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1. Introduction.

Dirichlet noted that, from the formula

$$\Gamma(s) = n^s \int_0^\infty x^{s-1} e^{-nx} dx = n^s \int_0^1 (-\log t)^{s-1} t^{n-1}\, dt,$$

we may obtain the identity

$$\begin{aligned}\Gamma(s) L\Big(s, \Big(\frac{\cdot}{p}\Big)\Big) &= \Gamma(s) \sum_{n\geqslant 1} \frac{(n/p)}{n^s} = \int_0^1 (-\log t)^{s-1} \sum_{n\geqslant 1} \Big(\frac{n}{p}\Big) t^{n-1}\, dt \\ &= \int_0^1 \frac{(-\log t)^{s-1}}{t} \, \frac{f_p(t)}{1-t^p}\, dt.\end{aligned} \tag{1.1}$$

Here $\big(\frac{\cdot}{p}\big)$ is the Legendre symbol and

$$f_p(t) := \sum_{a=0}^{p-1} \Big(\frac{a}{p}\Big) t^a. \tag{1.2}$$

Equation (1.1) allowed Dirichlet to define $L\big(s, \big(\frac{\cdot}{p}\big)\big)$ as a regular function for all complex s. Fekete observed that if $f_p(t)$ has no real zeros t with

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$0 < t < 1$, then $L\big(s, \big(\frac{\cdot}{p}\big)\big)$ has no real zeros $s > 0$; and the $f_p(t)$ are thus now known as *Fekete polynomials*. Indeed, if $L\big(s, \big(\frac{\cdot}{p}\big)\big) = 0$ then by (1.1) and the mean value theorem there is a t in $(0,1)$ with $\frac{(-\log t)^{s-1}}{t}\frac{f_p(t)}{1-t^p} = 0$, and so $f_p(t) = 0$ here.

Among small primes p, there are only a few for which the Fekete polynomial $f_p(t)$ has a real zero t in the range $0 < t < 1$. In fact, we may verify computationally that there are just 23 primes up to 1000 for which f_p has a zero in $(0,1)$. This implies that there are no positive real zeros of $L\big(s, \big(\frac{\cdot}{p}\big)\big)$ for most such primes p, and in particular no *Siegel zeros* (that is, real zeros "especially close to 1"). It is interesting to note that for those primes $p \equiv 3 \bmod 4$ for which $f_p(t)$ does have a zero in $(0,1)$, the class number of $Q(\sqrt{-p})$ is surprisingly small (for example $p = 43, 67, 163, \ldots$). Unfortunately this trend does not persist: Indeed Baker and Montgomery [1] proved that $f_p(t)$ has a large number of zeros in $(0,1)$ for almost all primes p (that is, the number of such zeros $\to \infty$ as $p \to \infty$, and it seems likely that there are, in fact, $\asymp \log\log p$ such zeros).

In this paper we shall study the complex zeros of $f_p(t)$. Using zero locating software one finds that, for primes p up to 1000, about half of the zeros lie on the unit circle; leading one to expect this to be the general phenomenon. It turns out to be fairly easy to prove that *at least* half of the zeros of $f_p(t)$ are on the unit circle (that is $|t| = 1$): First note that

$$F_p(z) := z^{-p/2} f_p(z) = \sum_{a=1}^{(p-1)/2} \left(\frac{a}{p}\right)\left(z^{a-p/2} + \left(\frac{-1}{p}\right) z^{p/2-a}\right)$$

by combining the a and $p-a$ terms[1]. Taking $z = e^{2i\pi t}$ we have

$$F_p\left(e^{2i\pi t}\right) = \begin{cases} 2\displaystyle\sum_{a=1}^{(p-1)/2} \left(\frac{a}{p}\right)\cos((2a-p)\pi t) & \text{if } p \equiv 1 \bmod 4 \\ 2i\displaystyle\sum_{a=1}^{(p-1)/2} \left(\frac{a}{p}\right)\sin((2a-p)\pi t) & \text{if } p \equiv 3 \bmod 4. \end{cases} \tag{1.3}$$

Define $H_p(t) = F_p(e^{2i\pi t})$ if $p \equiv 1 \pmod 4$, and $H_p(t) = -iF_p(e^{2i\pi t})$ if $p \equiv 3 \pmod 4$. By (1.3) we see that $H_p(t)$ is a periodic, continuous real-valued function when t is real.

Now if $\zeta_p = e^{2i\pi/p}$ then, for all k not divisible by p, $f_p(\zeta_p^k)$ is a *Gauss sum* and has absolute value $\sqrt{p}$ (see Section 2 of [2]); therefore

[1] Here $z = e^{2i\pi t}$ with $0 \leqslant t < 1$, so that there is no ambiguity in the meaning of $z^{-p/2}$.

$|F_p(\zeta_p^k)| = \sqrt{p}$. Moreover

$$F_p(\zeta_p^k) = (\zeta_p^k)^{-p/2} \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) \zeta_p^{ak} = (-1)^k \left(\frac{k}{p}\right) \sum_{a=1}^{p-1} \left(\frac{ak}{p}\right) \zeta_p^{ak}$$
$$= (-1)^k \left(\frac{k}{p}\right) F_p(\zeta_p).$$

Therefore if $(k/p) = ((k+1)/p)$ then $H_p(k/p)$ and $H_p((k+1)/p)$ have different signs. Since $H_p(t)$ is real-valued and continuous, it must have a zero in-between k/p and $(k+1)/p$, by the intermediate value theorem. Thus the number of zeros of $H_p(t)$ in $[0,1)$ (and so of $F_p(z)$ on the unit circle) is

$$\geqslant \#\left\{k : 1 \leqslant k \leqslant p-2 \ \text{ and } \left(\frac{k}{p}\right) = \left(\frac{k+1}{p}\right)\right\} = \frac{p-3}{2},$$

as we shall see in Lemma 2.

Other than possible zeros at $z = -1$ and at $z = 1$, this accounts for all the zeros on the unit circle for each prime $p < 500$. So the question is, is this all, for all p? The answer is "no" and indeed one finds more zeros when $p = 661$. In general one has the following:

THEOREM 1. — *There exists a constant κ_0, $1 > \kappa_0 > 1/2$ such that*

$$\#\{z : |z| = 1 \ \text{ and } f_p(z) = 0\} \sim \kappa_0 p \text{ as } p \to \infty.$$

We determine κ_0 in terms of another constant κ_1 defined as follows:

THEOREM 2. — *Let $\mathcal{F}_J$ be the set of rational functions*

$$g(x) = \frac{1}{x} + \frac{1}{1-x} + \sum_{\substack{|j|<J \\ j\neq 0,-1}} \frac{\delta_j}{x+j}$$

where we allow each δ_j to take value $+1$ or -1. There exists a constant $\kappa_1, 1/2 > \kappa_1 > 0$, such that

$$\#\{g \in \mathcal{F}_J : g(x) = 0 \ \text{ for some } \ x \in (0,1)\} \sim \kappa_1 \#\{g \in \mathcal{F}_J\}$$

as $J \to \infty$.

The constants κ_0 and κ_1 are related as follows:

THEOREM $1\frac{1}{2}$. — *In fact $\kappa_0 = 1/2 + \kappa_1$.*

It is still an open question to determine the value of κ_0. It is known that a "random" trigonometric polynomial of degree p has $p/\sqrt{3}$ zeros in

$[0,1)$ (see [7]), so one might guess that $\kappa_0 = 1/\sqrt{3} \approx 0.5773\ldots$ However this is not the case. We will show

$$0.500813 > \kappa_0 > 0.500668.$$

While it is theoretically easy to find the value of κ_0, we do not know a good practical way of achieving this.

As well as determining precisely the proportion, κ_0, of the zeros of $f_p(t)$ which lie on the unit circle, we would also like to understand the distribution of the set of zeros in the complex plane. There are several easy remarks to make: By (1.2) we have

$$t^p f_p(1/t) = \left(\frac{-1}{p}\right) f_p(t)$$

and so the zeros of $f_p(t)$, other than $t = 0$, are symmetric about the unit circle (*i.e.* they come in pairs other than at $t = 0, \pm 1$). We also note that, for $|t| > 1$,

$$|f_p(t)/t^{p-1}| = \left|\sum_{a=0}^{p-1} \left(\frac{a}{p}\right) \frac{1}{t^{p-1-a}}\right| \geqslant 1 - \sum_{a=0}^{p-2} \frac{1}{|t|^{p-1-a}} > 1 - \frac{1}{|t|-1}.$$

However if $|t| \geqslant 2$ then $1 - 1/(|t|-1) \geqslant 0$, and so $f_p(t)$ has no zeros in $|t| \geqslant 2$. By symmetry it has no zeros in $|t| \leqslant 1/2$ except 0. Thus

PROPOSITION 1. — *The zeros of $f_p(t)$, other than at $0, 1$ and -1 come in pairs $\alpha, 1/\alpha$. Moreover, other than 0, they all lie in the annulus $\{r \in \mathbb{C} : 1/2 < |r| < 2\}$.*

As for the distribution of the arguments of the roots of $f_p(t)$ we can use a beautiful result of Erdős and Turán (Theorem 1 of [3]), which immediately implies that, for any $0 \leqslant \alpha < \beta < 1$,

$$\#\{\tau \in \mathbb{C} :\ f_p(\tau) = 0,\ \alpha < \arg(\tau)/2\pi < \beta\} = (\beta - \alpha)p + O(\sqrt{p \log p}). \tag{1.4}$$

The arguments above, and those used in proving Theorems 1 and 2, focus on determining which arcs $(\zeta_p^K, \zeta_p^{K+1})$ of the unit circle contain a zero of $f_p(t)$. Evidently (1.4) cannot be used so precisely. However we can show that there are zeros of $f_p(t)$ near to such an arc, so long as $f_p(t)$ gets "small" on that arc.

