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CLASSIFICATION OF IRREDUCIBLE WEIGHT MODULES

by Olivier MATHIEU*

Contents.

Introduction.

Let K be an algebraically closed field of characteristic zero, let $\mathfrak{g}$ be a reductive Lie K-algebra and let $\mathfrak{h}$ be a Cartan subalgebra. A $\mathfrak{g}$-module

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M is called a *weight module* if and only if $M = \oplus_{\lambda \in \mathfrak{h}^*} M_\lambda$, where each weight space M_λ is finite dimensional (in the literature, the last condition is sometimes dropped). The main result of this paper is the classification of all simple weight $\mathfrak{g}$-modules.

The general strategy to get this classification is the following. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ and let $\mathfrak{u}$ be its nilradical, i.e. the smallest ideal such that $\mathfrak{p}/\mathfrak{u}$ is reductive. It is always assumed that $\mathfrak{p}$ contains $\mathfrak{h}$ and therefore $\mathfrak{h}$ can be viewed as a Cartan subalgebra of $\mathfrak{p}/\mathfrak{u}$. For any simple weight $\mathfrak{p}/\mathfrak{u}$-module S, the $\mathfrak{g}$-module $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} S$ has a unique simple quotient, denoted by $L_{\mathfrak{p}}(S)$. When $\mathfrak{p} \neq \mathfrak{g}$, the simple weight module $L_{\mathfrak{p}}(S)$ is called *parabolically induced.* Otherwise a simple weight module which is not parabolically induced is called *cuspidal.* By Fernando's Theorem [Fe], any simple weight module is cuspidal or is parabolically induced from a cuspidal module. Hence the general classification reduces to the classification of cuspidal modules (for a more precise statement, see Theorem 1.2). Moreover, for simple Lie algebras $\mathfrak{g}$, there are cuspidal modules only if $\mathfrak{g}$ has type A or C. Thus the main result of the paper is the explicit classification of cuspidal modules for Lie algebras of type A and C.

The main idea to investigate the cuspidal modules is the notion of coherent families. In order to define this notion, it should be noted that the weight spaces M_λ of any $\mathfrak{g}$-module M are $\mathcal{A}$-modules, where $\mathcal{A}$ is the commutant of $\mathfrak{h}$ in $U(\mathfrak{g})$. Then a *coherent family of degree d* is a weight module $\mathcal{M}$ such that

- $\dim \mathcal{M}_\lambda = d$ for all $\lambda \in \mathfrak{h}^*$, and
- the function $\lambda \mapsto \mathrm{Tr}\, u|_{\mathcal{M}_\lambda}$ is polynomial in λ, for all $u \in \mathcal{A}$.

Let $T^* = \mathfrak{h}^*/Q$ be the dual torus, where Q is the root lattice. For any $t \in T^*$, $\mathcal{M}[t] := \oplus_{\lambda \in t} \mathcal{M}_\lambda$ is a $\mathfrak{g}$-submodule of $\mathcal{M}$. Then it is shown that any cuspidal module is isomorphic to $\mathcal{M}[t]$ for a unique $t \in T^*$ and a unique irreducible and semi-simple coherent family $\mathcal{M}$ (these notions are defined in Section 4).

It turns out that any semi-simple irreducible coherent family $\mathcal{M}$ contains some infinite dimensional simple highest weight modules $L(\lambda)$ (Proposition 6.2). Moreover $\mathcal{M}$ is determined by these components (Proposition 4.8). Since any highest weight module $L(\lambda)$ occurring in $\mathcal{M}$ is *admissible* (i.e. the multiplicity of its weights is uniformly bounded), the first step is the classification of all admissible $L(\lambda)$. This can be done with the Kazhdan-Lusztig character formula. Indeed a more elementary approach is used because it gives quick proofs. From this, the classification of semi-simple irreducible coherent families follows (Theorems 8.6 for type A and Theorem 9.3 for type C).

The last step of the classification is Theorem 10.2 which determines in each irreducible coherent family $\mathcal{M}$ the full list of its cuspidal submodules. Equivalently, this result describes explicitly the set $\operatorname{Sing}\mathcal{M} := \{t \in T^* \mid \mathcal{M}[t] \text{ is not cuspidal}\}$. Indeed $\operatorname{Sing}\mathcal{M}$ is an union of finitely many codimension one cosets of T^*. In Sections 11 and 12, the classification of semi-simple coherent families is described more concretely. For $\mathfrak{g}=\mathfrak{sl}(n+1)$, irreducible coherent families are classified by some finite dimensional simple $\mathfrak{gl}(n)$-module and any such family is a space of tensors or a difference of two spaces of tensors (Theorem 11.4). For $\mathfrak{g}=\mathfrak{sp}(2n)$, irreducible coherent families are classified by the odd pairs of $\mathrm{Spin}(2n)$-modules and they are closely related to the Shale-Weil representation (Theorem 12.2).

In the last section, the character of any simple weight module is computed (Theorem 13.4). Besides Fernando's Theorem, there were two main results toward the classification of all simple weight modules in the previous literature. For $\mathfrak{sl}(2)$, the classification is usually attributed to Gabriel [Gab] and can be found in [Mi] and in [D] (7.8.16). More recently, Britten and Lemire classified all simple multiplicity free weight modules [BL], see also [BHL] [BFL] and Corollaries 11.5 and 12.3. Other interesting results have been obtained by Benkart, Britten, Cylke, Dimitrov, Futorny, Hooper, Gaillard, Joseph, Lemire, Ovsienko and Penkov (see the bibliography).

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0. General conventions and definitions.

The ground field K has characteristic zero and, for simplicity, it is assumed that K is algebraically closed. Throughout the whole paper, $\mathfrak{g}$ will be a given reductive Lie algebra and $\mathfrak{h}$ will be a given Cartan subalgebra of $\mathfrak{g}$. Moreover, after Theorem 1.2, it will be assumed that $\mathfrak{g}$ is simple. Unless assumed otherwise, set $l = \dim \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$. The set of roots relative to $\mathfrak{h}$ is denoted by Δ, the root lattice by Q and the Weyl group by W. For $\alpha \in \Delta$, let h_α be the corresponding coroot, let e_α be a basis of $\mathfrak{g}_\alpha$

and let $f_\alpha \in \mathfrak{g}_{-\alpha}$ be defined by the requirement $[e_\alpha, f_\alpha] = h_\alpha$. Viewed as a subgroup of $GL(\mathfrak{h}^*)$, W is generated by the hyperplane reflections $s_\alpha : \lambda \mapsto \lambda - \lambda(h_\alpha)\,\alpha$. Also set $P = \{\lambda \in \mathfrak{h}^* | \lambda(h_\alpha) \in \mathbb{Z}, \forall \alpha \in \Delta\}$. An element of $\mathfrak{h}^*$ is called a *weight* and an element of P is called an *integral weight.*

Let M be an $\mathfrak{h}$-module. When λ runs over $\mathfrak{h}^*$, the subspaces $M_\lambda := \{m \in M |\, h.m = \lambda(h)\, m, \forall h \in \mathfrak{h}\}$ are called the weight spaces of M. For any $t \subset \mathfrak{h}^*$, set $M[t] = \oplus_{\lambda \in t} M_\lambda$. Let $\mathfrak{a}$ be any Lie algebra containing $\mathfrak{h}$. A *weight* $\mathfrak{a}$*-module* is an $\mathfrak{a}$-module M such that $M = \oplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ and $\dim M_\lambda < \infty$ for all λ (when $\mathfrak{a}$ is tacitly assumed or when it does not matter, M is called a weight module). The *support* of the weight module M is the set $\operatorname{Supp} M := \{\lambda |\, M_\lambda \neq 0\}$.

A *basis of the root system* Δ is a set B of l roots such any root α belongs to the monoid $\mathbb{Z}_{\geqslant 0} B$ or to its opposite. For a basis B, the set of *positive roots relative to* B is $\Delta_B^+ := \Delta \cap \mathbb{Z}_{\geqslant 0} B$. Also set $\Delta_B^- = -\Delta_B^+$ and $\rho_B = 1/2 \sum_{\alpha \in \Delta_B^+} \alpha$. The set of *dominant integral weights relative to* B is the set $P_B^+ := \{\lambda \in P | \lambda(h_\alpha) \geqslant 0, \forall \alpha \in B\}$. Similarly, $L_B(\lambda)$ denotes the simple B-highest weight $\mathfrak{g}$-module with highest weight $\lambda \in \mathfrak{h}^*$. This simple module is characterized by the existence of a non-zero vector v of weight λ such that $e_\alpha.v = 0$ for any $\alpha \in B$. When there will be no ambiguity on B, the index B will be dropped from the previous notations.

In what follows, $< \mid >$ denotes a W-invariant bilinear form on $\mathfrak{h}^*$ such that $< \lambda|\lambda > \in \mathbb{Z}_{>0}$ for any $\lambda \in Q \setminus 0$. For $\lambda \in Q$, $\|\lambda\|$ is the real number $< \lambda|\lambda >^{1/2}$. Any basis B of the root system Δ has a natural structure of a graph, with edges the pairs of distinct roots $\alpha, \beta \in B$ such that $< \alpha|\beta > \neq 0$. Of course, a basis B of the root system is also a basis of the lattice Q. However, there are some subsets Σ of Δ which are bases of Q but which are not bases of Δ. Since these subsets Σ will play an important role in the paper, these two notions of bases should not be confused.

1. Reduction of the classification to cuspidal modules.

The reduction of the classification of all simple weight modules to the classification of cuspidal modules of some Levi factors is the main topic of the section. This idea is mainly due to Fernando [Fe]. Before this idea is explained, some notations are introduced and a useful lemma is stated.

Let $\mathfrak{g}$ be a reductive Lie algebra, let $\mathfrak{h}$ be a Cartan subalgebra. For any Lie algebra $\mathfrak{a}$ on which $\mathfrak{h}$ acts by derivations, let $\Delta(\mathfrak{a})$ be the set of non-zero weights of the $\mathfrak{h}$-module $\mathfrak{a}$ and let $Q(\mathfrak{a})$ the subgroup of $\mathfrak{h}^*$ generated by $\Delta(\mathfrak{a})$.

It is always assumed that any parabolic subalgebra of $\mathfrak{g}$ contains $\mathfrak{h}$. Let $\mathfrak{p}$ be a parabolic subalgebra and let $\mathfrak{u}$ be its nilradical. We have $\Delta = \Delta(\mathfrak{p}) \cup -\Delta(\mathfrak{p})$, $\Delta(\mathfrak{u}) = \{\alpha \in \Delta | \alpha \in \Delta(\mathfrak{p}) \text{ but } -\alpha \notin \Delta(\mathfrak{p})\}$ and $\mathfrak{u} = \oplus_{\alpha\in\Delta(\mathfrak{u})} \mathfrak{g}_\alpha$. The opposed nilradical of $\mathfrak{p}$ is the subalgebra $\mathfrak{u}^- := \oplus_{\alpha\in-\Delta(\mathfrak{u})} \mathfrak{g}_\alpha$ and there is a triangular decomposition $\mathfrak{g} \simeq \mathfrak{u} \oplus \mathfrak{p}/\mathfrak{u} \oplus \mathfrak{u}^-$. A basis B of the root system Δ is called $\mathfrak{p}$-*adapted* if $B \subset \Delta(\mathfrak{p})$. By convention, a *cone* is a finitely generated submonoid of Q containing 0. For instance, the *cone of* $\mathfrak{p}$, denoted by $C(\mathfrak{p})$, is the cone generated by $-\Delta(\mathfrak{u})$.

Let $u \subset \mathfrak{h}^*$ be a $K \otimes Q(\mathfrak{p}/u)$-coset. In the following, $\mathcal{C}_u$ denotes the category of all weight $\mathfrak{p}$-modules with support lying in u, where $u \subset \mathfrak{h}^*$ is a given $K \otimes Q(\mathfrak{p}/\mathfrak{u})$-coset. For $S \in \mathcal{C}_u$, set $M_\mathfrak{p}(S) = \mathrm{Ind}_\mathfrak{p}^\mathfrak{g} S$. Since we have $K \otimes Q(\mathfrak{p}/\mathfrak{u}) \cap C(\mathfrak{p}) = 0$, the Lie algebra $\mathfrak{u}$ acts trivially on S. Moreover $M_\mathfrak{p}(S)$ decomposes into $S \oplus M^+$, where $M^+ = \oplus_{\lambda\notin u} M_\mathfrak{p}(S)_\lambda$. Set $L_\mathfrak{p}(S) = M_\mathfrak{p}(S)/Z$, where Z is the biggest $\mathfrak{g}$-submodule of $M_\mathfrak{p} S$ included in M^+.

LEMMA 1.1. — *Let u be a $K \otimes Q(\mathfrak{p}/\mathfrak{u})$-coset.*

i) *The support of any simple weight $\mathfrak{p}$-module S lies in a single $Q(\mathfrak{p}/\mathfrak{u})$-coset.*

ii) *For any $S \in \mathcal{C}_u$, $L_\mathfrak{p}(S)$ is a weight module.*

iii) *For any simple weight $\mathfrak{p}$-module S, the module $L_\mathfrak{p}(S)$ is simple. Moreover it is the unique simple quotient of $M_\mathfrak{p}(S)$.*

iv) *For any submodule (respectively quotient, subquotient) S' of a $\mathfrak{p}$-module $S \in \mathcal{C}_u$, the $\mathfrak{g}$-module $L_\mathfrak{p}(S')$ is a submodule (respectively quotient, subquotient) of $L_\mathfrak{p}(S)$.*

Proof. — Proof of (i): Let $t \subset \mathfrak{h}^*$ be a $Q(\mathfrak{p}/\mathfrak{u})$-coset such that $S[t] \neq 0$. Set $v = t - C(\mathfrak{p})$ and $v^+ = v \setminus t$. It is clear that $S[v]$ and $S[v^+]$ are $\mathfrak{p}$-submodules of S. Since $S[t] \neq 0$, we have $S[v^+] \neq S[v]$. By simplicity of S, we have $S[v] = S$ and $S[v^+] = 0$. Therefore $S = S[t]$, which proves the first point.

Proof of (ii): By PBW Theorem, we have $M_\mathfrak{p}(S) \simeq U(\mathfrak{u}^-) \otimes S$. Therefore, for $\lambda \in \mathfrak{h}^*$ we have $M_\mathfrak{p}(S)_\lambda \simeq \oplus\, U(\mathfrak{u}^-)_{\lambda_1} \otimes S_{\lambda_2}$, where the sum runs over the set consisting of pairs (λ_1, λ_2) with $\lambda_1 \in C(\mathfrak{p})$, $\lambda_2 \in u$ and $\lambda_1 + \lambda_2 = \lambda$. Since $K \otimes Q(\mathfrak{p}/\mathfrak{u}) \cap C(\mathfrak{p}) = 0$, this set is finite. Hence

$\dim M_{\mathfrak{p}}(S)_\lambda < \infty$ and therefore $M_{\mathfrak{p}}(S)$ and its quotient $L_{\mathfrak{p}}(S)$ are weight modules.

Proof of (iii): By definition, the $\mathfrak{g}$-module $M_{\mathfrak{p}}(S)$ is generated by S and for any submodule N, we have $N \subset Z$ or $N \cap S \neq 0$. Since S is simple, any proper submodule is contained in Z. Therefore $L_{\mathfrak{p}}(S)$ is simple and it is the unique simple quotient of $M_{\mathfrak{p}}(S)$.

Proof of (iv): The functor $S \in \mathcal{C}_u \mapsto L_{\mathfrak{p}}(S)$ is not exact, but it preserves injective and surjective maps. Therefore $L_{\mathfrak{p}}(S')$ is a subquotient of $L_{\mathfrak{p}}(S)$, for any $\mathfrak{p}$-subquotient S' of S. Q.E.D.

For instance, let $\mathfrak{b}$ be a Borel subalgebra and let B be the unique $\mathfrak{b}$-adapted basis. Any weight λ can be identified with a one dimensional $\mathfrak{b}$-module. The module $M_{\mathfrak{b}}(\lambda)$ is the Verma module with B-highest weight λ and $L_{\mathfrak{b}}(\lambda)$ is its unique simple quotient, also denoted by $L_B(\lambda)$. When $\mathfrak{p} \neq \mathfrak{g}$ is any proper parabolic subalgebra and S is a simple weight $\mathfrak{p}$-module, the simple $\mathfrak{g}$-module $L_{\mathfrak{p}}(S)$ is called *parabolically induced*. A simple $\mathfrak{g}$-module L which is not parabolically induced is called *cuspidal*. Roughly speaking, Fernando's Theorem states that any simple weight $\mathfrak{g}$-module L is isomorphic to $L_{\mathfrak{p}}(S)$ for some parabolic subalgebra $\mathfrak{p}$ and some cuspidal $\mathfrak{p}$-module S. To clarify this statement, let us point out the following remarks:

• In this statement, $\mathfrak{p}=\mathfrak{g}$ is allowed. This occurs when L is cuspidal.

• By the previous lemma, any simple weight $\mathfrak{p}$-module S is a indeed a $\mathfrak{p}/\mathfrak{u}$-module, therefore the notion of a cuspidal $\mathfrak{p}$-module is well defined.

• Moreover for any simple $\mathfrak{p}$-module S, $\operatorname{Supp} S$ lies in a single $Q(\mathfrak{p}/\mathfrak{u})$-coset which is the condition required to define $L_{\mathfrak{p}}(S)$.

However the pair $(\mathfrak{p}, S)$ is not always uniquely determined by L. For instance, there is a well-known isomorphism $L_B(\lambda) \simeq L_{s_\alpha B}(s_\alpha \lambda)$ for any $\alpha \in B$ with $\lambda(h_\alpha) \in \mathbb{Z}_{\geqslant 0}$. In order to get an exact classification of simple weight modules in term of cuspidal modules, one also needs to determine when the modules $L_{\mathfrak{p}}(S)$ and $L_{\mathfrak{p}'}(S')$ are isomorphic. This requires the *small Weyl group* $W(S)$, which is now defined.

Let S be a simple weight $\mathfrak{p}$-module, let B be a $\mathfrak{p}$-adapted basis of Δ and let $B^\perp \subset B$ be the subset of all roots which are orthogonal to $\Delta(\mathfrak{p}/\mathfrak{u})$. Equivalently, $B^\perp$ is the set of all $\alpha \in B$ such that the image of h_α in $\mathfrak{p}/\mathfrak{u}$ is central. For $\alpha \in B^\perp$, h_α acts on the simple $\mathfrak{p}/\mathfrak{u}$-module S as some scalar l_α. Set $B^\perp_F = \{\alpha \in B^\perp \mid l_\alpha \in \mathbb{Z}_{\geqslant 0}\}$. The root system generated by $B^\perp_F$ is denoted by $\Delta(S)$ and its Weyl group by $W(S)$. Indeed the set $B^\perp$

is independent of the choice of the $\mathfrak{p}$-adapted basis B and therefore $\Delta(S)$ and $W(S)$ depend only on S.

Let $\mathfrak{p}$ be a parabolic subalgebra. For $w \in W$, let $\mathfrak{p}^w$ be the parabolic subalgebra such that $\Delta(\mathfrak{p}^w) = w.\Delta(\mathfrak{p})$. There is an automorphism $\overline{w}$ of the Lie algebra $\mathfrak{g}$ which stabilizes $\mathfrak{h}$ and which restricts to $\mathfrak{h}$ as w. The automorphism $\overline{w}$ conjugates $\mathfrak{p}$ and $\mathfrak{p}^w$. For any weight $\mathfrak{p}$-module S, the isomorphism class of the $\mathfrak{p}^w$-module $S^{\overline{w}}$ is independent of the choice of $\overline{w}$, and therefore it will be denoted by S^w.

THEOREM 1.2.— i) *(Fernando) Any simple weight $\mathfrak{g}$-module is isomorphic to $L_{\mathfrak{p}}(S)$ for some parabolic subalgebra $\mathfrak{p}$ and some cuspidal $\mathfrak{p}$-module S.*

ii) *Let S, S' be cuspidal modules for two parabolic subalgebras $\mathfrak{p}$, $\mathfrak{p}'$. Then $L_{\mathfrak{p}}(S)$ and $L_{\mathfrak{p}'}(S')$ are isomorphic if and only if we have $\mathfrak{p}'=\mathfrak{p}^w$ and $S' = S^w$, for some $w \in W(S)$.*

The main part of the previous theorem, namely Assertion (i), is due to Fernando [Fe], see also [Jo1], [Jo2], [Jo3] and for a quick proof [DMP] (Theorem 3.6). Assertion (ii) is easy, see Section 5 of [DMP]. Q.E.D.

Set $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \dots$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$ and where each $\mathfrak{g}_i$ is a simple component of $\mathfrak{g}$. Any simple weight module L decomposes into $Z \otimes L_1 \otimes L_2 \cdots$, where Z is a one dimensional representation of $\mathfrak{z}$ and each L_i is a simple weight $\mathfrak{g}_i$-module. Thus the general classification of all simple weight modules reduces to the problem of determining all cuspidal representations for the simple Lie algebras, which is the aim of the present paper. *So from now on, it is assumed that $\mathfrak{g}$ is simple.* Before reducing further the classification (see Proposition 1.6), we start with a useful proposition and its corollaries.

Let L be a simple weight $\mathfrak{g}$-module and let $\alpha \in \Delta$ be a root. It is easy to prove that the action of e_α on L is either injective or locally nilpotent. Following [DMP], the set Δ decomposes into four disjoint parts $\Delta_L^I, \Delta_L^F, \Delta_L^+$ and Δ_L^- as follows:

$$\Delta_L^I = \{\alpha \in \Delta;\ e_\alpha|_L \text{ and } f_\alpha|_L \text{ are injective}\}$$
$$\Delta_L^F = \{\alpha \in \Delta;\ e_\alpha|_L \text{ and } f_\alpha|_L \text{ are locally nilpotent}\}$$
$$\Delta_L^+ = \{\alpha \in \Delta;\ f_\alpha|_L \text{ is injective but } e_\alpha|_L \text{ is locally nilpotent}\}$$
$$\Delta_L^- = -\Delta_L^+.$$

By definition, the *cone of* L is the monoid $C(L)$ generated by $\Delta_L^I \cup \Delta_L^-$. The following statement is proved in Section 5 of [DMP].

PROPOSITION 1.3. — *Let* $\mathfrak{p}$ *be a parabolic subalgebra, let* S *be a cuspidal* $\mathfrak{p}$*-module and set* $L \simeq L_{\mathfrak{p}}(S)$.

i) *There is a finite set* Ω *such that* $\operatorname{Supp} L = \Omega + C(L)$.

ii) *We have* $\Delta_L^I = \Delta(\mathfrak{p}/\mathfrak{u})$, $\Delta_L^F = \Delta(S)$, *and* Δ_L^I *and* Δ_L^F *are orthogonal.*

iii) *There is a parabolic subalgebra* $\mathfrak{q} \supset \mathfrak{p}$ *with nilradical* $\mathfrak{v}$ *such that* $\Delta(\mathfrak{q}/\mathfrak{v}) = \Delta_L^I \cup \Delta_L^F$ *and* $\Delta(\mathfrak{v}) = \Delta_L^+$. *Moreover* $C(L) = Q(\mathfrak{p}/\mathfrak{u}) + C(\mathfrak{q})$.

Let L be a simple weight $\mathfrak{g}$-module. A basis B is called *L-adapted* if B is $\mathfrak{p}$-adapted for some parabolic subalgebra $\mathfrak{p}$ such that $L \simeq L_{\mathfrak{p}}(S)$ with S cuspidal. By the previous proposition, a basis B is adapted if and only if $B \cap \Delta_L^I$ is a basis of Δ_L^I and $\Delta_L^+ \subset \Delta_B^+$. (moreover $B \cap \Delta_L^F$ is always a basis of Δ_L^F).

It is obvious that the support of any simple weight $\mathfrak{g}$-module L lies in a single Q-coset. The following proposition, which follows from Proposition 1.3, was first proved by Fernando, see [Fe].

COROLLARY 1.4 (Fernando). — *For a simple weight* $\mathfrak{g}$*-module* L, *the following assertions are equivalent:*

i) L *is cuspidal,*

ii) e_α *acts injectively on* L *for all* $\alpha \in \Delta$,

iii) *The support of* L *is exactly one* Q*-coset.*

Remark. — A module satisfying (ii) is called torsion-free and a module with property (iii) is called dense. By the previous proposition, cuspidal, torsion-free and dense are equivalent properties, and in the literature the three terminologies are used simultaneously. For clarity, only the terminology "cuspidal" will be used.

COROLLARY 1.5. — *Let* L *be a cuspidal* $\mathfrak{g}$*-module.*

i) *For any* $\alpha \in \Delta$, f_α *acts bijectively on* L.

ii) *There exists an integer* d *such that* $\dim L_\lambda = d$ *for any* $\lambda \in \operatorname{Supp} L$.

Proof. — By Proposition 1.3, we have $\Delta_L^I = \Delta$. Therefore, for any $\lambda \in \operatorname{Supp} L$, the maps $e_\alpha : L_\lambda \to L_{\lambda+\alpha}$ and $f_\alpha : L_{\lambda+\alpha} \to L_\lambda$ are injective. Therefore f_α acts bijectively on L and the dimensions of the weight spaces of L are all the same. Q.E.D.

PROPOSITION 1.6 (Fernando).— *If there is at least one cuspidal $\mathfrak{g}$-module, then $\mathfrak{g}$ is of type A or C.*

Proof.— For the convenience of the reader, a quick proof follows. Therefore, assume that there exists one cuspidal $\mathfrak{g}$-module L.

Let $K = U_0 \subset U_1 \subset U_2 \subset \ldots \subset U = U(\mathfrak{g})$ be the canonical filtration of $U(\mathfrak{g})$. There exist some $C > 0$ such that $||\mu|| \leqslant Cn$ for any $n \geqslant 0$ and any weight μ of U_n. Hence we have Card Supp $U_n = O(n^l)$. By Corollary 1.5, the dimension of its weight spaces of L is uniformly bounded, hence we get $\dim U_n \cdot L_\lambda = O(n^l)$, for any λ. Therefore the characteristic variety $V(L)$ of L has dimension $\leqslant l$.

