Annales de l'institut Fourier

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Annales de l'institut Fourier, tome 50, nº 1 (2000), p. 35-65
<http://www.numdam.org/item?id=AIF_2000__50_1_35_0>

Ann. Inst. Fourier, Grenoble
50, 1 (2000), 35–65

GALOIS CO-DESCENT FOR ÉTALE WILD KERNELS AND CAPITULATION

by M. KOLSTER* and A. MOVAHHEDI

Introduction.

For a number field F, the classical wild kernel - denoted by $WK_2(F)$ - is the kernel of all local power norm residue symbols on $K_2(F)$, in other words it fits into Moore's exact sequence

$$0 \to WK_2(F) \to K_2(F) \to \bigoplus_v \mu(F_v) \to \mu(F) \to 0,$$

where v runs through all finite and real infinite primes of F, and $\mu(F_v)$ and $\mu(F)$ denote the groups of roots of unity of the local field F_v and of F, respectively. For a fixed prime number p, the p-primary part $WK_2(F)\{p\}$ of $WK_2(F)$ has another description in terms of étale cohomology: For any finite set S of primes in F containing the p-adic primes and the real infinite primes, we have

$$WK_2(F)\{p\} = \ker\,(H^2_{\text{ét}}(o_F^S, \mathbb{Z}_p(2)) \to \bigoplus_{v\in S} H^2(F_v, \mathbb{Z}_p(2))).$$

This property immediately leads to the definition of the higher *étale wild kernels* for $i \geqslant 2$:

$$WK^{\text{ét}}_{2i-2}(F) := \ker\,(H^2_{\text{ét}}(o_F^S, \mathbb{Z}_p(i)) \to \bigoplus_{v\in S} H^2(F_v, \mathbb{Z}_p(i))).$$

The étale wild kernels play a similar role in étale cohomology, étale K-theory and Iwasawa-theory as the p-primary parts A'_F of the S-class groups

*Research partially supported by NSERC grant #OGP 0042510.
Keywords : Capitulation – Iwasawa – K-theory – Wild kernel.
Math classification : 11R23 – 11R34 – 11R70.

of F. For these, Galois co-descent is classically described by genus theory. The main result of this paper proves an analogous genus formula for the étale wild kernels of a cyclic extension L/F of degree p, p odd. Let $G = \mathrm{Gal}(L/F)$. We first show that the transfer map $WK_{2i-2}^{\text{ét}}(L)_G \to WK_{2i-2}^{\text{ét}}(F)$ is onto except in a very special situation, and we determine its kernel as the cokernel of a certain cup-product which is obtained as follows: Let $E = F(\mu_p)$, where μ_p consists of the p-th roots of unity and let $\Delta = \mathrm{Gal}(E/F)$. We associate with the extension LE/E a certain set $T_{LE/E}$ of primes of E, consisting of all tamely ramified primes and some undecomposed p-adic primes. Let $\mathrm{Br}^T(E)$ denote the subgroup of the Brauer-group which is supported only at primes in $T_{LE/E}$, and let ${}_p\mathrm{Br}^T(E)$ denote the subgroup of all the elements in $\mathrm{Br}^T(E)$ of exponent p. The target of the cup-product is the $(1-i)-$ eigenspace ${}_p\mathrm{Br}^T(E)^{[1-i]}$, under the action of the Teichmüller character w. Now, let S be the set of primes in E consisting of the p-adic primes, the real infinite primes as well as all primes ramified in LE and denote by o_E^S the ring of S-integers in E. The étale cohomology group $H^1_{\text{ét}}(o_E^S, \mathbb{Z}_p(i))/p$ injects into the $(i-1)-$fold Tate twist of the module E^*/E^{*p} and hence is isomorphic to $D_E^{(i)}/E^{*p}(i-1)$, where $D_E^{(i)} \subset E^{*p}$ can be viewed as the analog of the Tate kernel $(i=2)$. The cup-product is now given by

$$(D_E^{(i)}/E^{*p})^{[1-i]} \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \to ({}_p\mathrm{Br}^T(E))^{[1-i]}.$$

We illustrate the method by finding all Galois p-extensions of $\mathbb{Q}$ for which the p-part of the classical wild kernel is trivial.

We also discuss the Galois co-descent situation for $p = 2$ in the classical case $i = 2$.

Let E_∞ denote the cyclotomic $\mathbb{Z}_p$-extension of E with finite layers E_n. If we assume the Gross Conjecture for E_n with n large, for instance if E is abelian over $\mathbb{Q}$, then the groups $D_{E_n}^{(i)}/E_n^{*p}$ can be described in terms of local conditions at p-adic primes, and are independent of i.