THEOREM 3. — *Suppose that $\epsilon > 0$ is a sufficiently small constant. If p is a sufficiently large prime and K an integer such that there exists a value of t on the unit circle in the arc from ζ_p^K to ζ_p^{K+1} with $|f_p(t)| < \epsilon\sqrt{p}$, then there exists $\tau = r\zeta_p^{K+\theta}$ with $f_p(\tau) = 0$ where $0 < \theta < 1$ and $1 - \epsilon^{1/3}/p < r \leqslant 1$.*

Remark. — Applying Proposition 1 we also have $f_p((1/r)\zeta_p^{K+\theta}) = 0$.

As we have already discussed, Gauss sums $\sum_{a=1}^{p-1} \left(\frac{a}{p}\right)\zeta_p^{ak}$ (and many generalizations) have the surprising property that they have absolute value exactly equal to $\sqrt{p}$. It is, we think, of interest to ask what happens when we replace the primitive p-th root of unity ζ_p^k in the expression for a Gauss sum above, by some primitive $2p$-th root of unity. These may be written as $\zeta_p^{k+1/2}$ or ζ_{2p}^{2k+1}, or $-\zeta_p^k$; so we must consider the values of $f_p(-\zeta_p^k)$. Do these all take on the same absolute value? The answer we now see is "no", as we evaluate the distribution of these absolute values:

THEOREM 4. — *For any fixed real number* ρ

$$\#\left\{k : 1 \leqslant k \leqslant p \text{ such that } H_p\left(\frac{k+\frac{1}{2}}{p}\right) < \rho\sqrt{p}\right\} \sim c_\rho p$$

as $p \to \infty$ *where*

$$c_\rho = \frac{1}{2} + \frac{1}{\pi}\int_{x=0}^{\infty} \sin(\rho\pi x) \prod_{\substack{n\geqslant 1 \\ n \text{ odd}}} \cos^2\left(\frac{2x}{n}\right)\frac{dx}{x}.$$

Moreover $c_{-\rho}$ *and* $1 - c_\rho = \exp(-\exp(\pi\rho/2 + O(1)))$ *for positive* ρ.

After proving this in Section 6, we indicate how our proof may be modified to establish several related results. First, to show that $\max_{|z|=1}|f_p(z)| \gg \sqrt{p}\log\log p$, so re-establishing a result of Montgomery [5]. Second to understand the distribution of the values of the Fekete polynomial at $(p-1)$-st roots of unity.

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2. First results.

Let χ be any character $\pmod p$ and let k be an integer not divisible by p. Note that

$$\sum_{a=1}^{p-1}\chi(a)\zeta_p^{ak} = \bar{\chi}(k)\sum_{a=1}^{p-1}\chi(ak)\zeta_p^{ak} = \bar{\chi}(k)\sum_{b=1}^{p-1}\chi(b)\zeta_p^{b}. \tag{2.1}$$

In particular we see that $f_p(\zeta_p^k) = \left(\frac{k}{p}\right) f_p(\zeta_p)$, whereas in contrast $f_p(1) = 0$. Recall that for a non-principal character $\chi \pmod p$, the Gauss sum $\tau(\chi)$ is $\sum_{a=1}^{p-1} \chi(a)\zeta_p^a$. Thus $f_p(\zeta_p)$ is the Gauss sum $\tau\left(\left(\frac{\cdot}{p}\right)\right)$. It is easy to determine the magnitude of $|f_p(\zeta_p)|$: Note that

$$\begin{aligned}(p-1)f_p(\zeta_p)^2 &= \sum_{k=0}^{p-1} f_p(\zeta_p^k)^2 = \sum_{k=0}^{p-1}\sum_{a,b=0}^{p-1}\left(\frac{ab}{p}\right)\zeta_p^{(a+b)k}\\ &= \sum_{a,b=1}^{p-1}\left(\frac{ab}{p}\right)\sum_{k=0}^{p-1}\zeta_p^{(a+b)k} = p\sum_{\substack{a=1\\ b=p-a}}^{p-1}\left(\frac{ab}{p}\right) = p\left(\frac{-1}{p}\right)(p-1).\end{aligned}$$

Hence we have $f_p(\zeta_p)^2 = (-1/p)p$, and so $|f_p(\zeta_p)| = \sqrt{p}$. Gauss showed more and determined that

$$f_p(\zeta_p) = \begin{cases} \sqrt{p} & \text{if } p \equiv 1 \pmod 4,\\ i\sqrt{p} & \text{if } p \equiv 3 \pmod 4.\end{cases}$$

Since $f_p(\zeta_p^k) = (k/p) f_p(\zeta_p)$, for $1 \leqslant k \leqslant p-1$, and $f_p(1) = 0$, we get by Lagrangian interpolation

$$f_p(z) = \sum_{k=0}^{p-1} f_p(\zeta_p^k) \prod_{\substack{j=0\\ j\neq k}}^{p-1}\left(\frac{z-\zeta_p^j}{\zeta_p^k-\zeta_p^j}\right).$$

Note that

$$\prod_{\substack{j=0\\ j\neq k}}^{p-1}(z-\zeta_p^j) = \frac{z^p-1}{z-\zeta_p^k},$$

and that

$$\prod_{\substack{j=0\\ j\neq k}}^{p-1}(\zeta_p^k-\zeta_p^j) = \zeta_p^{k(p-1)}\prod_{j=1}^{p-1}(1-\zeta_p^j) = p\zeta_p^{-k}.$$

Hence

$$\frac{p}{f_p(\zeta_p)}\,\frac{f_p(z)}{z^p-1} = \frac{p}{f_p(\zeta_p)}\,\frac{z^{-p/2}f_p(z)}{z^{p/2}-z^{-p/2}} = \sum_{k=1}^{p-1}\left(\frac{k}{p}\right)\frac{\zeta_p^k}{z-\zeta_p^k}. \tag{2.2}$$

If $|z| = 1$ then note that $z^{p/2} - z^{-p/2} \in i\mathbb{R}$, and from (1.3) and $f_p(\zeta_p)^2 = (-1/p)p$ we have $z^{\frac{-p}{2}} f_p(z)/f_p(\zeta_p) \in \mathbb{R}$. Thus the right side of (2.2) $\in i\mathbb{R}$

for all $|z| = 1$. To facilitate studying $f_p(z)$ as z goes around the unit circle from ζ_p^K to ζ_p^{K+1}, we write $z = \zeta_p^{K+x} = \zeta_p^K e^{2i\pi x/p}$ and then let

$$\begin{aligned}(2.3)\qquad g_{p,K}(x) &:= i\left(\frac{K}{p}\right)\frac{p}{f_p(\zeta_p)}\frac{f_p(z)}{z^p-1}\Big|_{z=\zeta_p^{K+x}} \\ &= i\left(\frac{K}{p}\right)\sum_{k=K-(\frac{p-1}{2})}^{K+(\frac{p-1}{2})}\left(\frac{k}{p}\right)\frac{1}{\zeta_p^{K-k+x}-1}.\end{aligned}$$

Thus $g_{p,K}(x)$ is a real valued function of $x \in [0,1]$.

PROPOSITION 2. — *If $0 \leqslant K \leqslant p-1$ is an integer with $\left(\frac{K}{p}\right) = \left(\frac{K+1}{p}\right)$ then $g_{p,K}(x)$ has exactly one zero in $(0,1)$. Equivalently, $f_p(z)$ has exactly one zero on the arc of the unit circle from ζ_p^K to ζ_p^{K+1}. If $\left(\frac{K}{p}\right) = -\left(\frac{K+1}{p}\right)$ then $g_{p,K}$ has either no zeros, or exactly two zeros in $(0,1)$. Equivalently, $f_p(z)$ has exactly 0 or 2 zeros on the arc from ζ_p^K to ζ_p^{K+1}.*

Remark. — In the above proposition, and henceforth, we count zeros with multiplicity.

Before proving the proposition, we evaluate $\sum_{k=1}^{p-1}\frac{1}{\sin^2(\pi k/p)}$.