Let G be the adjoint group of $\mathfrak{g}$, and set $\mathcal{O} = G.e_\alpha$, where α is any long root (i.e. any root if $\mathfrak{g}$ is simply laced). Since $\mathcal{O}$ is the G-orbit of an extremal weight vector, any non-zero G-invariant closed cone of $\mathfrak{g}$ contains $\mathcal{O}$. Thus for any $x \in \mathfrak{g}\backslash 0$, we have $\overline{G.Kx} \supset \mathcal{O}$. Therefore $\dim \mathcal{O}' \geqslant \dim \mathcal{O}$ for any G-orbit $\mathcal{O}' \neq 0$ in $\mathfrak{g}$. Since the G-modules $\mathfrak{g}$ and $\mathfrak{g}^*$ are isomorphic, the dimension of any non-zero symplectic leaf of $\mathfrak{g}^*$ is $\geqslant \dim \mathcal{O}$, and therefore we have $\dim V \geqslant \frac{1}{2}\dim \mathcal{O}$, for any coisotropic subvariety V of $\mathfrak{g}^* \setminus 0$.

Since $V(L)$ is coisotropic, we get that $\dim \mathcal{O} \leqslant 2l$. By direct calculations, it is easy to prove that $\dim \mathcal{O} > 2l$ if $\mathfrak{g}$ is not of type A or C.

Q.E.D.

2. Generalities about weight modules.

Recall that $\mathfrak{g}$ denotes a simple Lie algebra and $\mathfrak{h}$ is a fixed Cartan subalgebra. Set $U = U(\mathfrak{g})$, let $\mathcal{A} = U_0$ be the commutant of $\mathfrak{h}$ and let $\mathcal{B} = \oplus_{\lambda \neq 0} U_\lambda$ be its complement in U. For $\lambda \in \mathfrak{h}^*$, a $\mathcal{A}$-module of weight λ is a finite dimensional $\mathcal{A}$-module X such that $X = X_\lambda$ as an $\mathfrak{h}$-module. Set then $I(X) = U \otimes_{\mathcal{A}} X$ and $I^+(X) = \mathcal{B} \otimes_{\mathcal{A}} X$. We have $I(X) = \oplus_{\mu \in Q} I(X)_{\mu+\lambda}$, where $I(X)_\lambda = X$ and $I(X)_{\mu+\lambda} = \mathcal{B}_\mu \otimes_{\mathcal{A}} X$ for $\mu \neq 0$. Let $Z(X)$ be the biggest U-submodule of $I(X)$ which is contained in $I^+(X)$. Then set $I_!(X) = I(X)/Z(X)$.

LEMMA 2.1.— *Let X be a $\mathcal{A}$-module of weight λ.*

i) *$I(X)$ is a weight module generated by X.*

ii) *Let Y be a U-module generated by Y_λ. Any morphism of $\mathcal{A}$-modules $\pi : Y_\lambda \to X$ extends to a unique morphism of U-modules $Y \to I_!(X)$.*

Proof. — By Hilbert's Theorem, the right $\mathcal{A}$-module U_μ is finitely generated for any $\mu \in Q$, see e.g. [DMP]. Hence $I(X)_{\mu+\lambda}$ is finite dimensional, which proves (i). Let Y be a U-module generated by Y_λ and let $\pi^e : I(Y_\lambda) \to I(X)$ be the morphism extending π. We have $Y \simeq I(Y_\lambda)/Z$ for some submodule Z of $I(Y_\lambda)$ such that $Z_\lambda = 0$. Hence we have $\pi^e(Z) \subset Z(X)$ and π^e induces a morphism $Y \to I_!(X)$. Q.E.D.

LEMMA 2.2. — *Let L be a simple weight $\mathfrak{g}$-module and let $\lambda \in \operatorname{Supp} L$ be a weight.*

i) *The $\mathcal{A}$-module L_λ is simple.*

ii) *We have $L \simeq I_!(L_\lambda)$.*

Proof. — This is an immediate corollary of Lemma 2.1. Q.E.D.

Let M be a weight module. Its *trace* Tr^M is the function $(\lambda, u) \in \mathfrak{h}^* \times \mathcal{A} \mapsto \operatorname{Tr}_\lambda^M(u) = \operatorname{Tr} u|_{M_\lambda}$. A weight module is called *semi-simple* if it is a finite or infinite direct sum of simple modules.

LEMMA 2.3. — *Let M, N be two semi-simple weight modules. If $\operatorname{Tr}^M = \operatorname{Tr}^N$, then $M \simeq N$.*

Proof. — Let L be a simple weight module. By Proposition 1.3, we have $L \simeq I_!(L_\lambda)$, for any $\lambda \in \operatorname{Supp} L$. Hence the multiplicity of L in M as a U-module is the multiplicity of L_λ in M_λ as a $\mathcal{A}$-module. The last multiplicity is determined by the function $\operatorname{Tr}_\lambda^M$ and the lemma follows. Q.E.D.

3. Support of admissible modules.

A weight module M with support lying in a single Q-coset is called *admissible* if $\dim M_\lambda$ is uniformly bounded. The *degree* of an admissible module is the maximal dimension of its weight spaces. For an admissible module M of degree d, its *essential support* is the set $\operatorname{Supp}_{\text{ess}} M := \{\lambda \in \operatorname{Supp} M \mid \dim M_\lambda = d\}$. By Corollary 1.5, any cuspidal $\mathfrak{g}$-module L is admissible and $\operatorname{Supp} L = \operatorname{Supp}_{\text{ess}} L$.

LEMMA 3.1. — *Let L be an infinite dimensional simple weight $\mathfrak{g}$-module. Then the group generated by the monoid $C(L)$ is Q.*

Proof. — First consider the case where $\Delta_L^{\pm} = 0$. By Proposition 1.3, Δ is the disjoint union of the orthogonal subsets Δ_L^I and Δ_L^F. Therefore we have $\Delta = \Delta_L^I$ and $C(L) = Q$. So it can be assumed that $\Delta_L^{\pm} \neq 0$. By Proposition 1.3, $\Delta_L^{\pm}$ is the set of roots of the nilradicals $\mathfrak{v}^{\pm}$ of two opposed parabolic subalgebras. We have $\mathfrak{g}=\mathfrak{v}^{+}+\mathfrak{v}^{-}+[\mathfrak{v}^{+},\mathfrak{v}^{-}]$, hence Δ_L^{-} generates Q. As $C(L)$ contains Δ_L^{-}, it generates the group Q.

For any real number $x \geqslant 0$, set $\rho(x) = \operatorname{Card} B(x)$, where $B(x) = \{\mu \in Q | \ \|\mu\| \leqslant x\}$. Let M be a weight module with support lying in a single Q-coset, say $\lambda + Q$. The density of M is $\delta(M) := \liminf_{x\to\infty} \rho(x)^{-1} \sum_{\mu \in B(x)} \dim M_{\lambda+\mu}$ (the symbol "lim inf" stands for the infimum limit). Of course, $\delta(M)$ is independent of λ.

LEMMA 3.2. — *There exists $\epsilon > 0$ such that $\delta(L) > \epsilon$, for any infinite dimensional simple weight module L.*

Proof. — For any cone C, the number $\delta(C) = \liminf_{x\to\infty} \rho(x)^{-1} \operatorname{Card} C \cap B(x)$ is positive if C generates Q. For any infinite dimensional simple weight module L, we have $\operatorname{Supp} L \supset \lambda + C(L)$, for any $\lambda \in \operatorname{Supp} L$. Thus we get $\delta(L) \geqslant \delta(C(L))$ and $\delta(C(L)) > 0$ by Lemma 3.1. By Proposition 1.3, there are only finitely many different cones attached to all simple weight modules. Hence there exists $\epsilon > 0$ such that $\delta(C(L)) > \epsilon$ for any infinite dimensional simple weight module L. Hence $\delta(L) > \epsilon$ for any L. Q.E.D.

LEMMA 3.3. — *Any admissible weight module M has finite length.*

Proof. — As M is admissible, we have $\delta(M) < \infty$. For any exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$, we have $\delta(M_2) \geqslant \delta(M_1) + \delta(M_3)$. Let X be the set of all $\mu \in P$ such that $|\mu(h_\alpha)| \leqslant 1$ for any $\alpha \in \Delta$. For any simple module L of finite dimension, we have $L_\mu \neq 0$ for some $\mu \in X$, see [B1]. It follows easily that the length of M is finite, and bounded by $A + \delta(M)/\epsilon$, where the constant ϵ is defined by Lemma 3.2 and $A = \sum_{\lambda \in X} \dim M_\lambda$. Q.E.D.

LEMMA 3.4. — *Let C be a cone generating Q and let $S_1 \subset S_2$ be nonempty subsets of $\mathfrak{h}^*$ lying in a single Q-coset. Let us assume that $S_2 = C + \Omega$ for some finite set Ω and that $C + S_1 \subset S_1$. Then we have*

- *$S_2 \setminus S_1$ is contained in a finite union of affine hyperplanes.*
- *S_1 is Zariski dense in $\mathfrak{h}^*$.*

Proof. — It is clear that S_1 is Zariski dense. Since Ω is finite and C generates Q, we have $\Omega \subset \lambda + C$ for some $\lambda \in \mathfrak{h}^*$. After a convenient translation, it can be assumed that $S_2 \subset C$. Moreover we have $S_1 \supset \mu + C$

for some μ. Without loss of generality, it can be assumed that $S_2 = C$ and $S_1 = \mu + C$ for some $\mu \in C$.

Let Γ be a finite set of generators of C. Let $\mathbf{b}$ (respectively $\mathbf{c}$) the set of all subsets of Γ which are a basis of $\mathfrak{h}^*$ (respectively all subsets of rank $< l$). For any $T \in \mathbf{c}$, let H_T be a linear hyperplane containing T. For any $T \in \mathbf{b}$, there is an identity $m_T\mu = \sum_{\gamma\in T} m_\gamma^T\gamma$, where $m_T \in \mathbb{Z}_{>0}$ and all m_γ^T are integers. Set $N = \mathrm{Sup}_{T\in\mathbf{b},\gamma\in T}\, m_\gamma^T$ and let Γ' the set of all integral combinations $\sum_{\gamma\in\Gamma} m_\gamma\gamma$ with $0 \leqslant m_\gamma < N$ for all γ.

Any $\tau \in C$ can be written as $\sum_{\gamma\in\Gamma} m_\gamma\gamma$ where $m_\gamma \in \mathbb{Z}_{\geqslant 0}$ for all $\gamma \in \Gamma$. For such a decomposition of τ, set $T = \{\gamma |\, m_\gamma \geqslant N\}$. If T contains a basis T', then $\tau - m_{T'}\mu \in C$ and therefore τ belongs to S_1. Otherwise τ belongs to $\Gamma' + H_T$. Hence $S_2 \setminus S_1$ is contained in $\cup_{T\in\mathbf{c}}\Gamma' + H_T$. Q.E.D.

PROPOSITION 3.5.— *Let L be an infinite dimensional admissible simple module.*

i) *For any $\lambda \in \mathrm{Supp}_{\mathrm{ess}}\, L$, we have $\lambda + C(L) \subset \mathrm{Supp}_{\mathrm{ess}}\, L$,*

ii) *$\mathrm{Supp}_{\mathrm{ess}}\, L$ is Zariski dense in $\mathfrak{h}^*$, but $\mathrm{Supp}\, L \setminus \mathrm{Supp}_{\mathrm{ess}}\, L$ is contained in a finite union of affine hyperplanes.*

Proof.— Let $\lambda \in \mathrm{Supp}_{\mathrm{ess}}\, L$ and let $\alpha \in \Delta_L^I \cup \Delta_L^+$. As f_α acts injectively on L, we have $\dim L_{\lambda-\alpha} \geqslant \dim L_\lambda$. Hence $\lambda-\alpha \in \mathrm{Supp}_{\mathrm{ess}}\, L$, and Assertion (i) is proved. Set $C = C(L)$, $S_2 = \mathrm{Supp}\, L$ and $S_1 = \mathrm{Supp}_{\mathrm{ess}}\, L$. By Proposition 1.3 (i), Lemma 3.1, and Assertion (i), the triple (C, S_1, S_2) satisfies the hypotheses of Lemma 3.4. Hence Assertion (ii) follows from this lemma. Q.E.D.

4. Coherent families.

After the preparatory Sections 2 and 3, the present section is devoted to the introduction of the notion of coherent families. This notion, which is the main tool of the classification, will be also investigated in Sections 5 and 6.

A *coherent family* of degree d is a weight $\mathfrak{g}$-module $\mathcal{M}$ such that

- $\dim \mathcal{M}_\lambda = d$ for any $\lambda \in \mathfrak{h}^*$, and
- for any $u \in \mathcal{A}$, the function $\lambda \in \mathfrak{h}^* \mapsto \mathrm{Tr}\, u|_{\mathcal{M}_\lambda}$ is polynomial in λ.

The dual torus of $\mathfrak{g}$ is $T^* := \mathfrak{h}^*/Q$. As a $\mathfrak{g}$-module, any coherent family $\mathcal{M}$ decomposes into $\oplus_{t\in T^*}\mathcal{M}[t]$, and its $\mathfrak{g}$-submodules $\mathcal{M}[t]$ are all admissible.

Example of a coherent $\mathfrak{sl}(2)$*-family:* Let e, f, h be the usual basis of $\mathfrak{sl}(2)$. For $a \in K$, let $\mathcal{M}(a)$ be the $\mathfrak{sl}(2)$-module with basis $(x^s)_{s \in K}$ and where the action is given by the following formulas:

$$f \mapsto -\mathrm{d}/\mathrm{d}x + a/x, \qquad h \mapsto 2x\mathrm{d}/\mathrm{d}x, \qquad e \mapsto x^2\mathrm{d}/\mathrm{d}x + ax,$$

where $x^{\pm 1}$ denotes the operator $x^s \mapsto x^{s \pm 1}$ and $\mathrm{d}/\mathrm{d}x$ denotes the operator $x^s \mapsto sx^{s-1}$. Clearly, $\mathcal{M}(a)$ is an example of a coherent family. Except for one or two $t \in T^*$, the $\mathfrak{sl}(2)$-module $\mathcal{M}(a)[t]$ is simple and cuspidal. Otherwise, $\mathcal{M}(a)[t]$ has length $\leqslant 3$, i.e. it is a non-split extension of two or three simple modules. Therefore there exists a coherent family $\mathcal{M}(a)^{ss}$ such that $\mathcal{M}(a)^{ss}[t]$ is a finite sum of simple modules and has the same composition series as $\mathcal{M}(a)[t]$ for any $t \in T^*$. In what follows, we will define the notion of semi-simplicity and irreducibility of a coherent family. Indeed any semi-simple irreducible coherent $\mathfrak{sl}(2)$-family is isomorphic to $\mathcal{M}(a)^{ss}$ for some $a \in K$, see the remark at the end of Section 11.

A *commuting set of roots* is a subset $\Sigma \subset \Delta$ such that $[e_\alpha, e_\beta] = 0$ (or equivalently $[f_\alpha, f_\beta] = 0$) for any $\alpha, \beta \in \Sigma$. For any subset I of a basis B of Δ, Q_I denotes the subgroup of Q generated by I, Δ_I the root system generated by I, and Δ_I^+ the set of positive roots of Δ_I relative to its basis I.

LEMMA 4.1. — *Let B be a basis of Δ.*

i) Let $I \subset B$ and let $\alpha \in I$. There exists a set of commuting roots $\Sigma' \subset \Delta_I^+$ with $\alpha \in \Sigma'$ such that Σ' is a basis of Q_I.

ii) Let J, F be subsets of B with $F \neq B$. Let $\Sigma' \subset \Delta_J^+ \setminus \Delta_{J\cap F}^+$ be a set of commuting roots which is a basis of Q_J. There exists a set of commuting roots Σ which is a basis of Q such that $\Sigma' \subset \Sigma \subset \Delta_B^+ \setminus \Delta_F^+$.

Proof. — First let us prove Assertion (ii). If J is empty, let us choose any $\alpha \in B \setminus F$ and replace J and Σ' by $\{\alpha\}$. Therefore, it can be assumed that J is not empty. Set $J' = J \setminus F$, $p = \operatorname{Card} J'$, $q = \operatorname{Card} J$. Let $J_1, \dots J_k$ be the connected components of J and set $J_i' = J' \cap J_i$, $F_i = F \cap J_i$, and $\Sigma_i' = \Sigma' \cap \Delta_{J_i}$ for any $1 \leqslant i \leqslant k$. Since $\Sigma' \subset \Delta_J$ is a basis of Q_J, each Σ_i' is a basis of Q_{J_i}. Since Σ_i' lies in $\Delta_{J_i}^+ \setminus \Delta_{F_i}^+$, the set $J_i' = J_i \setminus F_i$ is not empty. Hence J' meets every connected component of J. Therefore we can write $J = \{\alpha_1, \dots, \alpha_q\}$ in such a way that $J' = \{\alpha_1, \dots, \alpha_p\}$ and, for any s with $p + 1 \leqslant s \leqslant q$, α_s is connected to α_i for some $i < s$. Since B is connected, we can write $B \setminus J = \{\alpha_{q+1}, \dots, \alpha_l\}$ in such a way that, for any $s \geqslant q + 1$, α_s is connected to α_i for some i with $1 \leqslant i < s$. Therefore the connected component C_s of α_s in $\{\alpha_1, \dots, \alpha_s\}$ contains at least one element in J', for any $s > 0$. For $s > q$, set $\gamma_s = \sum_{j \in C_s} \alpha_j$. By [B2], γ_s is a root, and since $C_s \not\subset F$ we have $\gamma_s \notin \Delta_F$.

Set $\Sigma' = \{\beta_1, \ldots, \beta_q\}$. Then the roots $\beta_{q+1}, \ldots, \beta_l$ are defined by induction. For $s \geqslant q$, assume that the set of commuting roots $\{\beta_1, ..., \beta_s\}$ is already given. Set $\mathfrak{m}_s = \oplus_{1 \leqslant j \leqslant s} K\, e_{\beta_j}$. Since $\mathfrak{m}_s$ is a commutative ad-nilpotent Lie subalgebra of $\mathfrak{g}$, there is a root γ such that $e_\gamma \in \mathrm{Ad}(U(\mathfrak{m}_s))(e_{\gamma_{s+1}})$ and $[\mathfrak{m}_s, e_\gamma] = 0$. Since $\gamma_{s+1} \notin \Delta_F$, the root $\beta_{s+1} := \gamma$ does no belongs to Δ_F^+. This inductive process defines a set $\Sigma := \{\beta_1, \ldots, \beta_l\}$ of commuting roots such that $\Sigma' \subset \Sigma \subset \Delta_B^+ \setminus \Delta_F^+$. Moreover for $s > q$, we have $\beta_s = \alpha_s + \sum_{k<s} m_{k,s}\alpha_k$ for some integers $m_{k,s}$. Hence $\{\alpha_1, \ldots, \alpha_q\} \cup \{\beta_{q+1}, \ldots, \beta_l\}$ is a basis of Q. Since Σ' and $\{\alpha_1, \ldots, \alpha_q\}$ generate the same lattice Q_J, Σ is a basis of Q.

To prove Assertion (i), it can be assumed that $I = B$. Thus Assertion (i) follows from Assertion (ii) with $J = \{\alpha\}$ and $F = \varnothing$. Q.E.D.

Let R be an associative algebra and let S be a multiplicative subset (by convention: $0 \notin S$). An element $s \in S$ satisfies *Ore's localizability conditions* if for any $r \in R$, there are $s', s'' \in S$, $r', r'' \in R$ such that $sr' = rs'$ and $r''s = rs''$. The multiplicative set S satisfies *Ore's localizability conditions* if any $s \in S$ satisfies these conditions.

LEMMA 4.2. — *Let R be an associative algebra and let S be a multiplicative subset generated by locally* ad*-nilpotent elements. Then S satisfies Ore's localizability conditions.*

Proof. — To check Ore's condition for S, it is enough to check them on any set of generators of S. Therefore, it is enough to check them for any ad-nilpotent element $s \in S$. For any $r \in R$, we have $\mathrm{ad}^N(s)(r) = 0$ for some $N > 0$. Then the identity $\sum_{0 \leqslant i \leqslant N} (-1)^i \binom{N}{i} s^i r s^{N-i} = 0$ can be written as $sr' = rs^N$ and $r''s = s^N r$ for some $r', r'' \in R$. Hence Ore's conditions are satisfied. Q.E.D.

For S and R as previously, R_S denotes the localization of R relative to S. For any R-module M, set $M_S = R_S \otimes_R M$. When S contains no zero divisors, the algebra morphism $R \to R_S$ is injective. Similarly the map $M \to M_S$ is injective whenever s acts injectively on M, for all $s \in S$. When S is generated by a single element s, it will be convenient to set $R_s = R_S$ and $M_s = M_S$.

LEMMA 4.3. — *Let R be an associative K-algebra and let S be a multiplicative subset generated by q commuting and locally ad-nilpotent* *elements $s_1, \dots, s_q$. Then there exists a unique q-parameters family of automorphisms $\Theta_{(x_1 \dots, x_q)} : R_S \to R_S$ such that*

- $\Theta_{(x_1 \dots, x_q)}(r) = s_1^{x_1} \dots s_q^{x_q}\, r\, s_q^{-x_q} \dots s_1^{-x_1}$ *if all x_i are integers, and*
- *the map $(x_1 \dots, x_q) \in K^q \mapsto \Theta_{(x_1 \dots, x_q)}(r)$ is polynomial in $(x_1 \dots, x_q)$,*

for any $r \in R_S$.

Proof. — By Lemma 4.2, the localized ring R_S is well defined. As S is commutative, the elements s_i are locally ad-nilpotent on R_S. For any $r \in R_S$, we have $\mathrm{ad}(s_i)^{N+1}(r) = 0$, for all i and some $N \geqslant 0$. The general identity: $s^m u = \sum_{0 \leqslant i \leqslant m} \binom{m}{i} \mathrm{ad}(s)^i(u) s^{m-i}$ implies that

$$s_1^{x_1} \dots s_q^{x_q}\, r\, s_q^{-x_q} \dots s_1^{-x_1} = \sum_{0 \leqslant i_1, \dots, i_q \leqslant N} \binom{x_1}{i_1} \dots \binom{x_q}{i_q} \mathrm{ad}(s_1)^{i_1} \dots \mathrm{ad}(s_q)^{i_q}(r)\, s_1^{-i_1} \dots s_q^{-i_q}$$

for any $x_1, \dots, x_q \in \mathbb{Z}_{\geqslant 0}$. Since the binomial coefficient $\binom{x}{i} = x(x-1)\dots(x-i+1)/i!$ can be extended to a polynomial function in $x \in K$, there is a polynomial $\Theta_{(x_1 \dots, x_q)}(r)$ in the variables $(x_1 \dots, x_q) \in K^q$ such that $\Theta_{(x_1 \dots, x_q)}(r) = s_1^{x_1} \dots s_q^{x_q}\, r\, s_q^{-x_q} \dots s_1^{-x_1}$ for any $x_1, \dots, x_q \in \mathbb{Z}_{\geqslant 0}$. By unicity of the polynomial extensions, $\Theta_{(x_1 \dots, x_q)}$ is a q-parameters family of automorphisms of R_S. Q.E.D.

For a set of commuting roots Σ, let F_Σ be the multiplicative subset of U generated by $(f_\alpha)_{\alpha \in \Sigma}$.

LEMMA 4.4. — *Let L be an infinite dimensional admissible simple module of degree d.*

i) *There is a set Σ of commuting roots which is a basis of Q such that $\Sigma \subset \Delta_L^I \cup \Delta_L^+$.*

ii) *For such a Σ, set $L' = L_{F_\Sigma}$. Then L' is a weight module containing L, $\operatorname{Supp} L' = Q + \operatorname{Supp} L$ and $\dim L'_\mu = d$ for all $\mu \in \operatorname{Supp} L'$.*

Proof. — Let B be an L-adapted basis of Δ and set $F = \Delta_L^F \cap B$. By Lemma 4.1, there exists a set $\Sigma \subset \Delta_B^+ \setminus \Delta_F^+$ of commuting roots which is a basis of Q. Thus $\Sigma \subset \Delta_L^I \cup \Delta_L^+$ and Assertion (i) is proved.

By Lemma 4.2, F_Σ satisfies Ore's conditions and the localized module L' is well defined. Since the elements $(f_\alpha)_{\alpha \in \Sigma}$ act injectively on L, we have $L \subset L'$. It is clear that $L' = \oplus_{\lambda \in t} L'_\lambda$, where $t = Q + \operatorname{Supp} L$.

We claim that $\dim L'_\lambda = d$ for any $\lambda \in t$. Let $X \subset L'_\lambda$ be any finite dimensional subspace. There exists $s \in F_\Sigma$ such that $s.X \subset L$. Since

$s.X \subset L_{\lambda+\beta}$, where β is the weight of s, we have $\dim X \leqslant d$. Since X is arbitrary, we get $\dim L'_\lambda \leqslant d$. Since Σ is a basis of Q, the dimension of the weight spaces L'_μ, for $\mu \in t$, are all equal. It follows that $\dim L'_\mu = d$ for all $\mu \in t$. Q.E.D.

Let Σ' be a set of commuting roots which are linearly independent. Let $Q_{\Sigma'}$ be the subgroup of Q generated by Σ'. Set $\mathfrak{h}^*_{\Sigma'} = K \otimes Q_{\Sigma'}$ and $T^*_{\Sigma'} = \mathfrak{h}^*_{\Sigma'}/Q_{\Sigma'}$. When $Q/Q_{\Sigma'}$ is torsion free, $T^*_{\Sigma'}$ is a subtorus of T^*. For any element $\nu = -\sum_{\alpha\in\Sigma'} x_\alpha\, \alpha$ in $\mathfrak{h}^*_{\Sigma'}$, the formal expression $\prod_{\alpha\in\Sigma'} f_\alpha^{x_\alpha}$ is denoted by $f^\nu_{\Sigma'}$. By Lemma 4.3, there is an automorphism of the algebra $U_{F_{\Sigma'}}$ which is formally defined by $u \mapsto f^{-\nu}_{\Sigma'}\, u\, f^\nu_{\Sigma'}$. For any $U_{F_{\Sigma'}}$-module M, let $f^\nu_{\Sigma'}.M$ be the module M twisted by this automorphism. For $m \in M$, it is convenient to denote by $f^\nu_{\Sigma'}.m$ the corresponding element of $f^\nu_{\Sigma'}.M$. With this convention, we have $u.f^\nu_{\Sigma'}.m = f^\nu_{\Sigma'}.(f^{-\nu}_{\Sigma'}\, u\, f^\nu_{\Sigma'}).m$ for any $u \in U_{F_{\Sigma'}}$. For instance, if m is a weight vector of weight μ, then $f^\nu_{\Sigma'}.m$ is a weight vector of weight $\mu+\nu$. If $\nu_1 - \nu_2 \in Q_{\Sigma'}$, the $U_{F_{\Sigma'}}$-modules $f^{\nu_1}_{\Sigma'}.M$ and $f^{\nu_2}_{\Sigma'}.M$ are isomorphic. Therefore for $\overline{\nu} \in T^*_{\Sigma'}$ the module $f^{\overline{\nu}}_{\Sigma'}.M$ is defined up to an isomorphism.

