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The structure of the tensor product of $\mathbb{F}_2[-]$ with a finite functor between $\mathbb{F}_2$-vector spaces

THE STRUCTURE OF THE TENSOR PRODUCT OF $\mathbb{F}_2[-]$ WITH A FINITE FUNCTOR BETWEEN $\mathbb{F}_2$-VECTOR SPACES

by Geoffrey M.L. POWELL

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1. Introduction.

The category $\mathcal{F}$ of functors from the category of finite-dimensional $\mathbb{F}_2$-vector spaces to the category of all $\mathbb{F}_2$-spaces has a rich structure; for example it 'contains' the $\mathbb{F}_2$-representations of all the symmetric groups as well as the representations of the finite general linear groups over $\mathbb{F}_2$. (Here, $\mathbb{F}_2$ denotes the prime field of characteristic two.) The locally finite objects in $\mathcal{F}$ are termed *analytic* functors, since they are colimits of their 'polynomial' sub-objects, where polynomial refers to the notion of a polynomial functor introduced by Eilenberg and MacLane. The full sub-category of analytic functors is written as $\mathcal{F}_\omega$.

Part of the interest of the category $\mathcal{F}_\omega$ is that it contains non-finite objects with finite socle. For example, the injective envelope in $\mathcal{F}$ of the first exterior power functor Λ^1 is an analytic, uniserial, non-finite functor, $\bar{I}$, with composition factors which are the exterior power functors Λ^n, each occurring once. The tensor products $\bar{I}^{\otimes s}$ are also injective in $\mathcal{F}$ and Lionel Schwartz has proposed, in conjunction with Nick Kuhn [K1], the conjecture that these functors are artinian, that is that every descending sequence of sub-objects stabilizes. This turns out to be a difficult question to address; the author has established the conjecture for $s = 2$ (see [P1]) but the

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arguments relied on knowledge of the structure of the functors $\bar{I} \otimes \Lambda^n$, localized away from the finite functors. A survey of the artinian conjecture is given in [P5].

The main result of this paper is the following theorem:

THEOREM 1. — *Suppose that F is a finite functor, then the functor $\bar{I} \otimes F$ is artinian.*

In fact, a stronger result concerning the structure of such functors is proved. A functor is said to be *simple modulo finites* if every proper sub-functor is finite. It is clear that examples of such functors are provided by finite functors and uniserial functors; however, the lattice structure of most such functors is more exotic. The following result is easily established and gives a characterization of such functors.

PROPOSITION 1.0.1. — *A functor F which takes finite dimensional values is simple modulo finites if and only if $F = \operatorname{colim} F_i$ is the colimit of a system of finite sub-objects $F_i \subset F$ and, for all n there exists an integer $N = N(n)$ such that, whenever $G \subset F$ such that $G \cap F_N \neq G$, then G contains F_n.*

Theorem 1 is shown to imply the stronger result:

THEOREM 2. — *Suppose that F is a finite functor; the functor $\bar{I} \otimes F$ admits a finite filtration, the filtration quotients of which are non-finite functors which are simple modulo finites.*

Remark 1.0.2. — In the terminology introduced in [P1], 'simple modulo finites' is equivalent to simple artinian of type one. A functor admitting a finite filtration of the form given in Theorem 2 is an artinian functor of type one. This terminology is related to the Krull codimension of objects in the category of locally finite functors.

The proof of Theorem 1 involves the construction of a non-finite simple modulo finites sub-functor:

THEOREM 3. — *Suppose that S is a simple functor, then $\bar{I} \otimes S$ contains a unique non-finite sub-functor which is simple modulo finites.*

The key ingredient to the proof of Theorem 1 is a detection result for non-finite sub-functors of $(\bar{I}/\pi_s) \otimes F$, where $\pi_s \subset \bar{I}$ denotes the sub-

functor $p_s\bar{I}$ of $\bar{I}$ of length s. This result uses the functors $\widetilde{\nabla}_n$ which were introduced in [P2], and is of independent interest.

THEOREM 4. — *Suppose that F is a finite functor such that $\widetilde{\nabla}_n G \neq 0$, for every non-trivial sub-functor $0 \neq G \subset F$. Suppose that $s \geq 0$ and that $H \hookrightarrow (\bar{I}/\pi_s) \otimes F$. If $\widetilde{\nabla}_{n+1} H = 0$, then H is a finite functor.*

This has the following corollary, which is the statement which is used in the proof of Theorem 3.

COROLLARY 1.0.3. — *Suppose that F satisfies the hypotheses of Theorem 4, and that $H \hookrightarrow \bar{I} \otimes F$ is a sub-functor. If $\widetilde{\nabla}_{n+1} H$ is a finite functor, then H is a finite functor.*

The hypotheses on the functor F appear rather technical; the reader should bear in mind the example of interest, where F is a simple functor. In this case, there is a maximal n for which $\widetilde{\nabla}_n S$ is non-trivial, and this is the value which should be used in applying Corollary 1.0.3.

The proof of Theorem 4, although the key to the whole paper, is postponed to Section 7. The essential input to this result is the knowledge derived from the paper [P3] on the structure of the functors $\bar{I} \otimes \Lambda^n$.

1.1. Outline of the paper.

The paper is organized as follows: Section 2 provides a brief review of the functor category $\mathcal{F}$ and a discussion of 'simple modulo finites' functors.

Section 3 introduces the strategy of proof; namely Proposition 3.0.3 gives a criterion for showing that a functor is artinian. The proof of Theorem 1 reduces to the study of $\bar{I} \otimes S$, where S is a simple functor. It suffices to construct an artinian sub-functor $X \subset \bar{I} \otimes S$ so that X satisfies the hypotheses of Proposition 3.0.3.

A sub-functor X_S of $\bar{I} \otimes S$ is defined in Section 4; the pair of functors $X_S, \bar{I} \otimes S$ plays the rôle of X, Y in Proposition 3.0.3. The main result of the section is Theorem 4.4.3, which shows that X_S is artinian. This argument depends upon Theorem 4.

In Section 5, the proof of Theorem 1 is completed by establishing the second of the hypotheses of Proposition 3.0.3. This involves the only calculational input concerning the simple functors, which is provided by Lemma 5.0.3. The proof of Theorem 2 is given in Section 6, using detection properties associated to the functor $\widetilde{\nabla}_2$.

Section 7 provides a proof of the detection result, Theorem 4, which was used in Section 4 to show that the functor X_S is artinian.

2. Review of $\mathcal{F}$ and 'simple modulo finites' functors.

For the convenience of the reader, some details concerning the category $\mathcal{F}$ are recalled; standard references for the basic theory are the articles of Kuhn [K1], [K2], [K3]. The shift functor $\widetilde{\Delta} : \mathcal{F} \to \mathcal{F}$ is defined by the equation $\widetilde{\Delta} F(V) := F(V \oplus \mathbb{F}_2)$; there is a natural inclusion $F \hookrightarrow \widetilde{\Delta} F$, the cokernel of which is written ΔF; this defines the *difference functor* Δ. A functor is said to be finite if it has a finite composition series; it is polynomial of degree $\leq d$ (in the sense of Eilenberg and MacLane) if $\Delta^{d+1} F = 0$; a functor F is finite if and only if it takes finite dimensional values and is polynomial.

There is an inclusion functor $\mathrm{pol}_d \, \mathcal{F} \hookrightarrow \mathcal{F}$, where $\mathrm{pol}_d \mathcal{F}$ is the full sub-category of functors of polynomial degree $\leq d$. This functor admits a right adjoint p_d and a left adjoint q_d, which may be regarded as functors to $\mathcal{F}$, by composition with the inclusion.

The category $\mathcal{F}$ has enough injectives; the injective functor I_V is defined by the equation $\mathrm{Hom}_{\mathcal{F}}(F, I_V) = DF(V)$, where D is the duality functor on $\mathcal{F}$ given by $DF(V) := F(V^*)^*$, '$*$' denoting vector space duality. In particular, the functor $I_{\mathbb{F}_2}$ decomposes as $I_{\mathbb{F}_2} \cong \bar{I} \oplus \mathbb{F}_2$, where $\bar{I}$ is the injective envelope of Λ^1. The functor $\bar{I}$ is uniserial and its polynomial filtration coincides with the socle filtration; the functor $p_s \bar{I} \subset \bar{I}$ denotes the largest sub-functor of degree s and has length s. For notational simplicity, the functor $p_s \bar{I}$ is written as π_s throughout this paper, with the convention that $\pi_{-1} = 0$.

The shift functor $\widetilde{\Delta}$ is left adjoint to the functor $- \otimes I$, whereas the difference functor Δ is left adjoint to the functor $- \otimes \bar{I}$.