LEMMA 1. — *For all integers $p \geqslant 2$,*

$$\sum_{k=1}^{p-1}\frac{1}{\sin^2(\frac{\pi k}{p})} = \frac{p^2-1}{3}.$$

Proof. — Put $A(z) = \prod_{k=1}^{p-1}(z-\zeta_p^k)$. Logarithmic differentiation shows that

$$\left\{z\left(\frac{A'(z)}{A(z)}\right)' + \frac{A'(z)}{A(z)}\right\}\Big|_{z=1} = -\sum_{k=1}^{p-1}\frac{\zeta_p^k}{(1-\zeta_p^k)^2} = \frac{1}{4}\sum_{k=1}^{p-1}\frac{1}{\sin^2\left(\frac{\pi k}{p}\right)}.$$

However, $A(z) = \frac{z^p-1}{z-1} = z^{p-1} + z^{p-2} + \ldots + 1$ and using this to evaluate the left side above, we get the lemma. □

Proof of Proposition 2. — Note that with $g = g_{p,K}$, we have $\lim_{x\to 0^+} g(x) = \infty$, and $\lim_{x\to 1^-} g(x) = -\left(\frac{K}{p}\right)\left(\frac{K+1}{p}\right)\infty$. Further observe that

$$\begin{aligned} g'(x) &= \frac{2\pi}{p}\Big(\frac{K}{p}\Big) \sum_{|k-K|<p/2} \Big(\frac{k}{p}\Big) \frac{\zeta_p^{K-k+x}}{(\zeta_p^{K-k+x}-1)^2} \\ &= -\frac{\pi}{2p}\Big(\frac{K}{p}\Big) \sum_{|k-K|<p/2} \Big(\frac{k}{p}\Big) \frac{1}{\sin^2(\frac{\pi}{p}(K-k+x))}. \end{aligned}$$

If $\left(\frac{K}{p}\right) = \left(\frac{K+1}{p}\right)$ then, by Lemma 1,

$$\begin{aligned} (2.4)\qquad |g'(x)| &\geqslant \frac{\pi}{2p}\left(\frac{1}{\sin^2(\frac{\pi}{p}x)} + \frac{1}{\sin^2(\frac{\pi}{p}(1-x))} - \sum_{\substack{j\neq 0,1\\ |j|<p/2}} \frac{1}{\sin^2(\frac{\pi}{p}(x-j))}\right) \\ &\geqslant \frac{\pi}{2p}\left(\frac{2}{\sin^2(\frac{\pi}{2p})} - \frac{p^2-1}{3}\right) > 0, \end{aligned}$$

since the sum of the first two terms is minimized when $x = 1/2$. Hence $g'(x) \neq 0$ for all $x \in (0,1)$, so that g is monotone decreasing in $[0,1]$ going from ∞ to $-\infty$. Thus g has exactly one zero in this interval.

Moreover

$$g''(x) = \frac{\pi^2}{p^2}\Big(\frac{K}{p}\Big) \sum_{|k-K|<p/2} \Big(\frac{k}{p}\Big) \frac{\cos\left(\frac{\pi}{p}(K-k+x)\right)}{\sin^3\left(\frac{\pi}{p}(K-k+x)\right)}.$$

Now if $\left(\frac{K}{p}\right) = -\left(\frac{K+1}{p}\right)$ then

$$|g''(x)| \geqslant \frac{\pi^2}{p^2}\left(\frac{\cos\left(\frac{\pi}{p}x\right)}{\sin^3\left(\frac{\pi}{p}x\right)} + \frac{\cos\left(\frac{\pi}{p}(1-x)\right)}{\sin^3\left(\frac{\pi}{p}(1-x)\right)} - \sum_{\substack{|j|<p/2\\ j\neq 0,1}} \frac{\cos\left(\frac{\pi}{p}(j-x)\right)}{|\sin\left(\frac{\pi}{p}(j-x)\right)|^3}\right).$$

Let μ be the minimum of $\cot(\frac{\pi}{p}t)$ over $t = x,\ 1-x$. Since $\cot t$ decreases rapidly as t goes from 0 to $\pi/2$ we see that the above is

$$\geqslant \frac{\pi^2}{p^2}\mu\left(\frac{1}{\sin^2(\frac{\pi}{p}x)} + \frac{1}{\sin^2(\frac{\pi}{p}(1-x))} - \sum_{\substack{j\neq 0,1\\ |j|<p/2}} \frac{1}{\sin^2(\frac{\pi}{p}(x-j))}\right) > 0,$$

as in (2.4). Thus $g'(x)$ is monotone increasing in $(0,1)$ going from $-\infty$ to $+\infty$. Thus there is a unique x_0 in $(0,1)$ with $g'(x_0) = 0$, and the minimum value of $g(x)$ is attained at x_0. Plainly g has 0 or 2 zeros depending on whether $g(x_0) > 0$, or $g(x_0) \leqslant 0$. This proves the proposition. □

From Proposition 2 we know that $f_p(z)$ has at least as many zeros on $|z| = 1$, as there are values $1 \leqslant K \leqslant p-1$ with $\left(\frac{K}{p}\right) = \left(\frac{K+1}{p}\right)$. We next determine the number of such values K.

LEMMA 2 (Gauss). — *For any non-principal character* $\chi \pmod p$, *we have*

$$\sum_{b=1}^{p-1} \chi(b)\bar{\chi}(b+k) = \begin{cases} p-1 & \text{if } p \mid k \\ -1 & \text{if } p \nmid k. \end{cases} \tag{2.5}$$

Hence

$$\#\left\{b \pmod p : \left(\frac{b}{p}\right) = \left(\frac{b+1}{p}\right)\right\} = \frac{p-3}{2},$$

and

$$\#\left\{b \pmod p : \left(\frac{b}{p}\right) = -\left(\frac{b+1}{p}\right)\right\} = \frac{p-1}{2}.$$

Proof. — If $p \mid k$ then the right side of (2.5) is $\sum_{b=1}^{p-1} |\chi(b)|^2 = p-1$. Suppose now that $p \nmid k$, and let $c = (b+k)/b = 1 + k/b$. As b runs over the non-zero residue classes $\pmod q$, note that c runs over all residue classes except the residue class $1 \pmod p$. Hence the right side of (2.5) is

$$\sum_{\substack{c \pmod p \\ c \not\equiv 1 \pmod p}} \bar{\chi}(c) = -1,$$

as desired. □

If $\left(\frac{K}{p}\right) = -\left(\frac{K+1}{p}\right)$ then we need to determine (in the notation of the proof of Proposition 2) whether $g(x_0) > 0$ or $\leqslant 0$. This depends heavily on the values of $\left(\frac{k}{p}\right)$ for k neighbouring K. The following lemma shows that these neighbouring values behave like independent random variables.

LEMMA 3 (Weil). — *Fix integer* J, *and then the numbers* $\delta_j \in \{-1, 1\}$ *for each* j *with* $|j| < J$. *We have, uniformly,*

$$\#\left\{x \pmod p : \left(\frac{x-j}{p}\right) = \delta_j \text{ for all } |j| < J\right\} = \frac{p}{2^{2J-1}} + O(J\sqrt{p}).$$

Proof. — The above equals

$$\sum_{x=1}^{p} \frac{1}{2^{2J-1}} \prod_{|j|<J} \left(1 + \delta_j\left(\frac{x-j}{p}\right)\right) + O(J)$$

$$= \frac{p}{2^{2J-1}} + O\left(\frac{1}{2^{2J-1}} \sum_{\substack{S \subseteq \{|j|<J\} \\ S \neq \emptyset}} \sum_{x=1}^{p} \left(\frac{\prod_{j\in S}(x-j)}{p}\right) + J\right).$$

By Weil's Theorem [8], if $f(x)$ is a squarefree polynomial $\pmod p$ then

$$\left|\sum_{x=1}^{p}\left(\frac{f(x)}{p}\right)\right| \ll (\text{degree } f)\sqrt{p}.$$

Hence the above is

$$= \frac{p}{2^{2J-1}} + O\left(\frac{\sqrt{p}}{2^{2J-1}}\sum_{m=1}^{2J-1}\binom{2J-1}{m} m + J\right),$$

and the result follows. □

We conclude this section by determining the order of the zeros of $f_p(z)$ at ± 1. In fact we shall determine the number of zeros of $f_p(z)$ on the arcs $\zeta_p^{(p-1)/2}$ to $\zeta_p^{(p+1)/2}$ (which contains -1), and ζ_p^{-1} to ζ_p (which contains 1).