LEMMA 4.5. — *Let L be an infinite dimensional admissible simple module of degree d.*

i) *Let $\Sigma \subset \Delta^I_L \cup \Delta^+_L$ be a set of commuting roots which is a basis of Q, let $\Sigma' \subset \Sigma$ and set $L' = L_{F_\Sigma}$. Then $\mathcal{M} := \oplus_{\overline{\nu}\in T^*}\, f^{\overline{\nu}}_\Sigma.L'$ is a coherent family of degree d which contains L and $f^{\nu'}_{\Sigma'}.L_{F_{\Sigma'}}$ for any $\nu' \in \mathfrak{h}^*_{\Sigma'}$.*

ii) *In particular, let B be an L-adapted basis and let $\alpha \in B$ with $\alpha \notin \Delta^F_L$. Then L is contained in a coherent family $\mathcal{M}$ of degree d such that f_α acts bijectively on $\mathcal{M}$ and $\mathcal{M}$ contains $f^x_\alpha.L_{f_\alpha}$, for all $x \in K$.*

Proof. — Proof of Assertion (i): Let $\lambda \in \operatorname{Supp} L'$ be a weight. By Lemma 4.4 (ii), we have $\dim L'_\lambda = d$ and $\mathcal{M}$ contains L. Since Σ is a basis of Q, we have $\mathcal{M}_{\lambda+\mu} = f^\mu_\Sigma.L'_\lambda$ for any $\mu \in \mathfrak{h}^*$. Therefore we have $\dim \mathcal{M}_{\lambda+\mu} = d$. Moreover, for any $u \in \mathcal{A}$ we have $\operatorname{Tr} u|_{\mathcal{M}_{\lambda+\mu}} = \operatorname{Tr} f^{-\mu}_\Sigma u f^\mu_\Sigma|_{L'_\lambda}$. By Lemma 4.3, the function $\mu \mapsto \operatorname{Tr} u|_{\mathcal{M}_{\lambda+\mu}}$ is polynomial and therefore $\mathcal{M}$ is a coherent family.

Proof of Assertion (ii): By Lemma 4.1, there exists a set Σ satisfying the previous conditions with $\alpha \in \Sigma$. Therefore the second assertion follows from the previous one. Q.E.D.

Remark. — Let T be the adjoint torus of $\mathfrak{g}$. The ring $K[T]$ can be identified with the Laurent polynomial algebra $K[(f^{\pm 1}_\alpha)_{\alpha\in\Sigma}]$. Let L and $\lambda \in$

$\operatorname{Supp} L'$ be as in Lemma 4.5. We have $L' \simeq K[T] \otimes L'_\lambda \simeq K[T]^d$. It is clear that $\mathfrak{g}$ acts over $K[T]^d$ by differential operators. Therefore the $\mathfrak{g}$-module structure on L' comes from an algebra morphism $\sigma : U_{F_\Sigma} \to \mathfrak{gl}(d, \operatorname{Diff}(T))$, such that L' is the restriction of the natural representation $K[T]^d$ of $\mathfrak{gl}(d, \operatorname{Diff}(T))$ (here $\operatorname{Diff}(T)$ denotes the ring of differential operators on T). Any closed one-form ω on T induces an automorphism Θ_ω of $\operatorname{Diff}(T)$, namely $\Theta_\omega(f) = f$ if $f \in K[T]$ and $\Theta_\omega(\xi) = \xi + i_\xi \omega$ if ξ is a vector field on T. Denote again by Θ_ω its extension to $\mathfrak{gl}(d, \operatorname{Diff}(T))$. Any $\mu \in \mathfrak{h}^*$ can be identified with a closed one-form on T, and it is easy to prove that the automorphism Θ_μ of $\mathfrak{gl}(d, \operatorname{Diff}(T))$ extends the automorphism $u \mapsto f_\Sigma^{-\mu} u f_\Sigma^{\mu}$ of U_{F_Σ} (however the morphism σ is not injective). Hence $\mathcal{M}$ is a direct sum of twists of the natural representation of $\mathfrak{gl}(d, \operatorname{Diff}(T))$. Unfortunately, it seems not easy to explicitly describe the corresponding morphisms σ, and we will follow another approach in Sections 11 and 12.

Let L be an infinite dimensional admissible simple module. By Proposition 3.5 (ii), $\operatorname{Supp}_{\mathrm{ess}} L$ is Zariski dense.

LEMMA 4.6.— *For any $u \in \mathcal{A}$, the map $\lambda \in \operatorname{Supp}_{\mathrm{ess}} L \mapsto \operatorname{Tr} u|_{L_\lambda}$ is polynomial.*

Proof. — By Lemma 4.5, there is a coherent family $\mathcal{M} \supset L$ of same degree. Since $\operatorname{Tr} u|_{L_\lambda} = \operatorname{Tr} u|_{\mathcal{M}_\lambda}$ for any $\lambda \in \operatorname{Supp}_{\mathrm{ess}} L$, the restriction of the trace to $\operatorname{Supp}_{\mathrm{ess}} L$ is polynomial. Q.E.D.

A coherent family $\mathcal{M}$ is called *irreducible* if the $\mathcal{A}$-module $\mathcal{M}_\lambda$ is simple for some λ.

LEMMA 4.7.— *Let $\mathcal{M}$ be an irreducible coherent family. Then the set Ω of all weights $\mu \in \mathfrak{h}^*$ such that the $\mathcal{A}$-module $\mathcal{M}_\mu$ is simple is a non-empty Zariski open subset of $\mathfrak{h}^*$.*

Proof. — Indeed the $\mathcal{A}$-module $\mathcal{M}_\mu$ is simple if and only if the bilinear map $B_\mu : (u, v) \in \mathcal{A} \times \mathcal{A} \mapsto \operatorname{Tr}(uv|_{\mathcal{M}_\lambda})$ has maximal rank d^2, where d is the degree of $\mathcal{M}$. For any finite dimensional subspace $E \subset \mathcal{A}$, the set Ω_E of all μ such that $B_\mu|_E$ has rank d^2 is open. Therefore $\Omega = \cup_E \Omega_E$ is open.

Q.E.D.

Let $\mathcal{M}$ be a coherent family. By Lemma 3.3, the $\mathfrak{g}$-module $\mathcal{M}[t]$ has finite length for any $t \in T^*$. Denote by $\mathcal{M}^{ss}$ the semi-simple coherent family such that $\mathcal{M}^{ss}[t]$ has the same composition series as $\mathcal{M}[t]$, for any $t \in T^*$. Roughly speaking, $\mathcal{M}^{ss}$ is the "semi-simplification" of $\mathcal{M}$. Let

L be an infinite dimensional admissible simple module of degree d. A *coherent extension* of L is a coherent family $\mathcal{M}$ of degree d containing L as a subquotient.

PROPOSITION 4.8.— *Let L be an infinite dimensional admissible simple module of degree d.*

i) *There exists a unique semi-simple coherent extension $\mathcal{EXT}(L)$ of L.*

ii) *The coherent family $\mathcal{EXT}(L)$ is irreducible. Any infinite dimensional submodule L' of $\mathcal{EXT}(L)$ is admissible of degree d, and we have $\mathcal{EXT}(L) = \mathcal{EXT}(L')$.*

iii) *The central characters of the simple submodules of $\mathcal{EXT}(L)$ are all the same.*

iv) *If $\mathcal{M}$ is any coherent extension of L, then $\mathcal{M}^{ss}$ is isomorphic to $\mathcal{EXT}(L)$.*

Proof.— Proof of Assertions (i) and (iv): By Lemma 4.4, L is contained in a coherent family $\mathcal{M}$ of degree d. Hence L is contained in the coherent extension $\mathcal{M}^{ss}$. As $\mathrm{Supp}_{\mathrm{ess}}\, L$ is Zariski dense (Proposition 3.5), the trace of any coherent extension is determined by the trace of L. The uniqueness of semi-simple coherent extensions follows from Lemma 2.3. Assertion (iv) follows as well.

Proof of Assertion (ii): For any $\lambda \in \mathrm{Supp}_{\mathrm{ess}}\, L$, $\mathcal{EXT}(L)_\lambda \simeq L_\lambda$ is a simple $\mathcal{A}$-module (Lemma 2.2). Hence the coherent family $\mathcal{EXT}(L)$ is irreducible. Let L' be any infinite dimensional simple submodule of $\mathcal{EXT}(L)$. By Lemma 4.7, $\mathcal{EXT}(L)_\mu$ is a simple $\mathcal{A}$-module for all μ in some non-empty Zariski open subset Ω. By Lemma 3.4, $\mathrm{Supp}_{\mathrm{ess}}\, L'$ is Zariski dense. Hence $\mathrm{Supp}_{\mathrm{ess}}\, L' \cap \Omega$ is not empty, and L' has same degree d. Moreover, we have $\mathcal{EXT}(L) = \mathcal{EXT}(L')$ by unicity of the semi-simple coherent extension.

Proof of Assertion (iii): Let $u \in U$ be a central element. It acts over L as some scalar c and we have and $\mathrm{Tr}\, u^n|_{L_\lambda} = dc^n$ for any $n \geqslant 0$ and any $\lambda \in \mathrm{Supp}_{\mathrm{ess}}\, M$. Since u belongs to $\mathcal{A}$, the map $\lambda \mapsto \mathrm{Tr}\, u^n|_{\mathcal{EXT}(L)_\lambda}$ is polynomial. Since $\mathrm{Supp}_{\mathrm{ess}}\, L$ is Zariski dense, we have $\mathrm{Tr}\, u^n|_{\mathcal{EXT}(L)_\lambda} = dc^n$ for any $n \geqslant 0$ and any $\lambda \in \mathfrak{h}^*$. Therefore $(u-c)^d$ acts trivially on $\mathcal{EXT}(L)$. Since it is semi-simple, u acts as the scalar c. Therefore the central characters of the simple submodules of $\mathcal{EXT}(L)$ are all the same.

Q.E.D.

5. Components of coherent families.

In this section, it is proved that any semi-simple coherent family contains an infinite dimensional highest weight module (relative to some basis of Δ).

LEMMA 5.1.— *Assume $\mathfrak{g} = \mathfrak{sl}(2)$. Let $\mathcal{M}$ be a coherent family of degree d. Then the function $\lambda \mapsto \det ef|_{\mathcal{M}_\lambda}$ is a non-zero polynomial of degree $2d$.*

Proof.— Let $\Omega = ef + h^2/4 - h/2$ be the Casimir operator. It is easy to see that the map $e : \mathcal{M}_{n\alpha} \to \mathcal{M}_{(n+1)\alpha}$ is bijective for any $n \geqslant N_0$ for some $N_0 \in \mathbb{Z}$. So the eigenvalues of $\Omega|_{\mathcal{M}_{n\alpha}}$ are all the same for $n \geqslant N_0$. Let $a_1, \ldots, a_d$ be these eigenvalues. Then we have $\det ef|_{\mathcal{M}_\lambda} = \prod_{1\leqslant i\leqslant d}(a_i + \lambda(h)/2 - \lambda(h)^2/4)$, for all λ of the form $n\alpha$, $n \geqslant N_0$. The two sides of the equality being polynomials in λ, it holds for all λ. Q.E.D.

From now on in the section, $\mathfrak{g}$ denotes an arbitrary simple Lie algebra. For any Zariski open subset Ω of $\mathfrak{h}^*$, set $T(\Omega) = \cap_{\mu\in Q}(\mu + \Omega)$. For any two open subsets Ω, Ω' we have $T(\Omega) \cap T(\Omega') = T(\Omega \cap \Omega')$. The collection of sets $T(\Omega)/Q$ is a basis of a certain (exotic) topology of the dual torus T^*.

LEMMA 5.2.— i) *For any non-empty Zariski open subset Ω of $\mathfrak{h}^*$, $T(\Omega)$ is not empty.*

ii) *Relative to this topology, any non-empty open subset of T^* is dense.*

Proof.— It can be assumed that Ω is the open set $\{f \neq 0\}$ for some polynomial function f on $\mathfrak{h}^*$. Relative to some basis of Q, this lattice is identified with $\mathbb{Z}^l$ and f is identified with a polynomial $f(x_1, \ldots, x_l)$ with coefficients in some finitely generated subfield k of K. Let $\xi_1, \ldots \xi_l$ be generators of l disjoint extensions of k of degree $> \deg f$ and let $\xi \in \mathfrak{h}^*$ be the element of coordinates $(\xi_1, \ldots, \xi_l)$. The monomials $(\xi_1 + y_1)^{m_1} \ldots (\xi_n + y_l)^{m_l}$, for $0 \leqslant m_i \leqslant \deg f$, are linearly independent over k, for any $y_1, \ldots, y_l \in \mathbb{Z}$. Therefore we have $f(\xi + \lambda) \neq 0$ for any $\lambda \in Q$, i.e. ξ belongs to $T(\Omega)$. Hence Assertion (i) is proved. The second assertion follows from the fact that $T(\Omega) \cap T(\Omega') = T(\Omega \cap \Omega')$ is not empty whenever Ω, Ω' are not empty. Q.E.D.

For a coherent family $\mathcal{M}$, let $\operatorname{Sing} \mathcal{M}$ be the set of all $t \in T^*$ such that $f_\alpha|_{\mathcal{M}[t]}$ is not injective for some $\alpha \in \Delta$. In Section 10, it will be proved that $\operatorname{Sing}\mathcal{M}$ is a finite union of codimension one cosets of T^*, i.e. $\operatorname{Sing}\mathcal{M} = \cup_j T_j + u_j$ where each T_j is a codimension one subtorus of T^*, $u_j \in T^*$ and j runs over a finite set of indices. In particular, it follows that $\operatorname{Sing}\mathcal{M}$ is a closed subsed of T^*, relative to its ordinary Zariski topology. At this stage of the paper, we can only prove a weaker statement about $\operatorname{Sing}\mathcal{M}$:

LEMMA 5.3. — i) *For any coherent family $\mathcal{M}$, $\operatorname{Sing}\mathcal{M}$ is a closed proper subset of T^*.*

ii) *If there is at least one non-zero coherent family, $\mathfrak{g}$ is of type A or C.*

Proof. — By Lemmas 5.1 and 5.2, the set $\Omega_\alpha := \{t \in T^* \mid e_\alpha f_\alpha|_{\mathcal{M}[t]}$ is bijective$\}$ is dense and open, for any $\alpha \in \Delta$. Hence by Lemma 5.2, $\Omega_{\mathcal{M}} := \cap_{\alpha\in\Delta}\Omega_\alpha$ is open and dense in T^*. Since $e_\alpha f_\alpha|_{\mathcal{M}[t]}$ is bijective if and only if $f_\alpha|_{\mathcal{M}[t]}$ and $f_{-\alpha}|_{\mathcal{M}[t]}$ are injective, $\operatorname{Sing}\mathcal{M}$ is the complement of $\Omega_{\mathcal{M}}$, and the first assertion is proved. By Lemma 3.3, for any non-zero coherent family $\mathcal{M}$ and any $t \in \Omega_{\mathcal{M}}$, $\mathcal{M}[t]$ contains a simple submodule L. Since $\Delta_L^I = \Delta$, L is cuspidal by Corollary 1.4. By Proposition 1.6, the type of $\mathfrak{g}$ is A or C. Q.E.D.

PROPOSITION 5.4. — *Let $\mathcal{M}$ be a irreducible coherent family, and let $t \in T^*$. The following assertions are equivalent:*

i) *$\mathcal{M}[t]$ is simple,*

ii) *$t \notin \operatorname{Sing}\mathcal{M}$,*

iii) *$\mathcal{M}[t]$ is cuspidal.*

Proof. — The facts that (i) and (iii) are equivalent, and that (iii) implies (ii) follow from Corollary 1.4. Assume that $t \notin \operatorname{Sing}\mathcal{M}$. By Lemma 3.3, $\mathcal{M}[t]$ contains a simple submodule L. Since $\Delta_L^I = \Delta$, L is cuspidal by Corollary 1.4. By Proposition 4.8, the degree of L equals the degree of $\mathcal{M}$. Hence $L = \mathcal{M}[t]$ and $\mathcal{M}[t]$ is simple. Q.E.D.

LEMMA 5.5. — *Let $\mathcal{M}$ be a semi-simple coherent family. Then $\mathcal{M}$ is a direct sum of irreducible coherent families, and such a decomposition is unique up to isomorphism.*

Proof. — By Lemma 5.3, there exists an element $t \in T^* \setminus \operatorname{Sing}\mathcal{M}$. By Lemma 3.3, the module $\mathcal{M}[t]$ is a finite sum of simple ones, say $L_1 \oplus \cdots \oplus L_k$ and by Corollary 1.4 each L_i is cuspidal. Since t is Zariski dense in $\mathfrak{h}^*$, it follows that $\mathcal{M}$ and $\mathcal{EXT}(L_1) \oplus \cdots \oplus \mathcal{EXT}(L_k)$ have the same trace. By Lemma 2.3, these two modules are isomorphic. Q.E.D.

LEMMA 5.6. — *Let L be a simple weight module, and let $\beta \in \Delta$. The following assertions are equivalent:*

i) *For any $\lambda \in \operatorname{Supp} L$, we have $\lambda + \mathbb{Z}\beta \subset \operatorname{Supp} L$,*

ii) *For some $\lambda \in \operatorname{Supp} L$, we have $\lambda + \mathbb{Z}\beta \subset \operatorname{Supp} L$.*

iii) *$\beta \in \Delta_I^L$.*

Proof. — See e.g. Corollary 3.4 of [DMP].

PROPOSITION 5.7. — *Any semi-simple irreducible coherent family $\mathcal{M}$ contains an infinite dimensional simple highest weight module relative to some basis B of Δ.*

Proof. — By Proposition 1.3, it is equivalent to show the existence of an infinite dimensional simple submodule L of $\mathcal{M}$ with $\Delta_L^I = \varnothing$. Thus it is enough to prove that for any infinite dimensional simple submodule L of $\mathcal{M}$ with $\Delta_L^I \neq \varnothing$, there is another one L' such that $\Delta_{L'}^I$ is properly included in Δ_L^I.

Let B be a basis adapted to L, let d be its degree and let $\alpha \in \Delta_L^I \cap B$. Set $e = e_\alpha$, $f = f_\alpha$ and let $\mathfrak{a} \simeq \mathfrak{sl}(2)$ be the Lie algebra generated by e and f. By Lemma 4.5, there exists a coherent family $\mathcal{N}$ of degree d on which f acts injectively and which contains the module $f^x.L$ for any $x \in K$ (note that $L = L_f$). Let $\lambda \in \operatorname{Supp}_{\mathrm{ess}} L$ and set $\mathcal{P} = \oplus_{x\in K} f^x.L_\lambda$. For any $x \in K$, $\mathcal{P}_{\lambda+x\alpha}$ has dimension d, so $\mathcal{P}_{\lambda+x\alpha} = \mathcal{N}_{\lambda+x\alpha}$. In particular, $\mathcal{P}$ is a coherent $\mathfrak{a}$-family. By Lemma 5.1, the function $x \mapsto \det fe|_{\mathcal{P}_{\lambda-x\alpha}}$ is a non-zero polynomial of degree $2d$. So there exist $y \in K$ and $v \in \mathcal{P}_{\lambda-y\alpha}$ with $v \neq 0$ and $fe.v = 0$. As f acts injectively on $\mathcal{N}$, we have $e.v = 0$.

Set $V = \{m \in f^y.L \,|\, e^N.m = 0 \text{ for some } N > 0\}$. Since $v \in V$ and $\operatorname{ad}(e)$ is nilpotent, V is a non-zero $\mathfrak{g}$-submodule of $f^y.L$. By Lemma 3.3, V has finite length. Thus it contains a simple submodule L'. As f acts injectively on $\mathcal{N}$, its submodule L' is infinite dimensional.

We claim that $\Delta_{L'}^I$ is a proper subset of Δ_L^I. We have $\operatorname{Supp} L' \subset \operatorname{Supp} f^y.L = -y\alpha + \operatorname{Supp} L$. For any $\beta \in \Delta_{L'}^I$, $\operatorname{Supp} L'$ is an union of $\mathbb{Z}\beta$-cosets, so $\operatorname{Supp} L$ contains at least one $\mathbb{Z}\beta$-coset. By Lemma 5.6, β belongs to Δ_L^I, which proves that $\Delta_{L'}^I \subset \Delta_L^I$. However e_α acts locally nilpotently on L' and therefore $\pm\alpha \notin \Delta_{L'}^I$. Hence $\Delta_{L'}^I$ is properly included in Δ_L^I. Since $\mathcal{N}^{ss} = \mathcal{M}$, L' is also a submodule of $\mathcal{M}$.

Q.E.D.

6. Admissible highest weight modules.

It is already shown that each semi-simple coherent family contains an infinite dimensional simple highest weight module relative to some basis (Proposition 5.8). In this section, the same result is proved relative to any basis of Δ. It also contains some lemmas in view to determine when two such highest weight modules occur in the same irreducible family.

LEMMA 6.1. — *Let B a basis of Δ, let $\alpha \in B$ and let $\lambda \in \mathfrak{h}^* \setminus P_B^+$ be a weight such that the $\mathfrak{g}$-module $L_B(\lambda)$ is admissible.*

i) *The module $L_{s_\alpha B}(s_\alpha \lambda)$ belongs to $\mathcal{EXT}(L_B(\lambda))$.*

ii) *Moreover if $\lambda(h_\alpha) \notin \mathbb{Z}_{\geqslant 0}$, the modules $L_B(s_\alpha(\lambda + \rho_B) - \rho_B)$ and $L_{s_\alpha B}(\lambda + \alpha)$ belong to $\mathcal{EXT}(L_B(\lambda))$.*

Proof. — It is clear that $L_{s_\alpha B}(s_\alpha \lambda) \simeq L_B(\lambda)$ whenever $\lambda(h_\alpha) \in \mathbb{Z}_{\geqslant 0}$, see e.g. Theorem 1.2 (ii). Thus Assertion (i) is obvious in this case, and it can be assumed from now on that $\lambda(h_\alpha) \notin \mathbb{Z}_{\geqslant 0}$. Set $e = e_\alpha$, $f = f_\alpha$, $h = h_\alpha$, let $\mathfrak{a} \simeq \mathfrak{sl}(2)$ be the Lie algebra generated by e and f. Let $\mathfrak{p}$ be the parabolic subalgebra $\mathfrak{h} \oplus Kf \oplus [\oplus_{\beta \in \Delta_B^+} \mathfrak{g}_\beta]$, let $\mathfrak{u}$ be its nilradical and let $\mathfrak{u}^-$ be the opposed nilradical. By definition, B is $L_B(\lambda)$-adapted, and $\alpha \notin \Delta^F_{L_B(\lambda)}$. By Lemma 4.5, there is a coherent extension $\mathcal{M}$ of $L_B(\lambda)$ such that f acts injectively on $\mathcal{M}$ and $\mathcal{M}$ contains the submodule $\mathcal{X} = \oplus_{\overline{x} \in K/\mathbb{Z}} f^{\overline{x}}.L_B(\lambda)_f$. Set $S = \oplus_{n \geqslant 0} L_B(\lambda)_{\lambda - n\alpha}$ and $\mathcal{S} = \mathcal{X}[u]$, where $u = \lambda + K\alpha$.

We claim that $\mathcal{X} = L_\mathfrak{p}(\mathcal{S})$. We have $\mathcal{S} = \oplus_{\overline{x} \in K/\mathbb{Z}} f^{\overline{x}}.S_f$. Since $\mathfrak{u}$ and $\mathfrak{p}$ are $\mathrm{ad}(f)$-invariant, we have $f^{-x}\mathfrak{u}f^x = \mathfrak{u}$ and $f^{-x}\mathfrak{p}f^x = \mathfrak{p}$ for any $x \in K$. Therefore $\mathcal{S}$ is a $\mathfrak{p}/\mathfrak{u}$-submodule of $\mathcal{X}$ and its support is the $K \otimes Q(\mathfrak{p}/\mathfrak{u})$-coset $\lambda + K\alpha$. Hence the $\mathfrak{g}$-module $L_\mathfrak{p}(\mathcal{S})$ is well defined. It is clear that $H^0(\mathfrak{u}, L_B(\lambda)) = S$. For any $\overline{x} \in K/\mathbb{Z}$, any $y \in f^{\overline{x}}.L_B(\lambda)_f$ can be written as $f^x.y'$ where $x \in \overline{x}$ and $y' \in L_B(\lambda)$. Since $f^{-x}\mathfrak{u}f^x = \mathfrak{u}$, we get $\mathfrak{u}.y = f^x.\mathfrak{u}.y'$. It follows that y is $\mathfrak{u}$-invariant if and only if y' is $\mathfrak{u}$-invariant, and therefore $H^0(\mathfrak{u}, \mathcal{X}) = \mathcal{S}$. Similarly, the natural map $U(\mathfrak{u}^-) \otimes S \to L_B(\lambda)$ is onto and it follows that $U(\mathfrak{u}^-) \otimes \mathcal{S} \to \mathcal{X}$ is onto. Hence the $\mathfrak{g}$-module $\mathcal{X}$ is generated by $\mathcal{S}$ and any non-zero submodule intersects $\mathcal{S}$, which proves the claim $\mathcal{X} = L_\mathfrak{p}(\mathcal{S})$.