The polynomial filtration of the difference functor was introduced in [P2], to which the reader is referred for details. There is a filtration of $\widetilde{\Delta}$ by left exact functors $[p_n \widetilde{\Delta}] \subset \widetilde{\Delta}$, which may be defined as the right adjoints to the functor $- \otimes D(\pi_n)$. The quotients $\widetilde{\nabla}_{n+1} := \widetilde{\Delta}/[p_n \widetilde{\Delta}]$ are not exact but preserve injections and surjections. These functors are of importance because they are calculable in many instances and preserve a reasonable amount of information on the action of $\widetilde{\Delta}$ on the category $\mathcal{F}$.

2.1. On 'simple modulo finites' functors.

It is important to know that non-finite analytic functors with finite socles usually *do* contain *non-finite* artinian sub-functors. The following general result confirms this; the proof given is chosen so as to be 'constructive':

PROPOSITION 2.1.1. — *Suppose that G is an analytic functor which has finite socle and which is non-finite. Then G contains a sub-functor F which is non-finite and simple modulo finites.*

Proof. — Construct a descending sequence of functors

$$\cdots F_n \subset F_{n-1} \subset \cdots \subset F_{-1} = G$$

such that, whenever $F' \subset F_n$ such that $p_n F' \neq p_n F_n$, then F' is finite. Start the chain by taking $F_{-1} : = G$; the functor p_{-1} should be taken to be trivial, so that the condition is vacuous.

Suppose now that F_n is specified and consider the set $\mathcal{S}: = \{H \mid H \subset F_n, H \text{ non-finite}\}$. This induces a set $\{p_{n+1}H \mid H \in \mathcal{S}\}$ of sub-functors of $p_{n+1}F_n$. The functor $p_{n+1}F_n$ is necessarily finite, so this set contains a (non-zero) minimal element under inclusion, corresponding to some $H \in \mathcal{S}$. Set $F_{n+1} : = H$; then, if $F' \subset F_{n+1}$ is non-finite, then $F' \in \mathcal{S}$, so that the minimality of $p_{n+1}F_{n+1}$ implies that $p_{n+1}F' = p_{n+1}F_{n+1}$. Thus, F_{n+1} satisfies the required property.

This completes the recursive construction of such a descending chain; define $F: = \bigcap F_n$. Suppose that $F' \subset F$ is a proper sub-functor, then there exists some n such that $p_n F' \subset p_n F$ is a proper inclusion. The above construction implies that $p_n F = p_n F_n$, hence it follows that F' is a finite functor, by the construction of the functors F_n. Namely, the constructed sub-functor F is simple modulo finites.

Finally, one must justify that the functor F is not finite. This follows from general considerations: namely, if G is a finite functor, there exists an integer $d = d(G)$ such that $\mathrm{Ext}^1_{\mathcal{F}}(S, G) = 0$ whenever S is a simple functor of polynomial degree $\geq d$. In particular, for any non-negative integer n there exists an integer $d(n) \geq n$ such that, if H is an analytic functor with finite socle and $p_d H = p_n H$ for some $d \geq d(n)$, then H is finite. Hence, suppose that F is finite, say $F = p_n F$. Choose $d = d(n)$; by construction F_d is not finite, hence $p_d F_d \neq p_n(F_d) = p_n F$, by the choice of d. This is a contradiction, since $p_d F_d = p_d F = p_n F$. $\qquad\square$

3. The reduction strategy.

The strategy of the paper is to replace the following elementary result, by a more subtle version (Proposition 3.0.3 below).

LEMMA 3.0.1. — *Suppose that F is an analytic functor such that $F(\mathbb{F}_2^t)$ is finite dimensional, where $t \geq 0$. The functor F is artinian if the functor $\Delta^{t+1} F$ is artinian.*

It is tempting to assert that the converse should be true; however, at the time of writing, the author believes it unlikely that the converse will be proven in full generality without first obtaining a proof of the Artinian Conjecture [P1], [P4].

It is equally unfortunate that Lemma 3.0.1 is seldom of direct use. For example, $\Delta(\bar{I} \otimes F) = (\bar{I} \otimes F) \oplus F \oplus (I \oplus \bar{I}) \otimes \Delta F$, so that $\bar{I} \otimes F$ is itself a direct summand of $\Delta(\bar{I} \otimes F)$; this means that any direct inductive argument is circular.

However, the functor $\widetilde{\nabla}_2(\bar{I} \otimes F)$ is a sum $G \oplus (\bar{I} \otimes H)$, where G, H are finite functors and $\deg H < \deg F$. Hence, if all functors of the form $\bar{I} \otimes H$ are known to be artinian, for $\deg H < \deg F$, then $\widetilde{\nabla}_2(\bar{I} \otimes F)$ is artinian. The problem is that it is not immediately clear that this implies that $\bar{I} \otimes F$ is artinian, since the functors $\widetilde{\nabla}_2$ do not a priori detect exterior powers.

This paper exploits the strong property of the functors $\bar{I} \otimes F$ given by the following statement.

LEMMA 3.0.2. — *Suppose that $t \geq 0$, then there is a direct sum decomposition*

$$\Delta^t(\bar{I} \otimes F) \cong (\bar{I} \otimes F) \oplus G_t \oplus (\bar{I} \otimes H_t),$$

where G_t, H_t are finite functors and $\deg H_t < \deg F$.

Suppose that Y is an analytic functor taking finite-dimensional values and that there exists a surjection $\Delta Y \twoheadrightarrow Y$, then for any $t \geq 0$, there is a surjection $\Delta^t Y \twoheadrightarrow Y$, obtained by iteration and the exactness of Δ. This motivates the hypotheses of the following result:

PROPOSITION 3.0.3. — *Suppose that Y is a functor taking finite-dimensional values and that there exists a surjection $\Delta Y \twoheadrightarrow Y$. Suppose*

that the kernel of the induced map $\Delta^t Y \twoheadrightarrow Y$ is artinian for all $t \geq 0$. If there exists a sub-object $X \hookrightarrow Y$, such that

    1) X is artinian,

    2) the composite $\Delta^T X \hookrightarrow \Delta^T Y \twoheadrightarrow Y$ is surjective, for some $T \geq 0$,

then the functor Y is artinian.

    Proof. — It suffices to show that Y/X is artinian and hence, by Lemma 3.0.1, that $\Delta^T Y / \Delta^T X$ is artinian. This is clear, since the commutative diagram

$$
\begin{array}{ccc}
\Delta^T X & \hookrightarrow & \Delta^T Y \\
\downarrow & & \downarrow \\
Y & = & Y
\end{array}
$$

in which the vertical arrows are surjections, exhibits $\Delta^T Y / \Delta^T X$ as a quotient of $\ker\{\Delta^T Y \twoheadrightarrow Y\}$, which is artinian, by hypothesis. $\square$

4. The sub-functor X_S of $\bar{I} \otimes S$, when S is simple.

    The proof of Theorem 1 reduces to the proof that the functors $\bar{I} \otimes S$ are artinian, where S is a simple functor. The purpose of this section is to establish Theorem 3. The proof of the theorem requires some basic facts about the simple functors in $\mathcal{F}$, which are reviewed below. However, the technical input is limited to the results of Proposition 4.1.3 and Lemma 5.0.3.

4.1. A review of simple functors and Weyl functors.

    The simple functors in $\mathcal{F}$ are indexed by the strictly decreasing partitions $\lambda = (\lambda_1 > \lambda_2 > \cdots \lambda_n > 0)$. The integer n is the length of the partition λ; write $|\lambda| := \sum \lambda_i$. The Weyl functor associated to the partition λ is a sub-functor $W_\lambda \hookrightarrow \Lambda^\lambda$, where Λ^λ denotes the functor $\bigotimes_{i=1}^n \Lambda^{\lambda_i}$. The Weyl functor has a unique top composition factor, $S_\lambda \cong W_\lambda / \mathrm{rad} W_\lambda$, which is the simple functor indexed by the partition λ, and has polynomial degree $|\lambda|$.

    For the convenience of the reader, the definition of the Weyl functor is recalled below; the information on the bases of Weyl modules is only required in the proof of Lemma 5.0.3. Standard references for this and related material are the books [J], [JK]; the paper [J2] is closer to the spirit

of the current article. The reader may also consult the paper by Piriou and Schwartz [PS], which treats the category $\mathcal{F}$, but should be forewarned that they have indexed the simple modules by the conjugate partitions.