LEMMA 4. — *If $p \equiv 1 \pmod 4$ then $f_p(z)$ has only a simple zero at $z=-1$, on the arc from $\zeta_p^{(p-1)/2}$ to $\zeta_p^{(p+1)/2}$, and $f_p(z)$ has only a double zero at $z=1$, on the arc from ζ_p^{-1} to ζ_p. If $p \equiv 3 \pmod 4$ then there are no zeros of $f_p(z)$ on the arc from $\zeta_p^{(p-1)/2}$ to $\zeta_p^{(p+1)/2}$, and $f_p(z)$ has only a simple zero at $z=1$ on the arc from ζ_p^{-1} to ζ_p.*

Proof. — We make free use of the fact that $(-1/p) = 1$, or -1 depending on whether $p \equiv 1 \pmod 4$, or $3 \pmod 4$. Let's begin with the arc from $\zeta_p^{(p-1)/2}$ to $\zeta_p^{(p+1)/2}$. We take $K=(p-1)/2$ in Proposition 2. Note that $\left(\frac{K}{p}\right) = \left(\frac{K+1}{p}\right)$ if $p \equiv 1 \pmod 4$, and $\left(\frac{K}{p}\right) = -\left(\frac{K+1}{p}\right)$ if $p \equiv 3 \pmod 4$. In the first case, Proposition 2 tells us that there's exactly one (simple) zero on this arc. Since

$$f_p(-1) = \sum_{a=1}^{p-1}(-1)^a\left(\frac{a}{p}\right) = \frac{1}{2}\sum_{a=1}^{p-1}(-1)^a\left(\left(\frac{a}{p}\right) - \left(\frac{p-a}{p}\right)\right) = 0$$

for $p \equiv 1 \pmod 4$, this simple zero is at -1. Now suppose $p \equiv 3 \pmod 4$. By Proposition 2, we know that there are 0 or 2 zeros on this arc, depending on whether $\min_x g_{p,K}(x) > 0$ or not. We now show that this minimum is attained at $x = 1/2$, and the minimum value is positive. Putting $j = K-k$ in (2.3) we have

$$\begin{aligned} g_{p,K}(x) &= i\left(\frac{K}{p}\right)\sum_{|j|\leqslant(p-1)/2}\left(\frac{K-j}{p}\right)\frac{1}{\zeta_p^{j+x}-1} \\ &= i\left(\frac{K}{p}\right)\sum_{j=0}^{(p-1)/2}\left(\frac{K-j}{p}\right)\left(\frac{1}{\zeta_p^{j+x}-1} - \frac{1}{\zeta_p^{-j-1+x}-1}\right), \end{aligned}$$

since $K+j+1 \equiv -(K-j) \pmod p$. Evidently $g_{p,K}(1-x) = \overline{g_{p,K}(x)}$, so $g_{p,K}(1-x) = g_{p,K}(x)$ since $g_{p,K}(x)$ is real-valued. However we see that the minimum of $g_{p,K}(x)$ is obtained at a unique point in $(0,1)$, so that must be at $x = 1/2$. Now

$$f_p(-1) = \sum_{a=1}^{p-1} (-1)^a \left(\frac{a}{p}\right) = \sum_{\substack{a=1 \\ a \text{ even}}}^{p-1} \left(\frac{a}{p}\right) - \sum_{\substack{b=1 \\ b \text{ even}}}^{p-1} \left(\frac{p-b}{p}\right)$$

where $a = p-b$ is odd in the second sum,

$$f_p(-1) = 2\sum_{d=1}^{(p-1)/2} \left(\frac{2d}{p}\right) = 2\left(\frac{2}{p}\right)\sum_{d=1}^{(p-1)/2}\left(\frac{d}{p}\right) = 2\left(2\left(\frac{2}{p}\right) - 1\right)h(-p),$$

where $h(-p)$ is the class number of $\mathbb{Q}(\sqrt{-p})$ (see Section 2 of [2]). By (2.3), and since $f_p(\zeta_p) = i\sqrt{p}$ by Gauss, we have

$$\begin{aligned} g_{p,K}\left(\frac{1}{2}\right) &= -\left(\frac{K}{p}\right)\frac{\sqrt{p}}{2} f_p(-1) = \sqrt{p}\left(-2\left(\frac{2K}{p}\right) + \left(\frac{K}{p}\right)\right)h(-p) \\ &= \sqrt{p}\left(2 + \left(\frac{K}{p}\right)\right)h(-p) > 0. \end{aligned}$$

This shows that $f_p(z)$ has no zeros on the arc from $\zeta_p^{(p-1)/2}$ to $\zeta_p^{(p+1)/2}$ when $p \equiv 3 \pmod 4$.

Now let's consider the arc from ζ_p^{-1} to ζ_p. Take $K = p-1$, and consider $g_{p,K}(x)$ as defined in (2.3). Usually $g_{p,K}(x)$ would have a discontinuity at 1, but here since $\left(\frac{K+1}{p}\right) = \left(\frac{0}{p}\right) = 0$ we do not have this problem. Thus $g_{p,K}$ is a continuous function on $(0,2)$, and we may study $f_p(z)$ on the arc from ζ_p^{-1} to ζ_p by studying $g_{p,K}(x)$ on $(0,2)$. Note that for any p, $f_p(1) = \sum_{a=1}^{p-1}\left(\frac{a}{p}\right) = 0$, so that there is at least a simple zero at $z = 1$. Also $f_p'(1) = -i(-1/p)f_p(\zeta_p)g_{p,p-1}(1)$ by (2.3). Since $f_p(z) = (-1/p)z^p f_p(\overline{z})$, we deduce that $g_{p,p-1}(x) = -(-1/p)g_{p,p-1}(2-x)$.

If $p \equiv 1 \pmod 4$ then $g_{p,p-1}(1) = 0$ and so $f_p'(1) = 0$. Now, as in the proof of (2.4), the first part of the proof of Proposition 2, we have $|g_{p,K}'(x)| > 0$ for all $x \in (0,2)$. Therefore g has only a simple zero at $x = 1$, and thus f_p has a double zero at 1.

If $p \equiv 3 \pmod 4$ then, as in the second part of the proof of Proposition 2, $|g_{p,K}''(x)| > 0$ for $x \in (0,2)$. Thus there is a unique minimum of $g_{p,K}(x)$ on $(0,2)$, but since $g_{p,p-1}(x) = g_{p,p-1}(2-x)$ this must be attained at $x = 1$. However, by (2.3), and as $f_p(\zeta_p) = i\sqrt{p}$ by Gauss,

$$g_{p,K}(1) = -\frac{f_p'(1)}{\sqrt{p}} = -\frac{1}{\sqrt{p}}\sum_{a=1}^{p-1} a\left(\frac{a}{p}\right) = \sqrt{p}h(-p) > 0,$$

(see [2], Section 2), and so $g_{p,K}(x) > 0$ and thus has no zeros in $(0,2)$. Therefore f_p has only a simple zero at $z = 1$ on this arc. □

3. Functions with random coefficients.

If $g \in \mathcal{F}_J$ then, for any $x \in (0,1)$, we have

$$\begin{aligned}\frac{1}{2}g''(x) &= \frac{1}{x^3} + \frac{1}{(1-x)^3} + \sum_{\substack{|j|<J \\ j\neq 0,-1}} \frac{\delta_j}{(x+j)^3} \\ &\geqslant \frac{1}{x^3} + \frac{1}{(1-x)^3} - \sum_{\substack{|j|<J \\ j\neq 0,-1}} \frac{1}{(x+j)^3} \\ &> 2\frac{1}{(1/2)^3} - 2\zeta(3) > 0.\end{aligned} \tag{3.1}$$

Since $\lim_{t\to 0^+} g'(t) = -\infty$ and $\lim_{t\to 1^-} g'(t) = \infty$ we deduce that $g'(x)$ has exactly one zero in $(0,1)$, call it x_0. Note that $g(x)$ attains its minimum value at x_0. If $0 \leqslant t < 1/\pi$ then

$$-g'(t) \geqslant \frac{1}{t^2} - 2\left(\frac{1}{(1/2)^2} + \frac{1}{(3/2)^2} + \frac{1}{(5/2)^2} + \dots\right) = \frac{1}{t^2} - \pi^2 > 0.$$

Similarly if $1 - 1/\pi < t \leqslant 1$ then $g'(t) > 0$. Thus

$$x_0 \in \left[\frac{1}{\pi}, 1 - \frac{1}{\pi}\right]. \tag{3.2}$$

We now show that few g are small in absolute value, at their minimum x_0.

PROPOSITION 3. — *We have $|g(x_0)| > J^{-1/4}$ for almost all $g \in \mathcal{F}_J$, where $g'(x_0) = 0$, uniformly as $J \to \infty$.*

Proof. — Consider the subset S of $\mathcal{F}_J$ with all the δ_j fixed given values, except when $j \in [I, I + I^{1/2}]$ where $I = J^{1/4}$. Let $f \in S$ with $\delta_j = -1$ for all $j \in [I, I + I^{1/2}]$. Suppose that $f'(x_1) = 0$ and let

$$\gamma = \sum_{\substack{|j|<J \\ j\notin[I,I+I^{1/2}]}} \frac{\delta_j}{x_1 + j}$$

where $\delta_0 = 1$, $\delta_{-1} = -1$. Let g be any element of S with $g'(x_0) = 0$.