Let F_∞ denote the cyclotomic $\mathbb{Z}_p$-extension of F with finite layers F_n and let A'_n denote the p-part of the p-class group of F_n. The classical capitulation kernel is defined as

$$\mathrm{Cap}_0(F_\infty) = \ker(A'_n \to A'_\infty) \qquad \text{for } n \text{ large.}$$

The study of capitulation kernels under Galois extensions is an essential ingredient in the more general problem of comparing Iwasawa-invariants (cp. e.g. [28]). In Section 3 we introduce similar capitulation kernels

$\mathrm{Cap}_{i-1}(F_\infty)$ for all $i \geqslant 2$ using étale K-theory, and show that they have properties similar to $\mathrm{Cap}_0(F_\infty)$.

Assume now that F is totally real, and let E^+ denote the maximal real subfield of $E = F(\mu_p)$. A conjecture of Greenberg predicts that $\varprojlim A'_n(E^+)$ is finite. Under this assumption we show that for all odd $i \geqslant 3$:

$$\mathrm{Cap}_{i-1}(F_\infty) \cong A'_n(E^+)^{[1-i]} \cong WK^{\text{ét}}_{2i-2}(F_n) \qquad \text{for } n \text{ large.}$$

Therefore the co-descent results from Section 2 imply similar results for $\mathrm{Cap}_{i-1}(F_\infty)$ and for the eigenspaces $A'_n(E^+)^{[1-i]}$, when n is large.

In Section 4, we briefly discuss how our approach can be applied to the simpler problem of Galois co-descent for étale tame kernels. This has already been studied by Assim ([1], [2]) under Leopoldt's conjecture.

Acknowlegdements. — The second-named author would like to thank McMaster University for its hospitality during March 1996, when a preliminary version of this paper was written, which was circulated as a McMaster preprint. The first-named author thanks the IHÉS for its hospitality from January 1999 to March 1999, when the final version was written. Thanks go also to Thong Nguyen Quang Do for helpful comments on the earlier version.

1. Preliminaries.

In this section we briefly recall some of the basic properties of étale K-theory and étale cohomology which are subsequently needed. A more detailed account can be found in [3], [20]. Let F be a number field and p a fixed prime number. Let S be a finite set of primes in F, containing the set S_p of primes above p and the set S_∞ of infinite primes. As usual, $G_S(F)$ denotes the Galois group over F of the maximal algebraic extension of F, which is unramified outside S. We note that the condition on infinite primes only intervenes if $p = 2$ and F is not totally imaginary. Let o_F^S denote the ring of S-integers of F. As is well-known, the étale cohomology groups $H^k_{\text{ét}}(\mathrm{spec}(o_F^S), \mathbb{Z}/p^n\mathbb{Z}(i))$ of $\mathrm{spec}(o_F^S)$ coincide with the Galois-cohomology groups $H^k(G_S(F), \mathbb{Z}/p^n\mathbb{Z}(i))$, and will be denoted by $H^k_{\text{ét}}(o_F^S, \mathbb{Z}/p^n\mathbb{Z}(i))$. Here, as usual, $\mathbb{Z}/p^n\mathbb{Z}(i)$ denotes the i-fold Tate twist of $\mathbb{Z}/p^n\mathbb{Z}$. Furthermore, let

$$H^k_{\text{ét}}(o_F^S, \mathbb{Z}_p(i)) = \varprojlim H^k_{\text{ét}}(o_F^S, \mathbb{Z}/p^n\mathbb{Z}(i))$$

and

$$H^k_{\text{ét}}(o_F^S, \mathbb{Q}_p/\mathbb{Z}_p(i)) = \varinjlim H^k_{\text{ét}}(o_F^S, \mathbb{Z}/p^n\mathbb{Z}(i)).$$

Assume now that p is either odd or that $p = 2$ and F contains $\sqrt{-1}$. Then for $i \geqslant 2$ and $k = 1, 2$ the étale cohomology groups $H^k_{\text{ét}}(o_F^S, \mathbb{Z}_p(i))$ are isomorphic to the higher étale K-theory groups $K^{\text{ét}}_{2i-k}(o_F^S)$, introduced by Dwyer-Friedlander ([8]). Moreover, the relation to Quillen's K-theory groups $K_{2i-k}(o_F^S)$ is provided by a Chern character, which yields split surjective maps with finite kernels

$$K_{2i-k}(o_F^S) \otimes \mathbb{Z}_p \to K^{\text{ét}}_{2i-k}(o_F^S)$$

(cp. [8], [15]), which conjecturally are isomorphisms (recall that for $p = 2$, F contains $\sqrt{-1}$). Borel's results (cp. [4]) then imply that the groups $K^{\text{ét}}_{2i-2}(o_F^S)$ are finite and that the groups $K^{\text{ét}}_{2i-1}(o_F^S)$ are finitely generated of rank $r_1 + r_2$ if i is odd, and of rank r_2 if i is even, where as usual r_1 and r_2 denote the number of real and pairs of conjugate complex embeddings of F, respectively. We note that the *odd* étale K-theory groups are independent of the choice of the set S of primes: If $H^*(F, \ \ )$ denotes the absolute Galois cohomology groups of F then, in fact, the localization sequence in étale cohomology (cp. [36, Proposition 1]) implies that

$$H^1_{\text{ét}}(o_F^S, \mathbb{Z}_p(i)) \cong H^1(F, \mathbb{Z}_p(i)) \qquad \forall i \geqslant 2.$$

We therefore simply denote the odd étale K-theory groups by $K^{\text{ét}}_{2i-1}(F)$. The torsion subgroup of $K^{\text{ét}}_{2i-1}(F)$ is isomorphic to $H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(i))$.