There is a Young diagram associated to the partition λ; namely, this is the array of boxes arranged in n columns, the i-th column having length λ_i. This is pictured as follows:

1	$\lambda_1 + 1$	·	$|\lambda| - \lambda_n + 1$
2	$\lambda_1 + 2$	·	
·	·	·	λ_n
·	·	·	
·	λ_2		
·			
λ_1			

The boxes are numbered from 1 to $|\lambda|$, consecutively down the columns, taking the columns from left to right.

The Weyl functor W_λ, when λ is a strictly decreasing partition, is defined as the functor $\overline{C}_\lambda R_\lambda (\Lambda^1)^{\otimes |\lambda|}$, where the symmetric group $\Sigma_{|\lambda|}$ on $|\lambda|$ letters operates on $(\Lambda^1)^{\otimes |\lambda|}$ by place permutations and $R_\lambda, \overline{C}_\lambda$ are the following elements of the symmetric group algebra $\mathbb{F}_2[\Sigma_{|\lambda|}]$: R_λ is the sum of the elements of the row stabilizer of the Young diagram associated to λ and $\overline{C}_\lambda$ is the signed sum (over $\mathbb{F}_2$ the sign is irrelevant) of the elements of the column stabilizer of the Young diagram.

A λ-tableau on d letters is a map $\tau \colon \{1, \ldots, |\lambda|\} \to \{1, \ldots, d\}$, which may be represented on the Young diagram in an obvious way. A λ-tableau τ gives rise to an associated element on $\bigotimes_{i=1}^n \Lambda^{\lambda_i}(\mathbb{F}_2^d)$, with respect to a choice of basis $\{y_j\}$ of $\mathbb{F}_2^d$. Namely:

$$[\tau] := \bigotimes_{i=1}^n \left(y_{\tau_i(1)} \wedge \ldots \wedge y_{\tau_i(\lambda_i)} \right)$$

where $\tau_i(k)$ is the k-th entry of the i-th column of τ.

A λ-tableau is semi-standard if entries increase strictly down the columns and are non-decreasing across the rows. The proof of Lemma 5.0.3 requires the following result.

PROPOSITION 4.1.1 (see [JK]). — *Suppose that λ is a strictly decreasing partition. A basis for $W_\lambda(\mathbb{F}_2^d)$ is indexed by the semi-standard λ-tableaux on d letters. If σ is a semi-standard λ-tableau on d letters, then the associated basis element, with respect to a basis $\{y_j\}$ of $\mathbb{F}_2^d$, is $\sum_{\tau \in \mathcal{O}(\sigma)}[\tau]$, where $\mathcal{O}(\sigma)$ denotes the set of λ-tableaux which are row equivalent to σ.*

Notation 4.1.2. — In the course of the paper, the following notation will be used. Suppose that $\lambda = (\lambda_1 > \lambda_2 > \cdots \lambda_n > 0)$, then write

1) $\hat{\lambda} := (\lambda_1 + 1 > \lambda_2 + 1 > \cdots > \lambda_n + 1 > 0)$;

2) $\bar{\lambda} := (\lambda_1 - 1 > \lambda_2 - 1 > \cdots > \lambda_n - 1 \geq 0)$.

Observe that the partition $\hat{\lambda}$ has length n, whilst $\bar{\lambda}$ has length $n - 1$ or n.

The following basic result on the action of the functor Δ on the simple functors is required:

PROPOSITION 4.1.3 (see [P2], [PS]). — *Suppose that λ is a partition of length n.*

1) $\widetilde{\nabla}_n S_\lambda = S_{\bar{\lambda}}$ *and* $\widetilde{\nabla}_t S_\lambda = 0$ *for* $t > n$.

2) $\Delta S_\lambda = S_{\bar{\lambda}} \oplus G$, *where G is a self-dual functor which embeds in a direct sum of tensor power functors, $T^a : V \mapsto V^{\otimes a}$, where $|\lambda| - n < a < |\lambda|$.*

4.2. Certain sub-functors of $\bar{I} \otimes W_\lambda$ and $\bar{I} \otimes S_\lambda$.

Theorem 3 involves an explicit sub-functor X_λ of $\bar{I} \otimes S_\lambda$, which is constructed in this section. Theorem 1 is proved by applying Proposition 3.0.3; in order to establish the hypotheses of the proposition, the functor X_λ is constructed as the image of a sub-functor $\widehat{X}_\lambda \hookrightarrow \bar{I} \otimes W_\lambda$, so that there is a commutative diagram:

$$\begin{array}{ccc} \widehat{X}_\lambda & \longrightarrow & \bar{I} \otimes W_\lambda \\ \downarrow & & \downarrow \\ X_\lambda & \hookrightarrow & \bar{I} \otimes S_\lambda \end{array}$$

in which the vertical arrows are surjections. Recall in the following that π_m is written for the functor $p_m \bar{I}$.

A key point in the construction of X_λ and $\widehat{X}_\lambda$ is the following observation:

LEMMA 4.2.1. — *Suppose that* $m > \lambda_1$, *then the functors* $\bar{I} \otimes W_\lambda$ *and* $\bar{I} \otimes S_\lambda$ *each contain a unique composition factor isomorphic to* $S_{(m,\lambda)}$, *which appears respectively in the sub-quotients* $\Lambda^m \otimes W_\lambda$ *and* $\Lambda^m \otimes S_\lambda$.

Proof. — It is well-known that the functors $\Lambda^m \otimes W_\lambda$ and $\Lambda^m \otimes S_\lambda$ contain a unique composition factor $S_{(m,\lambda)}$. For reasons of polynomial degree, $S_{(m,\lambda)}$ is not a composition factor of $\pi_{m-1} \otimes F$ for $F \in \{W_\lambda, S_\lambda\}$.

The functors $(\bar{I}/\pi_m) \otimes F$, for these F, are zero on $\mathbb{F}_2^m$, since $(\bar{I}/\pi_m)(\mathbb{F}_2^m) = 0$. Hence the functors $(\bar{I}/\pi_m) \otimes F$ do not contain a composition factor $S_{(m,\lambda)}$, since the latter is non-zero on $\mathbb{F}_2^m$. □

Notation 4.2.2. — Let P_μ denote the projective cover in $\mathcal{F}$ of the functor S_μ. Suppose that $\mu > \lambda_1$, then there are unique non-trivial maps $P_{(m,\lambda)} \to \bar{I} \otimes W_\lambda$ and $P_{(m,\lambda)} \to \bar{I} \otimes S_\lambda$, by Lemma 4.2.1. Write

1) $G_{(m,\lambda)} := \text{image}\{P_{(m,\lambda)} \to \bar{I} \otimes W_\lambda\}$,

2) $F_{(m,\lambda)} := \text{image}\{P_{(m,\lambda)} \to \bar{I} \otimes S_\lambda\}$.

By construction, there are canonical inclusions $G_{(m,\lambda)} \hookrightarrow \pi_m \otimes W_\lambda$ and $F_{(m,\lambda)} \hookrightarrow \pi_m \otimes S_\lambda$.

LEMMA 4.2.3. — *Suppose that* $m > \lambda_1$.

1) *The surjection* $\bar{I} \otimes W_\lambda \twoheadrightarrow \bar{I} \otimes S_\lambda$ *induces a surjection* $G_{(m,\lambda)} \twoheadrightarrow F_{(m,\lambda)}$.

2) *The composite* $G_{(m,\lambda)} \hookrightarrow \pi_m \otimes W_\lambda \twoheadrightarrow \Lambda^m \otimes W_\lambda$ *surjects onto* $W_{(m,\lambda)} \subset \Lambda^m \otimes W_\lambda$.

4.3. $\widetilde{\nabla}$ and the functors $F_{(m,\lambda)}$.

The arguments of this paper rely on exploiting the functors $\widetilde{\nabla}$ on the functors $F_{(m,\lambda)}$. It is straightforward to establish:

LEMMA 4.3.1. — *Suppose that* λ *is a partition of length* n, *then*

1) $\widetilde{\nabla}_{n+1}(\bar{I} \otimes S_\lambda) = S_{\bar{\lambda}} \oplus (\bar{I} \otimes S_{\bar{\lambda}})$;

2) $\widetilde{\nabla}_{n+1}(\pi_k \otimes S_\lambda) = S_{\bar{\lambda}} \oplus (\pi_{k-1} \otimes S_{\bar{\lambda}})$.

Suppose that $m > \lambda_1$, then there is a unique non-trivial map $F_{(m-1,\bar{\lambda})} \hookrightarrow \widetilde{\nabla}_{n+1}(\bar{I} \otimes S_\lambda)$; this factors through the canonical inclusion $\bar{I} \otimes S_{\bar{\lambda}} \hookrightarrow \widetilde{\nabla}_{n+1}(\bar{I} \otimes S_\lambda)$.

