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QUIVER VARIETIES AND WEYL GROUP ACTIONS

by George LUSZTIG (*)

Introduction.

Consider a finite graph of type ADE with set of vertices I. Nakajima [N1], [N2] associates to $\mathbf{v}, \mathbf{w} \in \mathbf{N}^I$ a smooth algebraic variety $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ ("quiver variety") and shows that the cohomology of $\sqcup_{\mathbf{v}} \mathfrak{M}(\mathbf{v}, \mathbf{w})$ has a natural module structure over the corresponding enveloping algebra; note that for fixed $\mathbf{w}$, $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ is empty for all but finitely many choices of $\mathbf{v}$. He also constructs [N1, Sec. 9] a Weyl group action on this cohomology space using techniques of hyper-Kähler geometry. In this paper we give an alternative construction of this Weyl group action, based not on hyper-Kähler geometry, but on techniques of intersection cohomology, analogous to those used in [L1] to construct Springer representations. This gives in fact a refinement of the Weyl group action (see 6.13, 6.14, 6.15). I wish to thank H. Nakajima for interesting conversations.

1. A non-linear W-action.

1.1. We fix a graph with finite set of vertices I. We assume that there is at most one edge joining two vertices of I and no edge joining a vertex with itself. Let H be the set of all ordered pairs i, j of vertices such that i, j are joined by an edge. For $h = (i, j)$, we set $\bar{h} = (j, i) \in H$, $j = h' \in I$, $i = h'' \in I$. We fix a function $\varepsilon : H \to \{1, -1\}$ such that $\varepsilon(h) + \varepsilon(\bar{h}) = 0$ for all h. We often write ε_h instead of $\varepsilon(h)$.

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Let $\mathbf{I}$ be the set of all sequences $i_1, i_2, \ldots, i_s$ (with $s \geq 1$) in I such that $(i_k, i_{k+1}) \in H$ for any $k \in [1, s-1]$. Let $\widetilde{\mathcal{F}}$ be the $\mathbf{C}$-vector space spanned by elements $[i_1, i_2, \ldots, i_s]$ corresponding to the various elements of $\mathbf{I}$ and by the elements u_i indexed by $i \in I$. Let $\mathcal{F}$ be the subspace of $\widetilde{\mathcal{F}}$ spanned by the elements of the form $[i_1, i_2, \ldots, i_s]$. We regard $\mathcal{F}$ as an algebra in which the product $[i_1, i_2, \ldots, i_s][j_1, j_2, \ldots, j_{s'}]$ is equal to $[i_1, i_2, \ldots, i_s, j_2, \ldots, j_{s'}]$ if $i_s = j_1$ and is zero, otherwise.

Let E be a $\mathbf{C}$-vector space with basis $\{\varpi_i | i \in I\}$. For $\lambda \in E$ we define $\lambda_i \in \mathbf{C}$ by $\lambda = \sum_i \lambda_i \varpi_i$. For $i \in I$ we define $s_i : E \to E$ by $s_i(\lambda) = \lambda'$ where $\lambda'_i = -\lambda_i$, $\lambda'_j = \lambda_j + \lambda_i$ if $(i,j) \in H$ and $\lambda'_j = \lambda_j$ if $j \neq i$ and $(i,j) \notin H$. Let W be the subgroup of $GL(E)$ generated by the $s_i : E \to E$ with $i \in I$. It is well known that W is a Coxeter group with generators s_i and relations $s_i^2 = 1$, $s_i s_j s_i = s_j s_i s_j$ if $(i,j) \in H$, $s_i s_j = s_j s_i$ if $i \neq j$ and $(i,j) \notin H$. For any $\lambda \in E$ we define a linear map $s_i^\lambda : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}}$ by

$$s_i^\lambda(u_j) = u_j \quad \text{for all } j;$$
$$s_i^\lambda[i] = [i] + \lambda_i u_i;$$
$$s_i^\lambda[i_1, i_2, \ldots, i_s] = \sum_{J; J \subset J_0} \prod_{t \in J} (-\varepsilon_{i_t, i_{t+1}} \lambda_i)[i_1, i_2, \ldots, i_s; \hat{J}]$$

if $[i_1, i_2, \ldots, i_s] \neq [i]$.

Here $J_0 = \{t \in [2, s-1] | i_t = i, i_{t-1} = i_{t+1}\}$; $[i_1, i_2, \ldots, i_s; \hat{J}]$ is the element of $\mathbf{I}$ obtained from $[i_1, i_2, \ldots, i_s]$ by omitting i_t, i_{t+1} for all $t \in J$.

It will be convenient to define $\bar{s}_i^\lambda : \mathcal{F} \to \mathcal{F}$ as the composition $\mathcal{F} \to \widetilde{\mathcal{F}} \xrightarrow{s_i^\lambda} \widetilde{\mathcal{F}} \to \mathcal{F}$ where the first map is the obvious imbedding and the third map is the projection with kernel $\sum_j \mathbf{C} u_j$. We define a map $s_i : E \times \widetilde{\mathcal{F}} \to E \times \widetilde{\mathcal{F}}$ by

(a) $$(\lambda, f) \to (s_i(\lambda), s_i^\lambda(f)).$$

Let $\hat{s}_i^\lambda = s_i^{-\lambda}$. We define $\hat{s}_i : E \times \widetilde{\mathcal{F}} \to E \times \widetilde{\mathcal{F}}$

(b) $$(\lambda, f) \to (s_i(\lambda), \hat{s}_i^\lambda(f)).$$

LEMMA 1.2. — *The map 1.1(a) is an involution.*

Assume first that $i_1, i_2, \ldots, i_s$ in $\mathbf{I}$ is other than i. Let J_0 be as in 1.1. We have

$$s_i s_i(\lambda, [i_1, i_2, \ldots, i_s])$$
$$= s_i\Big(s_i(\lambda), \sum_{J; J \subset J_0} \prod_{t \in J} (-\varepsilon_{i_t, i_{t+1}} \lambda_i)[i_1, i_2, \ldots, i_s; \hat{J}]\Big)$$

$$= \Big(s_i^2\lambda, \sum_{\substack{J,J' \\ J\subset J'\subset J_0}} \prod_{t\in J}(-\varepsilon_{i_t,i_{t+1}}\lambda_i) \prod_{t\in J'-J} (\varepsilon_{i_t,i_{t+1}}\lambda_i)[i_1,i_2,\dots,i_s;\hat{J'}]\Big)$$

$$= \Big(\lambda, \sum_{\substack{J' \\ J'\subset J_0}} \sum_{J;J\subset J'} (-1)^{|J|} \prod_{t\in J'} (\varepsilon_{i_t,i_{t+1}}\lambda_i)[i_1,i_2,\dots,i_s;\hat{J'}]\Big)$$

$$= (\lambda, [i_1,i_2,\dots,i_s;\hat{\emptyset}]) = (\lambda, [i_1,i_2,\dots,i_s]).$$

Next, we have $s_is_i(\lambda,[i]) = s_i(s_i(\lambda),[i]+\lambda_iu_i) = (s_is_i(\lambda),[i]+\lambda_iu_i-\lambda_iu_i) = (\lambda,[i])$. Clearly, $s_is_i(\lambda,u_j) = (\lambda,u_j)$ for any j. The lemma is proved.

LEMMA 1.3. — *If $(i,j)\in H$, then $s_is_js_i = s_js_is_j : E\times\widetilde{\mathcal{F}} \to E\times\widetilde{\mathcal{F}}$.*

It suffices to show that

(a) $$s_i^{s_js_i(\lambda)} s_j^{s_i(\lambda)} s_i^{\lambda}\phi = s_j^{s_is_j(\lambda)} s_i^{s_j(\lambda)} s_j^{\lambda}\phi$$

for any $\phi\in\widetilde{\mathcal{F}}$. The case where $\phi = u_j$ for some j is trivial. Hence it suffices to show that (a) holds for $\phi = [f]$ where $f\in\mathbf{I}$ is $i_1,i_2,\dots,i_s$. Note that (a) for $\phi=[f]$ implies

(a1) $$\bar{s}_i^{s_js_i(\lambda)} \bar{s}_j^{s_i(\lambda)} \bar{s}_i^{\lambda}[f] = \bar{s}_j^{s_is_j(\lambda)} \bar{s}_i^{s_j(\lambda)} \bar{s}_j^{\lambda}[f].$$

We prove (a) for $[f]$ by induction on s. Assume first that $[f]=[i]$. Both sides of (a) are in this case equal to $[i]+(\lambda_i+\lambda_j)u_i$. The same argument applies to $[f]=[j]$. If $s\le 2$ and $[f]$ is not $[i]$ or $[j]$, then (a) is obviously true for $\phi=[f]$. We now assume that $s\ge 3$.

Assume first that the first three entries of f are not of the form kik or ljl. Let $f'\in\mathbf{I}$ be $i_2,i_3,\dots,i_s$. We have $[f]=[i_1,i_2][f']$ and from the definition we see that

$$s_i^{s_js_i(\lambda)} s_j^{s_i(\lambda)} s_i^{\lambda}[f] = [i_1,i_2]\bar{s}_i^{s_js_i(\lambda)} \bar{s}_j^{s_i(\lambda)} \bar{s}_i^{\lambda}[f'],$$
$$s_j^{s_is_j(\lambda)} s_i^{s_j(\lambda)} s_j^{\lambda}[f] = [i_1,i_2]\bar{s}_j^{s_is_j(\lambda)} \bar{s}_i^{s_j(\lambda)} \bar{s}_j^{\lambda}[f'].$$

By the induction hypothesis, (a1) holds for $[f']$; hence (a) holds for $\phi=[f]$. Thus, we may assume that the first three entries of f are kik or ljl. Since i,j play a symmetrical role, we may assume that the first three entries are kik. Let u be the largest integer ≥ 3 such that $f_1=(i_1,i_2,\dots,i_u)$ is of the form $jijij\dots i$ (so u is even) or of the form $kikik\dots ik$ where k may or may not be j (so u is odd). If $u<s$, we have $[f]=[f_1][f_2]$ where $f_2\in\mathbf{I}$ is $i_u,i_{u+1},\dots,i_s$ and from the definitions we have

$$s_i^{s_js_i(\lambda)} s_j^{s_i(\lambda)} s_i^{\lambda}[f] = (\bar{s}_i^{s_js_i(\lambda)} \bar{s}_j^{s_i(\lambda)} \bar{s}_i^{\lambda}[f_1])(\bar{s}_i^{s_js_i(\lambda)} \bar{s}_j^{s_i(\lambda)} \bar{s}_i^{\lambda}[f_2]),$$
$$s_j^{s_is_j(\lambda)} s_i^{s_j(\lambda)} s_j^{\lambda}[f] = (\bar{s}_j^{s_is_j(\lambda)} \bar{s}_i^{s_j(\lambda)} \bar{s}_j^{\lambda}[f_1])(\bar{s}_j^{s_is_j(\lambda)} \bar{s}_i^{s_j(\lambda)} \bar{s}_j^{\lambda}[f_2]).$$

Since the induction hypothesis is applicable to f_1 and f_2, we see that (a) holds for f. Thus, we may assume that $u = s$. We must consider three cases:

(b) $f = f_u$ is $kik \cdots ik$ where $k \neq j$ and i appears u times;

(c) $f = f_u$ is $jij \cdots ji$ where i appears $u+1$ times;

(d) $f = f_u$ is $jij \cdots ij$ where i appears u times.

Assume that $f = f_u$ is as in (b) and $u \geq 1$. We must show

$$\sum_{u',u'';0\leq u''\leq u'\leq u} \binom{u}{u'}\binom{u'}{u''}(-\varepsilon_{ij}\lambda_j)^{u'-u''}(-\varepsilon_{ij}\lambda_i)^{u-u'} f_{u''} = \sum_{u'';0\leq u''\leq u} \binom{u}{u''}(-\varepsilon_{ij}(\lambda_i+\lambda_j))^{u-u''} f_{u''}$$

or that

$$\sum_{u';u''\leq u'\leq u} \binom{u-u''}{u-u'}(-\varepsilon_{ij}\lambda_j)^{u'-u''}(-\varepsilon_{ij}\lambda_i)^{u-u'} = (-\varepsilon_{ij}(\lambda_i+\lambda_j))^{u-u''}$$

which is clear.

Assume that $f = f_u$ is as in (c) and $u \geq 1$. We must show that $A = B$ where

$$A = \sum_{\substack{u_1,u_2,u_3\\ 0\leq u_3\leq u_2\leq u_1\leq u}} \binom{u}{u_1}\binom{u_1}{u_2}\binom{u_2}{u_3}(-\varepsilon_{ij}\lambda_j)^{u_2-u_3}(-\varepsilon_{ji}(\lambda_i+\lambda_j))^{u_1-u_2} \times(-\varepsilon_{ij}\lambda_i)^{u-u_1} f_{u_3},$$

$$B = \sum_{\substack{u_1,u_2,u_3\\ 0\leq u_3\leq u_2\leq u_1\leq u}} \binom{u}{u_1}\binom{u_1}{u_2}\binom{u_2}{u_3}(-\varepsilon_{ji}\lambda_i)^{u_2-u_3}(-\varepsilon_{ij}(\lambda_i+\lambda_j))^{u_1-u_2} \times(-\varepsilon_{ji}\lambda_j)^{u-u_1} f_{u_3}.$$

We have

$$\begin{aligned} A &= \sum_{u_3\in[0,u]} \sum_{\substack{a,b,c\\ a+b+c=u-u_3}} \frac{u!}{a!b!c!u_3!}(-\varepsilon_{ij}\lambda_j)^a(-\varepsilon_{ji}(\lambda_i+\lambda_j))^b(-\varepsilon_{ij}\lambda_i)^c f_{u_3} \\ &= \sum_{u_3\in[0,u]} \sum_{b\in[0,u-u_3]} \frac{u!}{(u-u_3-b)!b!u_3!}(-\varepsilon_{ij}(\lambda_i+\lambda_j))^{u-u_3-b} \times (-\varepsilon_{ji}(\lambda_i+\lambda_j))^b f_{u_3} \\ &= \sum_{u_3\in[0,u]} \sum_{b\in[0,u-u_3]} \frac{u!}{(u-u_3-b)!b!u_3!}(-1)^b(-\varepsilon_{ij}(\lambda_i+\lambda_j))^{u-u_3} f_{u_3} \\ &= \sum_{u_3\in[0,u]} \frac{u!}{u_3!}\delta_{u,u_3}(-\varepsilon_{ij}(\lambda_i+\lambda_j))^{u-u_3} f_{u_3} = f_0, \end{aligned}$$

$$B = \sum_{u_3\in[0,u]} \sum_{\substack{a,b,c\\ a+b+c=u-u_3}} \frac{u!}{a!b!c!u_3!}(-\varepsilon_{ji}\lambda_i)^a(-\varepsilon_{ij}(\lambda_i+\lambda_j))^b(-\varepsilon_{ji}\lambda_j)^c f_{u_3}$$
$$= \sum_{u_3\in[0,u]} \sum_{b\in[0,u-u_3]} \frac{u!}{(u-u_3-b)!b!u_3!}(-\varepsilon_{ji}(\lambda_i+\lambda_j))^{u-u_3-b} \times (-\varepsilon_{ij}(\lambda_i+\lambda_j))^b f_{u_3}$$
$$= \sum_{u_3\in[0,u]} \sum_{b\in[0,u-u_3]} \frac{u!}{(u-u_3-b)!b!u_3!}(-1)^b(-\varepsilon_{ji}(\lambda_i+\lambda_j))^{u-u_3} f_{u_3}$$
$$= \sum_{u_3\in[0,u]} \frac{u!}{u_3!}\delta_{u,u_3}(-\varepsilon_{ji}(\lambda_i+\lambda_j))^{u-u_3} f_{u_3} = f_0.$$