In the special case $i = 2$ more is known: There exist isomorphisms

$$K_2(o_F^S) \otimes \mathbb{Z}_p \to H^2_{\text{ét}}(o_F^S, \mathbb{Z}_p(2))$$

and

$$K_3^{nd}(F) \otimes \mathbb{Z}_p \to H^1(F, \mathbb{Z}_p(2))$$

without any restrictions on the prime p and the number field F (cp. [36], [22]). Here $K_3^{nd}(F)$ denotes the indecomposable K_3-group of F, i.e. $K_3(F)$ divided by the image of the Milnor group $K_3^M(F)$, which is 2-torsion.

More recently, Kahn ([18]) and Rognes-Weibel ([32]) have determined the kernel and cokernel of the 2-adic Chern character

$$K_{2i-k}(o_F) \otimes \mathbb{Z}_2 \to H^k_{\text{ét}}(o_F, \mathbb{Z}_2(i)),$$

which in general are non-trivial.

The following result is due to B. Kahn (cp. [16, Theorem 2.1, Proposition 6.1]):

THEOREM 1.1.— *Let L/F be a Galois extension of number fields with Galois group G and let S be a finite set of primes in F, containing the primes which are ramified in L. There is an exact sequence*

$$0 \to H^1(G, K_3^{nd}(L)) \to K_2(o_F^S) \to K_2(o_L^S)^G \to H^2(G, K_3^{nd}(L)) \to 0.$$

The following étale analog is well-known (cp. [1], [5]), however we include a proof, since the sources are not easily accessible:

THEOREM 1.2.— *Let p be an odd prime and let L/F be a Galois p-extension of number fields with Galois group G. Let S be a finite set of primes, containing the primes above p and the primes which ramify in L. Then for $i \geqslant 2$ there is an exact sequence*

$$0 \to H^1(G, K_{2i-1}^{\text{ét}}(L)) \to K_{2i-2}^{\text{ét}}(o_F^S) \to K_{2i-2}^{\text{ét}}(o_L^S)^G \to H^2(G, K_{2i-1}^{\text{ét}}(L)) \to 0.$$

Proof.— Consider the Hochschild-Serre spectral sequence

$$E_2^{pq} = H^p(G, H_{\text{ét}}^q(o_L^S, \mathbb{Z}_p(i))) \Rightarrow H_{\text{ét}}^{p+q}(o_F^S, \mathbb{Z}_p(i)).$$

Since $H_{\text{ét}}^0(o_L^S, \mathbb{Z}_p(i)) = 0$ ([36, Lemme 7]), all terms E_2^{p0} vanish. On the other hand, $cd_p(G_S(F)) = 2$ and hence $H_{\text{ét}}^q(o_F^S, \mathbb{Z}_p(i)) = H_{\text{ét}}^q(o_L^S, \mathbb{Z}_p(i)) = 0$ for all $q \geqslant 3$. The spectral sequence therefore yields

$$E^1 \cong E_\infty^{01} \cong E_2^{01},$$

i.e. an isomorphism

$$K_{2i-1}^{\text{ét}}(F) \cong K_{2i-1}^{\text{ét}}(L)^G,$$

as well as the exact sequence

$$0 \to E_2^{11} \to E^2 \to E_2^{02} \to E_2^{21} \to 0,$$

which is precisely the claim. □

As a by-product, we obtained the fact that the odd étale K-groups satisfy Galois descent. Note that this, in the form

$$H^1(F, \mathbb{Z}_p(i)) \cong H^1(L, \mathbb{Z}_p(i))^G,$$

remains true for $p = 2$.

On the other hand we have Galois co-descent for the even étale K-theory groups $K^{\text{ét}}_{2i-2}(o^S_F)$:

PROPOSITION 1.3. — *Let p be odd and L/F a Galois p-extension of number fields with Galois-group G. If S contains the primes above p and the ramified primes of L/F, then*

$$K^{\text{ét}}_{2i-2}(o^S_L)_G \cong K^{\text{ét}}_{2i-2}(o^S_F).$$

Proof. — This follows as above using the Tate spectral sequence (cp. [35], [17], [26]). □

Now $K^{\text{ét}}_{2i-2}(o^S_L)$ is finite, and hence this proposition together with Theorem 1.2 yields

COROLLARY 1.4. — *For any cyclic p-extension L/F (p odd) of number fields with Galois group G, the quotient*

$$\frac{|H^2(G, K^{\text{ét}}_{2i-1}(L)|}{|H^1(G, K^{\text{ét}}_{2i-1}(L)|}$$

is trivial.