The main result of this section is:

PROPOSITION 4.3.2. — *Suppose that $m > \lambda_1$ and that $H \hookrightarrow \bar{I} \otimes S_\lambda$ is a sub-functor. If $\widetilde{\nabla}_{n+1} H \subset S_{\bar{\lambda}} \oplus (\bar{I} \otimes S_{\bar{\lambda}})$ contains $F_{(m-1,\bar{\lambda})}$, then H contains $F_{(m,\lambda)}$.*

The proof appeals to the following lemma, which is a special case of a general argument:

LEMMA 4.3.3. — *Suppose that G is a sub-functor of $\bar{I} \otimes S_\lambda$ and that S_μ is a composition factor of $\widetilde{\nabla}_{n+1} G$ with $\mu_1 = m - 1$. Then S_μ is a composition factor of $\widetilde{\nabla}_{n+1} G'$, where $G' := G \cap (\pi_m \otimes S_\lambda)$.*

Proof. — The functor $(\bar{I}/\pi_m) \otimes S_\lambda$ is zero when evaluated on $\mathbb{F}_2^m$, hence $\Delta((\bar{I}/\pi_m) \otimes S_\lambda)$ is zero on $\mathbb{F}_2^{m-1}$. The cokernel of $\widetilde{\nabla}_{n+1} G' \hookrightarrow \widetilde{\nabla}_{n+1} G$ is a sub-quotient of $\Delta((\bar{I}/\pi_m) \otimes S_\lambda)$, hence vanishes on $\mathbb{F}_2^{m-1}$. In particular, since $S_\mu(\mathbb{F}_2^{m-1}) \neq 0$, by the hypothesis on μ, this shows that S_μ is not a composition factor of the cokernel of $\widetilde{\nabla}_{n+1} G' \hookrightarrow \widetilde{\nabla}_{n+1} G$. □

Proof of Proposition 4.3.2. — It suffices to show that if H does not contain $F_{(m,\lambda)}$ then $\widetilde{\nabla}_{n+1} H$ does not contain a composition factor of $S_{(m-1,\bar{\lambda})}$. We may suppose that $H \subset \pi_m \otimes S_\lambda$, by Lemma 4.3.3. There is a short exact sequence $H \hookrightarrow \pi_m \otimes S_\lambda \to C$, which defines the functor C.

Suppose that H does not contain $F_{(m,\lambda)}$; thus C contains a composition factor $S_{(m,\lambda)}$. Applying $\widetilde{\nabla}_{n+1}$ to the short exact sequence yields a sequence (*i.e.* the maps have zero composite but the sequence is not necessarily exact):

$$\widetilde{\nabla}_{n+1} H \hookrightarrow S_{\bar{\lambda}} \oplus (\pi_{m-1} \otimes S_{\bar{\lambda}}) \twoheadrightarrow \widetilde{\nabla}_{n+1} C.$$

Since $\widetilde{\nabla}_{n+1}$ preserves injections and surjections, $\widetilde{\nabla}_{n+1} C$ contains a composition factor $S_{(m-1,\bar{\lambda})}$, by Proposition 4.1.3; hence $\widetilde{\nabla}_{n+1} H$ does not contain a composition factor $S_{(m-1,\bar{\lambda})}$, by Lemma 4.2.1. This is the required result. □

A straightforward argument using $\widetilde{\nabla}_{n+1}$ establishes:

LEMMA 4.3.4. — *Suppose that $m > \lambda_1$, then there is a natural inclusion $F_{(m-1,\bar{\lambda})} \hookrightarrow \widetilde{\nabla}_{n+1} F_{(m,\lambda)}$.*

4.4. The construction and properties of $X_\lambda \subset \overline{I} \otimes S_\lambda$.

The equality $\widetilde{\nabla}_{n+1}(\overline{I} \otimes S_\lambda) = S_{\overline{\lambda}} \oplus (\overline{I} \otimes S_{\overline{\lambda}})$ is the basis for induction arguments, based on the polynomial degree of the functor S_λ. It is important to note that the second part of Proposition 4.4.1 below depends on Corollary 1.0.3 and hence upon Theorem 4.

PROPOSITION 4.4.1. — *Suppose that λ is a strictly decreasing partition of length n and that $m > \lambda_1$.*

1) *For any $M \geq m$, there is an inclusion $F_{(m,\lambda)} \subset F_{(M,\lambda)}$.*

2) *If $G \hookrightarrow \overline{I} \otimes S_\lambda$ is a non-finite functor, then G contains $F_{(m,\lambda)}$.*

Proof. — The proof of both statements is an induction upon the polynomial degree of S_λ; the induction is started by the case $S_\lambda = \mathbb{F}_2$, the constant functor, in which case both statements are straightforward, since the functor $\overline{I}$ is uniserial.

To prove the first statement, suppose that $M > m$. By induction, $F_{(M-1,\overline{\lambda})}$ contains $F_{(m-1,\overline{\lambda})}$. Lemma 4.3.4 implies that $F_{(M-1,\overline{\lambda})}$ is a subfunctor $\widetilde{\nabla}_{n+1}F_{(M,\lambda)}$, thus Proposition 4.3.2 shows that $F_{(M,\lambda)}$ contains $F_{(m,\lambda)}$, as required.

The second statement is proved by a similar argument: if $G \hookrightarrow \overline{I} \otimes S_\lambda$ is a non-finite functor then $\widetilde{\nabla}_{n+1}G$ is a non-finite functor, by Corollary 1.0.3. Hence, by induction, $\widetilde{\nabla}_{n+1}G$ contains the functor $F_{(m-1,\overline{\lambda})}$. Proposition 4.3.2 implies that G contains $F_{(m,\lambda)}$, as required. $\square$

Notation 4.4.2. — The functors $F_{(m,\lambda)}$, $m > \lambda_1$ form a directed system; write the colimit as $X_\lambda := \varinjlim_{m > \lambda_1} F_{(m,\lambda)}$. By construction, there is a unique inclusion $X_\lambda \hookrightarrow \overline{I} \otimes S_\lambda$.

Similarly, one may define $\widehat{X}_\lambda := \varinjlim_{m > \lambda_1} G_{(m,\lambda)}$. By construction, there is a unique inclusion $\widehat{X}_\lambda \hookrightarrow \overline{I} \otimes W_\lambda$.

The following result is Theorem 3 of the introduction.

THEOREM 4.4.3. — *The sub-functor $X_\lambda \hookrightarrow \overline{I} \otimes S_\lambda$ is the unique non-finite simple modulo finites sub-functor of $\overline{I} \otimes S_\lambda$.*

Proof. — The proof is a matter of collecting together results. The construction of X_λ ensures that it is not a finite functor, since the degrees of the $F_{(m,\lambda)}$ are unbounded. Proposition 4.4.1, part 2 shows that X_λ is

contained in any non-finite sub-functor of $\bar{I} \otimes S_\lambda$. Finally, Proposition 2.1.1 shows that X_λ contains a non-finite functor which is simple modulo finites; this must therefore identify with X_λ.   □

4.5. An alternative conjectural description of X_λ.

The author's original attempt at constructing a suitable X_λ was to generalize the approach of [P3]. Recall that, in the latter situation, there is a unique non-trivial map $\bar{I} \otimes \Lambda^{s+1} \to \bar{I} \otimes \Lambda^s$ and the image, $\overline{K}_s$ is a non-finite simple modulo finites functor. (This theory is reviewed in Theorem 7.0.1.)

PROPOSITION 4.5.1. — *Suppose that λ is a strictly decreasing partition. There is a unique non-trivial map $\phi_\lambda : \bar{I} \otimes S_{\hat{\lambda}} \to \bar{I} \otimes S_\lambda$.*

Notation 4.5.2. — Set $K_\lambda : = $ image ϕ_λ.

Theorem 4.4.3 shows that there is an inclusion $X_\lambda \subset K_\lambda$; it is natural to ask whether this is an isomorphism.

CONJECTURE 4.5.3. — *Suppose that λ is a strictly decreasing partition of length n. If $\bar{I} \otimes S_{\hat{\lambda}} \twoheadrightarrow Q$ is a non-trivial map, then $\widetilde{\nabla}_{n+1} Q \neq 0$.*

The usage of $\hat{\lambda}$ in the above conjecture is important, since $\hat{\lambda}$ has length n and $\hat{\lambda}_n > 1$. These conditions are necessary; for example, there is a surjection $\bar{I} \otimes \Lambda^1 \twoheadrightarrow \bar{I}$ which is not detected by $\widetilde{\nabla}_2$. This sort of example generalizes readily.

