathrm{supp}\alpha_\tau} |f|^2\, d^{2n}z \\ \leqslant 4S^{-2} e^{(4n+6)S} \|f\|^2_{A_S(w,D)}.$$

This shows, in particular, that the Alaoglu-Bourbaki theorem applies to the family of L^2-functions defined by

$$v_\tau := \begin{cases} e^{-\Phi_\tau/2} u_\tau & \text{on} \quad D_\tau \\ 0 & \text{on} \quad D \setminus D_\tau. \end{cases}$$

Hence there is a sequence $(\tau_k)_k$ of parameters converging to 0 and a function $v \in L^2(D)$, such that

$$\int_D |v|^2\, d^{2n}z \leqslant 4S^{-2} e^{(4n+6)S} \|f\|^2_{A_S(w,D)}$$

and $v_k \longrightarrow v$ in the weak-$*$-topology on $L^2(D)$ as $k \longrightarrow \infty$.

We, now, claim, that the function $u := e^{\Phi/2} v$ has the following properties:

i) The L^2-estimate

$$\int_D |u|^2\, d^{2n}z \leqslant e^{1+R^2} \int_D |u|^2 e^{-\Phi}\, d^{2n}z \leqslant 4S^{-2} e^{(4n+6)S+R^2} \|f\|^2_{A_S(w,D)}$$

holds.

ii) There is a holomorphic function $\tilde{f}$ on D with $\tilde{f} = \chi \circ G_D(\cdot, w) \cdot f - u$ almost everywhere.

iii) One has $\tilde{f}(w) = f(w)$, $\partial \tilde{f}(w) = \partial f(w)$.

Namely, the first claim follows from the definition of v together with (4.6). In order to prove the second assertion we put $\tilde{f}_1 := \chi \circ G_D(\cdot, w) \cdot f - u$. Since $u \in L^2(D)$, so is $\tilde{f}_1$. It, therefore, suffices to verify $\overline{\partial} \tilde{f}_1 = 0$ on $D \setminus \{w\}$ in the distributional sense. This can be done exactly as in the proof of Proposition 3.6 in [12]. For the convenience of the reader we mention the essential steps also here. For this we denote by $\langle \cdot, \cdot \rangle$ the inner product on $L^2(D)$ resp. on (p, q)-forms with coefficients in this space. We choose a smooth $(0,1)$-form β with compact support in $D \setminus \{w\}$ and calculate for large k with ϑ meaning the formal adjoint of $\overline{\partial}$

$$\begin{aligned}\langle e^{\Phi/2} v_{\tau_k}, \vartheta\beta\rangle &= \langle e^{\frac{1}{2}\Phi_{\tau_k}} v_{\tau_k}, e^{\frac{1}{2}(\Phi-\Phi_{\tau_k})}\vartheta\beta\rangle \\ &= \langle e^{\frac{1}{2}\Phi_{\tau_k}} v_{\tau_k}, \vartheta\beta\rangle + \langle e^{\frac{1}{2}\Phi_{\tau_k}} v_{\tau_k}, \left(e^{\frac{1}{2}(\Phi-\Phi_{\tau_k})} - 1\right)\vartheta\beta\rangle \\ &= \langle \alpha_{\tau_k}, \beta\rangle + \langle u_{\tau_k}, \left(e^{\frac{1}{2}(\Phi-\Phi_{\tau_k})} - 1\right)\vartheta\beta\rangle \\ &= \langle \chi \circ V_{\tau_k} \cdot f, \vartheta\beta\rangle + \langle u_{\tau_k}, \left(e^{\frac{1}{2}(\Phi-\Phi_{\tau_k})} - 1\right)\vartheta\beta\rangle.\end{aligned} \tag{4.11}$$