Let v be a highest weight vector of $L_B(\lambda)$. Using the identity $ef^{m+1} = f^{m+1}e + (m+1)f^m h - m(m+1)f^m$ for any $m \in \mathbb{Z}_{\geqslant 0}$, we get $e.f^{x+1}.y = f^x.[fe.y + (x+1)(h-x).y]$, for any $y \in L_B(\lambda)$ and $x \in K$. Set $v' = f^{\lambda(h)+1}.v$ and let V' be the $\mathfrak{p}$-module generated by v'. Since $e.v = 0$ and $h.v = \lambda(h)v$, the previous identity implies $e.v' = 0$ and v' is a weight vector of weight $s_\alpha(\lambda + \rho_B) - \rho_B$. Since f acts injectively, the $\mathfrak{a}$-module V' is the Verma $\mathfrak{sl}(2)$-module with highest weight $s_\alpha(\lambda + \rho_B) - \rho_B$. Therefore the simple highest weight $\mathfrak{p}$-module with highest weight $s_\alpha(\lambda+\rho_B)-\rho_B$ is a quotient of V' and $L_B(s_\alpha(\lambda+\rho_B)-\rho_B)$ is a quotient of $L_\mathfrak{p}(V')$. Therefore by Lemma 1.1 (iv), $L_B(s_\alpha(\lambda + \rho_B) - \rho_B)$ is a subquotient of $\mathcal{X}$.

Set $v'' = f^{\lambda(h)}.v$ and let V'' be the $\mathfrak{p}$-module generated by v''. Since $f.v'' = v'$, V'' contains V'. Similarly, it is proved that V''/V' is the simple $s_\alpha B$-highest weight $\mathfrak{p}$-module with highest weight $s_\alpha\lambda$ and $L_\mathfrak{p}(V''/V') = L_{s_\alpha B}(s_\alpha \lambda)$. Therefore by Lemma 1.1 (iv), $L_{s_\alpha B}(s_\alpha \lambda)$ is a subquotient of $\mathcal{X}$.

Set $v''' = f^{-1}.v$ and let V''' be the $\mathfrak{p}$-module generated by v'''. Since $f.v''' = v$, V''' contains S. Similarly, it is proved that V'''/S is the simple $s_\alpha B$-highest weight $\mathfrak{p}$-module with highest weight $\lambda + \alpha$ and $L_\mathfrak{p}(V'''/S) = L_{s_\alpha B}(\lambda+\alpha)$. Therefore by Lemma 1.1 (iv), $L_{s_\alpha B}(\lambda+\alpha)$ is a subquotient of $\mathcal{X}$.

By Lemma 4.8 (iv), we have $\mathcal{M}^{ss} = \mathcal{EXT}(L_B(\lambda))$ and by definition we have $\mathcal{X} \subset \mathcal{M}$. Hence the three modules $L_B(s_\alpha(\lambda+\rho_B)-\rho_B)$, $L_{s_\alpha B}(s_\alpha \lambda)$ and $L_{s_\alpha B}(\lambda+\alpha)$ are all contained in $\mathcal{EXT}(L_B(\lambda)$. Q.E.D.

PROPOSITION 6.2. — i) *Let $\mathcal{M}$ be a semi-simple coherent family. Then $\mathcal{M}^w \simeq \mathcal{M}$ for any $w \in W$.*

ii) *Assume that $\mathcal{M}$ is irreducible. Then for any basis B of Δ, there exists $\lambda \notin P_B^+$ such that $\mathcal{M} \simeq \mathcal{EXT}(L_B(\lambda))$.*

Proof. — By Lemma 5.5, it can be assumed that $\mathcal{M}$ is irreducible. By Proposition 5.7, there exists a basis B_0 of Δ such that $\mathcal{M} \simeq \mathcal{EXT}(L_{B_0}(\lambda))$ for some weight $\lambda \notin P_B^+$. By Lemma 6.1, $\mathcal{EXT}(L_{B_0}(\lambda)) \simeq \mathcal{EXT}(L_{B_0}(\lambda)^{s_\alpha})$ for any $\alpha \in B_0$. Hence the isomorphism class of $\mathcal{M}$ is W-invariant and the first assertion is proved. The second one follows from the fact that any basis B is W-conjugated to B_0. Q.E.D.

LEMMA 6.3. — *Let $\mathcal{M}$ be an irreducible coherent family and let M, N be two distinct simple submodules of $\mathcal{M}[t]$, for some $t \in T^*$. Then $C(M) \cap C(N)$ is contained in a hyperplane.*

Proof. — Assume that $C(M)\cap C(N)$ is not contained in a hyperplane and therefore that M and N are infinite dimensional. Thus $C(M) \cap C(N)$ generates a subgroup Q' of finite index in Q. By Lemma 3.1, the cone $C(M)$ generates the group Q. Since the image of $C(M)$ in Q/Q' is a subgroup, we have $C(M) + Q' = Q$. It follows that $C(M) - C(N) = Q$. Since any weights $\lambda \in \operatorname{Supp}_{\text{ess}} M$ and $\mu \in \operatorname{Supp}_{\text{ess}} N$ are in the same Q-coset, we have $\lambda + C(M) \cap \mu + C(N) \neq \varnothing$. It follows from Proposition 3.5 (i) that $\operatorname{Supp}_{\text{ess}} M$ and $\operatorname{Supp}_{\text{ess}} N$ intersect. By Proposition 4.8, M and N are admissible of same degree as $\mathcal{M}$. Hence we have $M_\nu = \mathcal{M}_\nu = N_\nu$, for any $\nu \in \operatorname{Supp}_{\text{ess}} M \cap \operatorname{Supp}_{\text{ess}} N$. Therefore $M = N$. Q.E.D.

Let B be a basis of Δ. For a weight $\lambda \in \mathfrak{h}^*$, set $A_B(\lambda) = \{\alpha \in B \mid \lambda(h_\alpha) \notin \mathbb{Z}_{\geqslant 0}\}$. For a root $\gamma \in B$, let $D_B(\gamma)$ be the set of all positive roots $\beta = \sum_{\alpha \in B} m_\alpha \alpha$ such that $m_\gamma > 0$.

LEMMA 6.4. — *Assume that $L_B(\lambda)$ is admissible.*

i) *If $\lambda(h_\alpha) \in \mathbb{Z}_{<0}$, for some $\alpha \in B$, then $A_B(\lambda) = \{\alpha\}$ and λ is integral.*

ii) *In general, $A_B(\lambda)$ is connected and* $\operatorname{Card} A_B(\lambda) \leqslant 2$.

Proof. — The sets $D_B(\gamma)$ satisfy the following elementary properties:

- $D_B(\gamma)$ generates Q, for any $\gamma \in B$,
- $s_\beta D_B(\gamma) = D_B(\gamma)$ for any $\beta, \gamma \in B$ with $\beta \neq \gamma$,
- $\gamma \in s_\beta B$ and $D_B(\gamma) = D_{s_\beta B}(\gamma)$ for any $\beta, \gamma \in B$ with $<\beta|\gamma>= 0$,
- $\Delta^+_{L_B(\mu)} = \cup_{\gamma \in A_B(\lambda)} D_B(\gamma)$, for any $\mu \in \mathfrak{h}^*$.

First assume that $\lambda(h_\alpha) \in \mathbb{Z}_{<0}$. By Lemma 6.1, $L_B(\lambda)$ and $L_{s_\alpha B}(s_\alpha \lambda)$ belong to the same irreducible coherent family. Moreover their supports are in the same Q-coset. By Lemma 6.3, $C(L_B(\lambda)) \cap s_\alpha C(L_B(\lambda))$ lies in a hyperplane. The cone of $L_B(\lambda)$ is generated by $\Delta^+_{L_B(\lambda)}$, therefore $\Delta^+_{L_B(\lambda)}$ does not contains $D_B(\gamma)$ for any $\gamma \in B$ with $\gamma \neq \alpha$. Hence $\gamma \notin A_B(\lambda)$ which proves the first assertion.

To prove the second assertion, it can be assumed that $\operatorname{Card} A_B(\lambda) \geqslant 2$. We claim that any two elements $\beta, \gamma \in A_B(\lambda)$ are connected. Assume otherwise. Then γ belongs to $s_\beta B$ and $D_B(\gamma) = D_{s_\beta B}(\gamma)$. It follows from the first assertion that $\lambda(h_\gamma)$ is not an integer, so is $(\lambda + \beta)(h_\gamma)$. Therefore both $\Delta^+_{L_B(\lambda)}$ and $\Delta^+_{L_{s_\beta B}(\lambda+\beta)}$ contain $D_B(\gamma)$. Since $\lambda + \beta$ is not a weight of $L_B(\lambda)$, the two modules $L_B(\lambda)$ and $L_{s_\beta B}(\lambda + \beta)$ are distinct. However by Lemma 6.1 these two modules belong to the same irreducible coherent family, their supports are in the same Q-coset and the intersection of their cones contains $D_B(\gamma)$, which contradicts Lemma 6.3. Hence β and γ are connected. Since B is simply connected, $A_B(\lambda)$ contains at most two elements. Q.E.D.

LEMMA 6.5.— *Let $\mathcal{M}$ be an irreducible coherent family, let B be a basis, let $\alpha \in B$. Then $\mathcal{M}$ contains at most one module $L_B(\lambda)$ with $\lambda(h_\alpha) \in \mathbb{Z}_{<0}$.*

Proof.— Let $L_B(\mu)$, $L_B(\lambda)$ be two such submodules in $\mathcal{M}$. It follows from Lemma 6.4 that λ and μ are integral and their cones are the same, both generated by $D_B(\alpha)$. By Proposition 4.8, their central character is the same. It follows that λ and μ are in the same Q-coset. By Lemma 6.3, we have $\lambda = \mu$. Q.E.D.

7. Rank two computations.

In order to classify all admissible simple highest weight modules, the first step is the classification for rank two Lie algebras. For rank two Lie algebras, the character formulas have been computed by Jantzen [Ja] (before the general Kazhdan-Lusztig formula has been conjectured and proved), and the following lemmas are based on his work. In this section, $\mathfrak{g}$ will be a rank two Lie algebra of type A_2 or C_2, namely $\mathfrak{g} = \mathfrak{sl}(3)$ or $\mathfrak{g} = \mathfrak{sp}(4)$. In the section, let us fix a basis B of the root system. The character of a weight module M is denoted by $\mathrm{ch}\, M$ and the Verma module with highest weight λ by $M(\lambda)$.

LEMMA 7.1.— *Assume $\mathfrak{g} = \mathfrak{sl}(3)$. Let $\lambda \notin P^+$ be a weight. The module $L(\lambda)$ is admissible if and only if $(\lambda+\rho)(h_\alpha) \in \mathbb{Z}_{>0}$, for at least one root $\alpha \in \Delta^+$.*

Proof.— It is easy to show that the Verma module $M(\lambda)$ is not admissible. So $(\lambda+\rho)(h_\alpha) \in \mathbb{Z}_{>0}$ for some $\alpha \in \Delta^+$ whenever $L(\lambda)$ is admissible. Conversely, assume that $n = (\lambda+\rho)(h_\alpha) \in \mathbb{Z}_{>0}$ for some $\alpha \in \Delta^+$. Then $M(\lambda - n\alpha)$ is a submodule of $M(\lambda)$ and set $L = M(\lambda)/M(\lambda-n\alpha)$. Let β, γ be the other two roots in Δ^+. We have

$$\begin{aligned}\mathrm{ch}\, L &= \mathrm{ch}\, M(\lambda) - \mathrm{ch}\, M(\lambda - n\alpha)\\ &= e^{\lambda}(1 + e^{-\alpha} + \ldots + e^{(1-n)\alpha})/(1-e^{-\beta})(1-e^{-\gamma}).\end{aligned}$$

Hence L is admissible. As $L(\lambda)$ is a quotient of L, it is also admissible.

Q.E.D.

Now, the case $\mathfrak{g} = \mathfrak{sp}(4)$ is investigated. Denote by Δ_l^+ (respectively Δ_s^+) the set of long (respectively short) positive roots. Recall that α is a long root if and only if h_α is a short coroot.

LEMMA 7.2. — *Assume* $\mathfrak{g} = \mathfrak{sp}(4)$. *Let* $\lambda \notin P^+$ *be a weight. The module* $L(\lambda)$ *is admissible if and only if we have* $(\lambda+\rho)(h_\gamma) \in \mathbb{Z}_{>0}$ *for any* $\gamma \in \Delta_s^+$ *and* $\lambda(h_\beta) \in 1/2 + \mathbb{Z}$ *for any* $\beta \in \Delta_l^+$.

Proof. — Assume that $L(\lambda)$ is admissible. Let $Z(\lambda)$ be the unique maximal submodule of $M(\lambda)$. It follows from [Ja] that $Z(\lambda) = 0$, or $Z(\lambda)$ is a Verma submodule, or $Z(\lambda)$ is the sum of two distinct maximal Verma submodules. However, the Verma module $M(\lambda)$, and any quotient $M(\lambda)/M(\mu)$ are not admissible. Therefore $Z(\lambda)$ is the sum of two distinct maximal Verma submodules. Following [Ja], this implies that $\lambda+\rho$ is regular and there exist at least two positive roots $\gamma_1 \neq \gamma_2$ with $(\lambda+\rho)(h_{\gamma_i}) \in \mathbb{Z}_{>0}$ for $i = 1, 2$.

We claim that λ is not integral. Assume otherwise. Since $\lambda + \rho$ is regular, we have $\lambda = x(\mu+\rho) - \rho$ for some $\mu \in P^+$ and some $x \in W$, $x \neq 1$. Let $x = s_{i_1} \dots s_{i_k}$ be a reduced decomposition of x. Using inductively Lemma 6.1, we get that $L(s_{i_k}(\mu+\rho) - \rho))$ is admissible. Therefore $L(s(\mu+\rho) - \rho))$ is admissible for some simple reflection s. Let s' be the other simple reflection. Following [Ja], we have

$$\operatorname{ch} L(s(\mu+\rho) - \rho) = -\sum_{v \geqslant s} \epsilon(v) \operatorname{ch} M(v(\mu+\rho) - \rho).$$

Since any element $v \in W \setminus \{1, s'\}$ is $\geqslant s$, we get

$$\begin{aligned}\operatorname{ch} L(s(\mu+\rho) - \rho) &= \operatorname{ch} M(\mu) - \operatorname{ch} M(s'(\mu+\rho) - \rho) \\ &\qquad - \sum_{v \in W} \epsilon(v) \operatorname{ch} M(v(\mu+\rho) - \rho) \\ &= \operatorname{ch} M(\mu)/M(s'(\mu+\rho) - \rho) - \operatorname{ch} L(\mu).\end{aligned}$$

Hence $L(s(\mu+\rho) - \rho)$ is not admissible, which proves the claim $\lambda \notin P$.

Since λ is not integral, one coroot does not belong to $\mathbb{Z}h_{\gamma_1} \oplus \mathbb{Z}h_{\gamma_2}$. This implies $\{\gamma_1, \gamma_2\} = \Delta_s^+$ and $(\lambda+\rho)(h_\beta) \in 1/2 + \mathbb{Z}$ for any $\beta \in \Delta_l^+$.

Conversely assume that λ satisfies the conditions. Set $\Delta_s^+ = \{\gamma_1, \gamma_2\}$ and $\Delta_l^+ = \{\beta_1, \beta_2\}$. Following $[Ja]$, we have

$$\begin{aligned}\operatorname{ch} L(\lambda) &= \operatorname{ch} M(\lambda) - \operatorname{ch} M(s_{\gamma_1}(\lambda+\rho) - \rho) \\ &\qquad - \operatorname{ch} M(s_{\gamma_2}(\lambda+\rho) - \rho) + \operatorname{ch} M(s_{\gamma_1} s_{\gamma_2}(\lambda+\rho) - \rho) \\ &= e^\lambda (1 + e^{-\gamma_1} + \ldots + e^{(1-m_1)\gamma_1})(1 + e^{-\gamma_2} + \ldots + e^{(1-m_2)\gamma_2}) \\ &\qquad /(1 - e^{-\beta_1})(1 - e^{-\beta_2}),\end{aligned}$$

where $m_i = (\lambda+\rho)(h_{\gamma_i})$. Hence $L(\lambda)$ is admissible. Q.E.D.

8. Classification of coherent families for $\mathfrak{sl}(n+1)$.

The section is devoted to the classification of all semi-simple irreducible families for $\mathfrak{sl}(n+1)$. More precisely, each semi-simple irreducible families $\mathcal{M}$ will be characterized by the list of all infinite dimensional highest weight modules occurring in $\mathcal{M}$. By Proposition 6.2, any semi-simple family is invariant by W, therefore we can fix a basis B of Δ. A more concrete version of the classification will be given in Section 11.

In this section, $\mathfrak{g}$=$\mathfrak{sl}(n+1)$. The simple roots $\alpha_1, \dots, \alpha_n$ are indexed in such a way that two consecutive roots are connected. Let $h_1, \dots, h_n$ be the corresponding simple coroots. There exists a spanning set $\epsilon_1, \dots, \epsilon_{n+1}$ of $\mathfrak{h}^*$ such that $\sum_i \epsilon_i = 0$ and $\alpha_i = \epsilon_i - \epsilon_{i+1}$. For $\lambda \in \mathfrak{h}^*$, set $A(\lambda) = \{i \mid \lambda(h_i) \notin \mathbb{Z}_{\geqslant 0}\}$.

LEMMA 8.1. — *Here $\mathfrak{g}$=$\mathfrak{sl}(n+1)$. Assume that $L(\lambda)$ is infinite dimensional and admissible. Then one of the following three assertions holds:*

i) $A(\lambda) = \{1\}$ *or* $A(\lambda) = \{n\}$.

ii) $A(\lambda) = \{i\}$ *for some* $1 < i < n$ *and* $(\lambda + \rho)(h_{i-1} + h_i)$ *or* $(\lambda + \rho)(h_i + h_{i+1})$ *is a positive integer.*

iii) $A(\lambda) = \{i, i+1\}$ *for some* $1 \leqslant i < n$ *and* $(\lambda + \rho)(h_i + h_{i+1}) \in \mathbb{Z}_{>0}$.

Proof. — First assume that $\operatorname{Card} A(\lambda) \geqslant 2$. By Lemma 6.4, we have $A(\lambda) = \{i, i+1\}$ for some $1 \leqslant i < n$. Let $\mathfrak{s}$ be the rank two Lie subalgebra of $\mathfrak{g}$ corresponding to the simple roots α_i, α_{i+1}. Denote by $L(\lambda,\mathfrak{s})$ be the simple highest weight $\mathfrak{s}$-module with highest weight λ. As $L(\lambda,\mathfrak{s})$ is a subquotient of $L(\lambda)$ (indeed a submodule), it is admissible. By Lemma 7.1, $(\lambda+\rho)(h_i + h_{i+1})$ is a positive integer, hence the third assertion holds.

Next assume that $\operatorname{Card} A(\lambda) \leqslant 1$. Since $L(\lambda)$ is infinite dimensional, the set $A(\lambda)$ is not empty. Then it can be assumed that $A(\lambda) = \{i\}$ for some $1 < i < n$ (otherwise Assertion (i) holds). Set $\lambda' = s_i(\lambda + \rho) - \rho$. By Lemma 6.1, the module $L(\lambda')$ is admissible. It is already proved that $A(\lambda')$ cannot contain both $i-1$ and $i+1$. So we have $(\lambda' + \rho)(h_{i-1}) \in \mathbb{Z}_{>0}$ or $(\lambda'+\rho)(h_{i+1}) \in \mathbb{Z}_{>0}$, i.e. $(\lambda+\rho)(h_{i-1}+h_i) \in \mathbb{Z}_{>0}$ or $(\lambda+\rho)(h_i+h_{i+1}) \in \mathbb{Z}_{>0}$. Hence the second assertion holds. Q.E.D.

LEMMA 8.2. — *Here $\mathfrak{g}$=$\mathfrak{sl}(n+1)$. Let λ be a weight such that $A(\lambda) = \{1\}$ or $A(\lambda) = \{n\}$. Then the module $L(\lambda)$ is admissible.*

Proof. — By symmetry of the Dynkin diagram, it can be assumed that $A(\lambda) = \{n\}$ and therefore $\lambda(h_i) \in \mathbb{Z}_{\geqslant 0}$ for all $i \neq n$. Let $\mathfrak{p}$ be the parabolic subalgebra such that $\Delta(\mathfrak{p}) = \Delta \setminus \{\beta_i | 1 \leqslant i \leqslant n\}$ where $\beta_i = \epsilon_{n+1} - \epsilon_i$ (its Levi component is $\mathfrak{gl}(n)$). Let S be the simple $\mathfrak{p}$-module with highest weight λ. By hypothesis, the $\mathfrak{p}$-module S is finite dimensional. Since the roots β_i are linearly independent, the generalized Verma module $M_{\mathfrak{p}}(S)$ is admissible. Therefore its simple quotient $L(\lambda)$ is admissible Q.E.D.

A $k-$*tuple* $m = (m_1, \ldots, m_k) \in K^k$ is called *ordered* if all differences $m_i - m_{i+1}$ are positive integers (in such a case, there will be no ambiguity to write $m_i > m_{i+1}$ to mean that $m_i - m_{i+1}$ is a positive integer). A $\mathfrak{sl}(n+1)$-*sequence* is a $n+1$-tuple $m = (m_1, \ldots, m_{n+1})$ of scalars such that $\sum_{1\leqslant i\leqslant n+1} m_i = 0$. Any weight λ of $\mathfrak{sl}(n+1)$ is identified with the $\mathfrak{sl}(n+1)$-sequence $m(\lambda)$ defined by the requirements $(\lambda+\rho)(h_i) = m_i(\lambda) - m_{i+1}(\lambda)$ and $\sum_{1\leqslant i\leqslant n+1} m_i(\lambda) = 0$. Thus P^+ is identified with the set of ordered $\mathfrak{sl}(n+1)$-sequences. Let $\mathcal{P}$ the set of all $\mathfrak{sl}(n+1)$-sequences which are not ordered but which become ordered after removing one term. Indeed a weight λ satisfies one of the assertion (i) (ii) (iii) of Lemma 8.1 if and only if $m(\lambda) \in \mathcal{P}$.

The Weyl group $W \simeq S_{n+1}$ acts on the set of $\mathfrak{sl}(n+1)$-sequences by permuting the indices. Two highest weight $\mathfrak{sl}(n+1)$-modules $L(\lambda)$ and $L(\mu)$ have the same central character if and only if $m(\lambda)$ and $m(\mu)$ are in the same S_{n+1}-orbit. Therefore, the S_{n+1}-orbit of a $\mathfrak{sl}(n+1)$-sequence is called a *central character*. A central character m is called *integral* (respectively *regular*, *singular*) if all differences $m_i - m_j$ are integers (respectively are not zero, at least one is zero). A central character of $\mathcal{P}$ is a non-empty intersection of a S_{n+1}-orbit with $\mathcal{P}$.

There is a structure of oriented graph on $\mathcal{P}$, which is defined as follows. By definition, there is an arrow $m \to m'$ between two distinct elements $m, m' \in \mathcal{P}$ if there is an index i such that $m_i - m_{i+1} \notin \mathbb{Z}_{>0}$ and $m' = s_i.m$, where s_i is the transposition exchanging i and $i+1$. When $m_i - m_{i+1} \notin \mathbb{Z}$ there will be also an oriented edge from m' to m and these two opposite edges are denoted by $m \leftrightarrow m'$.

Let $\mathcal{P}^+$ (respectively $\mathcal{P}^-$) be the set of all sequences $m \in \mathcal{P}$ such that the subsequence $m_2, m_3, \ldots m_{n+1}$ is ordered (respectively $m_1, m_2, \ldots m_n$ is ordered). For $n = 1$, we have $\mathcal{P}^+ = \mathcal{P}^-$, but otherwise $\mathcal{P}^+ \cap \mathcal{P}^- = \varnothing$. For $1 \leqslant i \neq k \leqslant n+1$, denote by $c_{i,k}$ the unique element of S_{n+1} such that $c_{i,k}(i) = k$ and such that its restriction to $\{1, \ldots, n+1\} \setminus \{i\}$ is increasing. More precisely, we have $c_{i,k} = s_k\, s_{k+1} \dots s_{i-1}$ if $k < i$ and $c_{i,k} = s_{k-1}\, s_k \dots s_i$ if $i < k$.

Clearly, there are three types of central characters occurring in $\mathcal{P}$:

i) the regular and integral central characters,

ii) the singular central characters: in this case the character is integral, and there are exactly two distinct indices i, j such that $m_i = m_j$ $(i \neq j)$,

iii) the non-integral characters: in this case the character is regular and, if $n \neq 1$, there exists a unique index i such that $m_j - m_k \in \mathbb{Z}$ for any $j, k \neq i$.

Let χ be a regular and integral central character. It can be represented by a sequence $m_1 > m_2 \dots > m_{n+1}$ (however this sequence is not in $\mathcal{P}$). For any i, k with $1 \leqslant k, i \leqslant n$, set $\chi(i,k) = c_{i+1,k}.m$ if $i \geqslant k$ and $\chi(i,k) = c_{i,k+1}.m$ if $i < k$.

Let χ be a singular character occuring in $\mathcal{P}$. It can be represented by a sequence $m_1 > m_2 \dots > m_i = m_{i+1} > m_{+2} \dots > m_{n+1}$. Then i is called the singularity of χ and for any k with $1 \leqslant k \leqslant n$, set $\chi(k) = c_{i,k}.m$ if $k < i$, $\chi(i) = m$ and $\chi(k) = c_{i+1,k+1}.m$ if $k > i$.

Let χ be a non-integral character occuring in $\mathcal{P}$. If $n \neq 1$, it can be represented by a sequence m with $m_2 > m_3 \dots > m_{n+1}$ (here $m_1 - m_i \notin \mathbb{Z}$ for $i \neq 1$). Then set $\chi(1) = m$ and $\chi(k) = c_{i,k}.m$ for any $1 < k \leqslant n+1$. In the special case $n = 1$ there is no canonical choice for m_1 and m_2: after an arbitrary choice, set $\chi(1) = (m_1, m_2)$ and $\chi(2) = (m_2, m_1)$.