PROPOSITION 4.5.4. — *If Conjecture 4.5.3 holds, then there is equality $X_\lambda = K_\lambda$.*

Remark 4.5.5. — The idea of the proof is precisely the same as the strategy employed in [P3]; the reader is invited to supply details for themselves. Unfortunately, the author's attempt to establish Conjecture 4.5.3 seemed to lead to feasible but detailed calculations. Given that this may be viewed as only a small step towards the Artinian Conjecture, such explicit calculations seem to be undesirable.

5. The proof of Theorem 1.

Theorem 1 is proved by applying Proposition 3.0.3. Lemma 3.0.2 has established that there is a natural surjection $\Delta^t(\bar{I} \otimes S_\lambda) \twoheadrightarrow \bar{I} \otimes S_\lambda$, for

any $t \geq 0$; an evident induction on the degree of S_λ means that we may suppose that the kernel of this map is artinian. The proof of Theorem 1 will therefore be completed by establishing:

PROPOSITION 5.0.1. — *There exists $t \geq 0$ so that the composite $\Delta^t X_\lambda \to \Delta^t(\bar{I} \otimes S_\lambda) \twoheadrightarrow \bar{I} \otimes S_\lambda$ is surjective.*

Implicit in the work of this section is the naturality of the map $\Delta(\bar{I} \otimes S_\lambda) \twoheadrightarrow \bar{I} \otimes S_\lambda$. The required facts are summarized as follows:

LEMMA 5.0.2. — *For non-negative integers $m \geq t$, there is a natural commutative diagram in which the vertical maps are surjections:*

$$
\begin{array}{ccccc}
\Delta^t(\bar{I} \otimes H) & \longleftarrow & \Delta^t(\pi_m \otimes H) & \longrightarrow & \Delta^t(\Lambda^m \otimes H) \\
\downarrow & & \downarrow & & \downarrow \\
I \otimes H & \longleftarrow & (\mathbb{F}_2 \oplus \pi_{m-t}) \otimes H & \longrightarrow & \Lambda^{m-t} \otimes H.
\end{array}
$$

Some calculational input is required; this is best achieved by lifting the arguments to $\bar{I} \otimes W_\lambda$, where explicit calculations may be performed.

LEMMA 5.0.3. — *Suppose that λ is a strictly decreasing partition and that $m > \lambda_1$. The inclusion $W_{(m,\lambda)} \hookrightarrow \Lambda^m \otimes W_\lambda$ induces a composite*

$$\Delta^t W_{(m,\lambda)} \longrightarrow \Delta^t(\Lambda^m \otimes W_\lambda) \twoheadrightarrow \Lambda^{m-t} \otimes W_\lambda,$$

which is surjective when $t \geq \lambda_1$.

Proof. — A basis for $W_\lambda(\mathbb{F}_2^d)$ is indexed by semi-standard λ-tableaux on d letters, by Proposition 4.1.1. One may add t ordered letters $y_1 < \cdots < y_t$ and totally order the resulting $d+t$ letters by $y_t < 1$. Suppose that $t \geq \lambda_1$; given a semi-standard λ-tableau τ on the d letters and an increasing sub-sequence $a_1 < \cdots < a_{m-t}$ of the same letters, consider the following (m, λ)-tableau:

$$
\begin{array}{|c|c|}
\hline
\begin{array}{c} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_t \end{array} & \tau \\
\hline
\begin{array}{c} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{m-t} \end{array} & \\
\hline
\end{array}
$$

By construction, this tableau is semi-standard when $t \geq \lambda_1$. Regarding the letters y_i as a basis of $\mathbb{F}_2^t$ and $\Delta^t F(V)$ as a direct summand of $F(V \oplus \mathbb{F}_2^t)$, this shows that the given map is surjective. The details of the argument are left to the reader. $\qquad\square$

There is a commutative diagram:

$$
\begin{array}{ccc}
G_{(m,\lambda)} & \longhookrightarrow & \pi_m \otimes W_\lambda \\
\downarrow & & \downarrow \\
W_{(m,\lambda)} & \longhookrightarrow & \Lambda^m \otimes W_\lambda.
\end{array}
$$

For $m > t$, this yields a commutative diagram, in which the vertical arrows are surjective:

$$
\begin{array}{ccccc}
\Delta^t G_{(m,\lambda)} & \longhookrightarrow & \Delta^t(\pi_m \otimes W_\lambda) & \longrightarrow\!\!\!\rightarrow & \pi_{m-t} \otimes W_\lambda \\
\downarrow & & \downarrow & & \downarrow \\
\Delta^t W_{(m,\lambda)} & \longhookrightarrow & \Delta^t(\Lambda^m \otimes W_\lambda) & \longrightarrow\!\!\!\rightarrow & \Lambda^{m-t} \otimes W_\lambda.
\end{array}
$$

One deduces:

LEMMA 5.0.4. — *If $m \gg \lambda_1$ and $t \geq \lambda_1$, then the composite*

$$
\Delta^t G_{(m,\lambda)} \longhookrightarrow \Delta^t(\pi_m \otimes W_\lambda) \longrightarrow\!\!\!\rightarrow \pi_{m-t} \otimes W_\lambda
$$

is surjective.

Proof. — One must justify surjectivity in small polynomial degree; this is the reason for the hypothesis $m \gg \lambda_1$. The argument is straightforward. $\qquad\square$

These results may be applied to the study of $\bar{I} \otimes S_\lambda$ by appealing to the diagram:

$$
\begin{array}{ccccc}
\Delta^t G_{(m,\lambda)} & \longhookrightarrow & \Delta^t(\pi_m \otimes W_\lambda) & \longrightarrow\!\!\!\rightarrow & \pi_{m-t} \otimes W_\lambda \\
\downarrow & & \downarrow & & \downarrow \\
\Delta^t F_{(m,\lambda)} & \longhookrightarrow & \Delta^t(\pi_m \otimes S_\lambda) & \longrightarrow\!\!\!\rightarrow & \pi_{m-t} \otimes S_\lambda,
\end{array}
$$

in which the vertical arrows are surjective. By Lemma 5.0.4, if $m \gg 0$ and $t \geq \lambda_1$, the composite $\Delta^t G_{(m,\lambda)} \to \pi_{m-t} \otimes S_\lambda$ is surjective. The kernel of the map $\Delta^t G_{(m,\lambda)} \to \Delta^t F_{(m,\lambda)}$ maps to zero in $\pi_{m-t} \otimes S_\lambda$, hence the map $\Delta^t F_{(m,\lambda)} \to \pi_{m-t} \otimes S_\lambda$ is surjective.

Finally, taking the direct limit as $m \to \infty$, one obtains:

PROPOSITION 5.0.5. — *The composite $\Delta^t X_\lambda \to \Delta^t(\overline{I} \otimes S_\lambda) \twoheadrightarrow \overline{I} \otimes S_\lambda$ is surjective, when $t \geq \lambda_1$.*

Remarks 5.0.6.

1) It seems likely that the bound required on t is rather larger than necessary. The author originally hoped to establish that $t \geq n$ would be sufficient, where n is the length of the partition λ. (Observe that $\lambda_1 \geq n$ for a strictly decreasing partition λ.) For example, the sub-functor $\overline{K}_k \subset \overline{I} \otimes \Lambda^k$ induces a surjection $\Delta \overline{K}_k \twoheadrightarrow \overline{I} \otimes \Lambda^k$, corresponding to $t = n = 1$.

2) An induction using the dominance order of partitions allows the direct application of Proposition 3.0.3 to prove that $\overline{I} \otimes W_\lambda$ is artinian; the fact that X_λ is artinian is still required.

6. The proof of Theorem 2.

The following theorem due to Vincent Franjou (unpublished) is required; a proof due to Lionel Schwartz is included as Theorem A.1 in the Appendix of [P1].