Remark 1.5. — The previous results depended only upon two facts:

$$cd_p(G_S(F)) \leqslant 2 \quad \text{and} \quad H^0(G_S(F), \mathbb{Z}_p(i)) = 0.$$

Therefore analogous results also hold for example for finite extensions of local fields, thus, for a Galois p-extension E/F of local fields with Galois group G, we have an exact sequence

$$\begin{aligned} 0 \to H^1(G, H^1(E, \mathbb{Z}_p(i))) &\to H^2(F, \mathbb{Z}_p(i)) \\ &\to H^2(E, \mathbb{Z}_p(i))^G \to H^2(G, H^1(E, \mathbb{Z}_p(i))) \to 0, \end{aligned}$$

and an isomorphism

$$H^2(E, \mathbb{Z}_p(i))_G \cong H^2(F, \mathbb{Z}_p(i)).$$

Again, in the case $i = 2$, more information on co-descent is available, i.e. no restrictions on F are necessary to also include results concerning the 2-primary part.

The following result is easily obtained from [16, Théorème 5.1] :

PROPOSITION 1.6. — *Let L/F be a finite Galois extension of number fields with Galois group G and let S be a finite set of primes in F, containing the primes which ramify in L/F. Then, there is a short exact sequence*

$$0 \to K_2(o_L^S)_G \to K_2(o_F^S) \to \bigoplus_{v \in S_\infty^r} \mu_2 \to 0,$$

where S_∞^r consists of the real infinite primes in F which ramify in L.

Finally, we recall the definition of the higher étale wild kernels (cp. [3], [20], [29]):

$$WK_{2i-2}^{\text{ét}}(F) = \ker\left(H_{\text{ét}}^2(o_F^S, \mathbb{Z}_p(i)) \to \bigoplus_{v \in S} H^2(F_v, \mathbb{Z}_p(i))\right).$$

The definition is independent of the choice of the set S containing S_p, and part of the Poitou-Tate duality sequence yields the exact sequence

$$0 \to WK_{2i-2}^{\text{ét}}(F) \to K_{2i-2}^{\text{ét}}(o_F^S) \\ \to \bigoplus_{v \in S} H^2(F_v, \mathbb{Z}_p(i)) \to H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^* \to 0,$$

where $*$ indicates the Pontrjagin dual. Moreover by local duality

$$H^2(F_v, \mathbb{Z}_p(i)) \cong H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^*.$$

The groups $H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$ and $H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$ are finite cyclic for $i \neq 1$.

The étale wild kernels are the analogs of the p-part of the classical wild kernel $WK_2(F)$ - defined for any number field F - which occurs in Moore's exact sequence of power norm symbols (cp. [23]):

$$0 \to WK_2(F) \to K_2(F) \to \bigoplus_v \mu(F_v) \to \mu(F) \to 0,$$

where v runs through all finite primes and all real infinite primes of F, and $\mu(F_v)$ and $\mu(F)$ denote the group of roots of unity of F_v and of F respectively. If S is a finite set of primes in F containing S_p and S_∞, then we obtain an exact sequence of finite groups

$$0 \to WK_2(F)\{p\} \to K_2(o_F^S)\{p\} \to \bigoplus_{v \in S} \mu(F_v)\{p\} \to \mu(F)\{p\} \to 0.$$

Here, for an abelian group A, we use the notation $A\{p\}$ for the p-primary part of A.

2. Galois co-descent for the étale wild kernel.

Let p be an odd prime and let L/F be a cyclic extension of number fields of degree p with Galois group G. In this section, for any local or global field K, we denote by K_∞ the cyclotomic $\mathbb{Z}_p$-extension of K with finite layers K_n. We also assume that $i \geqslant 2$. We obtain necessary and sufficient conditions for the étale wild kernel $WK_{2i-2}^{\text{ét}}(L)$ to satisfy Galois co-descent. This approach also yields a genus"-formula comparing the sizes of $WK_{2i-2}^{\text{ét}}(L)^G$ and $WK_{2i-2}^{\text{ét}}(F)$. Let S be the finite set of primes in F, containing the set S_p of all primes above p, as well as all primes which ramify in L. We denote by S_L the set of primes in L above S. Moreover, let $\tilde{\oplus}_{v\in S} H^2(F_v, \mathbb{Z}_p(i))$ be the kernel of the surjection

$$\bigoplus_{v\in S} H^2(F_v, \mathbb{Z}_p(i)) \to H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^*.$$

Then by definition of the étale wild kernel from section 1, we have the following short exact sequence:

$$0 \to WK_{2i-2}^{\text{ét}}(F) \to K_{2i-2}^{\text{ét}}(o_F^S) \to \tilde{\bigoplus_{v\in S}} H^2(F_v, \mathbb{Z}_p(i)) \to 0.$$