By (3.1) note that

$$|x_1 - x_0| \ll \left|\int_{x_0}^{x_1} f''(t)dt\right| = |f'(x_0) - f'(x_1)| = |f'(x_0)| \tag{3.3}$$

$$= |f'(x_0) - g'(x_0)| \leqslant 2 \sum_{j\in[I,I+I^{1/2}]} \frac{1}{(x_0+j)^2} \ll \frac{1}{I}.$$

Hence, keeping in mind x_0, $x_1 \in [1/\pi, 1-1/\pi]$,

$$g(x_0) - \gamma = \sum_{j\in[I,I+I^{1/2}]} \frac{\delta_j}{x_0+j} + O\Bigg(\sum_{\substack{|j|<J\\ j\notin[I,I+I^{1/2}]}} \left|\frac{1}{x_0+j} - \frac{1}{x_1+j}\right|\Bigg)$$

$$= \frac{1}{I} \sum_{j\in[I,I+I^{1/2}]} \delta_j + O\Bigg(\sum_{j\in[I,I+I^{1/2}]} \left|\frac{1}{I} - \frac{1}{x_0+j}\right| + |x_1 - x_0|\Bigg)$$

$$= \frac{1}{I} \sum_{j\in[I,I+I^{1/2}]} \delta_j + O\Big(\frac{1}{I}\Big),$$

since each $|1/I - 1/(x_0+j)| \ll 1/I^{3/2}$ and there are $I^{1/2}$ such terms. Therefore if $|g(x_0)| \leqslant 1/I$ then

$$\sum_{j\in[I,I+I^{\frac{1}{2}}]} \delta_j = -\gamma I + O(1). \tag{3.4}$$

Now, the δ_j are independent binomial random variables, so the distribution of their sum tends towards the normal distribution. Therefore the maximum probability for (3.4) to occur happens when $\gamma = 0$; and so (3.4) holds with probability $O(I^{-1/4})$, for any γ, implying Proposition 3. □

4. Proof of Theorem 2.

Suppose that $g \in \mathcal{F}_J$ and $f \in \mathcal{F}_K$, with $J < K$, such that the δ_j are the same in each for $|j| < J$. Select $x_0, x_1 \in (0,1)$ so that $g'(x_0) = 0$ and $f'(x_1) = 0$. Now

$$|f(x_1) - f(x_0)| \leqslant \sum_{|j|<K} \left|\frac{1}{x_1+j} - \frac{1}{x_0+j}\right| \ll \sum_{|j|<K} \frac{|x_1 - x_0|}{j^2+1} \ll |x_1 - x_0|,$$

since x_0, $x_1 \in [1/\pi, 1-1/\pi]$. Arguing exactly as in (3.3), we see that $|x_0 - x_1| \ll 1/J$, and so we have

$$|f(x_1) - f(x_0)| \ll \frac{1}{J}. \tag{4.1}$$

We next consider the mean-square of

$$|f(x_0) - g(x_0)| = \Big| \sum_{J \leqslant |j| < K} \frac{\delta_j}{x_0 + j} \Big|.$$

To do so we will need to sum over all $\delta = \{\delta_j\}_{J \leqslant |j| < K} \in \Delta_{J,K}$, that is the set of all possibilities with each $\delta_j = -1$ or 1 (note that there are 2 possible values for each δ_j so the set $\Delta_{J,K}$ has 2^{2K-2J} elements). With this notation, the mean square is

$$\begin{aligned}
\frac{1}{2^{2K-2J}} \sum_{\delta \in \Delta_{J,K}} \Big| \sum_{J \leqslant |j| < K} \frac{\delta_j}{x_0 + j} \Big|^2 \\
&= \sum_{J \leqslant |j_1|, |j_2| < K} \frac{1}{(x_0 + j_1)(x_0 + j_2)} \frac{1}{2^{2K-2J}} \sum_{\delta \in \Delta_{J,K}} \delta_{j_1} \delta_{j_2} \\
&= \sum_{J \leqslant |j| < K} \frac{1}{(x_0 + j)^2} \asymp \frac{1}{J}.
\end{aligned}$$

Thus if $\psi_J \to \infty$ as $J \to \infty$ then

$$\Big| \sum_{J \leqslant |j| < K} \frac{\delta_j}{x_0 + j} \Big| < \frac{\psi_J}{J^{1/2}}, \tag{4.2}$$

for almost all choices of the δ_j.

Combining (4.1) and (4.2), we see that for almost all choices of δ_j $(J \leqslant |j| < K)$ we have

$$|f(x_1) - g(x_0)| \leqslant |f(x_1) - f(x_0)| + |f(x_0) - g(x_0)| < \frac{2\psi_J}{J^{1/2}}. \tag{4.3}$$

Taking $\Psi_J = J^{1/4}/2$, and combining this with Proposition 3 we see that for almost all $g \in \mathcal{F}_J$, and almost all extensions f of g to $\mathcal{F}_K$, $f(x_1)$ has the same sign as $g(x_0)$. Summing up over all $g \in \mathcal{F}_J$ we deduce that $\omega_K = \omega_J + o(1)$, where

$$\omega_J := \frac{\#\{g \in \mathcal{F}_J : g(x) = 0 \ \text{ for some } x \in (0,1)\}}{\#\{g \in \mathcal{F}_J\}},$$

and the "$o(1)$" term depends only on J. Therefore $\lim_{J\to\infty} \omega_J$ exists, and equals κ_1 say.

Strong bounds on κ_1, which imply those in the statement of Theorem 2, are given in Proposition 6 in Section 8.

Theorem 2 follows.

5. Proofs of Theorems 1 and $1\frac{1}{2}$.

Let $1 \leqslant K \leqslant p-1$ be an integer. If $\left(\frac{K}{p}\right) = \left(\frac{K+1}{p}\right)$ then by Proposition 2 there is exactly one zero of $f_p(z)$ on the arc from ζ_p^K to ζ_p^{K+1}; by Lemma 2 this happens for $\sim p/2$ values of K. Suppose now that $\left(\frac{K}{p}\right) = -\left(\frac{K+1}{p}\right)$ so that $f_p(z)$ has either 0 or 2 zeros on the arc from ζ_p^K to ζ_p^{K+1} depending on whether $\min_{x\in(0,1)} g_{p,K}(x)$ is positive or not. To decide this question we need the following proposition:

PROPOSITION 4. — *Suppose $J \leqslant \sqrt{p}$, and $J \to \infty$ as $p \to \infty$. For almost all $1 \leqslant K \leqslant p-1$ we have*

$$g_{p,K}(x) = \frac{p}{2\pi}\left(\frac{K}{p}\right) \sum_{|j|<J} \left(\frac{K-j}{p}\right)\frac{1}{j+x} + O\left(\frac{p}{J^{1/3}}\right),$$

uniformly for all $x \in (0,1)$.

Proof. — Note that for $J \leqslant |j| < p/2$,

$$\left|\frac{1}{\zeta_p^{j+x}-1} - \frac{1}{\zeta_p^j - 1}\right| = \left|\frac{\zeta_p^x - 1}{(\zeta_p^{j+x}-1)(\zeta_p^j-1)}\right| \asymp \frac{px}{j(j+x)} \ll \frac{p}{j^2},$$

and, for $|j| < J$,

$$\frac{1}{\zeta_p^{j+x}-1} = \frac{p}{2i\pi}\ \frac{1}{(j+x)} + O(1).$$

Hence, putting $j = K-k$ in (2.3), we have

$$\begin{aligned} g_{p,K}(x) &= i\left(\frac{K}{p}\right) \sum_{|j|<p/2} \left(\frac{K-j}{p}\right)\frac{1}{\zeta_p^{j+x}-1} \\ &= \frac{p}{2\pi}\left(\frac{K}{p}\right) \sum_{|j|<J} \left(\frac{K-j}{p}\right)\frac{1}{j+x} \\ &\qquad + i\left(\frac{K}{p}\right) \sum_{J\leqslant|j|<p/2} \left(\frac{K-j}{p}\right)\frac{1}{\zeta_p^j-1} + O\left(J + \frac{p}{J}\right). \end{aligned}$$

We now show that the mean-square of the second term above is small, which proves the proposition. By Lemma 2,

$$
\begin{aligned}
&\sum_{K=1}^{p}\Big|\sum_{J\leqslant|j|<p/2}\Big(\frac{K-j}{p}\Big)\frac{1}{\zeta_p^j-1}\Big|^2\\
&\quad=\sum_{J\leqslant|j_1|,\ |j_2|<p/2}\frac{1}{(\zeta_p^{j_1}-1)(\zeta_p^{-j_2}-1)}\sum_{K=1}^{p}\Big(\frac{K-j}{p}\Big)\Big(\frac{K-j_2}{p}\Big)\\
&\quad=p\sum_{J\leqslant|j|<p/2}\frac{1}{|\zeta_p^j-1|^2}-\Big|\sum_{J\leqslant|j|<p/2}\frac{1}{\zeta_p^j-1}\Big|^2\\
&\quad\ll p\sum_{J\leqslant|j|<p/2}\Big(\frac{p}{j}\Big)^2+\Big(\sum_{J\leqslant|j|<p/2}\frac{p}{j}\Big)^2\ll\frac{p^3}{J}+p^2\log^2 p.
\end{aligned}
$$