Thus, $A = B$ as desired. Next we assume that $f = f_u$ is as in (d) and $u \geq 1$.We must show that $A = B$ where

$$A = \sum_{\substack{u_1,u_2,u_3\\ 0\le u_3\le u_2\\ 1\le u_2\le u_1\le u}} \binom{u}{u_1}\binom{u_1}{u_2}\binom{u_2}{u_3}(-\varepsilon_{ij}\lambda_j)^{u_2-u_3}(-\varepsilon_{ji}(\lambda_i+\lambda_j))^{u_1-u_2} \times(-\varepsilon_{ij}\lambda_i)^{u-u_1} f_{u_3} + (-\varepsilon_{ij}\lambda_i)^u(f_0+(\lambda_i+\lambda_j)u_j),$$

$$B = \sum_{\substack{u_1,u_2,u_3\\ 1\le u_3\le u_2\le u_1\le u}} \binom{u}{u_1}\binom{u_1}{u_2}\binom{u_2-1}{u_3-1}(-\varepsilon_{ji}\lambda_i)^{u_2-u_3}(-\varepsilon_{ij}(\lambda_i+\lambda_j))^{u_1-u_2}$$
$$\times(-\varepsilon_{ji}\lambda_j)^{u-u_1} f_{u_3} + \sum_{\substack{u_1\\ 1\le u_1\le u}} \binom{u-1}{u_1-1}(-\varepsilon_{ij}(\lambda_i+\lambda_j))^{u_1}(-\varepsilon_{ji}\lambda_j)^{u-u_1}(f_0+\lambda_i u_j).$$

We have $A = \sum_{u_3\in[0,u]} A_{u_3} f_{u_3} + \alpha u_j$, $B = \sum_{u_3\in[0,u]} B_{u_3} f_{u_3} + \beta u_j$, where

$$A_{u_3} = \sum_{\substack{a,b,c\\ a+b+c=u-u_3}} \frac{u!}{a!b!c!u_3!}\,\frac{u_3+a}{u_3+a+b}(-\varepsilon_{ij}\lambda_j)^a(-\varepsilon_{ji}(\lambda_i+\lambda_j))^b(-\varepsilon_{ij}\lambda_i)^c,$$

$$B_{u_3} = \sum_{\substack{a,b,c\\ a+b+c=u-u_3}} \frac{(u-1)!}{a!b!c!(u_3-1)!}\,\frac{u_3+a+b}{u_3+a}(-\varepsilon_{ji}\lambda_i)^a(-\varepsilon_{ij}(\lambda_i+\lambda_j))^b \times(-\varepsilon_{ji}\lambda_j)^c$$

for $u_3 \in [1,u]$,

$$A_0 = \sum_{\substack{a,b,c\\ a+b+c=u\\ a\ge 1}} \frac{u!}{a!b!c!}\,\frac{a}{a+b}(-\varepsilon_{ij}\lambda_j)^a(-\varepsilon_{ji}(\lambda_i+\lambda_j))^b(-\varepsilon_{ij}\lambda_i)^c + (-\varepsilon_{ij}\lambda_i)^u,$$

$$B_0 = (-\varepsilon_{ij}(\lambda_i+\lambda_j))(-\varepsilon_{ij}\lambda_i)^{u-1},$$

$$\alpha = (-\varepsilon_{ij}\lambda_i)^u(\lambda_i + \lambda_j),$$
$$\beta = \sum_{u_1; 1 \le u_1 \le u} \binom{u-1}{u_1 - 1}(-\varepsilon_{ij}(\lambda_i + \lambda_j))^{u_1}(-\varepsilon_{ji}\lambda_j)^{u-u_1}\lambda_i.$$

It is enough to show that

(e) $$A_{u_3} = B_{u_3}$$

for $u_3 \in [0, u]$. (The equality $\alpha = \beta$ is obvious.) We have

$$A_0 = (-\varepsilon_{ij}\lambda_j) \sum_{\substack{a,b,c \\ a+b+c=u \\ a \ge 1}} \frac{u!}{(a-1)!b!c!}\frac{1}{a+b}(-\varepsilon_{ij}\lambda_j)^{a-1}$$
$$\times (-\varepsilon_{ji}(\lambda_i + \lambda_j))^b(-\varepsilon_{ij}\lambda_i)^c + (-\varepsilon_{ij}\lambda_i)^u$$
$$= (-\varepsilon_{ij}\lambda_j) \sum_{c \in [0,u-1]} \frac{u!}{(u-c)!c!}(-\varepsilon_{ji}\lambda_i)^{u-c-1}(-\varepsilon_{ij}\lambda_i)^c + (-\varepsilon_{ij}\lambda_i)^u$$
$$= (-\varepsilon_{ij}\lambda_j) \sum_{c \in [0,u-1]} \frac{u!}{(u-c)!c!}(-\varepsilon_{ji}\lambda_i)^{u-1}(-1)^c + (-\varepsilon_{ij}\lambda_i)^u$$
$$= (-\varepsilon_{ij}\lambda_j)(-\varepsilon_{ji}\lambda_i)^{u-1}(-1)^{u-1} + (-\varepsilon_{ij}\lambda_i)^u$$
$$= (-\varepsilon_{ij}\lambda_i)^{u-1}(-\varepsilon_{ij}(\lambda_i + \lambda_j)) = B_0.$$

This verifies (e) for $u_3 = 0$. Assume now that $u_3 \in [1, u]$. The identities

(f) $$\sum_{p_1,p_2; p_1+p_2=p} \frac{(-1)^{p_1}}{p_1!p_2!(x-p_2)} = \frac{1}{x(x-1)(x-2)\cdots(x-p)},$$

(h) $$\sum_{p_1,p_2; p_1+p_2=p} \frac{(-1)^{p_1}(x+p_2)}{p_1!p_2!} = \delta_{p,0}x + \delta_{p,1},$$

for $p \in \mathbf{N}$, are easily verified. We set $X = -\varepsilon_{ij}\lambda_j, Y = -\varepsilon_{ij}\lambda_i$. Then A_{u_3} equals

$$\sum_{\substack{a,b_1,b_2,c \\ a+b_1+b_2+c=u-u_3}} \frac{u!}{a!b_1!b_2!c!u_3!} \frac{u_3+a}{u_3+a+b_1+b_2}(-1)^{b_1+b_2}X^aX^{b_1}Y^{b_2}Y^c$$
$$= \sum_{\substack{a,b_1,b_2,c \\ a+b_1+b_2+c=u-u_3}} \frac{u!}{a!b_1!b_2!c!u_3!}\frac{u_3+a}{u_3+a+b_1+b_2}(-1)^{b_1+b_2}X^{a+b_1}Y^{c+b_2}$$
$$= \sum_{\substack{\alpha,\beta \\ \alpha+\beta=u-u_3}} \frac{u!}{u_3!} \sum_{a+b_1=\alpha} \frac{u_3+a}{a!b_1!}(-1)^{b_1} \sum_{c+b_2=\beta} \frac{(-1)^{b_2}}{b_2!c!(u-c)}X^\alpha Y^\beta$$
$$= \sum_{\substack{\alpha,\beta \\ \alpha+\beta=u-u_3}} \frac{u!}{u_3!}(u_3\delta_{\alpha,0} + \delta_{\alpha,1})\frac{1}{u(u-1)(u-2)\cdots(u-\beta)}X^\alpha Y^\beta$$

$$= \frac{u!}{u_3!}\frac{u_3}{u(u-1)(u-2)\cdots u_3}Y^{u-u_3} + \frac{u!}{u_3!}\frac{1}{u(u-1)(u-2)\cdots(u_3+1)}XY^{u-u_3-1}$$

$$= Y^{u-u_3} + XY^{u-u_3-1} = (X+Y)Y^{u-u_3-1},$$

and B_{u_3} equals

$$\sum_{\substack{a,b_1,b_2,c\\ a+b_1+b_2+c=u-u_3}} \frac{(u-1)!}{a!b_1!b_2!c!(u_3-1)!}\frac{u_3+a+b_1+b_2}{u_3+a}(-Y)^aY^{b_1}X^{b_2}(-X)^c$$

$$= \sum_{\substack{a,b_1,b_2,c\\ a+b_1+b_2+c=u-u_3}} \frac{(u-1)!}{a!b_1!b_2!c!(u_3-1)!}\frac{u_3+a+b_1+b_2}{u_3+a}(-1)^{a+c}Y^{a+b_1}X^{b_2+c}$$

$$= \sum_{\substack{\alpha,\beta\\ \alpha+\beta=u-u_3}} \frac{(u-1)!}{(u_3-1)!}\sum_{a+b_1=\alpha}\frac{(-1)^a}{a!b_1!(u_3+\alpha-b_1)} \times \sum_{c+b_2=\beta}\frac{(-1)^c(u_3+\alpha+b_2)}{c!b_2!}Y^\alpha X^\beta$$

$$= \sum_{\substack{\alpha,\beta\\ \alpha+\beta=u-u_3}} \frac{(u-1)!}{(u_3-1)!}\frac{1}{(u_3+\alpha)(u_3+\alpha-1)\cdots u_3}(\delta_{\beta,0}(u_3+\alpha)+\delta_{\beta,1})Y^\alpha X^\beta$$

$$= Y^{u-u_3} + XY^{u-u_3-1} = (X+Y)Y^{u-u_3-1}.$$

Thus, $A_{u_3} = B_{u_3}$. The lemma is proved.

LEMMA 1.4. — *If $i \neq j$, $(i,j) \notin H$, then $s_is_j = s_js_i : E \times \widetilde{\mathcal{F}} \to E \times \widetilde{\mathcal{F}}$.*

As in the proof of 1.3 it is enough to show that $s_j^{s_i(\lambda)}s_i^\lambda[f] = s_i^{s_j(\lambda)}s_j^\lambda[f]$ for $f \in \mathbf{I}$ of the form $kik\cdots ik$ where i appears u times. This follows immediately from the definitions.

1.5. From Lemmas 1.2, 1.3, 1.4 we see that there is a unique W-action on $E \times \widetilde{\mathcal{F}}$ in which the generators s_i acts by 1.1(a). This W-action is not a linear one.

Now 1.2, 1.3, 1.4 remain true if the s_i^λ are replaced by $\hat{s}_i^\lambda$; these new statements are obtained from 1.2, 1.3, 1.4 with u_i, ε_{ij} replaced by $-u_i$, $-\varepsilon_{ij}$. Hence there is a unique W-action on $E \times \widetilde{\mathcal{F}}$ in which s_i acts by 1.1 (b).

2. The set $Z_{\mathbf{D}}$.

2.1. Let $\mathcal{C}^0$ be the category whose objects are I-graded $\mathbf{C}$-vector spaces $\mathbf{V} = \oplus_{i \in I} \mathbf{V}_i$ with $\dim \mathbf{V}_i < \infty$ for all i. For $\mathbf{V} \in \mathcal{C}^0$, we set $G_{\mathbf{V}} = \prod_i GL(\mathbf{V}_i)$.

We fix $\mathbf{D} \in \mathcal{C}^0$. Let $\widetilde{\mathcal{F}}_{\mathbf{D}}$ be the vector space of all linear maps $\widetilde{\mathcal{F}} \to \mathrm{End}(\mathbf{D})$; here $\mathrm{End}(\mathbf{D})$ is understood in the ungraded sense.

For $i \in I$ and $\lambda \in E$ we define a map $\widetilde{\mathcal{F}}_{\mathbf{D}} \to \widetilde{\mathcal{F}}_{\mathbf{D}}$ by associating to $\pi : \widetilde{\mathcal{F}} \to \mathrm{End}(\mathbf{D})$ the composition $\widetilde{\mathcal{F}} \xrightarrow{s_i^\lambda} \widetilde{\mathcal{F}} \xrightarrow{\pi} \mathrm{End}(\mathbf{D})$. This map $\widetilde{\mathcal{F}}_{\mathbf{D}} \to \widetilde{\mathcal{F}}_{\mathbf{D}}$ is denoted again by s_i^λ. We now define $s_i : E \times \widetilde{\mathcal{F}}_{\mathbf{D}} \to E \times \widetilde{\mathcal{F}}_{\mathbf{D}}$ by $s_i(\lambda, \pi) = (s_i(\lambda), s_i^\lambda(\pi))$. Since $s_i^\lambda = (\hat{s}_i^\lambda)^{-1}$, from 1.5 it follows that there is a unique action of W on $E \times \widetilde{\mathcal{F}}_{\mathbf{D}}$ such that for any $i \in I$, $s_i \in W$ acts in the way just described. Following [L4, 2.4], we define a subset $Z_{\mathbf{D}}$ of $E \times \widetilde{\mathcal{F}}_{\mathbf{D}}$ as follows. An element $(\lambda, \pi) \in E \times \widetilde{\mathcal{F}}_{\mathbf{D}}$ is said to be in $Z_{\mathbf{D}}$ if it satisfies conditions (a), (b), (c) below. (For $\phi \in \widetilde{\mathcal{F}}$ we write π_ϕ instead of $\pi(\phi) : \mathbf{D} \to \mathbf{D}$).

(a) If $f \in \mathbf{I}$ is $i_1, \dots, i_s$, then $\pi_{[f]}$ maps $\mathbf{D}_{i_s}$ into $\mathbf{D}_{i_1}$ and maps $\mathbf{D}_j$ to 0, for $j \neq i_s$.

(b) For any $i \in I$, π_{u_i} is the identity map on $\mathbf{D}_i$ and is zero on $\mathbf{D}_j$, for $j \neq i$.

(c) For any $f, f' \in \mathbf{I}$ such that f ends and f' begins with the same $i \in I$, we have

$$\pi_{[f]}\pi_{[f']} = \sum_k \varepsilon_{ik} \pi_{[f][iki][f']} - \lambda_i \pi_{[f][f']}$$

where k runs over the set of vertices such that $(i,k) \in H$.