Each connected component of $\mathcal{P}$ belongs to the same central character. The connected components of each central character are computed by the following easy lemma:

LEMMA 8.3. — *Let χ be a central character occurring in $\mathcal{P}$.*

i) *If χ is regular and integral, it contains exactly n connected components. More precisely $\chi(i,k)$ and $\chi(i',k')$ are in the same connected component if and only if $i = i'$. The connected component of $\chi(i,i)$ is as follows:*

$$\chi(i,1) \to \chi(i,2) \to \dots \to \chi(i,i) \leftarrow \chi(i,i+1) \leftarrow \dots \leftarrow \chi(i,n).$$

Moreover $\chi(i,1)$ is in $\mathcal{P}^+$ and $\chi(i,n)$ is in $\mathcal{P}^-$.

ii) *If χ is singular, it contains a unique connected component which is as follows:*

$$\chi(1) \to \chi(2) \to \dots \to \chi(i) \leftarrow \chi(i+1) \leftarrow \dots \leftarrow \chi(n),$$

where i is the singularity of χ. Moreover $\chi(1)$ is in $\mathcal{P}^+$ and $\chi(n)$ is in $\mathcal{P}^-$.

iii) *If χ is non-integral, it contains a unique connected component which is as follows:*

$$\chi(1) \leftrightarrow \chi(2) \leftrightarrow \ldots \leftrightarrow \chi(i) \leftrightarrow \chi(i+1) \leftrightarrow \ldots \leftrightarrow \chi(n+1).$$

Moreover $\chi(1)$ is in $\mathcal{P}^+$ and $\chi(n+1)$ is in $\mathcal{P}^-$.

LEMMA 8.4.— *Here $\mathfrak{g} = \mathfrak{sl}(n+1)$. Let $\lambda \notin P^+$ be a weight such that the $\mathfrak{g}$-module $L(\lambda)$ is admissible.*

i) *We have $m(\lambda) \in \mathcal{P}$.*

ii) *Let μ be another weight. If $m(\mu)$ belongs to $\mathcal{P}$ and there is an oriented arrow $m(\lambda) \to m(\mu)$ in $\mathcal{P}$, then $L(\mu)$ is admissible and $\mathcal{EXT}(L(\lambda)) = \mathcal{EXT}(L(\mu))$.*

Proof.— The first assertion is indeed equivalent to Lemma 8.1. Assume that there is an arrow $m(\lambda) \to m(\mu)$ in $\mathcal{P}$. By Lemma 6.1, $L(\mu)$ belongs to $\mathcal{EXT}(L(\lambda))$. In particular it is admissible. Moreover, by Proposition 4.8 we have $\mathcal{EXT}(L(\lambda)) = \mathcal{EXT}(L(\mu))$. Q.E.D.

The converse of Lemma 8.1 holds:

PROPOSITION 8.5.— *Here $\mathfrak{g} = \mathfrak{sl}(n+1)$. Let $\lambda \notin P^+$. The module $L(\lambda)$ is admissible if and only if $m(\lambda) \in \mathcal{P}$.*

Proof.— By the previous lemma, it is already proved that $L(\lambda)$ is admissible only if $m(\lambda)$ belongs to $\mathcal{P}$. Conversely, assume that $m(\lambda) \in \mathcal{P}$. By Lemma 8.3, there is always a weight μ such that $m(\mu) \in \mathcal{P}^+ \cup \mathcal{P}^-$ and an oriented path from $m(\mu)$ to $m(\lambda)$ in $\mathcal{P}$, say $m(\mu) = m(\mu_0) \to m(\mu_1) \to \cdots \to m(\mu_k) = m(\lambda)$. By Lemma 8.2, $L(\mu) = L(\mu_0)$ is admissible. Using inductively Lemma 8.4, each $L(\mu_i)$ is admissible. Therefore $L(\mu_k) = L(\lambda)$ is admissible.

For any semi-simple irreducible coherent family $\mathcal{M}$, let $m(\mathcal{M})$ be the set of all $\mathfrak{sl}(n+1)$-sequences $m := m(\lambda)$, where $\lambda \notin P^+$ and $L(\lambda)$ occurs in $\mathcal{M}$.

THEOREM 8.6.— *Here $\mathfrak{g} = \mathfrak{sl}(n+1)$.*

i) *For any semi-simple irreducible coherent family $\mathcal{M}$, $m(\mathcal{M})$ is exactly one connected component of $\mathcal{P}$.*

ii) *The map $m : \mathcal{M} \mapsto m(\mathcal{M})$ is a bijection from the set of all irreducible semi-simple coherent families to the set of connected components of $\mathcal{P}$.*

Proof. — First we claim that $m(\mathcal{M})$ is an union of connected components, for any semi-simple irreducible coherent family $\mathcal{M}$. Let λ, λ' be two weights such that $m(\lambda) \in m(\mathcal{M})$, $m(\lambda') \in \mathcal{P}$ and $m(\lambda)$ and $m(\lambda')$ are connected, i.e. there is an edge $m(\lambda) \to m(\lambda')$ or an edge $m(\lambda) \leftarrow m(\lambda')$. By Lemma 8.4, we have $\mathcal{M} = \mathcal{EXT}(L(\lambda)) = \mathcal{EXT}(L(\lambda'))$. Hence $m(\lambda')$ belongs to $m(\mathcal{M})$ and the claim is proved.

Proof of Assertion (i): by Proposition 6.2, $m(\mathcal{M})$ is not empty. By Proposition 4.8, $\mathcal{M}$ admits a central character. Therefore all connected components of $m(\mathcal{M})$ are in the same central character. By Lemma 8.3, any central character of $\mathcal{P}$ is connected except if it is regular and integral. Thus it can be assumed that the central character of $\mathcal{M}$ is integral. By Lemma 6.5, $m(\mathcal{M})$ contains at most one module $L(\lambda)$ with $\lambda(h_1) < 0$. Hence $m(\mathcal{M})$ contains at most one element in $\mathcal{P}^+$. However, each connected component of $\mathcal{P}$ meets $\mathcal{P}^+$ (see Lemma 8.3). Hence Assertion (i) is proved.

Proof of Assertion (ii): by Proposition 4.8 for non-isomorphic $\mathcal{M}$ and $\mathcal{M}'$, the sets $m(\mathcal{M})$ and $m(\mathcal{M}')$ are disjoint, therefore the map m is injective. The surjectivity results also from Propositions 4.8 and 8.5.

Q.E.D.

9. Classification of coherent families for $\mathfrak{sp}(2n)$.

In this section, $\mathfrak{g}=\mathfrak{sp}(2n)$. In this notation, it is always assumed that $n \geqslant 2$. This section is devoted to the classification of all semi-simple irreducible coherent $\mathfrak{g}$-families, which are characterized by the list of all highest weight modules occurring in such a family. It turns out that the result is easier than in the $\mathfrak{sl}(n+1)$-case, because there are always exactly two highest weight modules in each irreducible semi-simple coherent family. A concrete version of the classification theorem will be given in Section 12.

Let B be a basis of the root system. We write $B = \{\alpha_1, \cdots, \alpha_n\}$ in a such way that two consecutive roots are connected, $\alpha_1, \cdots, \alpha_{n-1}$ are the short roots and α_n is the long root of the basis. There is an orthonormal basis $\epsilon_1, \ldots, \epsilon_n$ of $\mathfrak{h}^*$ suct that $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i < n$ and $\alpha_n = 2\epsilon_n$, see e.g. [$B2$]. The corresponding coroots are $h_i = \epsilon_i^* - \epsilon_{i+1}^*$ for $i < n$ and $h_n = \epsilon_n^*$.

LEMMA 9.1.— *Here* $\mathfrak{g} = \mathfrak{sp}(2n)$. *Let* $\lambda \notin P^+$. *If* $L(\lambda)$ *is admissible, we have:*

i) $\lambda(h_i) \in \mathbb{Z}_{\geqslant 0}$ *for any* $i \neq n$,

ii) $\lambda(h_n) \in 1/2 + \mathbb{Z}$,

iii) $\lambda(h_{n-1} + 2h_n) \in \mathbb{Z}_{\geqslant -2}$.

Proof. — Let $i < n$. Let $\mathfrak{s}$ be the rank two subalgebra to which the simple roots are α_i and $\beta_i = 2\epsilon_{i+1}$ and let ρ' be the half sum of positive roots of $\mathfrak{s}$. The Lie algebra $\mathfrak{s}$ is of type C_2. Let $L(\lambda,\mathfrak{s})$ be the simple highest weight $\mathfrak{s}$-module with highest weight λ. Since $L(\lambda,\mathfrak{s})$ is a subquotient of $L(\lambda)$, it is admissible. Hence by Lemma 7.2, we have $(\lambda + \rho')(h_i) \in \mathbb{Z}_{>0}$. As α_i is a simple root in $\mathfrak{s}$, we have $\rho'(h_i) = \rho(h_i) = 1$. Hence $\lambda(h_i) \in \mathbb{Z}_{\geqslant 0}$. Assume now $i = n - 1$. In such a case, β_i is the simple root α_n, i.e. the restrictions of ρ and ρ' to the coroots of $\mathfrak{s}$ are the same. Set $\beta = \epsilon_{n-1} + \epsilon_n$. By Lemma 7.2, $(\lambda + \rho)(h_n)$ is a half integer, so is $\lambda(h_n)$. Moreover we have $(\lambda + \rho)(h_\beta) \in \mathbb{Z}_{>0}$, i.e. $\lambda(h_{n-1} + 2h_n) + 3 \in \mathbb{Z}_{>0}$ which proves Assertion (iii). Q.E.D.

The *Shale-Weil representation* $\mathbf{V}$ is an explicit action of $\mathfrak{sp}(2n)$ on $\mathbf{V} = K[x_1, \ldots, x_n]$. To describe this action, note that $\mathbf{V}$ has a natural grading. The Lie algebra $\mathfrak{g}$=$\mathfrak{sp}(2n)$ can be realized as a graded subalgebra of differential operators as follows: we have $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_{-2} = \oplus_{i,j} K.\partial^2/\partial x_i \partial x_j$, $\mathfrak{g}_0 = \oplus_{i,j} K.(2x_i\partial/\partial x_j + \delta_{i,j})$ (where $\delta_{i,j}$ is the Kronecker symbol) and $\mathfrak{g}_2 = \oplus_{i,j} K.x_i x_j$. In this setting $\mathfrak{h} = \oplus_i K.(x_i\partial/\partial x_i + 1/2)$ and the simple coroots are $h_i = -x_i\partial/\partial x_i + x_{i+1}\partial/\partial x_{i+1}$ for $1 \leqslant i \leqslant n-1$ and $h_n = -x_n\partial/\partial x_n - 1/2$. As an $\mathfrak{h}$-module, $\mathbf{V}$ is multiplicity-free. As a $\mathfrak{sp}(2n)$-module, we have $\mathbf{V} = \mathbf{V}^{\text{even}} \oplus \mathbf{V}^{\text{odd}}$, where $\mathbf{V}^{\text{even}}$, $\mathbf{V}^{\text{odd}}$ are the subspaces of even degree, odd degree polynomials. Indeed these two subspaces are simple highest weight modules, with highest weight vectors 1 and x_n respectively. More precisely, we have $\mathbf{V}^{\text{even}} = L(\omega^+)$, $\mathbf{V}^{\text{odd}} = L(\omega^-)$, where the weights $\omega^\pm$ are defined by $\omega^+(h_n) = -1/2$, $\omega^-(h_{n-1}) = 1$, $\omega^-(h_n) = -3/2$ and $\omega^\pm(h_i) = 0$ otherwise. Therefore $L(\omega^+)$ and $L(\omega^-)$ are admissible.

LEMMA 9.2.— *Here* $\mathfrak{g}$=$\mathfrak{sp}(2n)$. *Let* $\lambda \notin P^+$. *The module* $L(\lambda)$ *is admissible if and only if* λ *satisfies the conditions (i),(ii) and (iii) of Lemma 9.1.*

Proof. — In view of the previous lemma, it is enough to prove that for any weight λ satisfying the three conditions, the module $L(\lambda)$ is admissible. First assume that the half integer $\lambda(h_n)$ is $\geqslant -1/2$. We have $\lambda = \Lambda + \omega^+$, where $\Lambda \in P^+$. Since the module $L(\omega^+)$ is admissible, so is $L(\Lambda) \otimes L(\omega^+)$. Since $L(\lambda)$ is a subquotient (indeed a direct factor) of $L(\Lambda) \otimes L(\omega^+)$, $L(\lambda)$ is admissible.

Assume now that $\lambda(h_n) < -1/2$. Set $\lambda' = s_n(\lambda + \rho) - \rho$. We have $s_n\, h_{n-1} = 2\, h_n + h_{n-1}$, hence λ' satisfies (i), (ii), (iii), and $\lambda'(h_n) \geqslant -1/2$. Hence $L(\lambda')$ is admissible. By Lemma 6.1, $L(\lambda)$ belongs to $\mathcal{EXT}(L(\lambda'))$, hence $L(\lambda)$ is admissible. Q.E.D.

A $\mathfrak{sp}(2n)$*-sequence* is an arbitrary n-tuple $(m_1, \dots, m_n)$. Any weight λ of $\mathfrak{sp}(2n)$ is identified with the sequence $m(\lambda)$ defined by $m_i(\lambda) =< \lambda + \rho | \epsilon_i >$. The Weyl group $W = S_n \times (\mathbb{Z}/2\mathbb{Z})^n$ acts on the set of $\mathfrak{sp}(2n)$-sequence by permuting the indices and multiplying by ± 1 the m_i. Two highest weight modules $L(\lambda)$ and $L(\mu)$ have the same central character if and only if $m(\lambda)$ and $m(\mu)$ are W-conjugated. Therefore a $\mathfrak{sp}(2n)$-sequence modulo the action of W will be called a *central character*.

Let $\mathcal{Q}$ be the set of $\mathfrak{sp}(2n)$-sequences $(m_1, \dots, m_n)$ such that $m_i \in 1/2 + Z$ for any i and $m_1 > m_2 > \dots > m_{n-1} > \pm m_n$. Then λ satisfies the conditions (i), (ii) and (iii) of Lemma 9.1 if and only if $m(\lambda) \in \mathcal{Q}$. Any central character χ occurring in $\mathcal{Q}$ can be represented by a sequence m with $m_1 > m_2 > \dots > m_{n-1} > m_n \geqslant 1/2$. For such a χ, let $\chi^\pm$ be the two $\mathfrak{sp}(2n)$-sequences defined by $\chi^+ = m$ and $\chi^- = (m_1, \dots, m_{n-1}, -m_n)$. There is a structure of oriented graph on $\mathcal{Q}$ such that the only arrows are $\chi^+ \leftrightarrow \chi^-$.

For an irreducible semi-simple coherent family $\mathcal{M}$, denote by $m(\mathcal{M})$ the set of all $\mathfrak{sp}(2n)$-sequences m with $m = m(\lambda)$ for some λ such that $L(\lambda)$ occurs in $\mathcal{M}$.

THEOREM 9.3.— *(for* $\mathfrak{g}$=$\mathfrak{sp}(2n)$*)*

i) *For any irreducible semi-simple coherent family* $\mathcal{M}$, $m(\mathcal{M})$ *is a unique connected component of* $\mathcal{Q}$.

ii) *The map* $\mathcal{M} \to m(\mathcal{M})$ *is a bijection from the set of all irreducible semi-simple coherent families to the set of connected components of* $\mathcal{Q}$.

Proof.— The result follows easily from Proposition 4.8, Lemmas 6.1, 6.2 and 9.2.

10. Classification of cuspidal representations.

In the previous sections, it has been proved that any cuspidal module L is isomorphic to $\mathcal{M}[t]$ for a unique semi-simple irreducible coherent family $\mathcal{M}$ and a unique $t \in T^*$. By Lemma 5.3, there are coherent families only for Lie algebras of type A or C and the coherent families have been classified for these two types in Sections 8 and 9. To finish the classification of cuspidal modules, it is only necessary to determine when $\mathcal{M}[t]$ is cuspidal, or equivalently when $\mathcal{M}[t]$ is simple (Proposition 5.4). This is the aim of Theorem 10.2, which determines the set $\operatorname{Sing}\mathcal{M} := \{t \in T^* \mid \mathcal{M}[t] \text{ is not cuspidal}\}$. For the simplicity of the statements, it is always assumed that $\mathfrak{g} \neq \mathfrak{sl}(2)$. The $\mathfrak{sl}(2)$-case will be treated as a remark at the end.

Let B be a basis of Δ and let χ be a central character. Let $HW_B(\chi)$ be the set of all $\lambda \in \mathfrak{h}^*$ such that χ is the central character of $L_B(\lambda)$. Let $\overline{HW}(\chi)$ be its image in T^*. We have $HW_B(\chi) = \{w(\lambda+\rho_B) - \rho_B | w \in W\}$ for some $\lambda \in \mathfrak{h}^*$. Since $w\rho_B - \rho_B \in Q$, $\overline{HW}(\chi)$ consists of a single W-orbit and it is independent of B (as suggested by the notation). For an irreducible coherent family $\mathcal{M}$, let $HW_B(\mathcal{M})$ be the set of all weights $\lambda \notin P_B^+$ such that $L_B(\lambda)$ occurs as subquotient of $\mathcal{M}$. Let $\overline{HW}_B(\mathcal{M})$ be its image in T^*.

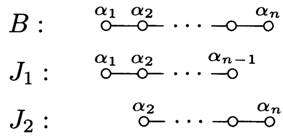
LEMMA 10.1.— *Let χ be the central character of an irreducible coherent family $\mathcal{M}$ and let B be a basis of Δ. We have $\overline{HW}_B(\mathcal{M}) = \overline{HW}(\chi)$. In particular, $\overline{HW}_B(\mathcal{M})$ is independent of B.*

Proof.— We have $\overline{HW}_B(\mathcal{M}) \subset \overline{HW}(\chi)$ and by Proposition 6.2, $\overline{HW}_B(\mathcal{M})$ is not empty. Since $\overline{HW}(\chi)$ consists of a single W-orbit, it is enough to prove that $\overline{HW}_B(\mathcal{M})$ is invariant by any simple reflexion s_α. Let $\lambda \in HW_B(\mathcal{M})$. If $\lambda(h_\alpha) \in \mathbb{Z}$, the image of λ in T^* is s_α-invariant. Otherwise by Lemma 6.1, $L_B(s_\alpha(\lambda+\rho_B)-\rho_B)$ also occurs in $\mathcal{M}$. Since $(s_\alpha(\lambda+\rho_B)-\rho_B)(h_\alpha) \notin \mathbb{Z}$, this module is infinite dimensional and $s_\alpha(\lambda+\rho_B)-\rho_B$ belongs to $HW_B(\mathcal{M})$. However, $s_\alpha(\lambda+\rho_B)-\rho_B \equiv s_\alpha\lambda$ modulo Q, hence $\overline{HW}_B(\mathcal{M})$ is s_α-invariant. Q.E.D.

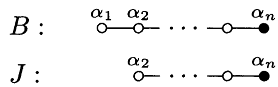
Since $\overline{HW}_B(\mathcal{M})$ is independent of B, it will be denoted by $\overline{HW}(\mathcal{M})$. From now on, it is assumed that $\mathfrak{g} = \mathfrak{sl}(n+1)$ or $\mathfrak{sp}(2n)$ (and $\mathfrak{g} \neq \mathfrak{sl}(2) = \mathfrak{sp}(2)$). Let B be a basis of Δ. The notations of Section 4 will be used, namely for any subset I of B, Q_I denotes the subgroup of Q generated by I, Δ_I denotes the root system generated by I, and Δ_I^+ denotes the set of positive roots of Δ_I relative to its basis I.

Let $J \subset B$ be a connected subset of cardinality $n-1$, with the additional requirement that J contains a long root if $\mathfrak{g}=\mathfrak{sp}(2n)$, see the following pictures:

Picture for $\mathfrak{g} = \mathfrak{sl}(n+1)$*:*



Picture for $\mathfrak{g} = \mathfrak{sp}(2n)$ *(where $\bullet$ denotes the long root):*



Denote by $\mathcal{T}$ the set of all sublattices of the form $w.Q_J$, where w runs over W. If $\mathfrak{g}=\mathfrak{sp}(2n)$, the subset J is uniquely determined by B. If $\mathfrak{g}=\mathfrak{sl}(n+1)$, there are two connected subsets of cardinality $n-1$, say J_1 and J_2, and we have $w_0 J_1 = -J_2$ and therefore $w_0 Q_{J_1} = Q_{J_2}$ where w_0 is the longest element of W. Hence, the set $\mathcal{T}$ is independent of the choice of a basis B and its subset J. Indeed $\mathcal{T}$ contains exactly $n+1$ elements if $\mathfrak{g}=\mathfrak{sl}(n+1)$ and n elements if $\mathfrak{g}=\mathfrak{sp}(2n)$. Roughly speaking, $\mathcal{T}$ is the set of all sublattices of type X_{n-1} if $\mathfrak{g}$ is of type X_n. For any sublattice $Q' \subset Q$ such that Q/Q' is torsion-free, set $T^*_{Q'} = (K \otimes Q')/Q'$ the corresponding subtorus in T^*.

Let $\tau \in T^*$ and let $Q' \in \mathcal{T}$. Let us say that τ *is Q'-non-integral* if there exists $\alpha \in \Delta \cap Q'$ such that $\lambda(h_\alpha)$ is not integral for some (or equivalently for all) $\lambda \in \tau$. Let χ be a central character. If χ is integral, let τ be the unique element of $HW(\chi)$ and set $\operatorname{Sing}(\chi) = \cup_{Q' \in \mathcal{T}}\ \tau + T^*_{Q'}$. Otherwise set $\operatorname{Sing}(\chi) = \cup_{(\tau, Q')}\ \tau + T^*_{Q'}$, where the union runs over the set of pairs (τ, Q') with $\tau \in HW(\chi)$, $Q' \in \mathcal{T}$ and τ is Q'-non-integral.

THEOREM 10.2.— *Here $\mathfrak{g} \neq \mathfrak{sl}(2)$. Let $\mathcal{M}$ be an irreducible coherent family and let χ be its central character. We have* $\operatorname{Sing}\mathcal{M} = \operatorname{Sing}(\chi)$.

Proof.— By Proposition 5.4, it can be assumed that $\mathcal{M}$ is semi-simple. Let d be its degree.

First, we prove that $\operatorname{Sing}\mathcal{M} \subset \operatorname{Sing}(\chi)$. Let t be any element of $\operatorname{Sing}\mathcal{M}$. By Proposition 5.4, $\mathcal{M}[t]$ is not simple and by Lemma 3.3 it contains an infinite dimensional simple submodule M. By Proposition 4.8, M is admissible of degree d. Since $M \neq \mathcal{M}[t]$, we have $\operatorname{Supp}_{\text{ess}} M \neq t$ and therefore M is not cuspidal. By Theorem 1.2, there is a proper parabolic subalgebra $\mathfrak{p}$ with nilradical $\mathfrak{u}$ and a cuspidal $\mathfrak{p}/\mathfrak{u}$-module S such that $M \simeq L_{\mathfrak{p}}(S)$. Let B be a $\mathfrak{p}$-adapted basis of Δ and let I be the corresponding basis of the root system of $\mathfrak{p}/\mathfrak{u}$.

Let $I_1, ..., I_k$ be the connected components of I. We have $\mathfrak{p}/\mathfrak{u} = \mathfrak{z} \oplus \mathfrak{a}_1 \oplus \ldots \oplus \mathfrak{a}_k$, where $\mathfrak{z}$ is the center of $\mathfrak{p}/\mathfrak{u}$ and each $\mathfrak{a}_i$ is the simple Lie algebra with root system Δ_{I_i}. We have $S = Z \otimes S_1 \otimes \ldots\ldots \otimes S_k$, where each S_i is a cuspidal $\mathfrak{a}_i$-module and Z is a one dimensional $\mathfrak{z}$-module. Set $F = B \cap \Delta_M^F$. For any i, by Lemma 4.1 (i), there is a set of commuting roots $\Sigma_i \subset \Delta_{I_i}^+$ which is a basis of Q_{I_i}. Set $\Sigma' = \cup_i \Sigma_i$. We have $I \cap F = \varnothing$, therefore by Lemma 4.1 (ii) there is a set Σ of commuting roots which is a basis of Q such that $\Sigma' \subset \Sigma \subset \Delta^+ \setminus \Delta_F^+$.

By Lemmas 4.1 and 4.5, the module $\mathcal{M}' := \oplus_{\nu \in T^*} f_\Sigma^\nu.M_{F_\Sigma}$ is a coherent extension of M and we have $\mathcal{M}'^{ss} = \mathcal{M}$. This coherent extension contains the submodule $\mathcal{X} = \oplus_{\nu \in T^*_{Q_I}} f_{\Sigma'}^\nu.M$ (note that $M = M_{F_{\Sigma'}}$). Set $\mathcal{S} = \oplus_{\nu \in T^*_{Q_I}} f_{\Sigma'}^\nu.S$. Similarly $\mathcal{S} = Z \otimes \mathcal{S}_1 \otimes \ldots \otimes \mathcal{S}_k$, where each $\mathcal{S}_i$ is the coherent $\mathfrak{a}_i$-family $\oplus_{\nu \in T^*_{Q_{I_i}}} f_{\Sigma_i}^\nu.S_i$. The support of $\mathcal{S}$ is the $K \otimes Q(\mathfrak{p}/\mathfrak{u})$-coset containing $\operatorname{Supp} S$. As in the proof of Lemma 6.1, it is easy to show that $\mathcal{X} = L_{\mathfrak{p}}(\mathcal{S})$ (see also Lemma 13.2 (i)).

By Proposition 6.2, each coherent $\mathfrak{a}_i$-family $\mathcal{S}_i^{ss}$ contains an infinite dimensional I_i-highest weight module L_i. Set $L' = Z \otimes L_1 \ldots \otimes L_k$. Therefore L' is a $\mathfrak{p}/\mathfrak{u}$-subquotient of $f_{\Sigma'}^\nu.S$ for some $\nu \in T^*_{Q_I}$. We have $L_{\mathfrak{p}}(L') = L_B(\lambda)$ for some λ and by Lemma 1.1 (iv), it is a subquotient of $\mathcal{X}$. Since $\mathcal{X}^{ss} \subset \mathcal{M}'^{ss} = \mathcal{M}$, the $\mathfrak{g}$-module $L_B(\lambda)$ is a submodule of $\mathcal{M}$. Since $\lambda \in \nu + \operatorname{Supp} S$, we have $\lambda \in t + K \otimes Q_I$. Hence we have $t \in \tau + T^*_{Q_I}$ where τ is the image of λ in T^*.