By Proposition 1.3 the group $K_{2i-2}^{\text{ét}}(o_L^S)$ satisfies Galois co-descent. The following commutative diagram:

$$\begin{array}{ccccccc} & WK_{2i-2}^{\text{ét}}(L)_G & \to & K_{2i-2}^{\text{ét}}(o_L^S)_G & \to & \left(\tilde{\bigoplus_{w\in S_L}} H^2(L_w, \mathbb{Z}_p(i))\right)_G & \to 0 \\ & \downarrow & & \downarrow \wr & & \downarrow & \\ 0 \to & WK_{2i-2}^{\text{ét}}(F) & \to & K_{2i-2}^{\text{ét}}(o_F^S) & \to & \tilde{\bigoplus_{v\in S}} H^2(F_v, \mathbb{Z}_p(i)) & \to 0 \end{array}$$

then shows that

$$\operatorname{coker}(WK_{2i-2}^{\text{ét}}(L)_G \to WK_{2i-2}^{\text{ét}}(F)) \cong \ker\left((\tilde{\bigoplus_{w\in S_L}} H^2(L_w, \mathbb{Z}_p(i))_G \to \tilde{\bigoplus_{v\in S}} H^2(F_v, \mathbb{Z}_p(i))\right)$$

and

$$\ker(WK_{2i-2}^{\text{ét}}(L)_G \to WK_{2i-2}^{\text{ét}}(F)) \cong \operatorname{coker}\left(K_{2i-2}^{\text{ét}}(o_L^S)^G \to \left(\tilde{\bigoplus_{w\in S_L}} H^2(L_w, \mathbb{Z}_p(i))\right)^G \right).$$

Before we compute the first group we need a preliminary result: Let M/N be a cyclic extension of degree p, p odd, of global or local fields of characteristic $\neq p$, and let G denote the Galois group of M/N. Furthermore, let N_∞ denote the cyclotomic $\mathbb{Z}_p$-extension of N.

There are two maps relating the cohomology groups $H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k))$ and $H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))$, where we assume $k \in \mathbb{Z}$, $k \neq 0$: The natural map $H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)) \to H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))$ and the norm map $H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)) \to H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))$. The first one induces an isomorphism

$$H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)) \xrightarrow{\sim} H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))^G,$$

which implies immediately that either both groups are trivial or both groups are non-trivial. Assume now that $H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k))$ is non-trivial. Then the order of $H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k))$ is the maximal power p^m, such that the Galois group $\mathrm{Gal}(N(\mu_{p^m})/N)$ has exponent k. If $M \not\subset N_\infty$, then $[M(\mu_{p^m}) : M] = [N(\mu_{p^m}) : N]$, and therefore G acts trivially on $H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))$. Therefore, in this case, the natural map is an isomorphism, and hence the norm map has both kernel and cokernel of order p. On the other hand, if $M \subset N_\infty$ and say $H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)) \cong (\mathbb{Z}/p^m\mathbb{Z})(k)$, then $H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k)) \cong (\mathbb{Z}/p^{m+1}\mathbb{Z})(k)$, and - p being odd - the norm

$$(\mathbb{Z}/p^{m+1}\mathbb{Z})(k) \to (\mathbb{Z}/p^m\mathbb{Z})(k)$$

is surjective, and therefore induces an isomorphism

$$H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))_G \xrightarrow{\sim} H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)).$$

We summarize:

LEMMA 2.1.— *Let $k \in \mathbb{Z}$, $k \neq 0$ and $H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)) \neq 0$.*

i) *If $M \not\subset N_\infty$, then G acts trivially on $H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))$, and hence the natural map*

$$H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)) \to H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))$$

is an isomorphism, whereas the norm map has kernel and cokernel of order p.

ii) *If $M \subset N_\infty$, then G acts non-trivially on $H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))$ and the norm induces an isomorphism*

$$H^0(M, \mathbb{Q}_p/\mathbb{Z}_p(k))_G \xrightarrow{\sim} H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)).$$

The non-vanishing of $H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k))$ can be characterized as follows: Let $d = [N(\mu_p) : N]$. Then

$$H^0(N, \mathbb{Q}_p/\mathbb{Z}_p(k)) \neq 0 \Leftrightarrow k \equiv 0 \bmod d.$$

Let us now study the question of co-descent for $\tilde{\oplus}_{w \in S_L} H^2(L_w, \mathbb{Z}_p(i))$. Using local duality the problem is equivalent to computing the cokernel of the map

$$\tilde{\bigoplus_{v \in S}} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \to \Big(\tilde{\bigoplus_{w \in S_L}} H^0(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i))\Big)^G.$$

As we noted above, we have isomorphisms

$$H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \cong H^0(L, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^G$$

and

$$\bigoplus_{v \in S} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \cong \Big(\bigoplus_{w \in S_L} H^0(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i))\Big)^G,$$

hence the above cokernel is isomorphic to the kernel of

$$H^0(L, \mathbb{Q}_p/\mathbb{Z}_p(1-i))_G \to \Big(\bigoplus_{w \in S_L} H^0(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i))\Big)_G.$$