PROPOSITION 2.2. — *$Z_{\mathbf{D}}$ is a W-stable subset of $E \times \widetilde{\mathcal{F}}_{\mathbf{D}}$.*

Let $(\lambda, \pi) \in Z_{\mathbf{D}}$. Let $i \in I$ and let $(\lambda', \pi') = s_i(\lambda, \pi) \in E \times \widetilde{\mathcal{F}}_{\mathbf{D}}$. It is enough to show that $(\lambda', \pi') \in Z_{\mathbf{D}}$. It is clear that (λ', π') satisfies conditions 2.1(a),(b). To verify condition 2.1(c), we consider nine cases. In the other cases, the result is trivial. In the following formulas, an expression like

$$\pi_{[**ui]}\pi_{[iu**]} - \sum_j \varepsilon_{ij} \pi_{[**uijiu**]} - 2\varepsilon_{iu}(\varepsilon_{iu}\lambda'_i)\pi_{[**uiu**]}$$
$$-\varepsilon_{iu}(\varepsilon_{iu}\lambda'_i)^2 \pi_{[**u**]} + \lambda'_i \pi_{[**uiu**]} + \lambda'_i(\varepsilon_{iu}\lambda'_i)\pi_{[**u**]}$$

should be interpreted as follows: each of

$$\pi_{[**ui]}\pi_{[iu**]}, \pi_{[**uijiu**]}, \pi_{[**u**]}, \pi_{[**uiu**]}$$

stands for a linear combination of $\pi_{[\cdots ui]}\pi_{[iu\cdots]}, \pi_{[\cdots uijiu\cdots]}, \pi_{[\cdots u\cdots]}, \pi_{[\cdots uiu\cdots]}$ over the same index set, with the same coefficients.

Case 1.

$$\begin{aligned}
&\pi'_{[i]}\pi'_{[i]} - \sum_j \varepsilon_{ij}\pi'_{[iji]} + \lambda'_i\pi'_{[i]} \\
&\quad = (\pi_{[i]} - \lambda'_i\pi_{u_i})(\pi_{[i]} - \lambda'_i\pi_{u_i}) - \sum_j \varepsilon_{ij}\pi_{[iji]} + \lambda'_i(\pi_{[i]} - \lambda'_i\pi_{u_i}) \\
&\quad = -\lambda_i\pi_{[i]} - \lambda'_i\pi_{[i]} = 0.
\end{aligned}$$

Case 2. Assume that $(u, i) \in H$.

$$\begin{aligned}
&\pi'_{[u]}\pi'_{[u]} - \sum_j \varepsilon_{uj}\pi'_{[uju]} + \lambda'_u\pi'_{[u]} \\
&\quad = \pi_{[u]}\pi_{[u]} - \sum_j \varepsilon_{uj}\pi_{[uju]} - \varepsilon_{ui}(\varepsilon_{iu}\lambda'_i)\pi_{[u]} + \lambda'_u\pi_{[u]} \\
&\quad = -\lambda_u\pi_{[u]} + \lambda'_i\pi_{[u]} + \lambda'_u\pi_{[u]} = 0.
\end{aligned}$$

Case 3.

$$\begin{aligned}
&\pi'_{[i]}\pi'_{[ik\cdots]} - \sum_j \varepsilon_{ij}\pi'_{[ijik\cdots]} + \lambda'_i\pi'_{[ik\cdots]} \\
&\quad = (\pi_{[i]} - \lambda'_i\pi_{u_i})\pi_{[ik**]} - \sum_j \varepsilon_{ij}\pi_{[ijik**]} - \varepsilon_{ik}(\varepsilon_{ik}\lambda'_i)\pi_{[ik**]} + \lambda'_i\pi_{[ik**]} \\
&\quad = -\lambda_i\pi_{[ik**]} - \lambda'_i\pi_{[ik**]} = 0.
\end{aligned}$$

Case 4. Assume that $(u, i) \in H$.

$$\begin{aligned}
&\pi'_{[u]}\pi'_{[u\cdots]} - \sum_j \varepsilon_{uj}\pi'_{[uju\cdots]} + \lambda'_u\pi'_{[u\cdots]} \\
&\quad = \pi_{[u]}\pi_{[u**]} - \sum_j \varepsilon_{uj}\pi_{[uju**]} - \varepsilon_{ui}\varepsilon_{iu}\lambda'_i\pi_{[u**]} + \lambda'_u\pi_{[u**]} \\
&\quad = -\lambda_u\pi_{[u**]} + \lambda'_i\pi_{[u**]} + \lambda'_u\pi_{[u**]} = 0.
\end{aligned}$$

Case 5.

$$\begin{aligned}
&\pi'_{[\cdots ki]}\pi'_{[i]} - \sum_j \varepsilon_{ij}\pi'_{[\cdots kiji]} + \lambda'_i\pi'_{[\cdots ki]} \\
&\quad = \pi_{**ki}(\pi_{[i]} - \lambda'_i\pi_{u_i}) - \sum_j \varepsilon_{ij}\pi_{[**kiji]} - \varepsilon_{ik}(\varepsilon_{ik}\lambda'_i)\pi_{[**ki]} + \lambda'_i\pi_{[**ki]} \\
&\quad = -\lambda_i\pi_{[**ki]} - \lambda'_i\pi_{[**ki]} = 0.
\end{aligned}$$

Case 6. Assume that $(u,i)\in H$.

$$\begin{aligned}&\pi'_{[\cdots u]}\pi'_{[u]}-\sum_j \varepsilon_{uj}\pi'_{[\cdots uju]}+\lambda'_u\pi'_{[\cdots u]}\\&=\pi_{**u}\pi_{[u]}-\sum_j \varepsilon_{uj}\pi_{[**uju]}-\varepsilon_{ui}(\varepsilon_{iu}\lambda'_i)\pi_{[**u]}+\lambda'_u\pi_{[**u]}\\&=-\lambda_u\pi_{[**u]}+\lambda'_i\pi_{[**u]}+\lambda'_u\pi_{[**u]}=0.\end{aligned}$$

Case 7.

$$\begin{aligned}&\pi'_{[\cdots ui]}\pi'_{[iu\cdots]}-\sum_j \varepsilon_{ij}\pi'_{[\cdots uijiu\cdots]}+\lambda'_i\pi'_{[\cdots uiu\cdots]}\\&=\pi_{[**ui]}\pi_{[iu**]}-\sum_j \varepsilon_{ij}\pi_{[**uijiu**]}-2\varepsilon_{iu}(\varepsilon_{iu}\lambda'_i)\pi_{[**uiu**]}\\&\qquad-\varepsilon_{iu}(\varepsilon_{iu}\lambda'_i)^2\pi_{[**u**]}+\lambda'_i\pi_{[**uiu**]}+\lambda'_i(\varepsilon_{iu}\lambda'_i)\pi_{[**u**]}\\&=-\lambda_i\pi_{[**uiu**]}-\lambda'_i\pi_{[**uiu**]}=0.\end{aligned}$$

Case 8. Assume that $u\neq v$.

$$\begin{aligned}&\pi'_{[\cdots ui]}\pi'_{[iv\cdots]}-\sum_j \varepsilon_{ij}\pi'_{[\cdots uijiv\cdots]}+\lambda'_i\pi'_{[\cdots uiv\cdots]}\\&=\pi_{[**ui]}\pi_{[iv**]}-\sum_j \varepsilon_{ij}\pi_{[**uijiv**]}-\varepsilon_{iu}(\varepsilon_{iu}\lambda'_i)\pi_{[**uiv**]}\\&\qquad-\varepsilon_{iv}(\varepsilon_{iv}\lambda'_i)\pi_{[**uiv**]}+\lambda'_i\pi_{[**uiv**]}\\&=-\lambda_i\pi_{[**uiv**]}-\lambda'_i\pi_{[**uiv**]}=0.\end{aligned}$$

Case 9. Assume that $(u,i)\in H$.

$$\begin{aligned}&\pi'_{[\cdots u]}\pi'_{[u\cdots]}-\sum_j \varepsilon_{uj}\pi'_{[\cdots uju\cdots]}+\lambda'_u\pi'_{[\cdots u\cdots]}\\&=\pi_{[**u]}\pi_{[u**]}-\sum_j \varepsilon_{uj}\pi_{[**uju**]}-\varepsilon_{ui}(\varepsilon_{iu}\lambda'_i)\pi_{[**u**]}+\lambda'_u\pi_{[**u**]}\\&=-\lambda_u\pi_{[**u**]}+\lambda'_i\pi_{[**u**]}+\lambda'_u\pi_{[**u**]}=0.\end{aligned}$$

The proposition is proved.

2.3. Consider the action of $\mathbf{C}^*$ on $E\times\widetilde{\mathcal{F}}_{\mathbf{D}}$ given by

$$t:(\lambda,\pi)\mapsto(t^2\lambda,\pi')$$

where $\pi'_{[f]}=t^{s+1}\pi_{[f]}$ for $f\in\mathbf{I}$ of form $i_1,\ldots,i_s$ and $\pi'_{u_i}=\pi_{u_i}$ for $i\in I$. It is easy to check that this restricts to an action of $\mathbf{C}^*$ on $Z_{\mathbf{D}}$ which commutes with the W-action.

2.4. Consider the action of $G_{\mathbf{D}}$ on $Z_{\mathbf{D}}$ given by

$$(g_i) : (\lambda, \pi) \mapsto (\lambda, \pi')$$

where $\pi'_{[f]} = g_{i_1}\pi_{[f]}g_{i_s} \subset$ for $f \in \mathbf{I}$ of form $i_1, \dots, i_s$ and $\pi'_{u_i} = \pi_{u_i}$ for $i \in I$. It is easy to check that this action commutes with the $\mathbf{C}^*$-action and with the W-action.

3. The varieties $\Lambda_{\mathbf{D},\mathbf{V}}$.

3.1. Given $\mathbf{D}, \mathbf{V} \in \mathcal{C}^0$, let $M_{\mathbf{D},\mathbf{V}}$ be the vector space consisting of all triples (x, p, q) where

$$x = (x_h)_{h \in H}, x_h \in \mathrm{Hom}(\mathbf{V}_{h'}, \mathbf{V}_{h''}),$$
$$p = (p_j)_{j \in I}, p_j \in \mathrm{Hom}(\mathbf{D}_j, \mathbf{V}_j),$$
$$q = (q_j)_{j \in I}, q_j \in \mathrm{Hom}(\mathbf{V}_j, \mathbf{D}_j).$$

Following [N1], let $\Lambda_{\mathbf{D},\mathbf{V}}$ be the affine variety consisting of all $((x, p, q), \lambda)$ in $M_{\mathbf{D},\mathbf{V}} \times E$ such that

(a) $$\sum_{h;h'=i} \varepsilon_{\bar{h}} x_{\bar{h}} x_h - p_i q_i - \lambda_i = 0 : \mathbf{V}_i \to \mathbf{V}_i$$

for all $i \in I$. For any $\lambda \in E$ let $\Lambda_{\mathbf{D},\mathbf{V},\lambda}$ be the affine variety consisting of all $(x, p, q) \in M_{\mathbf{D},\mathbf{V}}$ such that (a) holds; this may be naturally identified with the fibre at λ of the fourth projection $\Lambda_{\mathbf{D},\mathbf{V}} \to E$.

The group $G_{\mathbf{V}}$ acts on $M_{\mathbf{D},\mathbf{V}}$ in a natural way (see [L4, 1.2]); this induces an action of $G_{\mathbf{V}}$ on $\Lambda_{\mathbf{D},\mathbf{V}}$ and on $\Lambda_{\mathbf{D},\mathbf{V},\lambda}$ for any $\lambda \in E$.

3.2. In the remainder of this section we fix $i \in I$, $\mathbf{D}, \mathbf{V}, \mathbf{V}' \in \mathcal{C}^0$, $\lambda, \lambda' \in E$ such that $\lambda' = s_i(\lambda)$ and $\mathbf{V}_j = \mathbf{V}'_j$ for $j \in I - \{i\}$; $\dim \mathbf{V}_i + \dim \mathbf{V}'_i = \dim \mathbf{D}_i + \sum_{h;h'=i} \dim \mathbf{V}_{h''}$. Let $U = \mathbf{D}_i \oplus \oplus_{h;h'=i} \mathbf{V}_{h''}$. Let F be the affine variety whose points are the pairs $((x, p, q); (x', p', q')) \in M_{\mathbf{D},\mathbf{V}} \times M_{\mathbf{D},\mathbf{V}'}$ such that conditions (a)–(d2) below are satisfied:

(a) the sequence $0 \to \mathbf{V}'_i \xrightarrow{a} U \xrightarrow{b} \mathbf{V}_i \to 0$ is exact; here, $a = (q'_i, (x'_h)_{h;h'=i})$ and $b = (p_i, (\varepsilon_{\bar{h}} x_h)_{h;h''=i})$;

(b1) we have $\widetilde{b}b - a\widetilde{a} = \lambda'_i : U \to U$ where $\widetilde{a} = (p'_i, (\varepsilon_{\bar{h}} x'_h)_{h;h''=i}) : U \to \mathbf{V}'_i$ and $\widetilde{b} = (q_i, (x_h)_{h;h'=i}) : \mathbf{V}_i \to U$;

(b2) $\varepsilon_{\bar{h}}(x_{\widetilde{h}} x_h - x'_{\widetilde{h}} x'_h) = \delta_{\bar{h},\widetilde{h}} \lambda'_i : \mathbf{V}_{h'} \to \mathbf{V}_{\widetilde{h}''}$ for any $h, \widetilde{h}$ such that $h'' = i, \widetilde{h}' = i$; $q_i p_i - q'_i p'_i = \lambda'_i : \mathbf{D}_i \to \mathbf{D}_i$; $x_h p_i - x'_h p'_i = 0$ for any h such that $h' = i$; $q_i x_h - q'_i x'_h = 0$ for any h such that $h'' = i$;

(c) $x_h = x'_h$ if $h' \neq i, h'' \neq i$; $p_j = p'_j, q_j = q'_j$ if $j \neq i$;

(d1) $\sum_{h;h'=j} \varepsilon_{\bar{h}} x_{\bar{h}} x_h - p_j q_j = \lambda_j : \mathbf{V}_j \to \mathbf{V}_j$ if $j \neq i$;

(d2) $\sum_{h;h'=j} \varepsilon_{\bar{h}} x'_{\bar{h}} x'_h - p'_j q'_j = \lambda'_j : \mathbf{V}_j \to \mathbf{V}_j$if $j \neq i$.

Remarks.

(i) Conditions (b1), (b2) are equivalent.

(ii) In the presence of (b2), (c), conditions (d1), (d2) are equivalent. Indeed, let δ be the difference of the left hand sides of the equalities in (d1), (d2). We have $\delta = \sum_{h;h'=j} \varepsilon_{\bar{h}}(x_{\bar{h}} x_h - x'_{\bar{h}} x'_h) - p_j q_j + p'_j q'_j$. Using (c) we see that $\delta = \sum_{h;h'=j,h''=i} \varepsilon_{\bar{h}}(x_{\bar{h}} x_h - x'_{\bar{h}} x'_h)$ and, by (b2), this is $\lambda'_i \sharp(h; h' = j, h'' = i)$, which equals $\lambda_j - \lambda'_j$.