This proves the proposition. □

By Proposition 4 we know that for almost all K with $\left(\frac{K}{p}\right)=-\left(\frac{K+1}{p}\right)$ the minimum value of $\frac{2\pi}{p}g_{p,K}(x)$ equals the minimum of $\left(\frac{K}{p}\right)\sum_{|j|<J}\left(\frac{K-j}{p}\right)\frac{1}{j+x}+O(J^{-\frac{1}{3}})$. For such K the minimum value of $g_{p,K}(x)$ is non-positive if and only if the minimum of $\left(\frac{K}{p}\right)\sum_{|j|<J}\left(\frac{K-j}{p}\right)\frac{1}{j+x}$ is non-positive, unless

$$
\Big(\frac{K}{p}\Big)\sum_{|j|<J}\Big(\frac{K-j}{p}\Big)\frac{1}{j+x}\ll\frac{1}{J^{\frac{1}{3}}}. \tag{5.1}
$$

Now choose $J=[\log p/10]$. Given any choice of $\delta_j\in\{-1,1\}$, $0<|j|<J$ with $\delta_0=1$, and $\delta_{-1}=-1$, by Lemma 3 there are $\sim p/2^{2J-2}$ values of K with $\left(\frac{K}{p}\right)\left(\frac{K-j}{p}\right)=\delta_j$ for each j. Therefore (5.1) fails, for almost all K, by Proposition 3. Appealing now to Theorem 2 we have proved that for $\sim\kappa_1 p/2$ values of K with $\left(\frac{K}{p}\right)=-\left(\frac{K+1}{p}\right)$, the minimum of $g_{p,K}(x)$ is <0. For such K, $f_p(z)$ has two zeros on the arc from ζ_p^K to ζ_p^{K+1}, so that the total number of such zeros is $\sim\kappa_1 p$. Theorems 1 and $1\frac{1}{2}$ follow.

6. Pseudo-Gauss Sums: Proof of the first part of Theorem 4.

In this section, we wish to study the distribution of $f_p(\zeta_p^{K+1/2})$. By (2.3) and Proposition 4 we have (if $(\sqrt{p}>)J\to\infty$ as $p\to\infty$) for almost all $1 \leqslant K \leqslant p-1$,

$$f_p(\zeta_p^{K+\frac{1}{2}}) = \frac{i f_p(\zeta_p)}{\pi}\Big(\sum_{|j|<J}\Big(\frac{K-j}{p}\Big)\frac{1}{j+\frac{1}{2}} + O\Big(\frac{1}{J^{1/3}}\Big)\Big) \tag{6.1}$$
$$= \eta\frac{\sqrt{p}}{\pi}\Big(\sum_{|j|<J}\Big(\frac{K-j}{p}\Big)\frac{1}{j+\frac{1}{2}} + O\Big(\frac{1}{J^{1/3}}\Big)\Big),$$

where $\eta = \pm 1$ or $\pm i$ is fixed. Thus, by Lemma 3, we have that for any fixed real number ρ

$$\lim_{p\to\infty}\frac{1}{p}\#\Big\{K : 1 \leqslant K \leqslant p \text{ and } H_p\Big(\frac{K+\frac{1}{2}}{p}\Big) < \rho\sqrt{p}\Big\}$$

exists and equals

$$\lim_{J\to\infty}\ \operatorname{Prob}\Big(\sum_{|j|<J}\frac{\delta_j}{j+\frac{1}{2}} < \pi\rho : \ \delta \in \Delta_{0,J}\Big) \tag{6.2}$$

(using the notation $\Delta_{J,K}$ of Section 4). One may obtain an expression for this probability as follows: Recall that

$$\int_0^\infty \frac{\sin y}{y}dy = \frac{\pi}{2},$$

and so for any $k \neq 0$

$$\frac{2}{\pi}\int_0^\infty \frac{\sin(kx)}{x}dx = \operatorname{sgn}(k)\frac{2}{\pi}\int_0^\infty \frac{\sin(|k|x)}{x}dx$$
$$= \operatorname{sgn}(k)\frac{2}{\pi}\int_0^\infty \frac{\sin y}{y}dy = \operatorname{sgn}(k),$$

where $\operatorname{sgn}(k)$ is the sign of k ($=1$ if $k>0$ and -1 if $k<0$). Hence the probability (6.2) equals

$$\frac{1}{2^{2J-1}}\sum_{\delta\in\Delta_{0,J}}\Big(\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \sin\Big(\Big(\sum_{|j|<J}\frac{\delta_j}{j+\frac{1}{2}} - \pi\rho\Big)x\Big)\frac{dx}{x}\Big)$$
$$= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{1}{2^{2J-1}}\sum_{\delta\in\Delta_{0,J}}\left(\frac{e^{ix\left(\sum_{|j|<J}\frac{\delta_j}{j+\frac{1}{2}}-\pi\rho\right)} - e^{-ix\left(\sum_{|j|<J}\frac{\delta_j}{j+\frac{1}{2}}-\pi\rho\right)}}{2i}\right)\frac{dx}{x}$$
$$= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \prod_{|j|<J}\left(\frac{e^{\frac{ix}{j+\frac{1}{2}}} + e^{-\frac{ix}{j+\frac{1}{2}}}}{2}\right)\left(\frac{e^{-ix\pi\rho} - e^{ix\pi\rho}}{2i}\right)\frac{dx}{x}$$
$$= \frac{1}{2} + \frac{1}{\pi}\int_{x=0}^\infty \sin(\rho\pi x)\prod_{|j|<J}\cos\Big(\frac{2x}{2j+1}\Big)\frac{dx}{x}.$$

Letting $J \to \infty$, we get

$$c_\rho = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \sin(\rho\pi x)C(x)\frac{dx}{x} \quad \text{where } C(x) := \prod_{\substack{n\geqslant 1\\ n \text{ odd}}} \cos^2\left(\frac{2x}{n}\right),$$

and thus Theorem 4 is proved. Note that this integral does converge: For any $x > 0$ we have

$$C(x) \ll \frac{1}{2^{\frac{3x}{\pi}}}$$

since this estimate is trivial for $x \leqslant 1$, and otherwise we note that $|\cos(2x/n)| < 1/2$ if $3x/\pi < n < 6x/\pi$. Thus the part of the integral with $x \geqslant 1$ is easily bounded. Since $\sin(\rho\pi x) \ll \rho\pi x$, the portion of the integral from 0 to 1 is also easily bounded.

Remark 1. — We use the above to study the multiplicative average size of $f_p(\zeta_p^{k+1/2})$. Due to the symmetry of c_ρ we have that

$$\frac{1}{p-1}\log\bigg(\prod_{k=1}^{p-1}\frac{f_p(\zeta_p^{k+1/2})}{\sqrt{p}}\bigg) = 2\int_0^\infty \log\rho \,\mathrm{d}\Big(c_\rho - \frac{1}{2}\Big).$$

Using our expression for c_ρ one can show that this is

$$= \gamma + \log\pi - \int_0^1 \frac{C(x)-1}{x}\,dx - \int_1^\infty \frac{C(x)}{x}\,dx.$$

All of these integrals converge, though we do not know their exact values.

Remark 2. — The expansion given in (6.1) for f_p, and the general technique involved, is very similar to that used by Montgomery [5] in showing that

i) $|f_p(z)| \ll \sqrt{p}\log p$ for all $|z| = 1$.

ii) If p is sufficiently large then there exists some value of z with $|z| = 1$ for which $|f_p(z)| > \frac{2}{\pi}\sqrt{p}\log\log p$.

Indeed to prove a result like that in (ii) we note that we may select each δ_j equal to the sign of j for $|j| < J = \varepsilon\log p$. By Lemma 3 there are many such K and we proceed as before with the expansion in (6.1), but now taking a little more care over the set of excluded K.