THEOREM 6.0.1. — *If F is a finite functor then $\mathrm{Ext}^*_\mathcal{F}(\overline{I}, F) = 0$.*

COROLLARY 6.0.2. — *If $a, b \geq 0$, then $\mathrm{Ext}^1_\mathcal{F}(\overline{I}/\pi_a, \overline{I}/\pi_b) = 0$.*

Proof. — There is an isomorphism

$$\mathrm{Ext}^1_\mathcal{F}(\overline{I}/\pi_a, \overline{I}/\pi_b) \cong \mathrm{Ext}^2_\mathcal{F}(\overline{I}/\pi_a, \pi_b),$$

from the long exact sequence associated to $\pi_b \to \overline{I} \to \overline{I}/\pi_b$. The short exact sequence $\pi_a \to \overline{I} \to \overline{I}/\pi_a$ gives rise to a long exact sequence which, together with Theorem 6.0.1, yields an isomorphism $\mathrm{Ext}^2_\mathcal{F}(\overline{I}/\pi_a, \pi_b) \cong \mathrm{Ext}^1_\mathcal{F}(\pi_a, \pi_b)$. This is zero; to see this, use the long exact sequence associated to $\pi_b \to \overline{I} \to \overline{I}/\pi_b$ and the injectivity of $\overline{I}$.   □

PROPOSITION 6.0.3 (see [P1], Prop. 7.4). — *Suppose that G is a non-finite analytic functor with finite socle such that $\widetilde{\nabla}_2 G = 0$. There exists a non-trivial map $\overline{I} \to G$.*

This has the immediate consequence:

COROLLARY 6.0.4. — *Suppose that G is a non-finite functor which is simple modulo finites. If $\widetilde{\nabla}_2 G = 0$, then G is a quotient of $\bar{I}$.*

This is used in conjunction with the following result, which is of interest in its own right:

PROPOSITION 6.0.5. — *Suppose that G is a non-finite analytic functor which is simple modulo finites. Either $\widetilde{\nabla}_2 G = 0$ or $\widetilde{\nabla}_2 G$ is non-finite.*

Proof. — First one excludes the case $G(\mathbb{F}_2) \neq 0$; in this case, there is a non-trivial map $G \to \bar{I}$, since G is clearly constant-free. Since G is simple modulo finites it follows that this is a surjection with finite kernel. In particular, Theorem 6.0.1 implies that the surjection splits; hence, as G is simple modulo finites, the kernel is zero, so that $G \cong \bar{I}$. In this case, $\widetilde{\nabla}_2 G = 0$.

Suppose that G is such a functor and that $\widetilde{\nabla}_2 G$ is finite and non-trivial; there exists a surjection $\widetilde{\nabla}_2 G \twoheadrightarrow S_\lambda$, where $S_\lambda \neq \mathbb{F}_2$, since $\widetilde{\nabla}_2 G(0) = 0$ via the exclusion of the case $G(\mathbb{F}_2) = 0$. The adjoint to the composite $\Delta G \to \widetilde{\nabla}_2 G \twoheadrightarrow S_\lambda$ is a non-trivial map $f : G \to \bar{I} \otimes S_\lambda$. Since G is simple modulo finites and non-finite, the image of f is non-finite. Hence, the image of f contains the functor $X_\lambda \subset \bar{I} \otimes S_\lambda$. The construction of X_λ shows immediately that X_λ contains an infinite number of composition factors which are not exterior powers, hence that $\widetilde{\nabla}_2 X_\lambda$ is non-finite. This implies that $\widetilde{\nabla}_2(\text{image} f)$ is non-finite, since $\widetilde{\nabla}_2$ preserves injections. Finally, since $\widetilde{\nabla}_2$ preserves surjections, it follows that $\widetilde{\nabla}_2 G$ is non-finite, a contradiction. □

It is worth noting the following related result, which is not required in the proof of Theorem 2:

COROLLARY 6.0.6. — *Suppose that F is a finite functor. If G is a functor represented by a class in $\mathrm{Ext}^1_{\mathcal{F}}(\bar{I}/\pi_a, F)$, then*

1) *there is a non-trivial map $\bar{I} \to G$,*

2) *there is a unique non-finite simple modulo finites sub-functor of G, which is isomorphic to $\bar{I}/\pi_c$, for some c.*

Proof. — The second statement clearly follows from the first. Prove the first statement by an induction upon the length of F. The induction is started by the case $F = 0$, which is trivial.

The short exact sequence $\pi_a \to \bar{I} \to \bar{I}/\pi_a$ and Theorem 6.0.1 induce an isomorphism $\mathrm{Ext}^1_{\mathcal{F}}(\bar{I}/\pi_a, F) \cong \mathrm{Hom}_{\mathcal{F}}(\pi_a, F)$. Hence, if G is represented by a non-zero class, there exists a non-trivial map $\pi_a \to F$; write π' for the image of this map and form the short exact sequence $\pi' \to F \to F'$. By construction, F' has length less than F.

Consider $G' : = G/\pi'$; by the inductive hypothesis, there is a non-trivial map $\bar{I} \to G'$, this induces a non-trivial map $\bar{I} \to G$, from the short exact sequence $\pi' \to G \to G'$, together with Theorem 6.0.1.   □

Notation 6.0.7. — For the purposes of this section, say that an artinian functor has a *type one filtration* if it has a finite filtration of which the filtration quotients are non-finite simple modulo finites.

Proof of Theorem 2. — The proof is an induction, with the hypothesis that every functor $\bar{I} \otimes F$ with the degree of F less than $|\lambda|$ has a type one filtration. (Observe that it is sufficient to suppose that F is simple in this statement.) The induction starts by taking $S_\lambda = \Lambda^1$, for which the result is proved in [P3].

For the inductive step, suppose that $\deg S_\lambda > 1$. Theorem 1 has established that $\bar{I} \otimes S_\lambda$ is artinian; Proposition 2.1.1 then implies that $\bar{I} \otimes S_\lambda$ admits an increasing filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_k \subset \cdots \subset \bar{I} \otimes S_\lambda$$

in which each quotient F_k/F_{k-1} is non-finite and simple modulo finites and such that the filtration is finite if and only if $F_k = \bar{I} \otimes S_\lambda$ for some k. (Implicit in this statement is the observation that $\bar{I} \otimes S_\lambda$ does not admit any finite quotients; hence, if the filtration is maximal and finite, then the top filtration quotient is non-finite and hence simple modulo finites.)

The functor $\widetilde{\nabla}_2(\bar{I} \otimes S_\lambda)$ is of the form $G \oplus (\bar{I} \otimes H)$, where G, H are finite and $\deg H < |\lambda|$. In particular, the inductive hypothesis implies that $\bar{I} \otimes H$ has a type one filtration. Moreover, this implies that any filtration of $\bar{I} \otimes H$ which has quotients which are non-finite simple modulo finites is a type one filtration (that is of finite length).

Proposition 6.0.5 and Corollary 6.0.4 imply that, for each $t \geq 1$, either $\widetilde{\nabla}_2(F_{t+1}/F_t)$ is non-finite or F_{t+1}/F_t is a non-trivial quotient of $\bar{I}$. The inductive hypothesis implies that only finitely many of the $\widetilde{\nabla}_2(F_{t+1}/F_t)$ may be non-trivial, hence that all but finitely many of the filtration quotients F_{t+1}/F_t are non-trivial quotients of $\bar{I}$.

Suppose that the filtration is not finite, then there exists k such that, for all $t \geq k$, the quotients F_{t+1}/F_t are quotients of $\bar{I}$. Lemma 6.0.2 shows that, for any $t > k$, $F_t/F_k \cong \bigoplus_{i=k}^{t-1} F_{i+1}/F_i$. However, the functor $\bar{I} \otimes S_\lambda$ is artinian, hence $(\bar{I} \otimes S_\lambda)/F_k$ has finite socle. Taking $t > \text{length}\{\text{soc}((\bar{I} \otimes S_\lambda)/F_k)\}$, this derives a contradiction to the hypothesis that the filtration is not finite. $\square$

7. The proof of Theorem 4.

The proof of Theorem 4 relies on the study of the structure of the functors $\bar{I} \otimes \Lambda^n$ carried out in [P3]. For the convenience of the reader, the main results are recalled briefly. The functor $\overline{K}_n$, for $n \geq 0$, is defined as the image of the unique non-trivial map $\bar{\phi}_{n+1} : \bar{I} \otimes \Lambda^{n+1} \to \bar{I} \otimes \Lambda^n$ (the notation is taken from [P3]).

THEOREM 7.0.1 (see [P3]).

1) *There are short exact sequences* $\overline{K}_n \to \bar{I} \otimes \Lambda^n \to \overline{K}_{n-1}$, *for* $n \geq 1$.

2) *The functor* $\overline{K}_n$ *is simple modulo finites and is non-finite.*

3) $\text{Hom}_{\mathcal{F}}(\bar{I}, (\bar{I}/\pi_s) \otimes \Lambda^n) = 0$ *for any* $s \geq 0$ *and* $n \geq 1$.

4) *If* $F \hookrightarrow \bar{I} \otimes \Lambda^n$, *then* F *is finite unless* $\overline{K}_n \subset F$.