(iii) For a point in F we have automatically $\sum_{h;h'=i} \varepsilon_{\bar{h}} x'_{\bar{h}} x'_h - p'_i q'_i = \lambda'_i$. Indeed, since a in (a) is injective, it is enough to show that

$$(*) \qquad \sum_{h;h'=i} \varepsilon_{\bar{h}} x'_{\tilde{h}} x'_{\bar{h}} x'_h - x'_{\tilde{h}} p'_i q'_i - \lambda'_i x'_{\tilde{h}} = 0 : \mathbf{V}'_i \to \mathbf{V}_{\tilde{h}''}$$

for any $\tilde{h}$ such that $\tilde{h}' = i$ and

$$(**) \qquad \sum_{h;h'=i} \varepsilon_{\bar{h}} q'_i x'_{\bar{h}} x'_h - q'_i p'_i q'_i - \lambda'_i q'_i = 0 : \mathbf{V}'_i \to \mathbf{D}'_i.$$

By (b2), the left hand side of $(*)$ is

$$\sum_{h,h'=i} \varepsilon_{\bar{h}} x_{\tilde{h}} x_{\bar{h}} x'_h - x_{\tilde{h}} p_i q'_i$$

and this is 0, by (a). Again by (b2), the left hand side of $(**)$ is

$$\sum_{h;h'=i} \varepsilon_{\bar{h}} q_i x_{\bar{h}} x'_h - q_i p_i q'_i$$

and this is 0, by (a).

(iv) For a point in F we have automatically $\sum_{h;h'=i} \varepsilon_{\bar{h}} x_{\bar{h}} x_h - p_i q_i = \lambda_i$. Indeed, since b in (a) is surjective, it is enough to show that

$$(*) \qquad \sum_{h;h'=i} \varepsilon_{\bar{h}} x_{\bar{h}} x_h x_{\tilde{h}} - p_i q_i x_{\tilde{h}} - \lambda_i x_{\tilde{h}} = 0 : \mathbf{V}_{\tilde{h}'} \to \mathbf{V}_i$$

for any $\tilde{h}$ such that $\tilde{h}'' = i$ and

$$(**) \qquad \sum_{h,h'=i} \varepsilon_{\bar{h}} x_{\bar{h}} x_h p_i - p_i q_i p_i - \lambda_i p_i = 0 : \mathbf{D}_i \to \mathbf{V}_i.$$

By (b2), the left hand side of $(*)$ is

$$\sum_{h;h'=i} \varepsilon_{\bar{h}} x_{\bar{h}} x'_h x'_{\tilde{h}} - \lambda'_i x_{\tilde{h}} - p_i q'_i x'_{\tilde{h}} - \lambda_i x_{\tilde{h}}$$

which by (a) equals$-\lambda'_i x_{\tilde{h}} - \lambda_i x_{\tilde{h}} = 0$. Again by (b2), the left hand side of (**) is

$$\sum_{h,h'=i} \varepsilon_{\bar{h}} x_{\bar{h}} x'_h p'_i - p_i q'_i p'_i - \lambda'_i p_i - \lambda_i p_i$$

which by (a) equals $-\lambda'_i p_i - \lambda_i p_i = 0$.

3.3. From Remarks (iii), (iv) in 3.2, we see that the first (resp. second) projection is a well defined map $r : F \to \Lambda_{\mathbf{D},\mathbf{V},\lambda}$ (resp. $r' : F \to \Lambda_{\mathbf{D},\mathbf{V}',\lambda'}$).

3.4. *In the remainder of this section we assume that* $\lambda_i \neq 0$. In this case, for any $(x, p, q) \in \Lambda_{\mathbf{D},\mathbf{V},\lambda}$, the map $b : U \to \mathbf{V}_i$ (as in 3.2(a)) is surjective. Indeed, the identity map of $\mathbf{V}_i$ is equal to $-\lambda_i^{-1} \subset b\tilde{b}$ with $\tilde{b}$ as in 3.2(b1).

Similarly, for any $(x', p', q') \in \Lambda_{\mathbf{D},\mathbf{V}',\lambda'}$, the map $a : \mathbf{V}'_i \to U$ (as in 3.2(a)) is injective. Indeed, the identity map of $\mathbf{V}'_i$ is equal to $-\lambda_i'^{-1} \subset \tilde{a}a$ with $\tilde{a}$ as in 3.2(b1). The group

$$G = GL(\mathbf{V}_i) \times GL(\mathbf{V}'_i) \times \prod_{j;j\neq i} GL(\mathbf{V}_j)$$

acts naturally on $F, \Lambda_{\mathbf{D},\mathbf{V},\lambda}, \Lambda_{\mathbf{D},\mathbf{V}',\lambda'}$ compatibly with the maps r, r', so that the G action on $\Lambda_{\mathbf{D},\mathbf{V},\lambda}$ factors through the $G_{\mathbf{V}}$-action in 3.1 and the G action on $\Lambda_{\mathbf{D},\mathbf{V}',\lambda'}$ factors through the analogous $G_{\mathbf{V}'}$-action.

PROPOSITION 3.5.

(a) *r is a principal $GL(\mathbf{V}'_i)$-bundle.*

(b) *r' is a principal $GL(\mathbf{V}_i)$-bundle.*

We prove (a). We fix $x = (x, p, q) \in \Lambda_{\mathbf{D},\mathbf{V},\lambda}$. Then $r^{-1} \subset (x)$ may be identified with the set of all pairs $(a, \tilde{a})$ where

$$a = (q'_i, (x'_h)_{h;h'=i}) \in \mathrm{Hom}(\mathbf{V}'_i, U), \quad \tilde{a} = (p'_i, (\varepsilon_{\bar{h}} x'_h)_{h;h''=i}) \in \mathrm{Hom}(U, \mathbf{V}'_i)$$

are such that 3.2(a),(b1) hold. (Then 3.2(d2) holds automatically by 3.2(ii).) We show that the first projection establishes a bijection

(*c*) $\{(a, \tilde{a}) |\ 3.2\text{(a),(b1) hold}\} \xrightarrow{\sim} \{a |\ 3.2\text{(a) holds}\}.$

Let $\phi = \tilde{b}b - \lambda'_i : U \to U$ where $\tilde{b} = (q_i, (x_h)_{h;h'=i}) : \mathbf{V}_i \to U$. Assume that a satisfies 3.2(a). We must show that there is a unique linear map $\tilde{a} : U \to \mathbf{V}'_i$ such that $a\tilde{a} = \phi$, or equivalently, that the image of ϕ is contained in the image of the imbedding a (that is, in the kernel of b). Thus, it is enough to show that $b\phi = 0$, or that $b\widetilde{b}b - \lambda'_i b = 0$. It is also enough to show that $b\widetilde{b} - \lambda'_i = 0$. This is clear.

We see that $r^{-1} \subset (x)$ may be identified with the set of all $a : \mathbf{V}'_i \to U$ such that 3.2(a) holds, or equivalently (since b is surjective) with the set of all isomorphisms of $\mathbf{V}'_i$ onto $\operatorname{Ker} b$. This is clearly isomorphic to $GL(\mathbf{V}'_i)$. This proves (a).

We prove (b). We fix $(x', p', q') \in \Lambda_{\mathbf{D},\mathbf{V}',\lambda'}$. Then $r'^{-1} \subset (x')$ may be identified with the set of all pairs $(b, \widetilde{b})$ where

$$b = (p_i, (\varepsilon_{\bar{h}} x_h)_{h;h''=i}) \in \operatorname{Hom}(U, \mathbf{V}_i), \quad \widetilde{b} = (q_i, (x_h)_{h;h'=i}) \in \operatorname{Hom}(\mathbf{V}_i, U)$$

are such that 3.2(a),(b) hold. (Then 3.2(d1) holds automatically by 3.2(ii).) We show that the first projection establishes a bijection

$$\text{(d)} \qquad \{(b, \widetilde{b}) | 3.2(a), (b1) \text{ hold}\} \xrightarrow{\sim} \{b | 3.2(a) \text{holds}\}.$$

Let $\psi = a\widetilde{a} + \lambda'_i : U \to U$ where $\widetilde{a} = (p'_i, (\varepsilon_{\bar{h}} x'_h)_{h;h''=i})$. Assume that b satisfies 3.2(a). We must show that there is a unique linear map $\widetilde{b} : \mathbf{V}_i \to U$ such that $\widetilde{b}b = \psi$, or equivalently, that the kernel of b (that is, the image of a) is contained in $\operatorname{Ker} \psi$. Thus, it is enough to show that $\psi a = 0$ or that $a\widetilde{a}a + \lambda'_i a = 0$. It is enough to show that $a\widetilde{a} + \lambda'_i = 0$. This is clear. We see that $r'^{-1} \subset (x')$ may be identified with the set of all $b : U \to \mathbf{V}_i$ such that 3.2(a) holds, or equivalently (since a is injective) with the set of all isomorphisms of the cokernel of a onto $\mathbf{V}_i$. This is clearly isomorphic to $GL(\mathbf{V}_i)$. This proves (b). The proposition is proved.

COROLLARY 3.6. — *The maps r, r' in 3.3 induce bijections*

$$\text{(a)} \qquad \Lambda_{\mathbf{D},\mathbf{V},\lambda}/G_{\mathbf{V}} \xleftarrow{\sim} F/G \xrightarrow{\sim} \Lambda_{\mathbf{D},\mathbf{V}',\lambda'}/G_{\mathbf{V}'}$$

where orbit spaces are taken in the set theoretical sense, and isomorphisms of affine algebraic varieties

$$\text{(b)} \qquad \Lambda_{\mathbf{D},\mathbf{V},\lambda}//G_{\mathbf{V}} \xleftarrow{\sim} F//G \xrightarrow{\sim} \Lambda_{\mathbf{D},\mathbf{V}',\lambda'}//G_{\mathbf{V}'}$$

where the orbit spaces are taken in the algebraic geometric sense.

COROLLARY 3.7. — *Assume that $\mathbf{D} = 0$ and let*

$$d_{\mathbf{V}} = (1/2) \sum_h \dim \mathbf{V}_{h'} \dim \mathbf{V}_{h''}.$$

Then $\Lambda_{0,\mathbf{V},\lambda}$ has pure dimension $d_{\mathbf{V}}$ if and only if $\Lambda_{0,\mathbf{V}',\lambda'}$ has pure dimension $d_{\mathbf{V}'}$.

We set $\nu_j = \dim \mathbf{V}_j, \nu'_j = \dim \mathbf{V}'_j$. The two conditions above are equivalent to the condition that F has pure dimension $d_{\mathbf{V}} + \nu_i'^2$, (resp. $d_{\mathbf{V}'} + \nu_i^2$). So it is enough to prove that $d_{\mathbf{V}} - d_{\mathbf{V}'} = \nu_i^2 - \nu_i'^2$. We have

$$d_{\mathbf{V}} - d_{\mathbf{V}'} = \sum_{h;h'=i} \nu_{h''}\nu_i - \sum_{h;h'=i} \nu_{h''}\nu'_i = \sum_{h;h'=i} \nu_{h''}(\nu_i - \nu'_i)$$
$$= (\nu_i + \nu'_i)(\nu_i - \nu'_i) = \nu_i^2 - \nu_i'^2.$$

The lemma is proved.

4. The subsets $Z_{\mathbf{D},\mathbf{V}}$ and ${}_{\mathbf{V}}Z_{\mathbf{D}}$ of $Z_{\mathbf{D}}$.

4.1. *In the remainder of this paper we assume that our graph is of finite type, that is, W is a finite group.*

Let $E' = \mathrm{Hom}(E, \mathbf{C})$. We shall regard $i \in I$ as an element of E' by $i(\varpi_j) = \delta_{ij}$. For any $i \in I$, we define $\alpha_i \in E$ by $\alpha_i = 2\varpi_i - \sum_{h;h'=i} \varpi_{h''}$. Then $\{\alpha_i, i\}$ form a root datum and W is its Weyl group. For $\nu \in E'$ we define $\nu_i \in \mathbf{C}$ by $\nu = \sum_i \nu_i i$. The action of W on E induces an action of W on E', given by $s_i : \xi \mapsto s_i(\xi) = \xi - \xi(\alpha_i)i$. Let $\check{R}$ be the set of vectors in E' of the form $w(i)$ for some $i \in I, w \in W$. For $\lambda \in E$ let

$$\check{R}_\lambda = \{\check{\alpha} \in \check{R} | \check{\alpha}(\lambda) = 0\}.$$

4.2. Let E_0 be the set of all $\lambda \in E$ such that for any i we have either $\mathrm{Re}(\lambda_i) > 0$, or $\mathrm{Re}(\lambda_i) = 0$ and $\mathrm{Im}(\lambda_i) \geq 0$.

LEMMA 4.3. — *Any W-orbit in E meets E_0 in a unique point.*

This is well known.

LEMMA 4.4. — *Let $\mathbf{V} \in \mathcal{C}^0, \lambda \in E_0$. Assume that $\Lambda_{0,\mathbf{V},\lambda} \neq \emptyset$. Then for any $i \in I$ we have either $\lambda_i = 0$ or $\mathbf{V}_i = 0$.*

Let $((x_h), 0, 0) \in \Lambda_{0,\mathbf{V},\lambda}$. We have

$$\sum_i \lambda_i \dim \mathbf{V}_i = \sum_i \mathrm{Tr}(\lambda_i, \mathbf{V}_i) = \sum_i \mathrm{Tr}(\sum_{h;h'=i} \varepsilon_{\bar{h}} x_{\bar{h}} x_h, \mathbf{V}_i)$$
$$= \sum_h \varepsilon_{\bar{h}} \mathrm{Tr}(x_{\bar{h}} x_h, \mathbf{V}_{h'}).$$

In the last sum the term corresponding to $h, \bar{h}$ cancel out. Hence the sum is zero and we have

$$\sum_i \lambda_i \dim \mathbf{V}_i = 0.$$

Let $I' = \{i \in I | \mathbf{V}_i \neq 0\}$. Write $\mathrm{Re}(\lambda_i) = \lambda'_i, \mathrm{Im}(\lambda_i) = \lambda''_i$. Then $\sum_{i \in I'} \lambda_i \dim \mathbf{V}_i = 0$, hence

$$\sum_{i \in I'} \lambda'_i \dim \mathbf{V}_i = 0, \quad \sum_{i \in I'} \lambda''_i \dim \mathbf{V}_i = 0.$$

Since $\lambda'_i \in \mathbf{R}_{\geq 0}$ for all i, we deduce that for any $i \in I', \lambda'_i = 0$, hence $\lambda''_i \geq 0$. Hence the equality $\sum_{i \in I'} \lambda''_i \dim \mathbf{V}_i = 0$ implies $\lambda''_i = 0$ for all $i \in I'$. Thus, for $i \in I'$ we have $\lambda'_i = \lambda''_i = 0$ hence $\lambda_i = 0$. The lemma is proved.

PROPOSITION 4.5. — *Let $\mathbf{V} \in \mathcal{C}^0, \lambda \in E$ be such that $\Lambda_{0,\mathbf{V},\lambda} \neq \emptyset$. Then*

(a) $\Lambda_{0,\mathbf{V},\lambda}$ has pure dimension $d_{\mathbf{V}}$;

(b) $G_{\mathbf{V}}$ has a unique closed orbit in $\Lambda_{0,\mathbf{V},\lambda}$;

(c) if $\check{R}_\lambda = \emptyset$, then $\mathbf{V} = 0$.

Assume first that $\lambda \in E_0$. Let $I_0 = \{i \in I | \lambda_i = 0\}$. Clearly,$I_0 \subset \check{R}_\lambda$.