From (4.10) we get

$$\|u_{\tau_k}\|^2_{D_{\tau_k}} \leqslant 4S^{-2}e^{(4n+6)S}\|f\|^2$$

uniformly in k and Beppo-Levi gives $\|((\frac{1}{2}(\Phi - \Phi_{\tau_k})) - 1)\vartheta\beta\|_D \to 0$ as $k \to \infty$. Altogether the Cauchy-Schwarz inequality, therefore, gives as estimate for the second term of the right-hand side of (4.11)

$$\langle u_{\tau_k}, \left(e^{\frac{1}{2}(\Phi-\Phi_{\tau_k})} - 1\right)\vartheta\beta\rangle \to 0 \text{ for } k \to \infty.$$

This yields

$$\begin{aligned}\langle \tilde{f}_1, \vartheta\beta\rangle &= \langle \chi \circ G_D(\cdot, w) f, \vartheta\beta\rangle - \langle u, \vartheta\beta\rangle \\ &= \lim_{k\to\infty} \langle \chi \circ V_{\tau_k} \cdot f, \vartheta\beta\rangle - \lim_{k\to\infty} \langle e^{\Phi/2} v_{\tau_k}, \vartheta\beta\rangle \\ &= 0.\end{aligned}$$

This proves ii).

The equalities iii) follow immediately from i). Namely, on an open neighborhood U of w one has $G_D(\cdot, w) \leqslant -3S$, hence, $f - \tilde{f} = u$ almost everywhere on U. From i) we get

$$\int_U \left|f - \tilde{f}\right|^2 e^{-\Phi}\, d^{2n}z < \infty.$$

Therefore and because of the factor $(2n+2)$ in the definition of Φ, the holomorphic function $f - \tilde{f}$ must vanish to at least second order at w. This finishes the proof of Lemma 4.2. □

5. The boundary behavior of the sublevel sets of the Green function.

The next essential tool needed for the proof of Theorem 1.8 is the convergence property

$$\lim_{w\to w_0} \operatorname{diam}(A(w,D)) = 0 \tag{5.1}$$

for the sublevel sets of the pluricomplex Green function for any point $w_0 \in \partial D$. In order to formulate the exact hypothesis which allows us to show (5.1), we remind the reader of the following definition from [13]

DEFINITION 5.1.— *We say that a bounded domain $D \subset\subset \mathbb{C}^n$ satisfies property (P_*) at a point $w_0 \in \partial D$ if for any compact set $K \subset D$*

$$\lim_{w\to w_0} \sup_{z\in K} |G_D(z,w)| = 0.$$

Remark 5.2.— Theorem 1.2 as stated in the Introduction says, that any domain $D \subset\subset \mathbb{C}^n$ admitting a bounded uniformly Hölder continuous plurisubharmonic exhaustion function satisfies property (P_*) at any of its boundary points.

We will show in this section

PROPOSITION 5.3.— *Let $D \subset\subset \mathbb{C}^n$ be a domain satisfying property (P_*) at every boundary point. Assume, furthermore, that any point $w_0 \in \partial D$ is a peak point for the family $\mathcal{P}(D) := \mathcal{C}^0(\overline{D}) \cap PSH(D)$ of continuous functions on $\overline{D}$ plurisubharmonic on D. Then one has for all $w_0 \in \partial D$*

$$\lim_{w\to w_0} \operatorname{diam}(A(w,D)) = 0.$$

As a first step in the proof of this proposition we will show, that the sublevel sets $A(w,D)$ cannot come arbitrarily close to any point of $\partial D \setminus \{w_0\}$ as $w \to w_0$. For this we will need the hypothesis concerning plurisubharmonic peak functions, but not yet property (P_*). We claim:

LEMMA 5.4.— *Let $D \subset\subset \mathbb{C}^n$ be a pseudoconvex domain, such that any point $w_0 \in \partial D$ is a peak point for the family $\mathcal{P}(D) := \mathcal{C}^0(\overline{D}) \cap PSH(D)$. If then $K_1 \subset \partial D$ is compact and $L \subset D$ is a set with $\overline{L} \cap K_1 = \emptyset$, then, for every $s > 0$, there is an open neighborhood $U_s \supset K_1$ with $\overline{U}_s \cap \overline{L} = \emptyset$ and such that*

$$G_D(z,w) > -s$$

for all $(z,w) \in (U_s \cap D) \times L$.