Remark 3. — Fix $t \in (0,1)$. By the argument above, we have, for any fixed real number ρ,

$$\begin{aligned}c_{\rho,t} :&= \lim_{p\to\infty} \frac{1}{p}\#\left\{K : 1 \leqslant K \leqslant p \text{ and } H_p\left(\frac{K+t}{p}\right) < \rho\sqrt{p}\right\}\\ &= \lim_{J\to\infty} \operatorname{Prob}\left(\delta \in \Delta_{0,J} : \sum_{|j|<J} \frac{\delta_j}{j+t} < \frac{\pi\rho}{\sin(\pi t)}\right)\\ &= \frac{1}{2} + \frac{1}{\pi}\int_{x=0}^{\infty} \sin\left(\frac{\rho\pi x}{\sin(\pi t)}\right) \prod_{j\in\mathbb{Z}} \cos\left(\frac{x}{j+t}\right)\frac{dx}{x}.\end{aligned}$$

Remark 4. — We can also use these techniques to investigate the distribution of values of $H_p(t)$ at $t = a/(p-1)$ for $1 \leqslant a \leqslant p-1$. We note that if $K \sim \alpha p$ then $\zeta_{p-1}^K = \zeta_p^{K+\alpha}\{1+o(1/p)\}$. Therefore we can get an expression similar to (6.1) for almost all $F_p(\zeta_{p-1}^K)$, but now with $\sum_{|j|<J}\left(\frac{K-j}{p}\right)\frac{1}{j+\alpha}$ replacing the sum in (6.1), and multiplying the whole expression through by $\sin(\alpha\pi)$. Thus the density of those K, for which $H_p\left(\frac{K}{p-1}\right) \leqslant \rho\sqrt{p}$, is

$$\frac{1}{2} + \frac{1}{\pi}\int_{\alpha=0}^{1}\int_{x=0}^{\infty} \sin\left(\frac{\rho\pi x}{\sin(\alpha\pi)}\right)\prod_{m\in\mathbb{Z}}\cos\left(\frac{x}{m+\alpha}\right)\frac{dx}{x}d\alpha.$$

We cannot see how to obtain a simpler expression.

It is not hard to modify this technique to determine the distribution of values of the Fekete polynomial (or, in fact, $H_p(t)$) at any "reasonably" distributed set of values.

7. The distribution of $g(1/2)$ for $g \in \mathcal{F}_J$ as $J \to \infty$.

We now look at the limiting distribution of $g(1/2) - 4$ for $g \in \mathcal{F}_J$ as $J \to \infty$. Define, for $N \geqslant 1$,

$$S_N(\underline{\delta}) = \sum_{|j+1/2|>N} \frac{\delta_j}{j+\frac{1}{2}},$$

where each $\delta_j = 1$ or -1 with probability $1/2$. We will prove that the distribution function of $S_1(\underline{\delta})$ decays *double exponentially.*

THEOREM 5. — *As $x \to \infty$, we have*

$$\operatorname{Prob}(|S_1(\underline{\delta})| > x) = \exp(-e^{x/2+O(1)}).$$

Proof of the second part of Theorem 4. — Note that

$$\mathrm{Prob}(S_1(\underline{\delta}) > x) = \mathrm{Prob}(S_1(\underline{\delta}) < -x) = \exp(-e^{x/2+O(1)}),$$

by symmetry. Taking $x = \pi\rho$, the result follows from (6.2). □

To prove Theorem 5 we study the $2k$-th moment of $S_N(\underline{\delta})$, call it $M_N(k)$, that is, the expectation of $S_N(\underline{\delta})^{2k}$. For example

$$M_N(1) = \sum_{|j+1/2|>N} \frac{1}{(j+\frac{1}{2})^2}.$$

Our aim is to determine the asymptotic behaviour of $M_1(k)$ for large k.

PROPOSITION 5. — *For large k,*

$$M_1(k) = (2\log k - 2\log\log k + O(1))^{2k}.$$

Proof. — To establish the lower bound, consider $\underline{\delta}$ such that $\delta_j = 1$ for all $1 \leqslant |j+1/2| \leqslant k/\log k$; and such that $S_{k/\log k}(\underline{\delta}) > 0$. The probability of this happening is $\asymp 1/2^{2k/\log k}$, and $S_1(\underline{\delta}) \geqslant 2\log k - 2\log\log k + O(1)$ for such $\underline{\delta}$. Hence

$$\begin{aligned} M_1(k) &\gg \frac{1}{2^{2k/\log k}}(2\log k - 2\log\log k + O(1))^{2k} \\ &= (2\log k - 2\log\log k + O(1))^{2k}. \end{aligned}$$

Now

$$M_N(k) = \sum_{j_1,j_2,\dots j_{2k}} \mathbb{E}\left(\frac{\delta_{j_1}}{j_1+\frac{1}{2}}\,\frac{\delta_{j_2}}{j_2+\frac{1}{2}}\cdots\frac{\delta_{j_{2k}}}{j_{2k}+\frac{1}{2}}\right),$$

where $\mathbb{E}$ stands for the expectation. Observe that a summand above is non-zero only if each value of j appears an even number of times amongst $j_1, j_2, \dots j_{2k}$. In particular $j_\ell = j_1$ for some $\ell > 1$, and then $\mathbb{E}(\prod_{1\leqslant i\leqslant 2k}\delta_{j_i}) = \mathbb{E}(\prod_{1\leqslant i\leqslant 2k,\ i\neq 1,\ell}\delta_{j_i})$. Summing over all $2k-1$ possibilities for ℓ in the above, we deduce that

$$M_N(k) \leqslant (2k-1)\sum_{|j+1/2|>N}\frac{1}{(j+\frac{1}{2})^2}M_N(k-1), \tag{7.1}$$

for all $k \geqslant 1$ and all $N \geqslant 1$. Iterating this inequality, we obtain

$$\begin{aligned} M_N(k) &\leqslant (2k-1)\cdot(2k-3)\cdots 3\cdot 1\cdot\left(\sum_{|j+1/2|>N}\frac{1}{(j+\frac{1}{2})^2}\right)^k \\ &\leqslant \frac{(2k)!}{k!2^k}\left(\frac{2}{N-\frac{1}{2}}\right)^k = \frac{(2k)!}{k!(N-\frac{1}{2})^k}. \end{aligned} \tag{7.2}$$

Now

$$|S_1(\underline{\delta}) - S_N(\underline{\delta})| \leqslant 2\lambda_N, \quad \text{where} \quad \lambda_N := \sum_{N \geqslant j+1/2 \geqslant 1} \frac{1}{j+\frac{1}{2}} = \log N + O(1).$$

Evidently the odd moments of $S_N(\underline{\delta})$ are zero. Therefore, by the binomial theorem and (7.2),

$$\begin{aligned} M_1(k) &= \sum_{j=0}^{k} \binom{2k}{2j} M_N(j)\, \mathbb{E}(|S_1(\underline{\delta}) - S_N(\underline{\delta})|^{2k-2j}) \\ &\leqslant \sum_{j=0}^{k} \binom{2k}{2j} \frac{(2j)!}{j!(N-\frac{1}{2})^j} (2\lambda_N)^{2k-2j} \\ &\leqslant (2\lambda_N)^{2k} \sum_{j=0}^{k} \frac{1}{j!}\left(\frac{k^2}{(N-\frac{1}{2})\lambda_N^2}\right)^j \leqslant (2\lambda_N)^{2k} \exp\left(\frac{k^2}{(N-\frac{1}{2})\lambda_N^2}\right). \end{aligned}$$

Taking $N = k/\log k$ we obtain the upper bound of the proposition. □

Proof of Theorem 5. — Take $k = c_1 x e^{x/2} + O(1)$ for some $c_1 > 0$, and then $\text{Prob}(|S_1(\underline{\delta})| > x) \leqslant x^{-2k} M_1(k) \ll \exp(-c_2 e^{x/2})$ for some constant $c_2 > 0$, if c_1 is sufficiently small, by Proposition 5.

The lower bound is more involved. Select integer k so that $2\log k - 2\log\log k$ is as close as possible to x. The contribution to $M_1(k)$ of those $\underline{\delta}$ with $|S_1(\underline{\delta})| < x - c_3$ is $\leqslant (x-c_3)^{2k} \leqslant M_1(k)/4$ if c_3 is sufficiently large. The contribution to $M_1(k)$ of those $\underline{\delta}$ with $|S_1(\underline{\delta})| > x + c_3$ is $\leqslant \int_{t>x+c_3} \text{Prob}(|S_1(\underline{\delta})| > t) t^{2k} dt \ll \int_{t>x+c_3} \exp(-c_2 e^{t/2}) t^{2k} dt \leqslant M_1(k)/4$ if c_3 is sufficiently large, using the upper bound from the paragraph above. Thus $M_1(k)/2 \leqslant \text{Prob}(x - c_3 \leqslant |S_1(\underline{\delta})| \leqslant x + c_3)(x+c_3)^k$ which implies that $\text{Prob}(|S_1(\underline{\delta})| \geqslant x - c_3) \geqslant M_1(k)/2(x+c_3)^k \gg \exp(-c_4 e^{x/2})$ for some constant $c_4 > 0$, by Proposition 5. Replacing $x - c_3$ by x gives the lower bound and thus our result. □

Remark. — We follow up on Remark 3 of Section 6. The arguments above (Theorem 5 and Proposition 5) hold just as well with "1/2" replaced by any fixed $t \in (0,1)$. Thus $1 - c_{\rho,t}$ and $c_{-\rho,t} = \exp(-\exp(\pi\rho/2\sin(\pi t)) + O(1)))$ for $\rho > 0$.

8. Bounds on κ_1.

Applying the method of Section 6, we note that for any real λ,

$$\begin{aligned}\pi_\lambda &:= \lim_{J\to\infty} \operatorname{Prob}\Big\{g \in \mathcal{F}_J : g\Big(\frac{1}{2}\Big) < 4\lambda\Big\} \\ &= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \sin((1-\lambda)x) \prod_{\substack{n\geqslant 3\\ n \text{ odd}}} \cos^2\Big(\frac{x}{2n}\Big)\frac{dx}{x}.\end{aligned} \tag{8.1}$$

We can use this to obtain numerical bounds on κ_1 using the following result.