We prove (a). We may replace the datum $(I, H, \cdots)$ in 1.1 by $(I_0, H_0, \cdots)$ where $H_0 = \{h \in H | h' \in I_0, h'' \in I_0\}$. Using Lemma 4.4, we see that $\Lambda_{0,\mathbf{V},\lambda}$ may be identified with $\Lambda_{0,\mathbf{V}',0}$ defined in terms of $(I_0, H_0, \cdots)$ where $\mathbf{V}'$ is the I_0-graded vector space defined by $\mathbf{V}'_i = \mathbf{V}_i$ for $i \in I_0$. By [L3, 12.3], $\Lambda_{0,\mathbf{V}',0}$ has pure dimension $d_{\mathbf{V}'}$. Since $\mathbf{V}_i = 0$ for $i \notin I_0$, we have $d_{\mathbf{V}} = d_{\mathbf{V}'}$. This proves (a).

We prove (b). As in the proof of (a), we are reduced to the case where $\lambda = 0$. In that case the result is contained in [L4, 5.9].

We prove (c). Assume that $\check{R}_\lambda = \emptyset$ and $\mathbf{V}_i \neq 0$. By Lemma 4.4, we have $\lambda_i = 0$ hence $i \in \check{R}_\lambda$, a contradiction.

This completes the proof of the proposition under the assumption that $\lambda \in E_0$. We now consider the general case.

For any $\lambda \in E$, let $r = r_\lambda$ be the smallest integer ≥ 0 such that there exists a sequence $\lambda = \lambda^0, \lambda^1, \ldots, \lambda^r$ in E and a sequence $i_1, i_2, \ldots, i_r$ in I with the following properties:

$$\lambda^r \in E_0, \quad \lambda^1 = s_{i_1}(\lambda^0), \lambda^2 = s_{i_2}(\lambda^1), \ldots, \lambda^r = s_{i_r}(\lambda^{r-1}), \quad \lambda^j \neq \lambda^{j+1}$$

for $j = 0, 1, \ldots, r-1$. Note that r_λ is well defined by 4.3. We prove the proposition for $\lambda \in E$ (and any $\mathbf{V}$ such that $\Lambda_{0,\mathbf{V},\lambda} \neq \emptyset$) by induction on r_λ. If $r_\lambda = 0$, the result is clear by the first part of the proof. Assume now that $r_\lambda \geq 1$. By definition, we can find $i \in I$ such that $\lambda' = s_i(\lambda) \neq \lambda$ and $r_{\lambda'} = r_\lambda - 1$. Since $\lambda' \neq \lambda$, we have $\lambda_i \neq 0$. We define U, $b : \mathbf{V}_i \to U$ as in

3.2 (with $\mathbf{D} = 0$) in terms of some $(x, 0, 0) \in \Lambda_{0,\mathbf{V},\lambda}$. As in 3.4, from $\lambda_i \neq 0$ we deduce that b is surjective. Hence $\dim \mathbf{V}_i \geq \dim U = \sum_{h;h'=i} \dim \mathbf{V}_{h''}$. Hence there exists $\mathbf{V}' \in \mathcal{C}^0$ such that

$$\mathbf{V}_j = \mathbf{V}'_j \text{ for } j \in I - \{i\};$$
$$\dim \mathbf{V}_i + \dim \mathbf{V}'_i = \sum_{h;h'=i} \dim \mathbf{V}_{h''}.$$

By 3.6(a), $\Lambda_{0,\mathbf{V}',\lambda'} \neq \emptyset$. By the induction hypothesis, the proposition holds for $(\mathbf{V}', \lambda')$. Using 3.7, we see that (a) holds for $(\mathbf{V}, \lambda)$. Using 3.6(b), we see that (b) holds for $(\mathbf{V}, \lambda)$. Finally, assume that $\check{R}_\lambda = \emptyset$. Then $\check{R}_{\lambda'} = s_i(\check{R}_\lambda) = \emptyset$. Hence $\mathbf{V}' = 0$, by the induction hypothesis. Then the formulas above relating $\mathbf{V}_j, \mathbf{V}'_j$ show that $\mathbf{V}_j = 0$ for all j. The proposition is proved.

4.6. Let $\mathbf{D} \in \mathcal{C}^0$. For any $f \in \mathbf{I}$ of form $i_1, i_2, \ldots, i_s$ and any linear form $\chi : \mathrm{Hom}(\mathbf{D}_{i_s}, \mathbf{D}_{i_1}) \to \mathbf{C}$, we define a function $b_{f,\chi} : Z_\mathbf{D} \to \mathbf{C}$ by $b_{f,\chi}(\lambda, \pi) = \chi(\pi_{[f]})$. For any $i \in I$, let $\xi_i : Z_\mathbf{D} \to \mathbf{C}$ be defined by $\xi_i(\lambda, \pi) = \lambda_i$. Let B_1 be the $\mathbf{C}$-algebra with 1 of functions $Z_\mathbf{D} \to \mathbf{C}$ generated by the functions $b_{f,\chi}$ for various f, χ as above and by the functions ξ_i with $i \in I$. An argument almost identical to that in [L4, 5.3] shows that B_1 is a finitely generated algebra and that $Z_\mathbf{D}$ is naturally in bijection with the set of algebra homomorphisms $B_1 \to \mathbf{C}$. Thus, $Z_\mathbf{D}$ has a natural structure of affine variety.

Now let $\mathbf{V} \in \mathcal{C}^0$. As in [L4, 2.12], we define a map $\vartheta' : \Lambda_{\mathbf{D},\mathbf{V}} \to Z_\mathbf{D}$ by $(x, p, q, \lambda) \mapsto (\lambda, \pi)$ where $\pi \in \widetilde{\mathcal{F}}_\mathbf{D}$ is given by

$$\pi_{[f]} = q_{i_1} x_{i_1,i_2} x_{i_2,i_3} \cdots x_{i_{s-1},i_s} p_{i_s} : \mathbf{D}_{i_s} \to \mathbf{D}_{i_1}$$

for any $f \in \mathbf{I}$ of form $i_1, i_2, \ldots, i_s$ with $s \geq 2$,

$$\pi_{[j]} = q_j p_j : \mathbf{D}_j \to \mathbf{D}_j$$

for any $j \in I$ and π_{u_i} is as in 2.1(b) for any $i \in I$.

The map $\vartheta' : \Lambda_{\mathbf{D},\mathbf{V}} \to Z_\mathbf{D}$ is easily seen to be a morphism of algebraic varieties. Since this map is constant on the orbits of $G_\mathbf{V}$ on $\Lambda_{\mathbf{D},\mathbf{V}}$, it induces a morphism $\vartheta : \Lambda_{\mathbf{D},\mathbf{V}}//G_\mathbf{V} \to Z_\mathbf{D}$ of algebraic varieties.

THEOREM 4.7. — *ϑ is a finite, injective morphism. In particular, it is a homeomorphism onto its image (both in the Zariski and ordinary topology).*

The finiteness of ϑ is proved as in [L4, 5.8]. The injectivity is proved in [L4, 5.10] modulo the statement [L4, 5.9] which at the time of writing

[L4] was only known for $\lambda = 0$ but is now known without restriction, by 4.5(b).

4.8. Let $\Lambda^s_{\mathbf{D},\mathbf{V}}$ be the open subvariety of $\Lambda_{\mathbf{D},\mathbf{V}}$ consisting of all $(x,p,q,\lambda) \in \Lambda_{\mathbf{D},\mathbf{V}}$ such that the following stability condition holds: if $\mathbf{V}'$ is an I-graded subspace of $\mathbf{V}$ such that $x_h(\mathbf{V}'_{h'}) \subset \mathbf{V}'_{h''}$ for all $h \in H$ and $p_i(\mathbf{D}_i) \subset \mathbf{V}'_i$ forall i, then $\mathbf{V}' = \mathbf{V}$.

Let $\Lambda^{*s}_{\mathbf{D},\mathbf{V}}$ be the open subvariety of $\Lambda_{\mathbf{D},\mathbf{V}}$ consisting of all $(x,p,q,\lambda) \in \Lambda_{\mathbf{D},\mathbf{V}}$ such that the following stability condition holds: if $\mathbf{V}'$ is an I-graded subspace of $\mathbf{V}$ such that $x_h(\mathbf{V}'_{h'}) \subset \mathbf{V}'_{h''}$ for all $h \in H$ and $q_i(\mathbf{V}') = 0$ for all i, then $\mathbf{V}' = 0$. $\Lambda^s_{\mathbf{D},\mathbf{V}}, \Lambda^{*s}_{\mathbf{D},\mathbf{V}}$ were introduced in [N2], in a different notation.

For any $\lambda \in E$, let $\Lambda^s_{\mathbf{D},\mathbf{V},\lambda}$ (resp. $\Lambda^{*s}_{\mathbf{D},\mathbf{V},\lambda}$) be the open subvariety of $\Lambda_{\mathbf{D},\mathbf{V},\lambda}$ consisting of all triples $(x,p,q) \in \Lambda_{\mathbf{D},\mathbf{V},\lambda}$ such that (x,p,q,λ) belongs to $\Lambda^s_{\mathbf{D},\mathbf{V}}$ (resp. to $\Lambda^{*s}_{\mathbf{D},\mathbf{V}}$). Then $\Lambda^s_{\mathbf{D},\mathbf{V},\lambda}$ (resp. $\Lambda^{*s}_{\mathbf{D},\mathbf{V},\lambda}$) may be naturally identified with the fibre of the fourth projection $\Lambda^s_{\mathbf{D},\mathbf{V}} \to E$ (resp. $\Lambda^{*s}_{\mathbf{D},\mathbf{V}} \to E$) at λ.

By [N2], the natural action of $G_{\mathbf{V}}$ on $\Lambda^s_{\mathbf{D},\mathbf{V}}$ or $\Lambda^{*s}_{\mathbf{D},\mathbf{V}}$ is free; hence the orbit spaces $\Lambda^s_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}}, \Lambda^{*s}_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}}$ are well defined.

4.9. There is a natural isomorphism $\Lambda^s_{\mathbf{D},\mathbf{V}} \xrightarrow{\sim} \Lambda^{*s}_{\mathbf{D}^*,\mathbf{V}^*}$; it is given by $(x,p,q,\lambda) \mapsto ({}^t\bar{x}, {}^tq, {}^tp, \lambda)$. (Notation of [L4, 2.27].)

4.10. Let $\Lambda^{s,*s}_{\mathbf{D},\mathbf{V}} = \Lambda^s_{\mathbf{D},\mathbf{V}} \cap \Lambda^{*s}_{\mathbf{D},\mathbf{V}}$. For any $\lambda \in E$, let $\Lambda^{s,*s}_{\mathbf{D},\mathbf{V},\lambda} = \Lambda^s_{\mathbf{D},\mathbf{V},\lambda} \cap \Lambda^{*s}_{\mathbf{D},\mathbf{V},\lambda}$.

LEMMA 4.11. — *The following two conditions for $(x,p,q,\lambda) \in \Lambda_{\mathbf{D},\mathbf{V}}$ are equivalent:*

(i) $(x,p,q,\lambda) \in \Lambda^{s,*s}_{\mathbf{D},\mathbf{V}}$;

(ii) (x,p,q,λ) *has trivial isotropy group in $G_{\mathbf{V}}$ and its $G_{\mathbf{V}}$-orbit in $\Lambda_{\mathbf{D},\mathbf{V}}$ is closed.*

Assume that (ii) holds. Then $(x,p,q,\lambda) \in \Lambda^{*s}_{\mathbf{D},\mathbf{V}}$ by [N2, 3.24]; the same proof shows that $(x,p,q,\lambda) \in \Lambda^s_{\mathbf{D},\mathbf{V}}$, hence (i) holds.

Assume now that (i) holds. Then the first assertion of (ii) is proved in [N2, 3.10]. It remains to prove the second assertion. Let $\mathcal{O}$ be a $G_{\mathbf{V}}$-orbit in the closure of the orbit of (x,p,q,λ). By Hilbert's theorem, there exists a one parameter subgroup ζ_t of $G_{\mathbf{V}}$ $(t \in \mathbf{C}^*)$ such that $\lim_{t\to\infty} \zeta_t(x,p,q,\lambda) =$

$(x',p',q',\lambda) \in \mathcal{O}$. We can write $\mathbf{V} = \oplus_k \mathbf{V}^k$ where $\zeta_t v = t^k v$ for all $v \in \mathbf{V}^k$. Let $\mathbf{V}^{(k)} = \oplus_{k';k' \le k} \mathbf{V}^k$. We have $x_h(v) = \sum_{k'} x_{h;k,k'} v$ for $v \in \mathbf{V}^k_{h'}$ where $x_{h;k,k'}; \mathbf{V}^k_{h'} \to \mathbf{V}^{k'}_{h''}$. We have $p(d) = \sum_k p^{(k)}(d)$ for $d \in \mathbf{D}$ where $p^k : \mathbf{D} \to \mathbf{V}^k$. For $t \in \mathbf{C}^*$, write $\zeta_t(x,p,q) = (x(t),p(t),q(t))$. We have

$$x(t)_h = \sum_{k'} x_{h;k,k'} t^{k'-k}, \quad p(t) = \sum_k t^k p^{(k)}$$

and $q(t)(v) = t^{-k} q(v)$ for $v \in \mathbf{V}^k$. Since $\lim_{t\to\infty} \zeta_t(x,p,q,\lambda)$ exists, it follows that $x_{h;k,k'} = 0$ for $k' > k$, $p^{(k)} = 0$ for $k > 0$, $q\big|_{\mathbf{V}^k} = 0$ for $k < 0$. Hence $\mathbf{V}^{(-1)}$ is x-stable and contained in $\mathrm{Ker}(q)$. Since $(x,p,q,\lambda) \in \Lambda^{*s}_{\mathbf{D},\mathbf{V}}$, it follows that $\mathbf{V}^{(-1)} = 0$ and $p(\mathbf{D}) \subset \mathbf{V}^0$. (Up to this point, the argument is exactly as in [N2, 3.20].) Moreover, for $k \ge 0$, $\mathbf{V}^{(k)}$ is x-stable and contains $\mathrm{Im}(p)$. Since $(x,p,q,\lambda) \in \Lambda^{s}_{\mathbf{D},\mathbf{V}}$, it follows that $\mathbf{V}^{(k)} = \mathbf{V}$ for $k \ge 0$. Thus, $\mathbf{V} = \mathbf{V}^0$ hence $(x',p',q',\lambda) = (x,p,q,\lambda)$. This proves that the $G_{\mathbf{V}}$-orbit of (x,p,q,λ) is closed. The lemma is proved.

LEMMA 4.12.