Proof of Lemma 5.4. — We put $\delta := \frac{1}{4}\operatorname{dist}(K_1, \overline{L})$. Let $q \in K_1$ be arbitrary and $\psi \in \mathcal{P}(D)$ a plurisubharmonic function on D, continuous on $\overline{D}$, with $\psi(q) = 0$ and $\psi(z) < 0$ for $z \in \overline{D} \setminus \{q\}$. We then define for any $w \in L$ with the constant

$$C_q := 2\frac{\log 2R_D/\delta}{\min\limits_{|z-q|\geqslant 3\delta} |\psi(z)|}$$

the function

$$v_w(z) := \begin{cases} \max\left\{ C_q\psi(z), \log \dfrac{|z-w|}{2R_D} \right\} & \text{for} \quad |z-w| \geqslant \delta \\ \log \dfrac{|z-w|}{2R_D} & \text{for} \quad |z-w| < \delta. \end{cases}$$

Notice, that v_w is well-defined, since for $z \in D$ with $|z-w| = \delta$ it follows $C_q\psi(z) \leqslant \log \frac{|z-w|}{2R_D}$. Namely, for such z we get $|z-q| \geqslant |w-q| - |z-w| \geqslant 4\delta - \delta = 3\delta$ and, hence,

$$C_q\psi(z) = -C_q|\psi(z)| \leqslant -C_q \min_{\partial B(q,3\delta)\cap\overline{D}} |\psi| = 2\log\frac{\delta}{2R_D} < \log\frac{|z-w|}{2R_D}.$$

The function v_w, therefore, is for each $w \in L$ plurisubharmonic on D and, even, a candidate in the definition of $G_D(\cdot, w)$. Hence,

$$G_D(z,w) \geqslant C_q\psi(z)$$

if $|z-w| > \delta$ and $w \in L$. The function ψ being continuous, we can, after fixing a number $s > 0$, find an open neighborhood U_q^s of q, such that $\overline{U_s^q} \cap \overline{L} = \emptyset$ and $C_q\psi(z) > -s$ on $U_s^q \cap D$. The claim of the lemma now follows from the compactness of K_1, namely, we can cover K_1 by finitely many such neighborhoods U_s^q for a finite number of points $q \in K_1$. The union U_s of these neighborhoods has the desired property. □

We are now ready for the

Proof of Proposition 5.3. — Let $w_0 \in \partial D$. It suffices to prove:

Claim. — For any number $m > 0$ there is a number $\delta > 0$, such that $A(w, D) \subset B(w_0, \frac{1}{m})$ for all $w \in D$ with $|w - w_0| < \delta$.

We apply Lemma 5.4 to the compact set

$$K_1 := \partial D \cap \left\{ z \in \overline{D} : |z - w_0| \geqslant \frac{1}{m} \right\}$$

and the set

$$L := D \cap B\left(w_0, \frac{1}{2m}\right)$$

with $s = \frac{1}{4}$, obtaining an open neighborhood U_m of K_1 such that

$$A(w, D) \cap U_m = \emptyset \text{ for } w \in L.$$

The set

$$K_2 := \left\{ z \in D : |z - w_0| \geqslant \frac{1}{m},\ z \notin U_m \right\}$$

is a compact subset of D. By property (P_*) one has

$$\lim_{w \longrightarrow w_0} \sup_{z \in K_2} |G_D(z, w)| = 0$$

meaning, that for small enough $\delta' > 0$

$$A(w, D) \cap K_2 = \emptyset \text{ for } w \in D \cap B(w_0, \delta').$$

Hence, we obtain for all $w \in D$ with $|w - w_0| < \delta := \min\{\delta', \frac{1}{2m}\}$

$$A(w, D) \cap \left(D \setminus B\left(w_0, \frac{1}{m}\right)\right) = (A(w, D) \cap U_m) \cup (A(w, D) \cap K_2) = \emptyset.$$