PROPOSITION 6. — *We have* $\pi_{.013496\ldots} \geqslant \kappa_1 \geqslant \pi_0$.

Using Simpson's rule to compute the integrals in (8.1) we obtain $.000813 > \pi_{.013496\ldots} \geqslant \kappa_1 \geqslant \pi_0 > .000668$, from which we deduce the bounds on κ_0 in the introduction.

Proof. — Again selecting x_0 so that $g(x_0)$ is minimal, we have, by definition, that

$$\kappa_1 = \lim_{J\to\infty} \operatorname{Prob}\{g \in \mathcal{F}_J : g(x_0) \leqslant 0\}.$$

Since $g(x_0) \leqslant g(1/2)$ we deduce the lower bound on κ_1 above.

To get the upper bound, write $x_0 = 1/2 + \nu$ so that $|\nu| < 1/2$. If $g(x_0) \leqslant 0$ then

$$\begin{aligned}g\Big(\frac{1}{2}\Big) &\leqslant g\Big(\frac{1}{2}\Big) - g(x_0) \\ &= 4 - \frac{1}{x_0} - \frac{1}{1-x_0} + \sum_{\substack{|j|<J\\ j\neq 0,-1}} \frac{\delta_j(x_0-\frac{1}{2})}{(j+\frac{1}{2})(j+x_0)} \\ &\leqslant -\frac{4\nu^2}{\frac{1}{4}-\nu^2} + \sum_{j=1}^{\infty} \frac{|\nu|}{(j+\frac{1}{2})(j+\frac{1}{2}+\nu)} + \sum_{j=-\infty}^{-2} \frac{|\nu|}{(j+\frac{1}{2})(j+\frac{1}{2}+\nu)} \\ &= -\frac{4\nu^2}{\frac{1}{4}-\nu^2} + \sum_{j=1}^{\infty} \frac{2|\nu|}{(j+\frac{1}{2})^2-\nu^2} = -\frac{(2|\nu|+4\nu^2)}{\frac{1}{4}-\nu^2} + \pi\tan(\pi|\nu|).\end{aligned}$$

Using Maple to compute the $\max_\nu$, we obtain

$$g\Big(\frac{1}{2}\Big) \leqslant \max_{|\nu|\leqslant\frac{1}{2}} \Big(\pi\tan(\pi|\nu|) - \frac{(2|\nu| + 4\nu^2)}{\frac{1}{4} - \nu^2}\Big) = 0.053986\ldots,$$

the maximum being attained at $\nu = \pm.057052\ldots$. □

Remark. — One can refine the above to get better bounds for κ_1. First note that $g(x) = 1/x + 1/(1-x)$ is the only element in $\mathcal{F}_1$, and in this case $x_0 = 1/2$; thus "1/2" appears in the definition of π_λ. More generally, let J be some positive integer. For each $\gamma \in \mathcal{F}_J$ select χ_0 so that $\gamma(\chi_0)$ is minimal. We again have $g(x_0) \leqslant g(\chi_0)$, so if $g(\chi_0) \leqslant 0$ then $g(x_0) \leqslant 0$. On the other hand, if $g(x_0) \leqslant 0$ then we can again get an explicit upper bound on $g(\chi_0)$ and proceed as above. This can be used to give another proof that κ_1 exists.

9. Zeros off the unit circle.

Proof of Theorem 3. — Theorem 3 holds trivially if there is a zero of $f_p(t)$ on the unit circle in the arc from ζ_p^K to ζ_p^{K+1}. Thus we shall henceforth assume that there is no such zero. Let $h(x) := H_p((K+x)/p)/H_p(K/p)$, so that $|h(x)| = |f_p(\zeta_p^{K+x})/\sqrt{p}|$, and $h(x)$ is a continuous real-valued function. Now the hypothesis implies that $h(y) < \epsilon$ for some $y \in (0,1)$ (in fact, $t = \zeta_p^{K+y}$), while our assumption above implies that $h(x) \neq 0$ for all $x \in (0,1)$. By (2.3) we have, uniformly for $|x| \leqslant 2/3$,

$$\begin{aligned} h(x) &= \frac{\sin(\pi x)}{p}\Big(\frac{1}{\sin(\pi x/p)} + \Big(\frac{K}{p}\Big) \sum_{1\leqslant|K-k|<p/2} \frac{(k/p)}{\sin(\pi(x+K-k)/p)}\Big) \\ &= 1 - (C + O(1))x, \quad \text{where } C := -(K/p) \sum_{1\leqslant|K-k|<p/2} \frac{(k/p)}{K-k}. \end{aligned} \tag{9.1}$$

So if $h(y) < \epsilon$ for some sufficiently small y then $h(2y) = 2h(y) - 1 + O(y) < 0$, contradicting our assumption. Therefore we may assume that $y \gg 1$, and also $1 - y \gg 1$ by the symmetric argument. Thus $g_{p,K}(y) \ll \sqrt{p}|f_p(t)|/\sin(\pi y) \ll \epsilon p$ by (2.3), so that

$$g_{p,K}(x_0) \leqslant g_{p,K}(y) \ll \epsilon p$$

where x_0 is defined as in Section 3.

Let $x_1 = x_0 - \epsilon^{1/2}$, and $x_2 = x_0 + \epsilon^{1/2}$, and then $\alpha_j = \zeta_p^{x_j}$ for $j = 1, 2$. Let $R = 1 - \epsilon^{1/3}/p$. We shall consider the variation in argument of

$$G(z) := i\Big(\frac{K}{p}\Big)\frac{p}{f_p(\zeta_p)}\,\frac{f_p(z)}{z^p - 1} = i\Big(\frac{K}{p}\Big) \sum_{|K-k|<p/2} \Big(\frac{k}{p}\Big)\frac{1}{z\zeta_p^{-k} - 1},$$

as z goes around (in the anti-clockwise direction) the box bounded by the four curves, $\mathcal{C}_1$, the arc of the unit circle from α_1 to α_2, then $\mathcal{C}_2$, the straight line segment from α_2 to $R\alpha_2$, then $\mathcal{C}_3$, the arc of the circle of radius R, from $R\alpha_2$ to $R\alpha_1$, then finally $\mathcal{C}_4$, the straight line segment from $R\alpha_1$ back to α_1.

We know that $G(z)$ is real valued and positive on the arc $\mathcal{C}_1$. We shall show that $G(z)$ has positive imaginary part on $\mathcal{C}_2$, that $G(z)$ has negative real part on $\mathcal{C}_3$, and that $G(z)$ has negative imaginary part on $\mathcal{C}_4$, This shows that the change in argument of $G(z)$ is 2π as we go around our box, so that there is exactly one zero in our box. This implies a little more than Theorem 3.

To estimate $H(r,x) := G(r\zeta_p^{(K+x)/p})$ when $R \leqslant r \leqslant 1$, for a value of $x \in [x_1, x_2]$, we calculate the Taylor series expansion around $r = 1$, which is

$$H(r,x) = g_{p,K}(x) - \frac{(1-r)^2}{2r}\left(\frac{p}{2\pi}\right)^2 g''_{p,K}(x) \\ +i\,\frac{1-r^2}{2r}\;\frac{p}{2\pi} g'_{p,K}(x) + O\left(\frac{(1-r)^3}{r}p^4\right).$$

From the proof of Proposition 2 we have, since x is bounded away from 0 and 1,

$$g_{p,K}(x) = g_{p,K}(x_0) + O((x-x_0)^2 p),\;\; g'_{p,K}(x) \asymp (x-x_0)p \text{ and } g''_{p,K}(x) \asymp p.$$

Therefore

$$\begin{aligned} \mathrm{Im}(G(z)) &= \mathrm{Im}(H(r,x)) \asymp \epsilon^{1/2}p^2(1-r) + O((1-r)\epsilon^{2/3}p^2) > 0 \;\text{ on }\; \mathcal{C}_2,\\ \mathrm{Im}(G(z)) &= \mathrm{Im}(H(r,x)) \asymp -\epsilon^{1/2}p^2(1-r) + O((1-r)\epsilon^{2/3}p^2) < 0 \;\text{ on }\; \mathcal{C}_4,\\ \mathrm{Re}(G(z)) &= \mathrm{Re}(H(r,x)) \asymp -\epsilon^{2/3}p + O(\epsilon p) < 0 \;\text{ on }\; \mathcal{C}_3, \end{aligned}$$

as required. □

Remark. — By (9.1) we see that

$$\max_{|z|=1} |f_p(z)| \asymp \sqrt{p}\,\max_{K\in\mathbb{Z}} \sum_{j\neq 0} \frac{1}{j}\left(\frac{K+j}{p}\right).$$

This again allows us to recover the results of Montgomery [5], as in Remark 2 of Section 6.

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