(a) *The map $\Lambda^{s,*s}_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}} \to Z_{\mathbf{D}}$ induced by $\vartheta' : \Lambda_{\mathbf{D},\mathbf{V}} \to Z_{\mathbf{D}}$ is injective.*

(b) *Its image, $Z_{\mathbf{D},\mathbf{V}}$, depends only on the isomorphism class of $\mathbf{V}$ in $\mathcal{C}^0$.*

(c) *$Z_{\mathbf{D},\mathbf{V}}$ is a locally closed subvariety of $Z_{\mathbf{D}}$ and is homeomorphic (both for the Zariski and ordinary topology) to $\Lambda^{s,*s}_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}}$.*

(d) *The subsets $Z_{\mathbf{D},\mathbf{V}}$ (for $\mathbf{V}$ running through a set of representatives of the isomorphism classes of objects in $\mathcal{C}^0$) form a partition of $Z_{\mathbf{D}}$.*

We prove (a). Our map is the composition

$$\Lambda^{s,*s}_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}} \to \Lambda_{\mathbf{D},\mathbf{V}}//G_{\mathbf{V}} \xrightarrow{\vartheta} Z_{\mathbf{D}}$$

where the first map (the obvious one) is injective by 4.11 and ϑ is injective by 4.7. This proves (a). The proof of (b) is trivial.

We prove (c). From 4.11 we see that $\Lambda^{s,*s}_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}}$ is an open subvariety of $\Lambda_{\mathbf{D},\mathbf{V}}//G_{\mathbf{V}}$ and from 4.7 we see that$\Lambda_{\mathbf{D},\mathbf{V}}//G_{\mathbf{V}}$ is mapped by ϑ homeomorphically onto a closed subvariety of $Z_{\mathbf{D}}$. This proves (c).

We prove (d). Let $(\lambda,\pi) \in Z_{\mathbf{D}}$. Let $\mathbf{V} = \mathcal{E}^{\mathbf{D}}/\mathcal{K}^{\pi}$ (notation of [L4, 2.3, 2.8]). We have $\mathbf{V} \in \mathcal{C}^0$ by [L4, 5.12]. We define $(x,p,q) \in \Lambda_{\mathbf{D},\mathbf{V},\lambda}$ as in [L4, 2.18] (with $\mathcal{V} = \mathcal{K}^{\pi}$). As pointed out in [L4, 2.18], we have $(x,p,q) \in \Lambda^{s}_{\mathbf{D},\mathbf{V},\lambda}$ and $\vartheta'(x,p,q,\lambda) = (\lambda,\pi)$. From the definition of $\mathcal{K}^{\pi}$ we see also that $(x,p,q) \in \Lambda^{*s}_{\mathbf{D},\mathbf{V},\lambda}$. Thus, $(\lambda,\pi) \in Z_{\mathbf{D},\mathbf{V}}$. We see that the union of the subsets $Z_{\mathbf{D},\mathbf{V}}$ is the whole of $Z_{\mathbf{D}}$.

Now let $(\lambda,\pi) \in Z_{\mathbf{D},\mathbf{V}} \cap Z_{\mathbf{D},\mathbf{V}'}$ where $\mathbf{V},\mathbf{V}' \in \mathcal{C}^0$. We want to prove that $\mathbf{V},\mathbf{V}'$ are isomorphic in $\mathcal{C}^0$. We can find $(x,p,q,\lambda) \in \Lambda^{s,*s}_{\mathbf{D},\mathbf{V}}$ and $(x',p',q',\lambda) \in \Lambda^{s,*s}_{\mathbf{D},\mathbf{V}'}$ such that $\vartheta'(x,p,q,\lambda) = \vartheta'(x',p',q',\lambda) = (\lambda,\pi)$. By [L4, 2.20] we can assume that $\mathbf{V} = \mathcal{E}^{\mathbf{D}}/\mathcal{V}, \mathbf{V}' = \mathcal{E}^{\mathbf{D}}/\mathcal{V}'$ (notation of [L4, 2.3]) where $\mathcal{V},\mathcal{V}'$ are I-graded subspaces of $\mathcal{E}^{\mathbf{D}}$ containing $\mathcal{I}^\pi$ and contained in $\mathcal{K}^\pi$ (notation of [L4, 2.8]), that (x,p,q) is obtained from $\mathcal{V}$ as in [L4, 2.18] and that (x',p',q') is obtained in an analogous way from $\mathcal{V}'$. From the definition of $\mathcal{K}^\pi$ we see that the condition that $(x,p,q,\lambda) \in \Lambda^{*s}_{\mathbf{D},\mathcal{E}^{\mathbf{D}}/\mathcal{V}}$ is equivalent to the condition that $\mathcal{V} = \mathcal{K}^\pi$. Similarly, the condition that $(x',p',q',\lambda) \in \Lambda^{*s}_{\mathbf{D},\mathcal{E}^{\mathbf{D}}/\mathcal{V}'}$ is equivalent to the condition that $\mathcal{V}' = \mathcal{K}^\pi$. Hence we have $\mathcal{V} = \mathcal{V}' = \mathcal{K}^\pi$. It follows that $\mathcal{E}^{\mathbf{D}}/\mathcal{V} = \mathcal{E}^{\mathbf{D}}/\mathcal{V}'$ and our claim follows. The lemma is proved.

LEMMA 4.13.

(a) *The morphism $\Lambda^s_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}} \to Z_{\mathbf{D}}$ induced by $\vartheta' : \Lambda_{\mathbf{D},\mathbf{V}} \to Z_{\mathbf{D}}$ is proper. Hence its image, ${}_{\mathbf{V}}Z_{\mathbf{D}}$, is a closed subvariety of $Z_{\mathbf{D}}$.*

(b) *${}_{\mathbf{V}}Z_{\mathbf{D}}$ depends only on the isomorphism class of $\mathbf{V}$ in $\mathcal{C}^0$.*

Our map is the composition

$$\Lambda^s_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}} \to \Lambda_{\mathbf{D},\mathbf{V}}//G_{\mathbf{V}} \xrightarrow{\vartheta} Z_{\mathbf{D}}$$

where the first map (the obvious one) is proper by [N2, 3.18] and ϑ is proper by 4.7. This proves (a). The proof of (b) is trivial.

5. A computation of dimensions.

LEMMA 5.1. — *Let $\mathbf{V},\mathbf{D} \in \mathcal{C}^0$. The varieties $\Lambda^{*s}_{\mathbf{D},\mathbf{V}}, \Lambda^s_{\mathbf{D},\mathbf{V}}$ are smooth of pure dimension*

$$2d_{\mathbf{V}} + 2\sum_i \dim \mathbf{V}_i \dim \mathbf{D}_i - \sum_i \dim \mathbf{V}_i^2 + |I|$$

and the fourth projections $\Lambda^{*s}_{\mathbf{D},\mathbf{V}} \to E, \Lambda^s_{\mathbf{D},\mathbf{V}} \to E$ are submersions.

The fact that $\Lambda^{*s}_{\mathbf{D},\mathbf{V},0}$ is smooth is proved in [N2, 3.10]. That proof identifies the tangent space of $\Lambda^{*s}_{\mathbf{D},\mathbf{V},0}$ at (x,p,q) with the kernel of a certain linear map $m : M_{\mathbf{D},\mathbf{V}} \to \mathrm{Hom}_{\mathcal{C}^0}(\mathbf{V},\mathbf{V})$ (obtained by taking the derivative of the equation defining $\Lambda_{\mathbf{D},\mathbf{V},0}$). The main point is that m is surjective (which follows from the stability condition in the definition of $\Lambda^{*s}_{\mathbf{D},\mathbf{V},0}$). A similar argument shows that $\Lambda^{*s}_{\mathbf{D},\mathbf{V}}$ is smooth and that the tangent space of $\Lambda^{*s}_{\mathbf{D},\mathbf{V}}$ at (x,p,q,λ) is $\{(k,\lambda') \in M_{\mathbf{D},\mathbf{V}} \oplus E | m(k) + \lambda' = 0\}$ where m is as above and, in the last equation λ' is regarded as an element of $\mathrm{Hom}_{C^0}(\mathbf{V},\mathbf{V})$ whose i-component is multiplication by λ'_i. This tangent space maps to the tangent space of E at λ by $(k,\lambda')\mapsto\lambda'$. Using the fact that, as above, m is surjective, the assertions relative to $\Lambda^{*s}_{\mathbf{D},\mathbf{V}}$ follow. These assertions imply the assertions relative to $\Lambda^{s}_{\mathbf{D},\mathbf{V}}$, by 4.9.

5.2. Let $\lambda\in E$. Given $(x,p,q)\in\Lambda_{\mathbf{D},\mathbf{V},\lambda}$, let $\mathbf{V}'$ be the largest I-graded subspace of $\mathbf{V}$ such that $x_h(\mathbf{V}'_{h'})\subset\mathbf{V}'_{h''}$ for all h and $q_i(\mathbf{V}'_i)=0$ for all i. Clearly, $\mathbf{V}'$ is well defined. Note that(x,p,q) induces in an obvious way elements

$$(x',0,0)\in\Lambda_{0,\mathbf{V}',\lambda},\quad (x'',p'',q'')\in\Lambda^{*s}_{\mathbf{D},\mathbf{V}/\mathbf{V}',\lambda}.$$

Conversely, assume that we are given an I-graded subspace $\mathbf{V}'\subset\mathbf{V}$ and elements $(x',0,0)\in\Lambda_{0,\mathbf{V}',\lambda}$, $(x'',p'',q'')\in\Lambda^{*s}_{\mathbf{D},\mathbf{V}/\mathbf{V}',\lambda}$.

Let Φ be the set of all $(x,p,q)\in\Lambda_{\mathbf{D},\mathbf{V},\lambda}$ which give rise as above to $\mathbf{V}'$, $(x',0,0)$, (x'',p'',q'').

LEMMA 5.3. — *A choice of an I-graded complement $\mathbf{V}''$ of $\mathbf{V}'$ in $\mathbf{V}$ defines on Φ a structure of vector space of dimension*

(a) $$\sum_i \dim\mathbf{V}'_i\Big(\dim\mathbf{D}_i-\dim\mathbf{V}''_i+\sum_{h;h'=i}\dim\mathbf{V}''_{h''}\Big).$$

Let $\mathbf{V}''$ as above. We identify $\mathbf{V}/\mathbf{V}'=\mathbf{V}''$ in an obvious way. For $(x,p,q)\in\Phi$ we have

$$x_h(v'')=y_h(v'')+x''_h(v'')$$

for all $v''\in\mathbf{V}'_{h'}$, where

$$\begin{aligned}&y=(y_h)_{h\in H},\quad y_h:\mathbf{V}''_{h'}\to\mathbf{V}'_{h''},\\&x_h(v')=x'_h(v')\ \text{ for all }\ v'\in\mathbf{V}'_{h'},\\&p_i(d)=p'_i(d)+p''_i(d)\ \text{ for }\ d\in\mathbf{D}_i,\\&p'_i:\mathbf{D}_i\to\mathbf{V}'_i,\quad p''_i:\mathbf{D}_i\to\mathbf{V}''_i,\\&q_i(v'')=q''_i(v''),\ \text{ for all }\ v''\in\mathbf{V}''_i.\end{aligned}$$

By the change of variable $(x,p,q)\mapsto(y,p')$ the variety Φ becomes the set of all (y,p') as above such that

$$\sum_{h;h'=i}\varepsilon_{\bar h}x'_{\bar h}y_h+\varepsilon_{\bar h}y_{\bar h}x''_h-p'_iq''_i=0:\mathbf{V}''_i\to\mathbf{V}'_i$$

for all $i\in I$. The solutions of this system of equations (with fixed x',x'',q'') form a vector space. It remains to show that this vector space has dimension as in (a). This vector space is the kernel of the linear map

$$T:\oplus_h\mathrm{Hom}(\mathbf{V}''_{h'},\mathbf{V}'_{h''})\oplus\oplus_i\mathrm{Hom}(\mathbf{D}_i,\mathbf{V}'_i)\to\oplus_i\mathrm{Hom}(\mathbf{V}''_i,\mathbf{V}'_i),$$

$$(y,p') \mapsto \Big(\sum_{h;h'=i} \varepsilon_{\bar h} x'_{\bar h} y_h + \varepsilon_{\bar h} y_{\bar h} x''_h - p'_i q''_i \Big)_{i\in I}.$$

We will show that T is surjective; this implies that $\dim \operatorname{Ker} T$ is given by (a). To show the surjectivity of T, we consider the perfect bilinear pairing

$$\oplus_i \operatorname{Hom}(\mathbf{V}'_i, \mathbf{V}''_i) \times \oplus_i \operatorname{Hom}(\mathbf{V}''_i, \mathbf{V}'_i) \to \mathbf{C}$$

given by $((a_i),(b_i)) = \sum_i \operatorname{tr}(a_i b_i)$. It is enough to show that, if (a_i) is orthogonal to $\operatorname{Im} T$ under this pairing, then $(a_i) = 0$. Thus, we assume that

$$\sum_h \varepsilon_{\bar h} \operatorname{tr}(a_{h'} x'_{\bar h} y_h) + \varepsilon_{\bar h} \operatorname{tr}(a_{h'} y_{\bar h} x''_h) - \sum_i \operatorname{tr}(a_i p'_i q''_i) = 0$$

for any (y,p'). Equivalently,

$$\sum_h \varepsilon_{\bar h} \operatorname{tr}((a_{h'} x'_{\bar h} - x''_{\bar h} a_{h''}) y_h) - \sum_i \operatorname{tr}(q''_i a_i p'_i) = 0$$

for any (y,p'). It follows that

(∗) $$a_{h'} x'_{\bar h} - x''_{\bar h} a_{h''} = 0 \quad \text{for all } h,$$

(∗∗) $$q''_i a_i = 0 \quad \text{for all } i.$$

(∗) shows that $\operatorname{Im}(a)$ is an x'-stable I-graded subspace of $\mathbf{V}''$; (∗∗) shows that $\operatorname{Im}(a) \subset \operatorname{Ker}(q'')$. By the stability condition for (x'',p'',q''), we then have $\operatorname{Im}(a) = 0$ hence $a = 0$. (Compare with the argument in the proof of [N2, 3.10].) The lemma is proved.

5.4. Now let $\mathbf{D}, \mathbf{V}, \widetilde{\mathbf{V}} \in \mathcal{C}^0$ and let $(\lambda,\pi) \in {}_{\mathbf{V}}Z_{\mathbf{D}} \cap Z_{\mathbf{D},\widetilde{\mathbf{V}}}$. By definition, (λ,π) is in the image of the map

(a) $$\Lambda^s_{\mathbf{D},\mathbf{V}} \to Z_{\mathbf{D}}$$

(restriction of ϑ') and there is a unique $(\widetilde{x}, \widetilde{p}, \widetilde{q}, \lambda) \in \Lambda^{s,*s}_{\mathbf{D},\widetilde{\mathbf{V}}}$ which maps to (λ,π) under the map ϑ' defined in terms of $\widetilde{\mathbf{V}}$.

Let Ψ be the fibre of (a) at (λ,π).