We consider the commutative diagram

$$\begin{array}{ccccc} & & H^0(L, \mathbb{Q}_p/\mathbb{Z}_p(1-i))_G & \to & \Big(\bigoplus_{w \in S_L} H^0(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i))\Big)_G \\ & & \downarrow & & \downarrow \\ 0 & \to & H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) & \to & \bigoplus_{v \in S} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \end{array}$$

induced by the norm maps. It is now clear that the map in the top row is *not* injective, if and only if Galois co-descent fails globally for $H^0(L, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$, but holds locally for all $w \in S_L$, in which case the kernel is of order p. If $v \in S$ is decomposed in L, then obviously co-descent holds. We now define $T_{L/F}^{(i)}$ to be the set of *undecomposed* primes $v \in S$, such that Galois co-descent fails for $H^0(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$. By Lemma 2.1, an undecomposed prime v lies in $T_{L/F}^{(i)}$ if and only if $H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \neq 0$ and $L_w \not\subset F_{v,\infty}$. Let $d = [F(\mu_p) : F]$. Then it is clear from the definition that

$$T_{L/F}^{(i)} = T_{L/F}^{(j)} \quad \text{if} \quad i \equiv j \bmod d.$$

Let us analyze this set a little further:

LEMMA 2.2. — i) *$T^{(i)}_{L/F}$ contains all tamely ramified primes:*

$$S \setminus S_p \subset T^{(i)}_{L/F} \subset S.$$

ii) *Assume that $L \not\subset F_\infty$ and $i \equiv 1 \bmod d$. Then, for large n, the set $T^{(i)}_{L_n/F_n}$ contains all undecomposed p-adic primes.*

Proof. — Let v be any prime in $S \setminus S_p$. Then F_v contains the p-th roots of unity μ_p, which shows that $H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \neq 0$. Moreover, $F_{v,\infty}$ is the maximal unramified pro-p-extension of F_v, which shows that $L_w \not\subset F_{v,\infty}$. This proves i). To prove ii), it suffices to choose n large enough so that no p-adic prime of L_n decomposes in L_{n+1}. □

We can now formulate our first result in terms of the set $T^{(i)}_{L/F}$.

PROPOSITION 2.3. — *The canonical map*

$$WK^{\text{ét}}_{2i-2}(L)_G \to WK^{\text{ét}}_{2i-2}(F)$$

induced by the corestriction is surjective precisely in the following situations:

i) *$T^{(i)}_{L/F} \neq \varnothing$;*

ii) *$T^{(i)}_{L/F} = \varnothing$ and either $i \not\equiv 1 \bmod d$ or $L \subset F_\infty$.*

In the exceptional case where $T^{(i)}_{L/F} = \varnothing$, $i \equiv 1 \bmod d$ and $L \not\subset F_\infty$, the cokernel of $WK^{\text{ét}}_{2i-2}(L)_G \to WK^{\text{ét}}_{2i-2}(F)$ is cyclic of order p.

Remark 2.4. — The possibility of the failure of Galois co-descent in Proposition 2.3 was already observed in [2]. The situations where this happens are easily described: First of all we must have $i \equiv 1 \bmod d$ and $L \not\subset F_\infty$, in which case, for any n, the set $T^{(i)}_{L/F} = \varnothing$ if and only if $T^{(i)}_{L_n/F_n} = \varnothing$. Now, choose n large enough, such that no p-adic prime in L_n decomposes in L_{n+1}. By Lemma 2.2, the set $T^{(i)}_{L_n/F_n} = \varnothing$ precisely when L_n/F_n is unramified and all p-adic primes of F_n split in L_n. Thus, the exceptional case occurs for L/F if and only if the following two conditions hold:

i) $i \equiv 1 \bmod d$;

ii) $\varprojlim A'_n \neq 0$ and L_∞/F_∞ is an unramified cyclic extension of degree p, in which all primes above p split.

Example 2.5. — Assume that the prime p is irregular and let $F = \mathbb{Q}(\mu_p)$. Then F possesses a cyclic extension L of degree p inside the Hilbert p-class field, which is disjoint from F_∞. Therefore the canonical map $WK_{2i-2}^{\text{ét}}(L)_G \to WK_{2i-2}^{\text{ét}}(F)$ is not surjective for any $i \geqslant 2$.

We recall that by Lemma 2.1, the natural map

$$H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \to H^0(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$$

is an isomorphism for $v \in T_{L/F}^{(i)}$, and hence the norm map

$$H^0(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \to H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$$

can be identified with the p-th power map on $H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$, which is induced by the p-th power map on $\mathbb{Q}_p/\mathbb{Z}_p(1-i)$. Hence we have an exact sequence for $v \in T_{L/F}^{(i)}$:

$$\begin{aligned} 0 \to H^0(F_v, \mathbb{Z}/p\mathbb{Z}(1-i)) &\to H^0(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i))_G \\ &\to H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \to H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))/p \to 0. \end{aligned}$$

The dual sequence then reads:

$$\begin{aligned} 0 \to {}_pH^2(F_v, \mathbb{Z}_p(i)) &\to H^2(F_v, \mathbb{Z}_p(i)) \\ &\to H^2(L_w, \mathbb{Z}_p(i))^G \to H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i)) \to 0. \end{aligned}$$

If we compare this sequence with the one mentioned in Remark 1.5, we see that we have an isomorphism

$$H^2(G, H^1(L_w, \mathbb{Z}_p(i)) \cong H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i)).$$

We make this isomorphism more explicit in Proposition 2.9.