The following is a straightforward consequence, which is useful for calculations:

COROLLARY 7.0.2. — *A map* $f : (\bar{I}/\pi_s) \otimes \Lambda^n \to G$ *has finite kernel if and only if the composite* $f \circ (\text{proj}_s \otimes \Lambda^n) \circ \bar{\phi}_{n+1}$ *is non-trivial, where* $\text{proj}_s : \bar{I} \to \bar{I}/\pi_s$ *is the projection and* $\bar{\phi}_{n+1} : \bar{I} \otimes \Lambda^{n+1} \to \bar{I} \otimes \Lambda^n$ *is the unique non-trivial map.*

The argument of Corollary 7.4.2 is independent of the intervening results, hence implies:

COROLLARY 7.0.3. — *Suppose that* $K \subset (\bar{I}/\pi_s) \otimes \Lambda^n$, *for* $n \geq 1$, *such that* $\widetilde{\nabla}_2 K$ *is finite, then* K *is finite.*

This corollary implies the result which is required below:

PROPOSITION 7.0.4. — *Suppose that $K \subset (\bar{I}/\pi_s) \otimes \pi_n$ is a sub-functor such that $\widetilde{\nabla}_2 K$ is finite, then K is finite.*

Proof. — The proof is by induction upon n, starting with $n = 1$, where the result is given by 7.0.3. The inductive step is given by using the short exact sequence $\pi_{n-1} \to \pi_n \to \Lambda^n$. This induces a short exact sequence

$$K \cap ((\bar{I}/\pi_s) \otimes \pi_{n-1}) \hookrightarrow K \twoheadrightarrow K'$$

where K' embeds in $(\bar{I}/\pi_s) \otimes \Lambda^n$. The functor $\widetilde{\nabla}_2$ preserves injections and surjections, hence $\widetilde{\nabla}_2(K \cap ((\bar{I}/\pi_s) \otimes \pi_{n-1}))$ and $\widetilde{\nabla}_2 K'$ are both finite. By the inductive hypothesis and Corollary 7.0.3, one concludes that K is finite, as required.   □

7.1. Technical results for the functors $\widetilde{\nabla}$.

The natural surjection $\Delta \twoheadrightarrow \widetilde{\nabla}_n$ has an adjoint $F \to \widetilde{\nabla}_n F \otimes \bar{I}$, which was studied in [P2]. Suppose that F, G are functors such that $\widetilde{\nabla}_{a+1} F = 0 = \widetilde{\nabla}_{b+1} G$; there is an isomorphism $\widetilde{\nabla}_{a+b}(F \otimes G) \cong \widetilde{\nabla}_a F \otimes \widetilde{\nabla}_b G$ and the quotient map $\widetilde{\Delta}(F \otimes G) = \widetilde{\Delta} F \otimes \widetilde{\Delta} G \to \widetilde{\nabla}_{a+b}(F \otimes G)$ identifies as the tensor product of the individual maps $\widetilde{\Delta} F \to \widetilde{\nabla}_a F$ and $\widetilde{\Delta} G \to \widetilde{\nabla}_b G$.

LEMMA 7.1.1. — *Suppose that $f : \widetilde{\Delta} F \to X$, $g : \widetilde{\Delta} G \to Y$ are maps with adjoints $\tilde{f} : F \to X \otimes I$, $\tilde{g} : G \to Y \otimes I$ respectively; suppose that the functors take finite dimensional values. Then the adjoint of $f \otimes g$ identifies with the composite*

$$F \otimes G \xrightarrow{\tilde{f} \otimes \tilde{g}} X \otimes I \otimes Y \otimes I \longrightarrow X \otimes Y \otimes I$$

where the second map is induced by the natural product $I \otimes I \to I$.

Proof. — It is notationally simpler[1] to describe the argument for the dual adjunction, $(V \mapsto \mathbb{F}_2[V], \widetilde{\Delta})$. The adjoint to a map $\alpha : H \to \widetilde{\Delta} K$ is the map $\widetilde{\alpha} : \mathbb{F}_2[V] \otimes H(V) \to K(V)$ given by the composite

$$\mathbb{F}_2[V] \otimes H(V) \xrightarrow{1 \otimes \alpha} \mathbb{F}_2[V] \otimes K(V \oplus \mathbb{F}_2) \xrightarrow{\text{eval}} K(V \oplus V) \xrightarrow{\text{diag}} K(V),$$

in which the evaluation map is given by the identification $V \cong \text{Hom}_{\mathcal{F}}(\mathbb{F}_2, V)$.

---

[1] This is the reason for the assumption of finite-dimensional values. This assumption should have been included in some of the arguments used in [P2].

If $\beta : A \to \widetilde{\Delta}B$ is a second map, the adjoint of $\alpha \otimes \beta$ identifies with the composite

$$\mathbb{F}_2[V] \otimes A(V) \otimes H(V) \to \mathbb{F}_2[V] \otimes A(V) \otimes \mathbb{F}_2[V] \otimes H(V) \xrightarrow{\widetilde{\alpha} \otimes \widetilde{\beta}} B(V) \otimes K(V),$$

where the first arrow is the diagonal map $\mathbb{F}_2[V] \to \mathbb{F}_2[V] \otimes \mathbb{F}_2[V]$ together with a permutation of tensor factors. This identification follows from the commutative diagram:

$$
\begin{array}{ccc}
\mathbb{F}_2[V] \otimes B(W) \otimes K(W) & \longrightarrow & \mathbb{F}_2[V] \otimes B(W) \otimes \mathbb{F}_2[V] \otimes K(W) \\
\text{eval} \downarrow & & \downarrow \text{eval} \otimes \text{eval} \\
B(V \oplus V) \otimes K(V \oplus V) & = & B(V \oplus V) \otimes K(V \oplus V),
\end{array}
$$

where $W = V \oplus \mathbb{F}_2$, which is equivalent to the observation that the evaluation map acts diagonally on the tensor product. $\qquad\square$

This lemma has an evident counter-part for the $(\Delta, - \otimes \bar{I})$-adjunction. This applies to yield:

PROPOSITION 7.1.2. — *Suppose that F, G are analytic functors and that $\widetilde{\nabla}_{a+1}F = 0 = \widetilde{\nabla}_{b+1}G$. The adjunction map $F \otimes G \to \widetilde{\nabla}_{a+b}(F \otimes G) \otimes \bar{I}$ identifies with the composite*

$$F \otimes G \longrightarrow (\widetilde{\nabla}_a F \otimes \bar{I}) \otimes (\widetilde{\nabla}_b G \otimes \bar{I}) \longrightarrow \widetilde{\nabla}_a F \otimes \widetilde{\nabla}_b G \otimes \bar{I},$$

of the tensor product of the adjunction maps $F \to \widetilde{\nabla}_a F \otimes \bar{I}$, $G \to \widetilde{\nabla}_b G \otimes \bar{I}$ followed by the product $\bar{I} \otimes \bar{I} \to \bar{I}$.
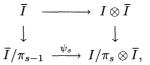
The importance of these adjunction maps is shown by:

PROPOSITION 7.1.3 (see [P2], Section 5). — *Suppose that F is an analytic functor. The map $F \hookrightarrow \widetilde{\nabla}_n F \otimes \bar{I}$ is injective if and only if $\widetilde{\nabla}_n K \neq 0$ whenever $0 \neq K \subset F$.*

Example 7.1.4. — Suppose that λ is a strictly decreasing partition of length $n > 0$ and S_λ is the associated simple functor. The adjunction map related to $\Delta \twoheadrightarrow \widetilde{\nabla}_n$ is the injection $S_\lambda \hookrightarrow S_{\bar{\lambda}} \otimes \bar{I}$.

Example 7.1.5. — Suppose that $s \geq 1$. The adjoint to the identity $\Delta(\bar{I}/\pi_s) \to \Delta(\bar{I}/\pi_s)$ (corresponding to $n = 1$) is a map $\psi_s : \bar{I}/\pi_s \to$

$I/\pi_{s-1} \otimes \bar{I}$. There is a unique such non-trivial map when $s > 0$. The map is described by the commutative diagram:

$$
\begin{array}{ccc}
\bar{I} & \longrightarrow & I \otimes \bar{I} \\
\downarrow & & \downarrow \\
\bar{I}/\pi_{s-1} & \xrightarrow{\psi_s} & I/\pi_s \otimes \bar{I},
\end{array}
$$

where the vertical arrows are induced by the canonical surjections and the map $\bar{I} \to I \otimes \bar{I}$ is the 'half-reduced' diagonal, namely the composite $\bar{I} \hookrightarrow I \to I \otimes I \twoheadrightarrow I \otimes \bar{I}$, in which $I \to I \otimes I$ is the usual diagonal map and $I \otimes I \twoheadrightarrow I \otimes \bar{I}$ is induced by the projection $I \twoheadrightarrow \bar{I}$ on the second factor.