PROPOSITION 5.5. — *Ψ has pure dimension equal to*

$$(1/2)(\dim \Lambda^s_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}} - \dim \Lambda^s_{\mathbf{D},\widetilde{\mathbf{V}}}/G_{\widetilde{\mathbf{V}}}) + \dim G_{\mathbf{V}}.$$

If $(x,p,q,\lambda) \in \Psi$ then, by attaching to it

(a) $\mathbf{V}', (x',0,0) \in \Lambda_{0,\mathbf{V}',\lambda}$, $\quad (x'',p'',q'') \in \Lambda^{*s}_{\mathbf{D},\mathbf{V}/\mathbf{V}',\lambda}$

as in 5.2, we have automatically $(x'',p'',q'') \in \Lambda^{s,*s}_{\mathbf{D},\mathbf{V}/\mathbf{V}',\lambda}$; moreover, from the definitions, ϑ' (relative to $\mathbf{V}/\mathbf{V}'$) carries (x'',p'',q'',λ) to $\vartheta'(x,p,q,\lambda) = (\lambda,\pi)$. Using now 4.12(a), we see that there exists an isomorphism (necessarily unique)

(b) $\iota : \mathbf{V}/\mathbf{V}' \xrightarrow{\sim} \widetilde{\mathbf{V}}$ which carries (x'',p'',q'') to $(\widetilde{x},\widetilde{p},\widetilde{q})$.
Thus, we have a map u from Ψ to the variety of all triples as in (a) such that (b) holds. Let Ψ' be the variety consisting of all $(x,p,q,\lambda) \in \Lambda_{\mathbf{D},\mathbf{V}}$ (without stability condition) such that the triple (a) attached to $(x,p,q,\lambda) \in \Lambda_{\mathbf{D},\mathbf{V}}$ satisfies (b).

Note that Ψ is an open subset of Ψ'. On the other hand, by Lemma 5.3, Ψ' is a vector bundle of dimension

(c) $\sum_i \dim \mathbf{V}'_i(\dim \mathbf{D}_i - \dim \widetilde{\mathbf{V}}_i + \sum_{h;h'=i} \dim \mathbf{V}'_{h''})$
over the variety of triples (a) satisfying (b). This variety of triples is itself a locally trivial fibration over the space of all surjective maps $\mathbf{V} \to \widetilde{\mathbf{V}}$ (in $\mathcal{C}^0$) with fibre isomorphic to $\Lambda_{0,\mathbf{V}',\lambda}$ (where $\dim \mathbf{V}'_i = \dim \mathbf{V}_i - \dim \widetilde{\mathbf{V}}_i$ for all i). Using now 4.5(a), we see that this variety of triples has pure dimension equal to

(d) $\dim \mathbf{V}_i \dim \widetilde{\mathbf{V}}_i + d_{\mathbf{V}'}$
where $\mathbf{V}'$ is as above. It follows that Ψ' (and hence also Ψ) has pure dimension equal to the sum of (c) and (d). This is equal to the expression in the proposition, by 5.1(b). The proposition is proved.

COROLLARY 5.6. — *The fibre of the map $\Lambda^s_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}} \to Z_{\mathbf{D}}$ induced by ϑ' at $(\lambda,\pi) \in {}_{\mathbf{V}}Z_{\mathbf{D}} \cap Z_{\mathbf{D},\widetilde{\mathbf{V}}}$ has pure dimension $(1/2)(\dim \Lambda^s_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}} - \dim \Lambda^s_{\mathbf{D},\widetilde{\mathbf{V}}}/G_{\widetilde{\mathbf{V}}})$.*

6. Small maps.

6.1. Let E_1 be the set of all $\lambda \in E$ such that $\check{R}_\lambda = \emptyset$. Let $Z^{E_1}_{\mathbf{D}}$, $Z^{E-E_1}_{\mathbf{D}}$, $Z^0_{\mathbf{D}}$ be the inverse images of $E_1, E - E_1, \{0\}$ under the canonical map $Z_{\mathbf{D}} \to E$.

LEMMA 6.2. — *If $\lambda \in E_1$, then $\Lambda_{\mathbf{D},\mathbf{V},\lambda} = \Lambda^s_{\mathbf{D},\mathbf{V},\lambda} = \Lambda^{*s}_{\mathbf{D},\mathbf{V},\lambda}$.*

Let $(x,p,q) \in \Lambda_{\mathbf{D},\mathbf{V},\lambda}$. We associate to (x,p,q)

$$\mathbf{V}', (x',0,0) \in \Lambda_{0,\mathbf{V}',\lambda}, \quad (x'',p'',q'') \in \Lambda^{*s}_{\mathbf{D},\mathbf{V}/\mathbf{V}',\lambda}$$

as in 5.2. We see that $\Lambda_{0,\mathbf{V}',\lambda} \neq \emptyset$ hence, by 4.5(c), we have $\mathbf{V}' = 0$. Hence $(x,p,q) = (x'',p'',q'') \in \Lambda^{*s}_{\mathbf{D},\mathbf{V},\lambda}$. Thus, $\Lambda_{\mathbf{D},\mathbf{V},\lambda} = \Lambda^{*s}_{\mathbf{D},\mathbf{V},\lambda}$. Passing to dual spaces we obtain $\Lambda_{\mathbf{D},\mathbf{V},\lambda} = \Lambda^s_{\mathbf{D},\mathbf{V},\lambda}$. The lemma is proved.

LEMMA 6.3.

(a) *$Z_{\mathbf{D},\mathbf{V}}$ is open dense in ${}_{\mathbf{V}}Z_{\mathbf{D}}$.*

(b) *The canonical map $\pi_{\mathbf{V}} : \Lambda^s_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}} \to {}_{\mathbf{V}}Z_{\mathbf{D}}$ induced by ϑ' restricts to a homeomorphism $\pi_{\mathbf{V}}^{-1} \subset (Z_{\mathbf{D},\mathbf{V}}) \xrightarrow{\sim} Z_{\mathbf{D},\mathbf{V}}$.*

Since the canonical map $\Lambda^s_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}} \to E$ is a submersion and E_1 is open dense in E, the inverse image of E_1 under this map is open dense in $\Lambda^s_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}}$. This inverse image is contained in $\Lambda^{s,*s}_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}}$ by 6.2. It follows that the open set $\Lambda^{s,*s}_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}}$ of $\Lambda^s_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}}$ is also dense. Applying the continuous surjective map $\Lambda^s_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}} \to {}_{\mathbf{V}}Z_{\mathbf{D}}$, we deduce that the image of $\Lambda^{s,*s}_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}}$, that is $Z_{\mathbf{D},\mathbf{V}}$, is dense in ${}_{\mathbf{V}}Z_{\mathbf{D}}$. It is open by 4.12. This proves (a).

We prove (b). It suffices to show that $\pi_{\mathbf{V}}^{-1} \subset (Z_{\mathbf{D},\mathbf{V}}) = \Lambda^{s,*s}_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}}$. Let $(x,p,q,\lambda) \in \pi_{\mathbf{V}}^{-1} \subset (Z_{\mathbf{D},\mathbf{V}})$. We associate to (x,p,q)

$$\mathbf{V}', (x',0,0) \in \Lambda_{0,\mathbf{V}',\lambda}, \quad (x'',p'',q'') \in \Lambda^{*s}_{\mathbf{D},\mathbf{V}/\mathbf{V}',\lambda}$$

as in 5.2. We have automatically $(x'',p'',q'') \in \Lambda^{s,*s}_{\mathbf{D},\mathbf{V}/\mathbf{V}',\lambda}$ and, as in the proof of 5.5, there exists an isomorphism $\iota : \mathbf{V}/\mathbf{V}' \xrightarrow{\sim} \mathbf{V}$ which carries (x'',p'',q'') to a triple in $\Lambda^{s,*s}_{\mathbf{D},\mathbf{V},\lambda}$. In particular, we must have $\mathbf{V}' = 0$ and $(x,p,q) \in \Lambda^{s,*s}_{\mathbf{D},\mathbf{V},\lambda}$. Thus, $\pi_{\mathbf{V}}^{-1} \subset (Z_{\mathbf{D},\mathbf{V}}) \subset \Lambda^{s,*s}_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}}$. The reverse inclusion is obvious. The lemma is proved.

LEMMA 6.4. — *Let $\mathbf{D}, \mathbf{V}, \widetilde{\mathbf{V}} \in \mathcal{C}^0$. If $\mathbf{V}, \widetilde{\mathbf{V}}$ are not isomorphic in $\mathcal{C}^0$, then $\dim({}_{\mathbf{V}}Z_{\mathbf{D}} \cap Z_{\mathbf{D},\widetilde{\mathbf{V}}}) < \dim \Lambda^s_{\mathbf{D},\widetilde{\mathbf{V}}}/G_{\widetilde{\mathbf{V}}}$.*

If $\lambda \in E_1$, we have (by 6.2) ${}_{\mathbf{V}}Z_{\mathbf{D}} \cap Z^{\lambda}_{\mathbf{D}} = Z_{\mathbf{D},\mathbf{V}} \cap Z^{\lambda}_{\mathbf{D}}$, hence

$${}_{\mathbf{V}}Z_{\mathbf{D}} \cap Z_{\mathbf{D},\widetilde{\mathbf{V}}} \cap Z^{\lambda}_{\mathbf{D}} = Z_{\mathbf{D},\mathbf{V}} \cap Z_{\mathbf{D},\widetilde{\mathbf{V}}} \cap Z^{\lambda}_{\mathbf{D}} = \emptyset.$$

(We use 4.12(d) and our hypothesis.) Thus,

$${}_{\mathbf{V}}Z_{\mathbf{D}} \cap Z_{\mathbf{D},\widetilde{\mathbf{V}}} \subset Z_{\mathbf{D},\widetilde{\mathbf{V}}} \cap Z^{E-E_1}_{\mathbf{D}}.$$

It is therefore enough to prove that

$$\text{(a)} \qquad \dim(Z_{\mathbf{D},\widetilde{\mathbf{V}}} \cap Z^{E-E_1}_{\mathbf{D}}) < \dim \Lambda^s_{\mathbf{D},\widetilde{\mathbf{V}}}/G_{\widetilde{\mathbf{V}}}.$$

By 4.12, the space in the left hand side of (a) is homeomorphic to the inverse image of $E - E_1$ under the canonical map $\Lambda^{s,*s}_{\mathbf{D},\widetilde{\mathbf{V}}}/G_{\widetilde{\mathbf{V}}} \to E$. Since this map is a submersion and $E - E_1$ is a proper closed subset of E, the dimension of the inverse image of $E - E_1$ is $< \dim \Lambda^s_{\mathbf{D},\widetilde{\mathbf{V}}}/G_{\widetilde{\mathbf{V}}}$. The lemma is proved.

THEOREM 6.5. — *The canonical map $\pi_{\mathbf{V}} : \Lambda^s_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}} \to {}_{\mathbf{V}}Z_{\mathbf{D}}$ is small.*

By 5.1, $\Lambda^s_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}}$ is smooth of pure dimension; by 4.13, $\pi_{\mathbf{V}}$ is proper. From 4.12 we see that the sets ${}_{\mathbf{V}}Z_{\mathbf{D}} \cap Z_{\mathbf{D},\widetilde{\mathbf{V}}}$ (for various $\widetilde{\mathbf{V}} \in \mathcal{C}^0$) form a partition of ${}_{\mathbf{V}}Z_{\mathbf{D}}$ into locally closed subvarieties. Only finitely many of these pieces are non-empty. One of them, $Z_{\mathbf{D},\mathbf{V}}$ is open dense in ${}_{\mathbf{V}}Z_{\mathbf{D}}$ and $\pi_{\mathbf{V}}$ is a homeomorphism over this open set. It is then enough to show that for any other piece, that is ${}_{\mathbf{V}}Z_{\mathbf{D}} \cap Z_{\mathbf{D},\widetilde{\mathbf{V}}}$ with $\widetilde{\mathbf{V}}, \mathbf{V}$ not isomorphic, twice the dimension of any fibre over a point in the piece plus the dimension of the piece is strictly less than $\dim {}_{\mathbf{V}}Z_{\mathbf{D}}$. Using 6.4 and 5.6 we see that this sum is strictly less than

$$\begin{aligned}\dim \Lambda^s_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}} - \dim \Lambda^s_{\mathbf{D},\widetilde{\mathbf{V}}}/G_{\widetilde{\mathbf{V}}} + \dim \Lambda^s_{\mathbf{D},\widetilde{\mathbf{V}}}/G_{\widetilde{\mathbf{V}}} &= \dim \Lambda^s_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}} \\ &= \dim {}_{\mathbf{V}}Z_{\mathbf{D}}.\end{aligned}$$

The theorem is proved.

6.6. The previous result should be compared with [N2, 10.11] which can be reformulated to say that $\Lambda^s_{\mathbf{D},\mathbf{V},0}/G_{\mathbf{V}} \to {}_{\mathbf{V}}Z_{\mathbf{D}} \cap Z^0_{\mathbf{D}}$ is semismall. That result is essentially equivalent to [N2, 7.2], which in turn is proved using the special case of Corollary 5.6 with $\lambda = 0$. The proof of this special case given in [N2, 7.2] is based on the method of proof of [L3, 12.3]. This proof does not generalize to the case $\lambda \neq 0$, where the arguments in Section 5 are needed.

6.7. If Y' is an irreducible complex algebraic variety, the intersection cohomology complex $IC(Y')$ is well defined. (We normalize it so that its restriction to an open dense subset of Y' is $\mathbf{C}$.) If Y is an arbitrary complex algebraic variety, the intersection cohomology complex of Y is defined as $IC(Y) = \oplus_{Y'} IC(Y')$ where Y' runs over the set of irreducible components of Y and $IC(Y')$ is extended to the whole of Y by 0 outside Y'. If Y is equidimensional and $\pi_1 : \widetilde{Y} \to Y$ is a small map then, from the definitions, $IC(Y) = (\pi_1)_*(\mathbf{C})$.

COROLLARY 6.8. — *We have canonically $IC({}_{\mathbf{V}}Z_{\mathbf{D}}) = (\pi_{\mathbf{V}})_*(\mathbf{C})$ (as complexes on ${}_{\mathbf{V}}Z_{\mathbf{D}}$).*

Note that ${}_{\mathbf{V}}Z_{\mathbf{D}}$ is equidimensional.

LEMMA 6.9.

(a) *For any $\mathbf{V} \in \mathcal{C}^0$, ${}_{\mathbf{V}}Z_{\mathbf{D}}$ is a union of irreducible components of $Z_{\mathbf{D}}$.*

(b) *Any irreducible component of* $Z_{\mathbf{D}}$ *is contained in some* ${}_{\mathbf{V}}Z_{\mathbf{D}}$ *and the isomorphism class of such* $\mathbf{V}$ *in* $\mathcal{C}^0$ *is uniquely determined.*

By 5.1, any irreducible component of $\Lambda^s_{\mathbf{D},\mathbf{V}}$ meets $\Lambda^s_{\mathbf{D},\mathbf{V},\lambda}$ for some $\lambda \in E_1$. Hence any irreducible component of ${}_{\mathbf{V}}Z_{\mathbf{D}}$ meets $Z^{E_1}_{\mathbf{D}}$. Since the closed subsets ${}_{\mathbf{V}}Z_{\mathbf{D}}$ cover $Z_{\mathbf{D}}$, it follows that any irreducible component of $Z_{\mathbf{D}}$ meets $Z^{E_1}_{\mathbf{D}}$. By the proof of 6.3(a), $Z^{E_1}_{\mathbf{D}}$ is open dense in $Z_{\mathbf{D}}$. Hence the irreducible components of $Z_{\mathbf{D}}$ are exactly the closures of the irreducible components of $Z^{E_1}_{\mathbf{D}}$.