As $L_B(\lambda)$ is admissible, it follows from Lemma 6.4 that the set $A_B(\lambda) = \{\alpha \in B | \lambda(h_\alpha) \notin \mathbb{Z}_{\geqslant 0}\}$ is connected and contains at most two roots. Since each L_i is an infinite dimensional highest weight module, $A_B(\lambda)$ intersects each I_i. Therefore either I is empty or I is connected and $I \cap A_B(\lambda) \neq \varnothing$. Since $\mathfrak{p}$ is a proper parabolic subalgebra, we have $I \neq B$. We have $t \in \tau + T^*_{Q_I}$ and by definition $\tau \in \overline{HW}(\chi)$. The argument splits into three cases:

- Assume $\mathfrak{g} = \mathfrak{sp}(2n)$ and let J be the unique connected subset of B of cardinality $n-1$ which contains the unique long root α of B. Then $A_B(\lambda) = \{\alpha\}$ and therefore $I \subset J$. Thus τ is Q_J-non-integral, $Q_J \in \mathcal{T}$ and $t \in \tau + T^*_{Q_J}$, which proves that $t \in \operatorname{Sing}(\chi)$.

• Assume $\mathfrak{g} = \mathfrak{sl}(n+1)$ and χ is not integral. If I is empty, let J be a maximal proper connected subset of B with $J \cap A_B(\lambda) \neq \varnothing$. If I is not empty, it is contained in a maximal proper connected subset J of B. In this case J intersects $A_B(\lambda)$, because $I \cap A_B(\lambda) \neq \varnothing$. It follows from Lemma 6.4 that for any $\alpha \in A_B(\lambda)$, we have $\lambda(h_\alpha) \notin \mathbb{Z}$. Therefore in both cases τ is Q_J-non-integral, $Q_J \in \mathcal{T}$ and $t \in \tau + T^*_{Q_J}$, which proves that $t \in \mathrm{Sing}(\chi)$.

• Assume $\mathfrak{g} = \mathfrak{sl}(n+1)$ and χ is integral. Since I is empty or connected, it is contained in a connected subset $J \subset B$ of cardinality $n-1$. We have $Q_J \in \mathcal{T}$ and $t \in \tau + T^*_{Q_J}$, which proves that $t \in \mathrm{Sing}(\chi)$.

Therefore it is proved that $\mathrm{Sing}\,\mathcal{M} \subset \mathrm{Sing}(\chi)$.

Conversely we prove that $\mathrm{Sing}(\chi) \subset \mathrm{Sing}\,\mathcal{M}$. Let t be any element of $\mathrm{Sing}(\chi)$. By Lemma 10.1, we have $t = \tau + \nu$ for some some $\tau \in \overline{HW}(\mathcal{M})$, for some $\nu \in T^*_{Q'}$ with $Q' \in \mathcal{T}$, and τ is Q'-non-integral if χ is not integral. By definition of $\mathcal{T}$, there is a basis B of Δ such that $B' = B \cap Q'$ is connected and $Q' = Q_{B'}$.

We claim that it is always possible to choose $\lambda \in \tau$ such that $L_B(\lambda)$ is an infinite dimensional submodule of $\mathcal{M}$, with the additional requirement that $\lambda(h_\alpha) \notin \mathbb{Z}_{\geqslant 0}$ for some $\alpha \in B'$ (or equivalently $A_B(\lambda) \cap B' \neq \varnothing$). If χ is not integral, the existence of such α follows from the fact that τ is Q'-non-integral. Otherwise, the central character is integral. In such a case $\mathfrak{g} = \mathfrak{sl}(n+1)$, $\overline{HW}(\mathcal{M}) = \{\tau\}$ and it follows from Lemma 8.3 and Proposition 8.4 that there are n distinct integral weights $\lambda_1, \dots \lambda_n \in \tau \setminus P^+_B$ such that $L_B(\lambda_i) \subset \mathcal{M}$. For each i, $A_B(\lambda_i)$ contains a single element, and $A_B(\lambda_i) \neq A_B(\lambda_j)$ for $i \neq j$. Therefore $A_B(\lambda_i) \not\subset B'$ for a unique i. Since $n \geqslant 2$, the claim is proved.

Set $L = L_B(\lambda)$. Set $F = \Delta^F_L \cap B$ and $F' = F \cap B'$. Since B' is connected and $F' \neq B'$ there is a set Σ' a commuting roots which is a basis of Q' such that $\Sigma' \subset \Delta^+_{B'} \setminus \Delta^+_{F'}$ (apply Lemma 4.1 (ii) to the basis B' with the subsets and $J' = \varnothing$ and F'). There is a set of commuting roots Σ which is a basis of Q such that $\Sigma' \subset \Sigma \subset \Delta^+_B \setminus \Delta^+_F$ (apply now Lemma 4.1 (ii) to the basis B, the subsets $J = B'$ and F and the basis Σ' of $Q_{B'}$).

By assumption, we have $\Sigma \subset \Delta^+_L \cup \Delta^I_L$. By Lemma 4.5, $\mathcal{L} := \oplus_{\mu \in T^*} f^\mu_\Sigma . L_{f_\Sigma}$ is a coherent family of degree d which contains the module $M := f^\nu_{\Sigma'} . L_{f_{\Sigma'}}$. We have $\mathrm{Supp}\, M = \nu + \mathbb{Z}B' + \mathrm{Supp}\, L$. Since $\mathrm{Supp}\, L \subset \lambda + \mathbb{Z}_{\leqslant 0}\, B$, the support of M is strictly included in the Q-coset t. Since $\mathcal{L}^{ss} = \mathcal{M}$, M^{ss} is a submodule of $\mathcal{M}[t]$. Hence the $\mathfrak{g}$-module $\mathcal{M}[t]$ is not cuspidal, i.e. $t \in \mathrm{Sing}\,\mathcal{M}$. Q.E.D.

It follows from the classification of coherent families, that $\overline{HW}(\mathcal{M})$ contains two elements if $\mathfrak{g}=\mathfrak{sp}(2n)$. If $\mathfrak{g}=\mathfrak{sl}(n+1)$, $\overline{HW}(\mathcal{M})$ contains a unique element if the central character is integral, and $n+1$ elements otherwise.

COROLLARY 10.3. — *Here $\mathfrak{g}\neq\mathfrak{sl}(2)$. Let $\mathcal{M}$ be an irreducible coherent family. The subset* Sing $\mathcal{M}$ *of T^* is an union of r distinct codimension one cosets, where $r=n+1$ if $\mathfrak{g}=\mathfrak{sl}(n+1)$ and $r=n$ if $\mathfrak{g}=\mathfrak{sp}(2n)$.*

Proof. — If $\mathfrak{g}=\mathfrak{sl}(n+1)$, the set $\mathcal{T}$ consists of $n+1$ lattices, say $Q_1,\cdots,Q_{n+1}$. If the central character of $\mathcal{M}$ is integral, then $\overline{HW}(\mathcal{M})=\{\tau\}$ and Sing $\mathcal{M}$ is the union of the $n+1$ codimension one cosets $\tau+T^*_{Q_i}$. If the central character is not integral, then $\overline{HW}(\mathcal{M})$ consists of $n+1$ elements and each of them is Q_i-non-integral for all i except one. Therefore there is a unique way to write $\overline{HW}(\mathcal{M})=\{\tau_1,\cdots,\tau_{n+1}\}$ in such a way that τ_i is Q_j-non-integral for all $i\neq j$. For $i\neq j$, the coset τ_i+Q_j contains all other τ_k, except τ_j. Hence Sing $\mathcal{M}$ is the union of exactly $n+1$ codimension one cosets.

If $\mathfrak{g}=\mathfrak{sp}(2n)$ the set $\mathcal{T}$ consists of n lattices, say $Q_1,\cdots,Q_n$ and $\overline{HW}(\mathcal{M})$ consists of two elements, say τ_1,τ_2. For any long root α, τ_1 and τ_2 differs by $\alpha/2$ and therefore $\tau_1+T^*_{Q_i}=\tau_2+T^*_{Q_i}$ for any i. Hence Sing $\mathcal{M}$ is the union of n codimension one cosets.

Remark. — Since the previous theorem does not hold for the Lie algebra $\mathfrak{sl}(2)$, this case is explained now. Let $\mathcal{M}$ be an irreducible coherent $\mathfrak{sl}(2)$-family and let $t\in T^*$. Then $\mathcal{M}[t]$ is irreducible if and only if $t\notin$ Sing $\mathcal{M}$. In particular Sing $\mathcal{M}$ consists of two elements if its central character is not integral, and one element otherwise.

11. Explicit realization of coherent families for $\mathfrak{sl}(n+1)$.

In this section, the irreducible coherent $\mathfrak{sl}(n+1)$-families will be explicitly described, and their degrees will be computed. Indeed, the Lie algebra $\mathfrak{sl}(n+1)$ contains a parabolic subalgebra $\mathfrak{p}$ with nilradical $\mathfrak{u}$ such that $\mathfrak{p}/\mathfrak{u}\simeq\mathfrak{gl}(n)$. Let $\mathcal{S}$ be the set of finite dimensional simple $\mathfrak{gl}(n)$-modules. For any finite dimensional simple $\mathfrak{sl}(n+1)$-module L, $H^0(\mathfrak{u},L)$ is a simple $\mathfrak{gl}(n)$-module. Let $\mathcal{H}^0$ be the set of all $S\in\mathcal{S}$ which are isomorphic to $H^0(\mathfrak{u},L)$ for some finite dimensional simple $\mathfrak{sl}(n+1)$-module L. The irreducible coherent $\mathfrak{sl}(n+1)$-families are parametrized by $\mathcal{S}\setminus\mathcal{H}^0$. Therefore, this reformulation of the classification theorem 8.6 is based on a more concrete set of parameters than the set of connected components of the graph $\mathcal{P}$.

Let X be a smooth affine algebraic variety of dimension n over K, let Ω_X^* be the space of algebraic differential forms and let W_X be the Lie algebra of vector fields on X. In order to quickly describe the *tensor representations of* W_X, it is simpler to assume that the tangent bundle TX is trivial, i.e. Ω_X^1 is a free $K[X]$-module. The definition of these W_X-modules requires some preparatory explanations.

Set $V = K^n$, let $v_1, \dots, v_n$ be its natural basis and let $(E_{i,j})_{1\leqslant i\leqslant n}$ be the natural basis of $\mathfrak{gl}(n)$. With these definitions, V is the natural representation of $\mathfrak{gl}(n)$, and we have $E_{i,j}.v_k = \delta_{j,k}.v_i$ and $E = \sum_i E_{i,i}$ is the identity matrix. Set $\mathfrak{G} = \mathfrak{gl}(n, K[X])$, $\mathfrak{G}' = K[X]$. Any element g of $\mathfrak{G}$ can be written as $g = \sum_{i,j} f_{i,j} \otimes E_{i,j}$, where $f_{i,j} \in K[X]$. Since the Lie algebra W_X acts by derivations on $K[X]$, it acts by derivations on $\mathfrak{G}$. The *extended current algebra* is the semi-direct product $\mathrm{Cur}(X) = W_X \ltimes (\mathfrak{G} \oplus \mathfrak{G}')$. For any $\mathfrak{gl}(n)$-module S, $K[X] \otimes S$ is obviously a $\mathrm{Cur}(X)$-module: indeed the general element $\xi + g + f$ of $\mathrm{Cur}(X)$ acts over $K[X] \otimes S$ by $\xi \otimes 1 + \sum_{i,j} f_{i,j} \otimes \sigma(E_{i,j}) + f \otimes 1$ (here $\sigma : \mathfrak{gl}(n) \to \mathrm{End}(S)$ denotes the $\mathfrak{gl}(n)$-action on S, $\xi \in W_X$, $g = \sum_{i,j} f_{i,j} \otimes E_{i,j} \in \mathfrak{G}$ and $f \in \mathfrak{G}'$). It should be noted that the elements $f \otimes E \in \mathfrak{G}$ and $f \in \mathfrak{G}'$ do not act in the same way on $K[X] \otimes S$: the first one acts as $f \otimes \sigma(E)$ and the second one by $f \otimes 1$. The restriction of this module to W_X is a direct sum of $\dim S$ copies of $K[X]$. In order to get more interesting W_X-modules, one needs to twist the natural embedding $W_X \to \mathrm{Cur}(X)$ by a non-abelian cocyle.

The notion of a *non-abelian cocycle* is defined as follows. Let $\mathfrak{L}$ be a Lie algebra and let $\mathfrak{W}$ be a Lie algebra of derivations of $\mathfrak{L}$. The elements of the semi-direct product $\mathfrak{W} \ltimes \mathfrak{L}$ will be denoted by $\xi + g$, $\xi \in \mathfrak{W}$, $g \in \mathfrak{L}$. In this setting, a non-abelian $\mathfrak{W}$-cocycle with value in $\mathfrak{L}$ is a linear map $c : \mathfrak{W} \to \mathfrak{L}$ such that the map $j : \mathfrak{W} \to \mathfrak{W} \ltimes \mathfrak{L}$, $\xi \mapsto \xi + c(\xi)$ is a morphism of Lie algebras. If $\mathfrak{L}$, $\mathfrak{L}'$ are Lie algebras on which $\mathfrak{W}$ acts by derivations and if $c : \mathfrak{W} \to \mathfrak{L}$ and $c' : \mathfrak{W} \to \mathfrak{L}'$ are non-abelian $\mathfrak{W}$-cocycles, it is obvious that their direct sum $c \oplus c'$ is a non-abelian $\mathfrak{W}$-cocycle with value in $\mathfrak{L} \oplus \mathfrak{L}'$.

Two non-abelian W_X-cocycles c_ω and c^b are now defined. Any vector field ξ acts over Ω_X^* by its Lie derivative $\mathcal{L}_\xi$, where $\mathcal{L}_\xi = [\mathrm{d}, i_\xi]$ (Cartan's formula). In this formula, d is the de Rham operator, i_ξ denotes the contraction by ξ and the bracket $[\mathrm{d}, i_\xi]$ should be understood as $\mathrm{d} \circ i_\xi + i_\xi \circ \mathrm{d}$ since the two operators d and i_ξ are odd. For any $\omega \in \Omega_X^1$, set $\mathcal{L}_\xi^\omega = \mathcal{L}_\xi + i_\xi \omega$.

We have $[\mathcal{L}^\omega_\xi, \mathcal{L}^\omega_\eta] = \mathcal{L}^\omega_{[\xi,\eta]} + i_{\xi\wedge\eta}\mathrm{d}\omega$ for any ξ, $\eta \in W_X$. From now on, it is assumed that ω is closed and therefore $\xi \in W_X \mapsto \mathcal{L}^\omega_\xi \in \mathrm{End}(\Omega^*_X)$ is a morphism of Lie algebras, i.e. the map $c_\omega : \xi \mapsto i_\xi\omega$ is a non-abelian W_X-cocycle with value in $\in \mathfrak{G}'$ (since $\mathfrak{G}'$ is abelian, c_ω is indeed an ordinary cocycle). In order to describe the second non-abelian W_X-cocycle with value in $\mathfrak{G}$, a $K[X]$-basis $\mathbf{b} = (\beta_1, \dots, \beta_n)$ of Ω^1_X is chosen. Let $(\xi_1, \dots, \xi_n)$ be the dual basis of W_X. We have

$$\begin{aligned}\mathcal{L}_\xi.f\beta_j &= (\xi.f)\beta_j + f\mathcal{L}_\xi.\beta_j \\ &= (\xi.f)\beta_j + f\sum_{1\leqslant i\leqslant n}(i_{\xi_i}\mathcal{L}_\xi.\beta_j)\beta_i \\ &= (\xi.f)\beta_j + f\sum_{1\leqslant i\leqslant n}(i_{[\xi_i,\xi]}\beta_j)\beta_i,\end{aligned}$$

for any $f \in K[X]$ and any j. Using its basis $\mathbf{b}$, the space Ω^1_X is identified with the $W_X \ltimes \mathfrak{G}$-module $K[X]\otimes V$ and $\mathcal{L}_\xi$ with the element $\xi + c^{\mathbf{b}}(\xi)$ of $W_X \ltimes \mathfrak{G}$, where

$$c^{\mathbf{b}}(\xi) = \sum_{1\leqslant i,j\leqslant n} i_{[\xi_i,\xi]}\beta_j \otimes E_{i,j}.$$

Therefore $c^{\mathbf{b}}$ is a non-abelian W_X-cocycle with value in $\mathfrak{G}$. Thus $c^{\mathbf{b}}_\omega := c^{\mathbf{b}} \oplus c_\omega$ is a non-abelian W_X-cocycle with value in $\mathfrak{G}\oplus\mathfrak{G}'$ and let $j^{\mathbf{b}}_\omega : W_X \to \mathrm{Cur}(X)$, $\xi \mapsto \xi + c^{\mathbf{b}}_\omega(\xi)$ the corresponding morphism of Lie algebras.

For any $\mathfrak{gl}(n)$-module S, $K[X]\otimes S$ is now a W_X-module, where each $\xi \in W_X$ acts as $j^{\mathbf{b}}_\omega(\xi)$. Viewed as a W_X-module, this space is denoted by $\mathrm{Tens}^{\mathbf{b}}_X(S,\omega)$. When S is a simple finite dimensional $\mathfrak{gl}(n)$-modules and ω is a closed one-form, the module $\mathrm{Tens}^{\mathbf{b}}_X(S,\omega)$ is called a *tensor W_X-modules.* To clarify this definition, let us start with general remarks about the tensor modules:

• Let $f \in K[X]^*$ be an invertible function on X. The multiplication by f induces an isomorphism $\mathrm{Tens}^{\mathbf{b}}_X(S,\omega) \simeq \mathrm{Tens}^{\mathbf{b}}_X(S,\omega{+}\mathrm{d}f/f)$. Hence the isomorphism class of the W_X-module $\mathrm{Tens}^{\mathbf{b}}_X(S,\omega)$ depends only on ω modulo a logarithmic differential. Set $Q(X) = \{\omega \in \Omega^1_X \,|\, \omega = \mathrm{d}f/f \text{ for some } f \in K[X]^*\}$ and $X^* = K\otimes Q(X)/Q(X)$. To explain these notations, it should be noted that when X is a torus, $Q(X)$ is isomorphic to the group of characters of X, $K\otimes Q(X)$ is the dual of its Lie algebra and X^* is the dual torus. Therefore for any $\omega \in X^*$, the module $\mathrm{Tens}^{\mathbf{b}}_X(S,\omega)$ is well defined up to an isomorphism.

• Let us investigate in which way the W_X-module $\mathrm{Tens}_X^{\mathbf{b}}(S,\omega)$ depends effectively on the trivialization of TX, i.e. in which way it depends on the choice of a $K[X]$-basis $\mathbf{b}$ of Ω_X^1. First consider the case where the $\mathfrak{gl}(n)$-action on S derives from a $GL(n)$-action (such a module will be called later a $GL(n)$-module). Since $K[X]\otimes S$ is a $GL(n,K[X])$-module, it is clear that $\mathrm{Tens}_X^{\mathbf{b}}(S,\omega)$ is independent of the trivialization of TX: in this case, the index $\mathbf{b}$ will be dropped from the notation $\mathrm{Tens}_X^{\mathbf{b}}(S,\omega)$. In general, any $S\in\mathcal{S}$ can be written as $S'\otimes K_s$, where S' is a $GL(n)$-module, $s\in K$ and K_s is the one dimensional $\mathfrak{gl}(n)$-module on which any $g\in\mathfrak{gl}(n)$ acts by $s\mathrm{Tr}\, g$. Let $\mathbf{b}'$ be another basis of Ω_X^1 and let f be the determinant of $\mathbf{b}'$ relative to $\mathbf{b}$. Then we have $\mathrm{Tens}_X^{\mathbf{b}'}(S,\omega)=\mathrm{Tens}_X^{\mathbf{b}}(S,\omega+s\mathrm{d}f/f)$. Therefore $\mathrm{Tens}_X^{\mathbf{b}}(S,\omega)$ only depends on the volume form $\beta_1\wedge\beta_2\cdots$ and the big W_X-module $\oplus_{\omega\in X^*}\mathrm{Tens}_X^{\mathbf{b}}(S,\omega)$ is independent of a particular basis of Ω_X^1. Hence it will be denoted by $\mathcal{TENS}_X(S)$.

• It follows from the previous remark that the tensor W_X-modules can be defined in the more general setting of an arbitrary smooth affine variety X (even when TX is not trivial). More precisely $\mathrm{Tens}_X(S,\omega)$ can be defined whenever one of the following two conditions is satisfied:

i) S is a $GL(n)$-module,

ii) X admits a volume form.

In the cases considered below, TX will be always trivial and therefore it is not necessary to define the tensor modules in this general setting.

• Let ω be a closed one-form. It is easy to show that $\mathrm{Tens}_X(\wedge^i V,\omega)$ is isomorphic to Ω_X^i where any $\xi\in W_X$ acts by $\mathcal{L}_\xi^\omega$, for any $1\leqslant 1\leqslant n$. Set $\mathrm{d}_\omega=\mathrm{d}+e(\omega)$, where $e(\omega)$ is the exterior product by ω. We have $\mathrm{d}_\omega^2=0$, $\mathcal{L}_\xi^\omega=[\mathrm{d}_\omega,i_\xi]$ and therefore $[\mathrm{d}_\omega,\mathcal{L}_\xi^\omega]=0$ for any $\xi\in W_X$. Hence d_ω is a W_X-equivariant map from $\mathrm{Tens}_X(\wedge^i V,\omega)$ to $\mathrm{Tens}_X(\wedge^{i+1}V,\omega)$. Therefore there is a W_X-equivariant map $D:\mathcal{TENS}_X(\wedge^* V)\to\mathcal{TENS}_X(\wedge^* V)$ such that its restriction on each component $\mathrm{Tens}_X(\wedge^* V,\omega)$ is d_ω. We have $D^2=0$.

After these general remarks, let us consider the case where the smooth affine variety X is a n-dimensional torus denoted by T. Let $\mathfrak{h}$ be its Lie algebra. Its dual $\mathfrak{h}^*$ is identified with the space of T-invariant one-forms. Let $\mathbf{b}$ be a basis of $\mathfrak{h}^*$ which is also viewed as a $K[T]$-basis of Ω_T^1. Then we have $c_\omega^{\mathbf{b}}(h)=\omega(h)$ for any $\omega\in\mathfrak{h}^*$ and $h\in\mathfrak{h}$. For any $S\in\mathcal{S}$, the element $h\in\mathfrak{h}$ acts on $\mathrm{Tens}_T(S,\omega)\simeq K[T]\otimes S$ as $(\mathcal{L}_h+\omega(h))\otimes 1$. Hence the W_T-module $\mathrm{Tens}_T(S,\omega)$ is a weight module (relative to $\mathfrak{h}$) and the dimension of any non-zero weight space is $\dim S$. From now on, T will be the group of diagonal matrices in $PSL(n+1)$. It will be identified with its unique open orbit on $\mathbb{P}^n$. Since $PSL(n+1)$ acts on $\mathbb{P}^n$, its Lie algebra $\mathfrak{sl}(n+1)$ can be identified with a Lie algebra of vector fields on T, therefore we have $\mathfrak{h} \subset \mathfrak{sl}(n+1) \subset W_T$. Viewed as a $\mathfrak{sl}(n+1)$-module, $\mathcal{TENS}_T(S)$ is a coherent family of degree equal to $\dim S$, for any $S \in \mathcal{S}$.

To get an explicit description of these coherent families, write $K[T] = K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. The Lie algebra $\mathfrak{sl}(n+1)$ decomposes into $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0 = \mathfrak{gl}(n)$, $\mathfrak{g}_{-1} = V$, $\mathfrak{g}_1 = V^*$. As a Lie algebra of vector fields on T, $\mathfrak{g}_{-1}$ has basis $(\partial/\partial x_i)_{1 \leqslant i \leqslant n}$, $\mathfrak{g}_0$ has basis $(E_{i,j} = x_j \partial/\partial x_i)_{1 \leqslant i,j \leqslant n}$ and $\mathfrak{g}_1$ has basis $(x_i.E)_{1 \leqslant i \leqslant n}$, where $E = \sum_{1 \leqslant i \leqslant n} x_i \partial/\partial x_i$. The subspace $\mathfrak{h}$ with basis $(x_i \partial/\partial x_i)_{1 \leqslant i \leqslant n}$ is the Cartan subalgebra of $\mathfrak{sl}(n+1)$. To describe the action of the Weyl group $W = S_{n+1}$ on $\mathfrak{h}$, it is convenient to set $E_{n+1,n+1} = -E$. Then S_{n+1} acts on the generating set $(E_{i,i})_{1 \leqslant i \leqslant n+1}$ by permuting the coordinates. In order to specify a basis of the root system, set $h_i = E_{i,i} - E_{i+1,i+1}$ for $i < n$ and $h_n = E_{n,n} + E$. There exists a unique basis of Δ relative to which the h_i are the simple coroots.

Example of a tensor family: Let $a \in K$ be a scalar. Recall that K_a denotes the one-dimensional $\mathfrak{gl}(n)$-module on which any element $g \in \mathfrak{gl}(n)$ acts as $a \operatorname{Tr} g$. For $\xi \in W_T$, its divergence $\operatorname{div} \xi$ is defined by the identity $\mathcal{L}_\xi v = (\operatorname{div} \xi) v$, where $v = \mathrm{d}x_1/x_1 \wedge \mathrm{d}x_2/x_2 \dots \mathrm{d}x_n/x_n$. Let $\mathcal{M}(a)$ be the W_T-module with basis $(x_1^{\alpha_1} \dots x_n^{\alpha_n})_{\alpha_1 \dots \alpha_n \in K}$, where any $\xi \in W_T$ acts by $\xi + a \operatorname{div} \xi$ (the conventions are the same as in the example of Section 4, e.g. $\partial/\partial x_1\, x_1^{\alpha_1} x_2^{\alpha_2} \dots = \alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2} \dots$). It is clear that $\mathcal{M}(a)$ is isomorphic with $\mathcal{TENS}_T(K_a)$, and this example generalizes the example of the coherent $\mathfrak{sl}(2)$-family occuring in Section 4.