Let us now consider the problem of the surjectivity of the homomorphism

$$K_{2i-2}^{\text{ét}}(o_L^S)^G \to \Big(\tilde{\bigoplus_{w \in S_L}} H^2(L_w, \mathbb{Z}_p(i))\Big)^G.$$

If $T_{L/F}^{(i)} = \varnothing$, then we assume that either $i \not\equiv 1 \bmod d$, or that $L \subset F_\infty$, so that we have Galois co-descent for $(\tilde{\oplus}_{w \in S_L} H^2(L_w, \mathbb{Z}_p(i)))$, i.e. $WK_{2i-2}^{\text{ét}}(L)_G \to WK_{2i-2}^{\text{ét}}(F)$ is surjective. In particular this implies that the map β in the following commutative diagram is surjective:

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \bigoplus_{v\in T^{(i)}_{L/F}} ({}_pH^2(F_v,\mathbb{Z}_p(i))) & \rightarrow & B & & \\
 & & \downarrow & & \downarrow & & \\
K^{\text{ét}}_{2i-2}(o_F^S) & \rightarrow & \bigoplus_{v\in S} H^2(F_v,\mathbb{Z}_p(i)) & \rightarrow & H^0(F,\mathbb{Q}_p/\mathbb{Z}_p(1-i))^* & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
K^{\text{ét}}_{2i-2}(o_L^S)^G & \stackrel{\alpha}{\rightarrow} & \bigoplus_{v\in S}(\bigoplus_{w|v} H^2(L_w,\mathbb{Z}_p(i)))^G & \stackrel{\beta}{\rightarrow} & (H^0(L,\mathbb{Q}_p/\mathbb{Z}_p(1-i))^*)^G & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H^2(G,K^{\text{ét}}_{2i-1}(L)) & \stackrel{\alpha'}{\rightarrow} & \bigoplus_{v\in T^{(i)}_{L/F}} H^2(F_v,\mathbb{Z}/p\mathbb{Z}(i)) & \stackrel{\beta'}{\rightarrow} & C & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}
$$

Here we define B and C to be the kernel and cokernel of the homomorphism

$$H^0(F,\mathbb{Q}_p/\mathbb{Z}_p(1-i))^* \rightarrow (H^0(L,\mathbb{Q}_p/\mathbb{Z}_p(1-i))^*)^G$$

respectively, hence both B and C are either trivial or of order p. More precisely, by Lemma 2.1, they are non-trivial if and only if $i\equiv 1 \bmod d$ and $L\not\subset F_\infty$. In this diagram the columns are exact and also the rows, except possibly at $\oplus_{v\in S}(\oplus_{w|v} H^2(L_w,\mathbb{Z}_p(i)))^G$ and $\oplus_{v\in T^{(i)}_{L/F}} H^2(F_v,\mathbb{Z}/p\mathbb{Z}(i))$. Note that

$$\ker\beta/\operatorname{im}\alpha = \operatorname{coker}\left(K^{\text{ét}}_{2i-2}(o_L^S)^G \rightarrow \left(\tilde{\bigoplus_{w\in S_L}} H^2(L_w,\mathbb{Z}_p(i))\right)^G\right)$$

is precisely the cokernel we want to study.

An easy diagram chase shows:

LEMMA 2.6. — *The surjection*

$$\ker\beta/\operatorname{im}\alpha \rightarrow \ker\beta'/\operatorname{im}\alpha'$$

is an isomorphism if the map

$$\bigoplus_{v\in T^{(i)}_{L/F}} {}_pH^2(F_v,\mathbb{Z}_p(i)) \rightarrow B$$

is surjective (otherwise, its kernel is of order at most p).

In particular, this settles the case $T^{(i)}_{L/F}=\varnothing$:

Corollary 2.7. — *If $T^{(i)}_{L/F} = \varnothing$, then $WK^{\text{ét}}_{2i-2}(L)_G \cong WK^{\text{ét}}_{2i-2}(F)$ if and only if either $i \not\equiv 1 \bmod d$ or $L \subset F_\infty$.*

Thus, for example, in the cyclotomic $\mathbb{Z}_p$-extension, the wild kernels satisfy Galois codescent, whereas, in general, the p-class groups do not.