The closed subsets ${}_{\mathbf{V}}Z_{\mathbf{D}} \cap Z^{E_1}_{\mathbf{D}}$ of $Z^{E_1}_{\mathbf{D}}$ coincide with the subsets $Z_{\mathbf{D},\mathbf{V}} \cap Z^{E_1}_{\mathbf{D}}$ of $Z^{E_1}_{\mathbf{D}}$ and these form a partition of $Z^{E_1}_{\mathbf{D}}$ by 4.12(d). Hence ${}_{\mathbf{V}}Z_{\mathbf{D}} \cap Z^{E_1}_{\mathbf{D}}$ are both open and closed in $Z^{E_1}_{\mathbf{D}}$ hence are unions of irreducible components of $Z^{E_1}_{\mathbf{D}}$. The lemma follows.

6.10. One expects that $\Lambda^s_{\mathbf{D},\mathbf{V}}$ is connected (if non-empty). This is equivalent to the property that $\Lambda^s_{\mathbf{D},\mathbf{V},0}$ is connected (if non-empty) which is stated in [N2, 6.2] but, as Nakajima informed me, the proof given there is incorrect. If we assume that this property holds, then 6.9 would have a simpler form, namely that *the* ${}_{\mathbf{V}}Z_{\mathbf{D}}$ *which are non-empty are precisely the irreducible components of* $Z_{\mathbf{D}}$.

6.11. Let $\widetilde{Z}_{\mathbf{D}}$ be the disjoint union $\sqcup_{\mathbf{V}}\Lambda^s_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}}$ where$\mathbf{V}$ runs over a set of representatives for the isomorphism classes of objects of $\mathcal{C}^0$. This is a finite union since $\Lambda^s_{\mathbf{D},\mathbf{V}}$ is empty for all but finitely many $\mathbf{V}$ (see [L4, 5.14]). Moreover, $\widetilde{Z}_{\mathbf{D}}$ is canonically defined (independent of the choice of representatives) due to the fact that we factor by $G_{\mathbf{V}}$. Let $\pi : \widetilde{Z}_{\mathbf{D}} \to Z_{\mathbf{D}}$ be the morphism whose restriction to $\Lambda^s_{\mathbf{D},\mathbf{V}}/G_{\mathbf{V}}$ is $\pi_{\mathbf{V}}$ for any $\mathbf{V}$. From 6.8 and 6.9 we deduce the following result.

COROLLARY 6.12. — *We have canonically* $IC(Z_{\mathbf{D}}) = \pi_*(\mathbf{C})$ *(as complexes on* $Z_{\mathbf{D}}$*).*

6.13. The action of W on $Z_{\mathbf{D}}$ given by 1.5, 2.2 is denoted by $w : z \mapsto w(z)$. From definitions one checks that this action is through morphisms of algebraic varieties. Since $IC(Z_{\mathbf{D}})$ is canonically attached to $Z_{\mathbf{D}}$, for any $w \in W$ we have a canonical isomorphism $\gamma_w : w^*IC(Z_{\mathbf{D}}) \xrightarrow{\sim} IC(Z_{\mathbf{D}})$. Moreover, for $w, w' \in W$, $\gamma_{ww'}$ is equal to the composition

$$w'^*w^*IC(Z_{\mathbf{D}}) \xrightarrow{w'^*\gamma_w} w'^*IC(Z_{\mathbf{D}}) \xrightarrow{\gamma_{w'}} IC(Z_{\mathbf{D}}).$$

In other words, the action of W on $Z_{\mathbf{D}}$ lifts canonically to an action of W on $IC(Z_{\mathbf{D}})$ hence (by 6.12) to an action of W on the complex

$\pi_*(\mathbf{C})$. In particular, by passage to stalks, we see that for any W-orbit $\mathcal{O}$ on $Z_{\mathbf{D}}$ we have a natural action of W on the cohomology spaces of $\sqcup_{z\in\mathcal{O}}\pi^{-1} \subset (z)$. Also, we have an induced W-action on the cohomology spaces of $\widetilde{Z}^0_{\mathbf{D}} = \pi^{-1} \subset (Z^0_{\mathbf{D}}) = \sqcup_{\mathbf{V}}\Lambda_{\mathbf{D},\mathbf{V},0}/G_{\mathbf{V}}$.

6.14. By 2.3, 2.4 we have an action of $G_{\mathbf{D}} \times \mathbf{C}^*$ on $Z_{\mathbf{D}}$. This is an algebraic group action. Moreover, $G_{\mathbf{D}}\times\mathbf{C}^*$ acts naturally on $\widetilde{Z}_{\mathbf{D}}$ so that π is $(G_{\mathbf{D}} \times \mathbf{C}^*)$-equivariant. The construction of the W-action in 6.13 extends automatically to the $(G_{\mathbf{D}} \times \mathbf{C}^*)$-equivariant setting in the same way as the construction [L1] of the Springer representation was extended to the equivariant setting in [L2]. This gives for example a natural W-action on

$$H_*^{G_{\mathbf{D}}\times\mathbf{C}^*}(\widetilde{Z}^0_{\mathbf{D}}) = \oplus_{\mathbf{V}} H_*^{G_{\mathbf{D}}\times\mathbf{C}^*}(\Lambda_{\mathbf{D},\mathbf{V},0}/G_{\mathbf{V}})$$

(equivariant homology) where $\widetilde{Z}^0_{\mathbf{D}} = \pi^{-1} \subset (Z^0_{\mathbf{D}})$.

6.15. Consider the fibre product $\widetilde{Z}^0_{\mathbf{D}} \times_{Z^0_{\mathbf{D}}} \widetilde{Z}^0_{\mathbf{D}}$. (This is homeomorphic to a variety in [N2, Sec.7].) Just as in [L1], from the W-action on $\pi_*(\mathbf{C})$ in 6.13, we obtain a $W \times W$-action on $H_*^{G_{\mathbf{D}}\times\mathbf{C}^*}(\widetilde{Z}^0_{\mathbf{D}} \times_{Z^0_{\mathbf{D}}} \widetilde{Z}^0_{\mathbf{D}})$.

7. Weight spaces.

7.1. From the definition of $\pi : \widetilde{Z}_{\mathbf{D}} \to Z_{\mathbf{D}}$, we have a canonical "weight" decomposition

$$\pi_*(\mathbf{C}) = \oplus_{\mathbf{V}}(\pi_{\mathbf{V}})_*(\mathbf{C})$$

where $\mathbf{V}$ runs over a set of representatives for the isomorphism classes of objects of $\mathcal{C}^0$ such that $\Lambda^s_{\mathbf{D},\mathbf{V}} \neq \emptyset$ and $(\pi_{\mathbf{V}})_*(\mathbf{C})$ is extended to the whole of $Z_{\mathbf{D}}$ by 0 outside ${}_{\mathbf{V}}Z_{\mathbf{D}}$. In this section we describe the relationship between the W-action on $\pi_*(\mathbf{C})$ (see 6.13) and this "weight decomposition".

LEMMA 7.2. — *Let $i{\in}I$ and let $\mathbf{V}{\in}\mathcal{C}^0$ be such that $Z_{\mathbf{D},\mathbf{V}} \cap Z^{E_1}_{\mathbf{D}} \neq \emptyset$. Then*

(a) *there exists $\mathbf{V}' \in \mathcal{C}^0$ such that $\mathbf{V}_j = \mathbf{V}'_j$ for $j \in I - \{i\}$ and $\dim \mathbf{V}_i + \dim \mathbf{V}'_i = \dim \mathbf{D}_i + \sum_{h;h'=i} \dim \mathbf{V}_{h''}$;*

(b) $s_i(Z_{\mathbf{D},\mathbf{V}} \cap Z^{E_1}_{\mathbf{D}}) = Z_{\mathbf{D},\mathbf{V}'} \cap Z^{E_1}_{\mathbf{D}}$.

We can find $(x, p, q, \lambda) \in \Lambda^{s,*s}_{\mathbf{D},\mathbf{V}}$ with $\lambda \in E_1$. Let $U, b : U \to \mathbf{V}_i$ be as in 3.2(a). Since $\lambda \in E_1$, we have $\lambda_i \neq 0$. By the argument in 3.4, b is surjective. Hence $\dim \mathbf{V}_i \geq \dim U = \dim \mathbf{D}_i + \sum_{h;h'=i} \dim \mathbf{V}_{h''}$ and (a) follows.

We prove (b). Since for $\lambda \in E_1$ we have $\Lambda^{s,*s}_{\mathbf{D},\mathbf{V},\lambda} = \Lambda_{\mathbf{D},\mathbf{V},\lambda}$ and $\Lambda^{s,*s}_{\mathbf{D},\mathbf{V}',\lambda'} = \Lambda_{\mathbf{D},\mathbf{V}',\lambda'}$ where $\lambda' = s_i(\lambda) \in E_1$,(see 6.1), it suffices to show that the following diagram of sets is commutative:

$$\begin{array}{ccccc} \Lambda_{\mathbf{D},\mathbf{V},\lambda} & \xleftarrow{r} & F & \xrightarrow{r'} & \Lambda_{\mathbf{D},\mathbf{V}',\lambda'} \\ \downarrow & & & & \downarrow \\ Z_{\mathbf{D}} & & \xrightarrow{s_i} & & Z_{\mathbf{D}} \end{array}$$

for $\lambda \in E_1$. Here the left vertical map is induced by ϑ', the right vertical map is the analogous map for $\mathbf{V}', \lambda'$, and r, r' are as in 3.3, 3.5.

Let $((x,p,q);(x',p',q')) \in F$. Let $\pi \in \widetilde{\mathcal{F}}_{\mathbf{D}}$ (resp. $\pi' \in \widetilde{\mathcal{F}}_{\mathbf{D}}$) be defined in terms of $(x,p,q) \in \Lambda_{\mathbf{D},\mathbf{V},\lambda}$ (resp. $(x',p',q') \in \Lambda_{\mathbf{D},\mathbf{V}',\lambda'}$) as in 4.6. We must show that $\pi' = s_i^\lambda(\pi)$. It is enough to show that

(c) $$\pi'_{[f]} = \pi_{s_i^\lambda[f]}$$

for any $f \in \mathbf{I}$ of form $i_1, i_2, \dots, i_s$. By definition,

(d) $$\pi'_{[f]} = q'_{i_1} x'_{i_1,i_2} x'_{i_2,i_3} \cdots x'_{i_{s-1},i_s} p'_{i_s}$$

where the product of the x' is taken to be 1 if $s = 1$. We wish to convert the right hand side of (d) into an expression involving only q_j, x_{kl}, p_j (rather than q'_j, x'_{kl}, p'_j). We will achieve this by a repeated use of the identities 3.2(b2),(c). Assume first that $s = 1$ so that f is j and $\pi'_{[f]} = q'_j p'_j$ for some j. If $j \neq i$, then by 3.2(c) we have $q'_j p'_j = q_j p_j = \pi_{[f]} = \pi_{s_i^\lambda[f]}$. If $j = i$, then by 3.2(b2) we have

$$q'_i p'_i = q_i p_i + \lambda_i = \pi_{[f]+\lambda_i u_i} = \pi_{s_i^\lambda[f]}.$$

Assume now that $s \geq 2$. In the right hand side of (d) we may

– replace any two consecutive factors $x'_{i_{t-1},i_t} x'_{i_t,i_{t+1}}$ such that $i_{t-1} = i_{t+1}, i_t = i$ by $x_{i_{t-1},i_t} x_{i_t,i_{t+1}} - \varepsilon_{i_t,i_{t+1}} \lambda_i$ (using 3.2(b2));

– replace any two consecutive factors $x'_{i_{t-1},i_t} x'_{i_t,i_{t+1}}$ such that $i_{t-1} \neq i_{t+1}, i_t = i$ by $x_{i_{t-1},i_t} x_{i_t,i_{t+1}}$ (using 3.2(b2)),

– if $i_1 = i$ we replace $q'_i x'_{i,i_2}$ by $q_i x_{i,i_2}$ (using 3.2(b2)),

– if $i_s = i$ we replace $x'_{i_{s-1},i} p_i$ by $x_{i_{s-1},i} p_i$ (using 3.2(b2)),

– the remaining factors will be of the form x'_{kl} or q'_k or p'_l with $k \neq i \neq l$ and can be replaced by x_{kl} or q_k or p_l (using 3.2(c)).

The resulting expression is clearly equal to $\pi_{s_i^\lambda[f]}$. This proves (c) hence also (b). The lemma is proved.

PROPOSITION 7.3. — *The subvarieties* ${}_{\mathbf{V}}Z_{\mathbf{D}}$ *of* $Z_{\mathbf{D}}$ *(for various* $\mathbf{V}$*) are permuted among themselves by the* W*-action on* $Z_{\mathbf{D}}$.

It suffices to show that, given $i \in I$ and $\mathbf{V} \in \mathcal{C}^0$ such that ${}_{\mathbf{V}}Z_{\mathbf{D}} \neq \emptyset$, we have $s_i({}_{\mathbf{V}}Z_{\mathbf{D}}) = {}_{\mathbf{V}'}Z_{\mathbf{D}}$ for some $\mathbf{V}'$. As in the proof of 6.3, $Z_{\mathbf{D},\mathbf{V}} \cap Z_{\mathbf{D}}^{E_1}$ is open dense in ${}_{\mathbf{V}}Z_{\mathbf{D}}$. Hence it is suffices to show that, given $i \in I$ and $\mathbf{V} \in \mathcal{C}^0$ such that $Z_{\mathbf{D},\mathbf{V}} \cap Z_{\mathbf{D}}^{E_1} \neq \emptyset$, we have $s_i(Z_{\mathbf{D},\mathbf{V}} \cap Z_{\mathbf{D}}^{E_1}) = Z_{\mathbf{D},\mathbf{V}'} \cap Z_{\mathbf{D}}^{E_1}$ for some $\mathbf{V}'$. But this follows from Lemma 7.2. The proposition is proved.

7.4. From the proof of 7.3 we see that if ${}_{\mathbf{V}}Z_{\mathbf{D}} \neq \emptyset$ and $w \in W$, then $w({}_{\mathbf{V}}Z_{\mathbf{D}}) = {}_{\widetilde{\mathbf{V}}}Z_{\mathbf{D}}$ where $\widetilde{\mathbf{V}} \in \mathcal{C}^0$ is characterized by the equation

$$\sum_j \dim \mathbf{D}_j \varpi_j - \sum_j \dim \widetilde{\mathbf{V}}_j \alpha_j = w\Big(\sum_j \dim \mathbf{D}_j \varpi_j - \sum_j \dim \mathbf{V}_j \alpha_j\Big)$$

in E at least if $w = s_i$; but then this automatically holds for general w. It follows that w carries the summand $(\pi_{\mathbf{V}})_*(\mathbf{C})$ of $\pi_*(\mathbf{C})$ onto the summand $(\pi_{\widetilde{\mathbf{V}}})_*(\mathbf{C})$ where $\widetilde{\mathbf{V}}$ is as above.

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