Any $S \in \mathcal{S}$ is determined by its highest weight $\lambda \in \mathfrak{h}^*$ which satisfies $\lambda(h_i) \in \mathbb{Z}_{\geqslant 0}$ for all $1 \leqslant i < n$. Set $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Then $\mathfrak{p}$ is a parabolic subalgebra with nilradical $\mathfrak{g}_1$. As before, any $S \in \mathcal{S}$ is viewed as a $\mathfrak{p}$-module with a trivial action of $\mathfrak{g}_1$. Set $M_\mathfrak{p}(S) = \operatorname{Ind}_\mathfrak{p}^\mathfrak{g} S$ and let $L_\mathfrak{p}(S)$ be its unique simple quotient. Indeed $L_\mathfrak{p}(S) = L(\lambda)$, where λ is the highest weight of S. It will be convenient to use these notations also for $S = 0$, in this case $M_\mathfrak{p}(S)$ and $L_\mathfrak{p}(S)$ are zero.

Let $\lambda \in P^+$. By a theorem of Kostant [K], $H^k(\mathfrak{g}_1, L(\lambda))$ is the simple $\mathfrak{gl}(n)$-module with highest weight $w_k(\lambda + \rho) - \rho$, where w_k is the unique element of length k which is minimal in its S_n-coset. More explicitly, we have $w_k(i) = i$ for $i < n+1-k$, $w_k(n+1-k) = n+1$ and $w_k(i) = i-1$ for $i > n+1-k$. Let $\mathcal{H}^k$ be the set of all $S \in \mathcal{S}$ which are isomorphic to $H^k(\mathfrak{g}_1, L)$ for some finite dimensional simple $\mathfrak{sl}(n+1)$-module L, and set

$\mathcal{H}^* = \cup_{0\leqslant k\leqslant n}\mathcal{H}^k$. The sets $\mathcal{H}^0, \mathcal{H}^1, \cdots$ are pairwise disjoint. For $S \in \mathcal{H}^*$, there is a unique finite dimensional simple $\mathfrak{sl}(n+1)$-module L and a unique k such that $S \simeq H^k(\mathfrak{g}_1, L)$. Hence for any $S \in \mathcal{H}^*$, the $\mathfrak{gl}(n)$-module $S[i] := H^{k+i}(\mathfrak{g}_1, L)$ is well defined, for all $i \in \mathbb{Z}$. For instance, let K be the trivial representation of $\mathfrak{gl}(n)$. Then K belongs to $\mathcal{H}^0$ and $K[k] = H^k(\mathfrak{g}_1)$ is the $\mathfrak{gl}(n)$-module $\wedge^k V$. Note that for any $S \in \mathcal{S}$, $L_{\mathfrak{p}}(S)$ is admissible and it is infinite dimensional unless $S \in \mathcal{H}^0$.

LEMMA 11.1.— *i) Here $\mathfrak{g} \neq \mathfrak{sl}(2)$. Let $S \neq S' \in \mathcal{S}$, and let λ, λ' be their highest weights. If $\lambda + \rho$ and $\lambda' + \rho$ are in the same W-orbit, then S and S' belong to $\mathcal{H}^*$ and $S' = S[i]$ for a unique $i \neq 0$.*

ii) *Let $S \in \mathcal{S}$. The central character of $L_{\mathfrak{p}}(S)$ is integral and regular if and only if S belongs to $\mathcal{H}^*$.*

Proof. — With the notation of Lemma 8.3, let $\mathcal{C}$ be the set of all $\mu \in \mathfrak{h}^*$ such that

i) $\mu + \rho \in W(\lambda + \rho)$,

ii) $m(\mu) \in \mathcal{P}^+$ or $\mu \in P^+$.

It follows from Lemma 8.3 that $\mathcal{C}$ is reduced to λ whenever the central character of $L(\lambda)$ is not integral and regular (unless $\mathfrak{g} = \mathfrak{sl}(2)$). Otherwise, it can be assumed that $\lambda \in P^+$. Then $\mathcal{C}$ contains $n+1$ elements, which are the highest weights of $S[k]$ for $0 \leqslant k \leqslant n$. Hence the two assertions are proved. Q.E.D.

LEMMA 11.2.— *Let $S \in \mathcal{S}$.*

i) *If $S \notin \mathcal{H}^*$, the $\mathfrak{sl}(n+1)$-module $M_{\mathfrak{p}}(S)$ is simple, and its degree is* $\dim S$.

ii) *If $S \in \mathcal{H}^*$, there is a exact sequence of $\mathfrak{sl}(n+1)$-modules:*

$$\cdots M_{\mathfrak{p}}(S[2]) \to M_{\mathfrak{p}}(S[1]) \to M_{\mathfrak{p}}(S[0]) \to L_{\mathfrak{p}}(S) \to 0.$$

Moreover if $S \notin \mathcal{H}^0$, the degree of $L_{\mathfrak{p}}(S)$ is $\sum_{i\geqslant 0}(-1)^i \dim S[i]$.

Proof. — Both assertions follow from Kazhdan-Lusztig character formulas. However, it is simpler to prove them directly. Since $M_{\mathfrak{p}}(S)$ is locally finite as a $\mathfrak{p}$-module, any simple submodule is isomorphic to $L_{\mathfrak{p}}(S')$, for some $S' \in \mathcal{S}$. Hence by Lemma 11.1, $M_{\mathfrak{p}}(S)$ is simple whenever $S \notin \mathcal{H}^*$ and $\mathfrak{g} \neq \mathfrak{sl}(2)$. For the $\mathfrak{sl}(2)$-case, this follows by direct computations.

To prove the second assertion, it is enough to consider the case where $S = K[k]$ for some $k \geqslant 0$, where K is the trivial representation of $\mathfrak{gl}(n)$: indeed the general case follows from the translation principle [Ja]. Set $X = \operatorname{Spec} K[x_1, \ldots, x_n]$. We have $T \subset X \subset \mathbb{P}^n$ and $\mathfrak{sl}(n+1)$ is indeed a subalgebra of W_X. Let $Z_X^k = \{\alpha \in \Omega_X^k \,|\, \mathrm{d}\alpha = 0\}$. Following [Gai], Z_X^k is a simple $\mathfrak{sl}(n+1)$-module. Moreover, it is easy to identify Ω_X^k and Z_X^k as the $\mathcal{O}$-duals of $M_{\mathfrak{p}}(K[k])$ and $L_{\mathfrak{p}}(K[k])$ respectively. Hence the required exact sequence

$$\cdots M_{\mathfrak{p}}(S[2]) \to M_{\mathfrak{p}}(S[1]) \to M_{\mathfrak{p}}(S[0]) \to L_{\mathfrak{p}}(S) \to 0$$

is the $\mathcal{O}$-dual of the truncated De Rham complex

$$0 \to Z_X^k \to \Omega_X^k \to \Omega_X^{k+1} \to \Omega_X^{k+2} \cdots.$$

Let $S \in \mathcal{S}$. It follows from the identity $\operatorname{ch} M_{\mathfrak{p}}(S) = \operatorname{ch} S \operatorname{ch} S\mathfrak{g}_{-1}$ that

$$\operatorname{ch} L_{\mathfrak{p}}(S) = \operatorname{ch} S \operatorname{ch} S\mathfrak{g}_{-1} \text{ if } S \notin \mathcal{H}^*, \text{ and}$$
$$\operatorname{ch} L_{\mathfrak{p}}(S) = \sum_{i \geqslant o} (-1)^i \operatorname{ch} S[i] \operatorname{ch} S\mathfrak{g}_{-1} \text{ otherwise.}$$

Since the set of roots of $\mathfrak{g}_{-1}$ is a basis of Q, the degree of $L_{\mathfrak{p}}(S)$ is as asserted. Q.E.D.

LEMMA 11.3.— *Let $S \in \mathcal{H}^0$ and let $\lambda \in P^+$ be its highest weight. There is a natural complex*

$$C(\lambda):\ 0 \to \mathcal{TENS}_T(S[0]) \to \mathcal{TENS}_T(S[1]) \to \mathcal{TENS}_T(S[2]) \cdots,$$

and its homology is $H^*_{DR}(T) \otimes L(\lambda)$.

Proof.— First consider the case $L = K$, i.e. $\lambda = 0$. Then $K[k] \simeq \wedge^k V$. In the last remark about tensor modules, a W_T-equivariant map $D : \mathcal{TENS}_T(\wedge^* V) \to \mathcal{TENS}_T(\wedge^* V)$ has been already defined. By definition $C(0)$ is the complex $(\mathcal{TENS}_T(\wedge^* V), D)$. As a complex, it is a direct sum of the complexes $(\Omega_T^*, \mathrm{d}_\omega)$ where $\omega \in \mathfrak{h}^*$ runs over a set of representatives of T^*. Since $[\mathrm{d}_\omega, i_E] = \mathcal{L}_E^\omega$ and $\mathcal{L}_E^\omega$ is diagonalizable, this complex is exact unless 0 is an eigenvalue of $\mathcal{L}_E^\omega$. This occurs only when ω belongs to Q. Therefore the homology of the full complex $C(0)$ is the De Rham cohomology $H^*_{DR}(T)$ of T.

The general case follows from the translation principle. Q.E.D.

Remark. — It should be noted that the differential of $C(0)$ is W_T-invariant, but for $\lambda \neq 0$ the corresponding differential on $C(\lambda)$ is only $\mathfrak{sl}(n+1)$-invariant.

By Theorem 8.6, any semi-simple irreducible coherent $\mathfrak{sl}(n+1)$-family $\mathcal{M}$ contains a unique infinite dimensional submodule $L(\lambda)$ with $\lambda(h_i) \geqslant 0$ for all $i < n$ and $\lambda(h_n) \notin \mathbb{Z}_{\geqslant 0}$, i.e. $\lambda \notin P^+$ but $m(\lambda) \in \mathcal{P}^+$ (here it is assumed that $n \geqslant 2$; indeed this result fails for $\mathfrak{sl}(2)$, because it is not possible to distinguish $\mathcal{P}^+$ from $\mathcal{P}^-$ for non-integral central characters). Set $S(\mathcal{M}) = H^0(\mathfrak{g}_1, L(\lambda))$. Thus $S(\mathcal{M})$ belongs to $\mathcal{S} \setminus \mathcal{H}^0$.

It follows from Lemma 5.5 that the formal coherent families (i.e. elements in a suitable K_0-group) can be defined and they are characterized by their trace. This is the meaning of the formal sum in the following Assertion (ii). The following theorem fails for $\mathfrak{sl}(2)$ and this case is treated in a remark.

THEOREM 11.4. — Here $\mathfrak{g}$=$\mathfrak{sl}(n+1)$, with $n \geqslant 2$. *The map* $\mathcal{M} \mapsto S(\mathcal{M})$ *is a bijection from the set of semi-simple irreducible coherent* $\mathfrak{sl}(n+1)$*-families to* $\mathcal{S} \setminus \mathcal{H}^0$.

Moreover, let $\mathcal{M}$ *be any semi-simple irreducible coherent* $\mathfrak{sl}(n+1)$*-family and set* $S = S(\mathcal{M})$. *Then one of the following two assertions hold:*

i) $S \notin \mathcal{H}^*$, $\mathcal{M}$ *is isomorphic to* $\mathcal{TENS}_T(S)^{ss}$, *and its degree is* $\dim S$.

ii) $S \in \mathcal{H}^*$, $\mathcal{M}$ *is equivalent to the formal coherent family* $\sum_{i\geqslant 0}(-1)^i$ $\mathcal{TENS}_T(S[i])$, *its degree is* $\sum_{i\geqslant 0}(-1)^i \dim S[i]$ *and there is an exact sequence*

$$0 \to \mathcal{M}[t] \to \mathcal{TENS}_T(S[0])[t] \to \mathcal{TENS}_T(S[1])[t] \to \mathcal{TENS}_T(S[2])[t] \cdots,$$

for any $t \in T^*$ *such that* $\mathcal{M}[t]$ *is cuspidal.*

Proof. — The fact that the map $\mathcal{M} \mapsto S(\mathcal{M})$ is a bijection is a reformulation of Theorem 8.6. Let $\mathcal{M}$ be a semi-simple irreducible coherent $\mathfrak{sl}(n+1)$-family, and set $S = S(\mathcal{M})$. Set $X = \operatorname{Spec} K[x_1, \cdots, x_n]$. We have $T \subset X \subset \mathbb{P}^n$ and $\mathfrak{sl}(n+1) \subset W_X \subset W_T$. The W_T-module $\mathcal{TENS}_T(S)$ contains $\mathrm{Tens}_X(S,0)$ as a W_X-submodule. As an $\mathfrak{sl}(n+1)$-module, $\mathrm{Tens}_X(S,0)$ is the $\mathcal{O}$-dual of $M_{\mathfrak{p}}(S)$.

First let us assume that $S \notin \mathcal{H}^*$ or $S \in \mathcal{H}^n$ (in such a case $S[1] = 0$). By Lemma 11.2, the $\mathfrak{sl}(n+1)$-module $\mathrm{Tens}_X(S,0)$ is simple and its degree equals the degree of the coherent family $\mathcal{TENS}_T(S)$. Hence the coherent family is irreducible by Proposition 4.8, and the assertions of the theorem follow.

Next let us assume that $S \in \mathcal{H}^k$, with $k < n$. Since $\mathrm{Tens}_X(S, 0)$ is the $\mathcal{O}$-dual of $M_{\mathfrak{p}}(S)$, it follows from Lemma 11.2 that the $\mathfrak{sl}(n+1)$-module $\mathrm{Tens}_X(L, 0)$ has two composition factors, namely $L_{\mathfrak{p}}(S)$ and $L_{\mathfrak{p}}(S[1])$. These factors are admissible and their supports are in the same Q-coset. It follows from Proposition 4.8 that $\mathcal{TENS}_T(S)^{ss} = \mathcal{EXT}(L_{\mathfrak{p}}(S)) \oplus \mathcal{EXT}(L_{\mathfrak{p}}(S[1]))$. Hence the formal coherent family $\sum_i (-1)^i \mathcal{TENS}_T(S[i])$ is effective and equals to $\mathcal{EXT}(L_{\mathfrak{p}}(S)) = \mathcal{M}$. Moreover for any $t \in T^*$ such that $\mathcal{M}[t]$ is cuspidal, the t-component of the sequence of Lemma 11.3 is exact, and the last assertion follows. Q.E.D.

It remains to describe $\mathrm{Sing}\,\mathcal{M}$ in terms of the geometry of $\mathbb{P}^n$. Under the action of T, $\mathbb{P}^n$ has a unique open orbit (identified with T), $n+1$ codimension one orbits, (say $\mathcal{O}_0, \mathcal{O}_1, \cdots, \mathcal{O}_n$) and some other orbits of higher codimension. For each $0 \leqslant i \leqslant n$, $T_i := T \cup \mathcal{O}_i$ is an affine open subset of $\mathbb{P}^n$ and we have $\mathfrak{sl}(n+1) \subset W_{T_i} \subset W_T$. These codimension one orbits can be indexed in such a way that

- for $i \neq 0$, $K[T_i]$ is the subalgebra of $K[T]$ generated by $x_j^{\pm 1}$ for $j \neq i$ and x_i,

- $K[T_0]$ is the subalgebra of $K[T]$ with a basis consisting of all monomials $x_1^{m_1} \cdots x_n^{m_n}$ such that $\sum_{1 \leqslant k \leqslant n} m_k \leqslant 0$.

It should be noted that each T_i^* is a codimension one torus of T^*. Therefore for any $S \in \mathcal{S}$, the image of $\mathrm{Supp}\,\mathcal{TENS}_{T_i}(S)$ in T^* is a codimension one coset.

COROLLARY 11.5. — *Let $\mathcal{M}$ be a semi-simple irreducible coherent $\mathfrak{sl}(n+1)$-family and set $S = S(\mathcal{M})$. We have*

$$\mathrm{Sing}\,\mathcal{M} = \cup_{0 \leqslant 1 \leqslant n} \mathrm{Supp}\,\mathcal{TENS}_{T_i}(S) \text{ modulo } Q.$$

Proof. — It follows from Theorem 11.4 that $t \notin \mathrm{Sing}\,\mathcal{M}$ if and only if $\mathcal{TENS}_T(S)[t]$ is cuspidal (if $S \notin \mathcal{H}^*$ or $S \in \mathcal{H}^n$) or is the extension of two cuspidal modules (if $S \in \mathcal{H}^*$ but $S \notin \mathcal{H}^n$). Let i be an integer with $0 \leqslant i \leqslant n$. Since the support of $\mathcal{TENS}_{T_i}(S)$ does not contain any Q-coset, $\mathcal{TENS}_{T_i}(S)$ contains no cuspidal subquotient. Moreover for any $t \in \mathrm{Supp}\,\mathcal{TENS}_{T_i}(S)$ modulo Q, the $\mathfrak{sl}(n+1)$-modules $\mathcal{TENS}_T(S)[t]$ and $\mathcal{TENS}_{T_i}(S)[t]$ have the same degree, namely $\dim S$. Since $\mathcal{TENS}_{T_i}(S)[t]$ is a $\mathfrak{sl}(n+1)$-submodule of $\mathcal{TENS}_T(S)[t]$, we have

$Q + \mathrm{Supp}\,\mathcal{TENS}_{T_i}(S) \subset \mathrm{Sing}\,\mathcal{M}$. Hence $\mathrm{Sing}\,\mathcal{M}$ contains the $n+1$ codimension one cosets $\mathrm{Supp}\,\mathcal{TENS}_{T_i}(S)$ modulo Q. By Corollary 10.3, $\mathrm{Sing}\,\mathcal{M}$ is the union of $n+1$ codimension one cosets, therefore $\mathrm{Sing}\,\mathcal{M}$ is exactly the union of these cosets. Q.E.D.

Remark. — Let us consider the special case $\mathfrak{g} = \mathfrak{sl}(2)$, for which the previous theorem fails. The set $\mathcal{S}$ of all simple $\mathfrak{gl}(1)$-modules is identified with $\mathfrak{h}^*$. For any two weights λ and μ, write $\lambda \equiv \mu$ if $\lambda + \rho = \pm(\mu + \rho)$. The map $S \in \mathcal{S} \mapsto \mathcal{TENS}_T(S)^{ss}$ induces a bijection from $\mathcal{S}/\equiv$ to the set of semi-simple irreducible coherent $\mathfrak{sl}(2)$-families.

Remark. — As mentionned before, Britten and Lemire have classified all multiplicity free simple $\mathfrak{sl}(n+1)$-modules [BL]. Of course it is possible to use the previous formula for the degree of a coherent family to recover their result. However there is a simpler way: indeed it is enough to determine for which $S \in \mathcal{S} \setminus \mathcal{H}^0$ the module $L := L_{\mathfrak{p}}(S)$ is multiplicity free.

Set $V = \mathfrak{g}_{-1}$ as before. Then L is a finitely generated torsion free SV-module of rank one. Therefore, the SV-module L can be identified with a SV-submodule of SV in such a way that the elements in L have no common divisors. It follows that $L = SV$ or L is an ideal defining a subscheme of codimension $\geqslant 2$. Since $\mathrm{Spec}\,SV$ is smooth, any differential operator $\theta : L \to L$ is the restriction of a unique differential operator from SV to itself (Cohen-Macaulay property). Since $\mathfrak{sl}(n+1)$ acts on L by differential operators, this action extends to SV. Then it is easy to identify SV with a generalized Verma module $M_{\mathfrak{p}}(K_a)$ for a certain $a \in K$. It follows that L is the unique simple submodule of $M_{\mathfrak{p}}(K_a)$. Therefore $\mathcal{EXT}(L)$ is the coherent family $\mathcal{M}(a)^{ss}$, where $\mathcal{M}(a)$ is the coherent family defined in the example at the begining of the section. Hence any multiplicity free semi-simple coherent family is isomorphic to $\mathcal{M}(a)^{ss}$ for some $a \in K$. This proves the following result:

COROLLARY 11.6 (Britten and Lemire [BL1]). — *Any infinite dimensional multiplicity free weight simple $\mathfrak{sl}(n+1)$-module is a submodule of $\mathcal{M}(a)^{ss}$, for some $a \in K$.*

12. Realization of coherent families for $\mathfrak{sp}(2n)$.

The Dynkin diagram of $\mathrm{Spin}(2n)$ has a non-trivial involution (it is unique unless $n = 4$) which induces an involution σ of $\mathrm{Spin}(2n)$. Also there is a unique central element $z \in \mathrm{Spin}(2n)$ such that $z^2 = 1$ and

$\mathrm{Spin}(2n)/\{1,z\} = SO(2n)$ (for $n=4$, z is uniquely determined by the additional requirement $\sigma(z)=z$). An odd pair of $\mathrm{Spin}(2n)$-modules is a pair $\{L, L'\}$ of simple $\mathrm{Spin}(2n)$-modules which are conjugated by σ and such that z acts as -1 on L and L'. In this section, it is shown that the irreducible semi-simple coherent $\mathfrak{sp}(2n)$-families are exactly parametrized by the odd pairs of $\mathrm{Spin}(2n)$-modules and their degrees can be computed in terms of the parameter. This statement is more concrete than Theorem 9.3, which involves the set of connected components of $\mathcal{Q}$. Unfortunately, I did not find a description of coherent families which is as explicit than the $\mathfrak{sl}(n+1)$-case.

Let X be a smooth affine variety of dimension n and let $\mathrm{Diff}(X)$ be the ring of differential operators on X. The ring $\mathrm{Diff}(X)$ is generated by the differential operators of order $\leqslant 1$, i.e. by the functions and the vector fields. Any closed one-form ω induces an automorphism Θ_ω of $\mathrm{Diff}(X)$ defined by $\Theta_\omega(f)=f$ if $f\in K[X]$ and $\Theta_\omega(\xi)=\xi+i_\xi\omega$ if $\xi\in W_X$. For any invertible function f, $\Theta_{df/f}$ is the conjugacy by f. Hence Θ_ω modulo the inner automorphisms of $\mathrm{Diff}(X)$ depends only on ω modulo the logarithmic differentials.

Let R be the subalgebra of $K[x_1^{\pm1},\dots,x_n^{\pm1}]$ generated by the monomials $x_1^{m_1}x_2^{m_2}\cdots$ with $\sum_i m_i\equiv 0$ modulo 2. Set $\hat{T}=\mathrm{Spec}\,K[x_1^{\pm1},\dots,x_n^{\pm1}]$ and $T=\mathrm{Spec}\,R$. In Section 9, $\mathfrak{sp}(2n)$ has been realized as a Lie algebra of differential operators on $\mathrm{Spec}\,K[x_1,\dots,x_n]$ and therefore on $\hat{T}$. Indeed we have $\mathfrak{sp}(2n)\subset\mathrm{Diff}(T)$. For any T-invariant one-form ω on T, let $K[T]^\omega$ be the natural $\mathrm{Diff}(T)$-module twisted by Θ_ω. Since this module depends only on ω up to a logarithmic differential, $\mathcal{V}:=\oplus_{\omega\in T^*}K[T]^\omega$ is a well defined $\mathrm{Diff}(T)$-module. Its restriction to $\mathfrak{sp}(2n)$ will be called the *Shale-Weil coherent family.*

Let $\mathcal{M}$ be a semi-simple irreducible coherent $\mathfrak{sp}(2n)$-family, and let χ be its central character. By Theorem 9.3, $\mathcal{M}$ contains exactly two highest weight modules $L(\lambda^\pm)$. With the notations of Section 9, $\lambda^\pm(h_n)$ are half integers and the weights $\lambda^\pm$ are defined by the requirement $\lambda^+(h_n)\geqslant -1/2$ and $\lambda^-(h_n)\leqslant -3/2$. For the Shale-Weil coherent family $\mathcal{V}$, these weights are denoted by $\omega^\pm$. We have $\omega^+(h_n)=-1/2$ and $\omega^+(h_i)=0$ otherwise. Therefore $\Lambda:=\lambda^+-\omega^+$ is integral and dominant. For any $\mathfrak{sp}(2n)$-module M, let $M^{(\chi)}$ be its maximal submodule with generalized central character χ.

PROPOSITION 12.1. — *Here $\mathfrak{g}$=$\mathfrak{sp}(2n)$. With the previous notations, we have*

$$\mathcal{M}\simeq(\mathcal{V}^{ss}\otimes L(\Lambda))^{(\chi)}.$$

Proof. — Let χ_0 be the central character of $\mathcal{V}$. The translation functor from the category of $\mathfrak{sp}(2n)$-modules with generalized central character χ_0 to the category of $\mathfrak{sp}(2n)$-modules with generalized central character χ is $M \mapsto (M \otimes L(\Lambda))^{(\chi)}$. By Theorem 9.3, $\mathcal{M}$ is the unique semi-simple irreducible coherent family with central character χ. Therefore the proposition follows from the fact that, in our setting, the translation functor is an equivalence of categories (see [Ja]). Q.E.D.

The notations specific to $\mathfrak{sp}(2n)$ have been introduced in Section 9. Let us recall that $\mathfrak{h}^*$ has a basis $(\epsilon_i)_{1\leqslant i\leqslant n}$, relative to which the simple roots are $\alpha_1 = \epsilon_1 - \epsilon_2, \cdots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n, \alpha_n = 2\epsilon_n$. The set Δ_s of short roots is the root system of Spin$(2n)$. A basis of this root system is $B' = \{\alpha'_i | 1 \leqslant i \leqslant n\}$, where $\alpha'_n = \alpha_n + \alpha_{n-1} = \epsilon_{n-1} + \epsilon_n$ and $\alpha'_i = \alpha_i$ otherwise. The involution of B' is the transposition which exchanges α'_{n-1} and α'_n. Let σ be the corresponding outer involution of Spin$(2n)$.

Let $\mathcal{M}$ be an irreducible coherent $\mathfrak{sp}(2n)$-family $\mathcal{M}$. Let $\lambda^{\pm}$ be the highest weights of the two highest weight modules occuring in $\mathcal{M}$. The weights $\lambda^{\pm} + \epsilon$ are integral and dominant with respect to B', where $\epsilon = \sum_{1\leqslant i\leqslant n} \epsilon_n$. Therefore let $S^{\pm}(\mathcal{M})$ be the simple Spin$(2n)$-modules with highest weights $\lambda^{\pm} + \epsilon$. Then $S^{\pm}(\mathcal{M})$ is an odd pair of Spin$(2n)$-modules. The common dimension of these modules is denoted by $\dim S^{\pm}(\mathcal{M})$.