7.2. Reduction of Theorem 4.

In this short section, Theorem 4 is reduced to the study of the kernel of certain explicit maps.

HYPOTHESIS 7.2.1. — *Suppose that F is a finite functor such that $\widetilde{\nabla}_{n+1}F = 0$ and the canonical map $F \to \widetilde{\nabla}_n F \otimes \bar{I}$ is an injection.*

LEMMA 7.2.2. — *If F is a finite functor, then the canonical map $F \to \widetilde{\nabla}_n F \otimes \bar{I}$ factors through $\widetilde{\nabla}_n F \otimes \pi_m \hookrightarrow \widetilde{\nabla}_n F \otimes \bar{I}$ for some m.*

The above material is of interest in studying functors of the form $\bar{I}/\pi_s \otimes F$.

LEMMA 7.2.3. — *Suppose that F satisfies Hypothesis 7.2.1. The adjoint to $\Delta(\bar{I}/\pi_s \otimes F) \twoheadrightarrow \widetilde{\nabla}_{n+1}(\bar{I}/\pi_s \otimes F)$ is a map*

$$
\bar{I}/\pi_s \otimes F \longrightarrow (I/\pi_{s-1} \otimes \bar{I} \otimes \widetilde{\nabla}_n F).
$$

This map has finite kernel if the composite

$$
\gamma_{s,m} : \bar{I}/\pi_s \otimes \pi_m \xrightarrow{\psi_s \otimes 1} (I/\pi_{s-1} \otimes \bar{I}) \otimes \pi_m \xrightarrow{1 \otimes \mu} I/\pi_{s-1} \otimes \bar{I}
$$

has finite kernel, where m is given by Lemma 7.2.2.

Proof. — The given map factors by Lemma 7.2.2 and Proposition 7.1.2 as

$$
\bar{I}/\pi_s \otimes F \longhookrightarrow \bar{I}/\pi_s \otimes \pi_m \otimes \widetilde{\nabla}_n F \xrightarrow{\gamma_{s,m} \otimes 1} (I/\pi_{s-1} \otimes \bar{I} \otimes \widetilde{\nabla}_n F).
$$

The first map is injective by Hypothesis 7.2.1, whereas the second is of the form $\gamma_{s,m} \otimes$ finite. If $\gamma_{s,m}$ has finite kernel, the second map has finite kernel. □

7.3. The completion of the proof.

The proof of Theorem 4 has been reduced to the proof of the following result:

PROPOSITION 7.3.1. — For any $s \geq 0, m \geq 1$, the map $\gamma_{s,m} :$ $\bar{I}/\pi_s \otimes \pi_m \to I/\pi_{s-1} \otimes \bar{I}$ has finite kernel, where $\gamma_{s,m}$ is the map defined in Lemma 7.2.3.

This result is a consequence of Theorem 7.0.1. The proof of the proposition given below appeals to the following general result:

LEMMA 7.3.2. — Suppose that α, β are maps $\bar{I} \to Y, X \to Y$ respectively, which have finite kernels. The kernel of the map $\alpha \oplus \beta :$ $\bar{I} \oplus X \to Y$ is either finite or splits as $\bar{I} \oplus$ finite. In particular, if $\mathrm{Hom}_{\mathcal{F}}(\bar{I}, X) = 0$, then the kernel of $\alpha \oplus \beta$ is finite.

Proof. — Write K for the kernel of $\alpha \oplus \beta$, so that there is a map $K \xrightarrow{(f,g)} \bar{I} \oplus X$. Consider the map $f : K \to \bar{I}$; $\ker f$ injects to X and the composite $\ker f \hookrightarrow X \xrightarrow{\beta} Y$ is zero. Since β has finite kernel, it follows that $\ker f$ is finite. Hence, there is an exact sequence

$$0 \to \ker f \longrightarrow K \longrightarrow \bar{I}.$$

If the image of f is finite, then this exhibits K as a finite functor. Otherwise the above sequence is short exact and represents a class of $\mathrm{Ext}^1_{\mathcal{F}}(\bar{I}, \ker f)$. This group is zero, by Theorem 6.0.1, hence the short exact sequence splits.

The last statement of the proposition is clear, by restricting the inclusion $K \hookrightarrow \bar{I} \oplus X$ to the summand of $\bar{I}$ in K. $\square$

Remark 7.3.3. — In order to show in the proof above that the kernel is finite, it is sufficient to know that $\mathrm{Hom}_{\mathcal{F}}(\bar{I}, X/G) = 0$ for any finite sub-functor $G \subset X$; thus, if sufficient is known about the structure of X, an appeal to Theorem 6.0.1 is not necessary.

Proof of Proposition 7.3.1. — The proof is by induction upon m. The case $m = 1$ is a direct consequence of the known structure of $\bar{I}/\pi_s \otimes \Lambda^1$; the map $\gamma_{(s,1)}$ has kernel which is zero under $\widetilde{\nabla}_2$. Proposition 6.0.5, Corollary 6.0.4 and Theorem 7.0.1 imply that this kernel is finite.

(A direct proof of the case $m = 1$ is given by applying Corollary 7.0.2; the relevant composite $\bar{I} \otimes \Lambda^2 \to (\bar{I}/\pi_s) \otimes \Lambda^1 \to (\bar{I}/\pi_{s-1}) \otimes \bar{I}$ may be calculated easily to be non-trivial.)

For the inductive step, consider the map $\bar{I}/\pi_s \otimes \pi_m \to I/\pi_{s-1} \otimes \bar{I}$. Applying $\widetilde{\nabla}_2$ yields a map

$$I/\pi_{s-1} \otimes (\pi_{m-1} \oplus \mathbb{F}_2) \longrightarrow I/\pi_{s-2} \otimes I.$$

By calculation, the restriction to the summand $\bar{I}/\pi_{s-1} \otimes \pi_{m-1}$ is precisely the map $\gamma_{s-1,m-1}$, which has finite kernel, by induction. Equally, the restriction to $I/\pi_{s-1} \otimes \mathbb{F}_2$ is the map $\widetilde{\nabla}_2 \gamma_{(s-1,1)}$ which is non-trivial and hence has finite kernel. To prove that the sum has finite kernel, one appeals to Lemma 7.3.2. This applies, since $\mathrm{Hom}_{\mathcal{F}}(\bar{I}, I/\pi_{s-1} \otimes \pi_{m-1}) = 0$, by the results of Theorem 7.0.1.

This argument establishes that $\widetilde{\nabla}_2 \gamma_{s,m}$ has finite kernel; this completes the proof by Proposition 7.0.4. $\square$

7.4. Conclusion.

The special case of Theorem 4 which is of interest here may be restated:

THEOREM 7.4.1. — *Suppose that λ is a strictly decreasing partition of length n. If $s \geq 0$ and $K \hookrightarrow \bar{I}/\pi_s \otimes S_\lambda$ is a sub-functor such that $\widetilde{\nabla}_{n+1} K = 0$, then K is finite.*

COROLLARY 7.4.2. — *Suppose that λ is a strictly decreasing partition of length n. Suppose that $K \hookrightarrow \bar{I} \otimes S_\lambda$ is a sub-functor such that $\widetilde{\nabla}_{n+1} K$ is finite, then K is finite.*

Proof of Corollary 7.4.2. — Suppose that $\widetilde{\nabla}_{n+1} K$ is finite. There exists an integer d such that $\widetilde{\nabla}_{n+1} K \subset \widetilde{\nabla}_{n+1}(K \cap (\pi_d \otimes S_\lambda))$. The commutative diagram:

$$
\begin{array}{ccccc}
K \cap (\pi_d \otimes S_\lambda) & \hookrightarrow & K & \longrightarrow\!\!\!\!\rightarrow & K/K \cap (\pi_d \otimes S_\lambda) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_d \otimes S_\lambda & \hookrightarrow & \bar{I} \otimes S_\lambda & \longrightarrow\!\!\!\!\rightarrow & \bar{I}/\pi_d \otimes S_\lambda
\end{array}
$$

shows that $K/K \cap (\pi_d \otimes S_\lambda)$ is a sub-functor of $\bar{I}/\pi_d \otimes S_\lambda$, such that $\widetilde{\nabla}_{n+1}(K/K \cap (\pi_d \otimes S_\lambda)) = 0$. Theorem 7.4.1 implies that the functor $K/K \cap (\pi_d \otimes S_\lambda)$ is finite, so that K is finite. $\square$

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