Since this implies $A(w, D) \subset D \cap B(w_0, \frac{1}{m})$ for such points, Proposition 5.3 has been proved. □

6. Proof of Theorem 1.8.

This is now very brief. Namely, take $(w, X) \in D \times \mathbb{C}^n$. Proposition 4.1 says that

$$B_D(w, X) \geqslant C_1 B_{A(w,D)}(w, X)$$

with some constant $C_1 > 0$ independent of w. Furthermore, a trivial argument yields

$$B_{A(w,D)}(w, X) \geqslant \frac{|X|}{\operatorname{diam}(A(w, D))}. \tag{6.1}$$

Since, however, Proposition 5.3 applies in the given situation, the right-hand side in (6.1) goes to infinity as w tends to w_0. This implies the theorem. □

7. Proof of Theorem 1.13.

Again, this proof is, of course, based on the $\overline{\partial}$-technique for the construction of good candidates of holomorphic L^2-functions contributing to the variational problems which give $K_D(w)$ and $b_D(w, X)$ (see (4.2) and (4.3)). The main tool obtained by this method is contained in the following lemma which is a suitable modification of Lemma 4.2.

LEMMA 7.1. — *Let $\tilde{D} \subset\subset \mathbb{C}^n$ be a pseudoconvex domain. Fix a point $w \in \tilde{D}$ and a unit vector $X \in \mathbb{C}^n$. Assume, we are given a negative plurisubharmonic function $V^{(w)}$ on $\tilde{D}$, such that on an open neighborhood of w the function $z \to V^{(w)}(z) - \log|z - w|$ is bounded from above, and denote for any number $S > 0$*

$$A_S\Big(w, V^{(w)}\Big) := \Big\{z : V^{(w)}(z) < -S\,\Big\}.$$

Then we can find for every function $f \in H^2(A_S(w, V^{(w)}))$ a function $\tilde{f} \in H^2(\tilde{D})$ with the following properties:

i) $\tilde{f}(w) = f(w)$

ii) $\partial\tilde{f}(w) = \partial f(w)$

iii) $\|\tilde{f}\|^2_{\tilde{D}} \leqslant (1 + 4S^{-2}e^{(4n+6)S+1+R^2_{\tilde{D}}})\|f\|^2_{A_S(w,V^{(w)})}$, where $\widetilde{R}_D := \max_{\tilde{D}} |z|$.

Because of the complete analogy to the proof of Lemma 4.2 we do not prove this lemma here.

We now come to the proof of Theorem 1.13.

Let $0 < t \ll 1$ be arbitrarily small and choose $\eta \geqslant 2$ such that D is pseudoconvexly extendable of order η at a given boundary point w^0. Let ρ be a corresponding extending function defined on an open neighborhood U of w^0. More precisely, suppose that ρ satisfies

$$-C\Big(\delta_D(z) + \big|z - w^0\big|^2\Big) \leqslant \rho(z) \leqslant -\big|z - w^0\big|^\eta \tag{7.1}$$

on $D \cap U$.

It suffices to show:

Claim. — One has

$$h_D\big(w^0\big) \geqslant \frac{1 - 2nt}{\eta}. \tag{7.2}$$

Let us next choose a pseudoconvex domain $D_1 \subset U \cap \{\rho < 0\}$ with $\mathcal{C}^3$-smooth boundary and the following properties:

1) There exists an open neighborhood $U_1 \subset\subset U$ of w^0 such that

$$\partial D_1 \cap U_1 = \{\rho = 0\} \cap U_1.$$

2) $D \cap U_1 \subset D_1 \cap U_1$.

3) On U_1 the inequality $|\rho| \leqslant C'\delta_{D_1}$ holds with some constant C'.