THEOREM 12.2. — i) *The map $\mathcal{M} \mapsto S^{\pm}(\mathcal{M})$ is a bijection from the set of semi-simple irreducible coherent $\mathfrak{sp}(2n)$-families to the set of odd pairs of* Spin$(2n)$*-modules.*

ii) *The degree of an irreducible coherent $\mathfrak{sp}(2n)$-family $\mathcal{M}$ is* $(1/2^{n-1})$ $\dim S^{\pm}(\mathcal{M})$.

iii) *Let $t = \sum_{1\leqslant i\leqslant n} a_i\epsilon_i \in T^*$. The module $\mathcal{M}[t]$ is cuspidal if and only if $a_i \notin 1/2 + \mathbb{Z}$ for all $1 \leqslant i \leqslant n$.*

Proof. — Assertions (i) and (iii) are just reformulations of Theorems 9.3 and 10.2. Let us prove Assertion (ii). Let $L(\lambda^{\pm})$ be the two highest weight representations of $\mathcal{M}$.

We claim that $\mathrm{ch}\, L(\lambda^{\pm}) = \sum_{w\in W'} \epsilon(w) e^{w(\lambda^{\pm}+\rho)-\rho} / \prod_{\alpha\in\Delta^+}(1-e^{-\alpha})$, where W' is the Weyl group of B'. Indeed this follows from [GJ]. More precisely, the proof of these authors is based on some conjectural facts about highest weight representations which have been proved later by Soergel [S]. However there is an elementary and direct proof for the special case considered here: indeed it is enough to check the character formula in the case where $\lambda^{\pm} = \omega^{\pm}$ and to use the translation functor.

Let $\Delta^+ = \Delta_s^+ \cup \Delta_l^+$ be the decomposition of Δ^+ into short and long roots. Since $\rho = \rho' + \epsilon$, where $\rho' = 1/2 \sum_{\alpha \in \Delta_s^+} \alpha$, Weyl's character formula implies

$$\operatorname{ch} L(\lambda^\pm) = \operatorname{ch} S^\pm(\mathcal{M}) / \prod_{\beta \in \Delta_l^+} (1 - e^{-\beta}).$$

Since $\Delta_l^+ = \{2\epsilon_i | 1 \leqslant i \leqslant n\}$ is the basis of a lattice of index 2^{n-1} in Q, the admissible modules $L(\lambda^\pm)$ have degree $(1/2^{n-1}) \dim S^\pm(\mathcal{M})$. Thus Assertion (iii) follows from Proposition 4.8. Q.E.D.

Remark. — There is only one odd pair of $\mathrm{Spin}(2n)$-modules of dimension 2^{n-1}, namely the pair consisting of the two fundamental representations $\mathrm{Spin}^\pm$. Hence $\mathcal{V}^{ss}$ is the unique semi-simple coherent family of degree one. Thus the following result is recovered:

COROLLARY 12.3 (Britten and Lemire [BL1]). — *Any infinite dimensional simple multiplicity free weight $\mathfrak{sp}(2n)$-module is a submodule of $\mathcal{V}^{ss}$.*

13. Character formula for simple weight modules.

Since the simple weight modules are classified, it remains to determine their characters, i.e. the dimension of their weight spaces. Indeed this question is reduced to a similar question for the category $\mathcal{O}$, which is fully understood. The reduction is based on a refinement of the notion of coherent families, namely the notion of *relative coherent families.* Of course, it would have been possible to treat at once the notion of coherent families in the relative context. For clarity, this notion has not been introduced before, which will cause some repetitions.

Let $\mathfrak{g}$ be a simple Lie algebra. Let us fix a Levi subalgebra $\mathfrak{a}$ containing $\mathfrak{h}$ and set $\mathfrak{h}^*_{\mathrm{rel}} = \mathfrak{h}^*/K \otimes Q(\mathfrak{a})$. For any weight $\mathfrak{h}$-module M, set $\deg_{\mathfrak{a}} M[t] = \operatorname{Sup}_{\mu \in t} \dim M_\mu$, for any $t \in \mathfrak{h}^*_{\mathrm{rel}}$. The function $\deg_{\mathfrak{a}} : t \mapsto \deg_{\mathfrak{a}} M[t]$ is called the *$\mathfrak{a}$-relative degree of M.* Assume now that $\operatorname{Supp} M$ is included in a single Q-coset. Then M is called $\mathfrak{a}$-admissible if $\deg_{\mathfrak{a}} M[t] < \infty$ for any $t \in \mathfrak{h}^*_{\mathrm{rel}}$. Moreover if M is $\mathfrak{a}$-admissible, its *$\mathfrak{a}$-relative essential support* is $\operatorname{Supp}_{\mathfrak{a}-\mathrm{ess}} M := \cup_{t \in \mathfrak{h}^*_{\mathrm{rel}}} \{\lambda \in t | \dim M_\lambda = \deg_{\mathfrak{a}} M[t]\}$. Then M is called *strictly $\mathfrak{a}$-admissible* if $\operatorname{Supp}_{\mathfrak{a}-\mathrm{ess}} M \cap t$ is Zariski dense in the vector space $K \otimes t$ for any $t \in \mathfrak{h}^*_{\mathrm{rel}}$.

From now on, $\mathfrak{p}$ is a parabolic subalgebra with nilradical $\mathfrak{u}$ such that $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{u}$. Let $\mathfrak{u}^-$ be the opposed nilradical and let B be a $\mathfrak{p}$-adapted basis of Δ. Set $B' = B \cap \Delta(\mathfrak{a})$. For any $\lambda \in \mathfrak{h}^*$, let $L_{B'}(\lambda)$ be the simple $\mathfrak{a}$-module with B'-highest weight λ. As usual $L_{B'}(\lambda)$ is viewed as a $\mathfrak{p}$-module with a trivial action of $\mathfrak{u}$. Recall that $A_B(\lambda) = \{\alpha \in B \mid \lambda(h_\alpha) \notin \mathbb{Z}_{\geqslant 0}\}$.

LEMMA 13.1.— i) *For any $\lambda \in \mathfrak{h}^*$, the $\mathfrak{a}$-module $L_{B'}(\lambda)$ is strictly $\mathfrak{a}$-admissible if and only if it is $\mathfrak{a}$-admissible and $A_B(\lambda)$ intersects each connected component of B'.*

ii) *Let S be a simple weight $\mathfrak{p}$-module. Then S is strictly $\mathfrak{a}$-admissible if and only if $L_\mathfrak{p}(S)$ is strictly $\mathfrak{a}$-admissible.*

iii) *Let S be a simple strictly $\mathfrak{a}$-admissible weight $\mathfrak{a}$-module. There exists a set $\Sigma \subset \Delta(\mathfrak{a})$ of commuting roots which is a basis of $Q(\mathfrak{a})$ such that f_α acts injectively on S for all $\alpha \in \Sigma$.*

Proof. — To prove the first assertion, it can be assumed that $L_{B'}(\lambda)$ is admissible. The Lie algebra $\mathfrak{a}$ decomposes into $\mathfrak{z} \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_2 \cdots$, where $\mathfrak{z}$ is the center of $\mathfrak{a}$, and each $\mathfrak{a}_i$ is a simple ideal. Then $L_{B'}(\lambda)$ decomposes into $Z \otimes L_1 \otimes L_2 \cdots$, where Z is a one dimensional $\mathfrak{z}$-module, and each L_i is an admissible simple highest weight $\mathfrak{a}_i$-module (here L_i is a highest weight module relative to the basis $B' \cap \Delta(\mathfrak{a}_i)$ of $\Delta(\mathfrak{a}_i)$). By Proposition 3.5, $\mathrm{Supp}_{\mathrm{ess}} L_i$ is Zariski dense in $K \otimes Q(\mathfrak{a}_i)$ whenever L_i is infinite dimensional and this condition is equivalent to $A_B(\lambda) \cap \Delta(\mathfrak{a}_i) \neq \varnothing$. The first assertion follows.

To prove the second assertion, it can be assumed that S is strictly $\mathfrak{a}$-admissible. Since $\mathrm{Supp}\, S$ lies in a single $Q(\mathfrak{a})$-coset by Lemma 1.1 and $M_\mathfrak{p}(S) \simeq U(\mathfrak{u}^-) \otimes S$, the $\mathfrak{g}$-module $M_\mathfrak{p}(S)$ is $\mathfrak{a}$-admissible, hence $L_\mathfrak{p}(S)$ is also $\mathfrak{a}$-admissible. As before there is a decomposition $S = Z \otimes S_1 \otimes S_2 \cdots$, where Z is a one dimensional $\mathfrak{z}$-module, and each S_i is an infinite dimensional simple weight $\mathfrak{a}_i$-module. Hence by Lemma 3.1, $C(S_i)$ generates $Q(\mathfrak{a}_i)$ for any i. Hence $C(S) := C(S_1) \oplus C(S_2) \oplus \cdots$ generates $Q(\mathfrak{a})$. Since $C(S) \subset C(L_\mathfrak{p}(S))$, we have $C(S) + \mathrm{Supp}_{\mathfrak{a}-\mathrm{ess}} L_\mathfrak{p}(S) \subset \mathrm{Supp}_{\mathfrak{a}-\mathrm{ess}} L_\mathfrak{p}(S)$. Therefore $L_\mathfrak{p}(S)$ is strictly $\mathfrak{a}$-admissible.

The last assertion follows from the previous decomposition $S = Z \otimes S_1 \otimes S_2 \cdots$ and Lemma 4.1. Q.E.D.

A $\mathfrak{a}$-*coherent family* is a weight $\mathfrak{g}$-module $\mathcal{M}$ such that

i) the function $\lambda \mapsto \dim \mathcal{M}_\lambda$ is constant on each $K \otimes Q(\mathfrak{a})$-coset,

ii) for any $u \in \mathcal{A}$, the function $\lambda \mapsto \mathrm{Tr}\, u|_{\mathcal{M}_\lambda}$ is polynomial on each $K \otimes Q(\mathfrak{a})$-coset.

Set $T^*_\mathfrak{a} = K \otimes Q(\mathfrak{a})/Q(\mathfrak{a})$.

LEMMA 13.2. — *Let S be a simple strictly $\mathfrak{a}$-admissible weight $\mathfrak{a}$-module, and let $\Sigma \subset \Delta(\mathfrak{a})$ be a set of commuting roots which is a basis of $Q(\mathfrak{a})$ such that f_α acts injectively on S for all $\alpha \in \Sigma$.*

i) *We have $f_\Sigma^\nu.L_\mathfrak{p}(S)_{F_\Sigma} = L_\mathfrak{p}(f_\Sigma^\nu.S_{F_\Sigma})$ for all $\nu \in K \otimes Q(\mathfrak{a})$.*

ii) *The $\mathfrak{g}$-module $\mathcal{M} := \oplus_{\nu \in T_\mathfrak{a}^*} L_\mathfrak{p}(f_\Sigma^\nu.S_{F_\Sigma})$ is a $\mathfrak{a}$-coherent family for which*

$$\deg_\mathfrak{a} \mathcal{M} = \deg_\mathfrak{a} L_\mathfrak{p}(S).$$

Proof. — It is clear that $\oplus_{\nu \in T_\mathfrak{a}^*} f_\Sigma^\nu.L_\mathfrak{p}(S)_{F_\Sigma}$ is a $\mathfrak{a}$-coherent coherent family and its $\mathfrak{a}$-relative degree equals $\deg_\mathfrak{a} L_\mathfrak{p}(S)$. Hence it is enough to prove the first assertion. Since $\mathrm{ad}(f_\alpha)\mathfrak{u} = \mathfrak{u}$ for any $\alpha \in \Sigma$, we have $f_\Sigma^\mu \mathfrak{u} f_\Sigma^{-\mu} = \mathfrak{u}$ for any $\mu \in K \otimes Q(\mathfrak{a})$. Any $y \in f_\Sigma^\nu.L_\mathfrak{p}(S)_{F_\Sigma}$ can be written as $f_\Sigma^{\nu'}.y'$, for some $\nu' \in \nu + Q(\mathfrak{a})$ and $y \in L_\mathfrak{p}(S)$. Therefore y is $\mathfrak{u}$-invariant if and only if y' is $\mathfrak{u}$-invariant. Hence $H^0(\mathfrak{u}, f_\Sigma^\nu.L_\mathfrak{p}(S)_{F_\Sigma}) = f_\Sigma^\nu.S_{F_\Sigma}$. A similar proof shows that $f_\Sigma^\nu.L_\mathfrak{p}(S)_{F_\Sigma}$ is generated by $f_\Sigma^\nu.S_{F_\Sigma}$ as a $U(\mathfrak{u}^-)$-module. Therefore $f_\Sigma^\nu.L_\mathfrak{p}(S)_{F_\Sigma} = L_\mathfrak{p}(f_\Sigma^\nu.S_{F_\Sigma})$. Q.E.D.

Let L be a simple weight $\mathfrak{g}$-module. Since $\mathfrak{p}$ is an arbitrary parabolic subalgebra of $\mathfrak{g}$, it can be assumed by Theorem 1.2 that $L = L_\mathfrak{p}(S)$ for some cuspidal $\mathfrak{p}/\mathfrak{u}$-module S. Let $\Sigma \subset \Delta(\mathfrak{a})$ be a set of commuting roots which is a basis of $Q(\mathfrak{a})$.

THEOREM 13.3. — *With the previous notations:*

i) *There exists $\lambda \in \mathfrak{h}^*$ such that*

- *the $\mathfrak{a}$-module $L_{B'}(\lambda)$ is strictly $\mathfrak{a}$-admissible, and*
- *$L_{B'}(\lambda)$ is a subquotient of $f_\Sigma^\nu.S_{F_\Sigma}$ for some $\nu \in K \otimes Q(\mathfrak{a})$.*

ii) *For any $\mu \in \mathfrak{h}^*$, set $t(\mu) = \mu + \nu + Q(\mathfrak{a})$. Then we have*

$$\dim L_\mathfrak{p}(S)_\mu = \mathrm{Sup}_{\mu' \in t(\mu)} \dim L_B(\lambda)_{\mu'}.$$

Proof. — Set $\mathcal{S} = \oplus_{\nu \in T_\mathfrak{a}^*} f_\Sigma^\nu.S_{F_\Sigma}$. The Lie algebra $\mathfrak{a}$ decomposes into $\mathfrak{z} \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_2 \cdots$, where $\mathfrak{z}$ is the center of $\mathfrak{a}$, and each $\mathfrak{a}_i$ is a simple ideal. The coherent $\mathfrak{a}$-family $\mathcal{S}$ decomposes into $Z \otimes \mathcal{S}_1 \otimes \mathcal{S}_2 \cdots$, where Z is a one dimensional $\mathfrak{z}$-module and each $\mathcal{S}_i$ is a coherent $\mathfrak{a}_i$-family. By Proposition 6.2, each coherent $\mathfrak{a}_i$-family $\mathcal{S}_i^{ss}$ contains an infinite dimensional highest weight module L_i relative to the basis $B' \cap \Delta(\mathfrak{a}_i)$ of $\Delta(\mathfrak{a}_i)$. Therefore the $\mathfrak{a}$-module $L_{B'}(\lambda) := Z \otimes L_1 \otimes L_2 \cdots$ is strictly admissible by Lemma 13.1 and it is a subquotient of $f_\Sigma^\nu.S_{F_\Sigma}$ for some $\nu \in K \otimes Q(\mathfrak{a})$.

By Lemma 13.1, there exists a set $\Sigma' \subset \Delta(\mathfrak{a})$ of commuting roots which is a basis of $Q(\mathfrak{a})$ such that f_α acts injectively on $L_{B'}(\lambda)$ for all $\alpha \in \Sigma'$. Set $\mathcal{S}' = \oplus_{\nu' \in T_\mathfrak{a}^*} f_{\Sigma'}^{\nu'}.L_{B'}(\lambda)_{F_{\Sigma'}}$, $\mathcal{M} = L_\mathfrak{p}(\mathcal{S})$ and $\mathcal{M}' = L_\mathfrak{p}(\mathcal{S}')$. By Lemma 13.2, $\mathcal{M}$ and $\mathcal{M}'$ are $\mathfrak{a}$-coherent families. By definition of $L_{B'}(\lambda)$, $\mathcal{S}$ and $\mathcal{S}'$ are coherent extensions of the $\mathfrak{a}$-module $L_{B'}(\lambda)$, hence by Proposition 4.8 $\mathcal{S}^{ss} = \mathcal{S}'^{ss}$. Therefore for any $Q(\mathfrak{a})$-coset u such that $\mathcal{S}[u]$ or $\mathcal{S}'[u]$ is a cuspidal $\mathfrak{a}$-module, we have $\mathcal{S}[u] \simeq \mathcal{S}'[u]$ and $\mathcal{M}[u+Q] \simeq \mathcal{M}'[u+Q]$. In particular, we have $\mathcal{M}[t] \simeq \mathcal{M}'[t]$ where $t = \operatorname{Supp} S + Q$. Therefore, we have

$$L_\mathfrak{p}(S) \simeq f_{\Sigma'}^{-\nu}.L_B(\lambda)_{F_{\Sigma'}},$$

from which the formula for $\operatorname{ch} L_\mathfrak{p}(S)$ follows. Q.E.D.

Appendix: Generalized Enright functors.

In the paper, some localizations with respect to some non-integral powers have been used often. Indeed this tool can be used to generalize the Enright functors. This appendix is devoted to one application of these functors, namely Proposition A.3. Indeed this proposition was known [S], but this approach provides an elementary proof.

Let $\alpha \in \Delta$. For simplicity set $e = e_\alpha$, $h = h_\alpha$, $f = f_\alpha$, and $s = s_\alpha$. For $u \in T^*$, let $\mathcal{C}_\alpha[u]$ be the category of all weight $\mathfrak{g}$-modules M such that $\operatorname{Supp} M \subset u$ and $e|_M$ is locally nilpotent. There is $x \in K$ such that $su = x\alpha + u$ (indeed x is unique modulo an integer). For $M \in \mathcal{C}_\alpha[u]$, set $F_u\, M = \{m \in f^{-x}.M_f \,|\, e^N.m = 0 \text{ for some } N >> 0\}$. It is clear that $F_u\, M$ belongs to $\mathcal{C}_\alpha[su]$ and $F_u : M \mapsto F_u\, M$ is a functor from $\mathcal{C}_\alpha[u]$ to $\mathcal{C}_\alpha[su]$.

LEMMA A.1.— *With the previous notations, assume that $x \notin \mathbb{Z}$. Then the functor $F_u : \mathcal{C}_\alpha[u] \to \mathcal{C}_\alpha[su]$ is an equivalence of categories, and its inverse is F_{su}.*

Proof.— Let $M \in \mathcal{C}_\alpha[u]$ be a $\mathfrak{g}$-module. It is clear that $F_{su} \circ F_u\, M = \{m \in M_f | e^N.m = 0 \text{ for some } N >> 0\}$. In particular, there is a natural map $M \to F_{su} \circ F_u\, M$. The Lie algebra $\mathfrak{a} := Ke \oplus Kh \oplus Kf$ is isomorphic to $\mathfrak{sl}(2)$. Since $x \notin \mathbb{Z}$, as an $\mathfrak{a}$-module M is a direct sum of simple Verma $\mathfrak{a}$-modules with non-integral highest weights. For such a Verma $\mathfrak{a}$-module V, the map $V \to V_f$ is injective and V_f/V is a simple infinite dimensional lowest weight module. It follows that $V = \{v \in V_f | e^N.v =$

0 for some $N >> 0\}$. Therefore the natural map $M \to F_{su} \circ F_u M$ is an isomorphism, which proves the lemma. Q.E.D.

From now on, let us fix a basis B of Δ, which allows us to define the category $\mathcal{O}$ (see e.g [Ja][D]). For a (non-necessarily integral) central character χ, let $\mathcal{O}^{(\chi)}$ be the category of all modules $M \in \mathcal{O}$ with generalized central character χ. For $u \in T^*$, let $\mathcal{O}[u]$ be the category of all modules $M \in \mathcal{O}$ such that $\operatorname{Supp} M \subset u$. Set $\mathcal{O}^{(\chi)}[u] = \mathcal{O}^{(\chi)} \cap \mathcal{O}[u]$.

LEMMA A.2. — *Let $u, v \in T^*$ be W-conjugated. Then the categories $\mathcal{O}^{(\chi)}[u]$ and $\mathcal{O}^{(\chi)}[v]$ are equivalent.*

Proof. — It is enough to prove the assertion when $u = s_\alpha.v$ for some simple root α. Let $x \in K$ be a scalar such that $v = x.\alpha + u$. It can be assumed that $u \neq v$ and therefore $x \notin \mathbb{Z}$. Set $\mathfrak{u} = \oplus_{\beta \in \Delta^+ \setminus \alpha} \mathfrak{g}_\beta$. Since $\mathfrak{u}$ is ad(f)-invariant, we have $f^{-x}\mathfrak{u}f^x = \mathfrak{u}$. Therefore the functor F_u sends $\mathfrak{u}$-locally nilpotent modules to $\mathfrak{u}$-locally nilpotent modules, and therefore $F_u\, \mathcal{O}[u] \subset \mathcal{O}[v]$. Since $f^{-x}zf^x = z$ for any $ad(f)$-invariant element $z \in U(\mathfrak{g})$, the functor F_u preserves the central characters. Hence we have $F_u\, \mathcal{O}^{(\chi)}[u] \subset \mathcal{O}^{(\chi)}[v]$. It follows from Lemma A.1 that F_u induces an equivalence of categories from $\mathcal{O}^{(\chi)}[u]$ to $\mathcal{O}^{(\chi)}[v]$. Q.E.D.

For a central character χ, let $HW(\chi)$ be the set of all weights λ such that $L(\lambda)$ belongs to $\mathcal{O}^{(\chi)}$ and let $\overline{HW}(\chi)$ be its image in T^*.

LEMMA A.3. — *Let χ be a central character. The blocks of $\mathcal{O}^{(\chi)}$ are the subcategories $\mathcal{O}^{(\chi)}[u]$, where u runs over $\overline{HW}(\chi)$. Moreover, they are all equivalent.*

Proof. — It is clear that $\mathcal{O}^{(\chi)}$ decomposes into $\oplus_{u \in \overline{HW}(\chi)} \mathcal{O}^{(\chi)}[u]$. Moreover, $\overline{HW}(\chi)$ consits of a single W-orbit and the subcategories $\mathcal{O}^{(\chi)}[u]$ are all equivalent by Lemma A.2. Therefore it is only necessary to prove that $\mathcal{O}^{(\chi)}[u]$ is indecomposable for any $u \in \overline{HW}(\chi)$. There exists a unique weight $\mu^+ \in u \cap HW(\chi)$ such that $(\mu^+ + \rho)(h_\alpha) \notin \mathbb{Z}_{<0}$ for all $\alpha \in \Delta^+$. The simple modules in $\mathcal{O}^{(\chi)}[u]$ are the highest weight modules $L(\lambda)$ where $\lambda \in u \cap HW(\chi)$ and moreover we have $u \cap HW(\chi) = \{w(\mu^+ + \rho) - \rho | w \in W_u\}$, where W_u is the stabilizer of u in W. It follows from [D] (ch. 7) that for any $\lambda \in u \cap HW(\chi)$ the Verma module $M(\lambda)$ is a submodule of $M(\mu^+)$. Therefore any simple module of $\mathcal{O}^{(\chi)}[u]$ is a subquotient of $M(\mu^+)$. Since $M(\mu^+)$ is indecomposable, $\mathcal{O}^{(\chi)}[u]$ is a block. Q.E.D.

PROPOSITION A.4 (Soergel [S]). — *Assume that $\mathfrak{g}$ is simply laced. Then any block in $\mathcal{O}$ is equivalent to a block with integral central character of the category $\mathcal{O}$ of some Levi subalgebra.*

Proof. — Let $\mathcal{B}$ be a block of $\mathcal{O}$. By Lemma A.3, we have $\mathcal{B} = \mathcal{O}^{(\chi)}[u]$ for some central character χ and some $u \in T^*$. Set $\Delta' = \{\alpha \in \Delta |\, u(h_\alpha) \equiv 0 \,\text{modulo}\, \mathbb{Z}\}$. Since $\mathfrak{g}$ is simply laced we have $h_{\alpha+\beta} = h_\alpha + h_\beta$, for all $\alpha, \beta \in \Delta$ such that $\alpha + \beta \in \Delta$. Hence the subspace $\mathfrak{g}' := \mathfrak{h} \oplus \oplus_{\alpha\in\Delta'} \mathfrak{g}_\alpha$ is a Lie subalgebra, and it is easy to see that $\mathfrak{g}$' is a Levi subagebra. Let B' be a basis of Δ'. There exists $w \in W$ such that $w(B') \subset B$. By Lemma A.3, $\mathcal{B}$ is equivalent to $\mathcal{O}^{(\chi)}[wu]$. Therefore it can be assumed that $w = 1$ and $B' \subset B$.

For any $\mu \in \mathfrak{h}^*$, let $L_{B'}(\mu)$ be the simple $\mathfrak{g}'$-module with highest weight μ relative to B'. Let $\lambda \in u \cap HW(\chi)$ be any weight and let χ' be the central character of the $\mathfrak{g}'$-module $L_{B'}(\lambda)$. Since χ' is an integral central character, $\mathcal{B}' := \mathcal{O}^{(\chi')}$ is a block of the category $\mathcal{O}$ for $\mathfrak{g}'$.

We claim that $\mathcal{B}$ and $\mathcal{B}'$ are equivalent. Set $u' = \lambda + Q(\mathfrak{g}')$, let W' be the Weyl group of $\mathfrak{g}'$, set $\mathfrak{u} = \oplus_{\alpha\in\Delta^+\setminus\Delta'} \mathfrak{g}_\alpha$, and $\mathfrak{p} = \mathfrak{g}' \oplus \mathfrak{u}$. Thus $\mathfrak{p}$ is a parabolic algebra, $\mathfrak{u}$ is its nilradical. Since $W' = W_u$, we have $HW(\chi) \cap u \subset u'$. Hence we have $\operatorname{Supp} M \subset u' + C(\mathfrak{p})$ and $H^0(\mathfrak{u}, M) = M[u']$ for any $M \in \mathcal{B}$. It follows that the functors $M \in \mathcal{B} \mapsto M[u'] \in \mathcal{B}'$ and $N \in \mathcal{B}' \mapsto \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}} N$ are inverse to each other. Therefore $\mathcal{B}$ and $\mathcal{B}'$ are equivalent. Q.E.D.

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