Because of the well-known localization property of the Bergman metric it suffices to do all work on the domain

$$\tilde{D} := D \cap U_1$$

instead of D.

Let, now, $U_2 \subset\subset U_1$ be another open neighborhood of w^0 and choose $w \in D \cap U_2$ arbitrarily. We define for any unit vector $X \in \mathbb{C}^n$ the function

$$f(z) := \langle z - w, X\rangle \frac{K_{\tilde{D}}(z,w)}{\sqrt{K_{\tilde{D}}(w)}}.$$

Lemma 7.1 with this domain $\tilde{D}$, the function $V^{(w)} := G_{D_1}(\cdot, w)$ and $S = 1$ applied to this function f yields a function $\tilde{f} \in H^2(D)$ such that

$$b_{\tilde{D}}(w,X) \geqslant \frac{\left|\langle \partial\tilde{f}(w), X\rangle\right|}{\left\|\tilde{f}\right\|_{\tilde{D}}} \geqslant C^* \frac{\sqrt{K_{\tilde{D}}(w)}}{\|f\|_{A(w,D_1)\cap\tilde{D}}}$$

and, hence,

$$B_{\tilde{D}}(w,X) \geqslant C^* \frac{1}{\|f\|_{A(w,D_1)\cap\tilde{D}}} \tag{7.3}$$

with an unimportant constant C^*.

It, therefore, remains to estimate the norm $\|f\|_{A(w,D_1)\cap\tilde{D}}$.

For this we notice, that on $\tilde{D} \subset U \cap D$

$$\left|z - w^0\right| \leqslant |\rho(z)|^{1/\eta} \leqslant C'\delta_{D_1}^{1/\eta}(z)$$

and, hence,

$$\begin{aligned}|f(z)| &\leqslant |z-w|\frac{|K_{\tilde D}(z,w)|}{\sqrt{K_{\tilde D}(w)}}\\
&\leqslant \left(|z-w^0|+|w-w^0|\right)\frac{|K_{\tilde D}(z,w)|}{\sqrt{K_{\tilde D}(w)}}\\
&\leqslant \left(C'\delta_{D_1}^{1/\eta}(z)+|w-w^0|\right)\frac{|K_{\tilde D}(z,w)|}{\sqrt{K_{\tilde D}(w)}}.\end{aligned}\tag{7.4}$$

By Lemma 1.6 we have with some number $\delta_t \ll 1$ for all $w \in D_1$ with $\delta_{D_1}(w) < \delta_t$ the estimate

$$A(w,D_1)\subset\left\{\delta_{D_1}<\delta_{D_1}^{1-2nt}(w)\right\}.$$

Inserting this into (7.4) we see that, on $\tilde D \cap A(w,D_1)$ one even has

$$|f(z)|\leqslant C''\left(\delta_{D_1}^{1-2nt/\eta}(w)+|w-w^0|\right)\frac{|K_{\tilde D}(z,w)|}{\sqrt{K_{\tilde D}(w)}}$$

if $\delta_{D_1}(w) < \delta_t$. By integration we obtain for such points w

$$\|f\|_{A(w,D_1)\cap\tilde D}\leqslant \tilde C\left(\delta_{D_1}^{1-2nt/\eta}(w)+|w-w^0|\right).$$

Finally, we observe that, on a cone within $\tilde D$ with vertex at w^0, we can estimate

$$\delta_{D_1}^{1-2nt/\eta}(w)+|w-w^0|\leqslant \hat C\delta_{\tilde D}^{1-2nt/\eta}(w).$$

This, together with (7.3) proves

$$B_{\tilde D}(w,X)\geqslant C^{**}\frac{1}{\delta_{\tilde D}^{1-2nt/\eta}(w)}$$

for all w sufficiently close to w^0 and inside a cone within $\tilde D$ with vertex at w^0. This proves the Claim (7.2) and, therefore, Theorem 1.13. □

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