Let us assume now that $T^{(i)}_{L/F} \neq \varnothing$. Then L is disjoint from F_∞, and therefore the kernel B is non-trivial if and only if $i \equiv 1 \bmod d$. In this case B is clearly isomorphic to ${}_pH^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^*$, and we can characterize the surjectivity of the map $\oplus_{v\in T^{(i)}_{L/F}}\, {}_pH^2(F_v, \mathbb{Z}_p(i)) \to {}_pH^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^*$ as follows:

Lemma 2.8. — *If $T^{(i)}_{L/F} \neq \varnothing$ and $i \equiv 1 \bmod d$, then*

$$\bigoplus_{v\in T^{(i)}_{L/F}} {}_pH^2(F_v, \mathbb{Z}_p(i)) \to {}_pH^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^*$$

is surjective if and only if at least one of the primes in $T^{(i)}_{L/F}$ is undecomposed in the first layer F_1 of the cyclotomic $\mathbb{Z}_p$-extension F_∞/F.

Proof. — It is clear that the map in question is surjective if and only if $|H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))| = |H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))|$ for at least one prime $v \in T^{(i)}_{L/F}$. On the other hand $|H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))| > |H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(1-i))|$ if and only if v splits in F_1. □

We note that any finite place v in F is finitely decomposed in F_∞. Therefore, if n is large enough, all the primes in $T^{(i)}_{L_n/F_n}$ will be undecomposed in F_{n+1}. If $i \equiv 1 \bmod d$, we will assume that $T^{(i)}_{L/F}$ contains at least one prime, which is undecomposed in F_1. We are then left with the determination of $|\ker \beta'/\operatorname{im} \alpha'|$.

The order of $\ker \beta'$ is clearly equal to

$$|\ker \beta'| = \frac{p^{|T^{(i)}_{L/F}|}}{|H^0(F, \mathbb{Z}/p\mathbb{Z}(1-i))|}.$$

To determine the order of $\operatorname{im} \alpha'$ we construct a canonical homomorphism

$$H^2(G, H^1(L, \mathbb{Z}_p(i))) \to H^2(F, \mathbb{Z}/p\mathbb{Z}(i)),$$

which gives rise to a commutative diagram

$$\begin{array}{ccc} H^2(G, H^1(L, \mathbb{Z}_p(i))) & \to & \bigoplus_{v\in T^{(i)}_{L/F}} H^2(G, H^1(L_w, \mathbb{Z}_p(i))) \\ \downarrow & & \wr\downarrow \\ H^2(F, \mathbb{Z}/p\mathbb{Z}(i)) & \to & \bigoplus_{v\in T^{(i)}_{L/F}} H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i)) \end{array}$$

and will factor the map α'.

PROPOSITION 2.9.— *Let M/N be a cyclic extension of degree p of local or global fields of characteristic $\neq p$, where p is an arbitrary prime. Let $G = \mathrm{Gal}(M/N)$. There is a canonical map*

$$H^2(G, H^1(M, \mathbb{Z}_p(i))) \to H^2(N, \mathbb{Z}/p\mathbb{Z}(i))$$

with kernel isomorphic to

$$(H^1(N, \mathbb{Z}_p(i))/p \cap N_{M/N}(H^1(M, \mathbb{Z}/p\mathbb{Z}(i)))) \,/\, N_{M/N}(H^1(M, \mathbb{Z}_p(i))/p).$$

Proof.— We first note that the exact sequence

$$0 \to \mathbb{Z}_p(i) \to \mathbb{Z}_p(i) \to \mathbb{Z}/p\mathbb{Z}(i) \to 0$$

induces an injection

$$H^1(N, \mathbb{Z}_p(i))/p \hookrightarrow H^1(N, \mathbb{Z}/p\mathbb{Z}(i)),$$

and therefore we can view $H^1(N, \mathbb{Z}_p(i))/p$ as a subgroup of $H^1(N, \mathbb{Z}/p\mathbb{Z}(i))$, and similarly for M. Since G is cyclic, we have a canonical isomorphism

$$H^2(G, H^1(M, \mathbb{Z}_p(i))) \cong \hat{H}^0(G, H^1(M, \mathbb{Z}_p(i))) \otimes H^2(G, \mathbb{Z}_p)$$

given by the cup-product. Here $\hat{H}$ denotes Tate-cohomology. Now the group $H^1(M, \mathbb{Z}_p(i))$ satisfies Galois descent as we have seen in the proof of Theorem 1.2, even in the case $p = 2$. Hence

$$\begin{aligned}\hat{H}^0(G, H^1(M, \mathbb{Z}_p(i))) &\cong H^1(N, \mathbb{Z}_p(i))/N_{M/N}(H^1(M, \mathbb{Z}_p(i))) \\ &\cong (H^1(N, \mathbb{Z}_p(i))/p)/N_{M/N}(H^1(M, \mathbb{Z}_p(i))/p).\end{aligned}$$

Now $H^2(G, \mathbb{Z}_p) \cong H^1(G, \mathbb{Q}_p/\mathbb{Z}_p) \cong H^1(G, \mathbb{Z}/p\mathbb{Z})$, since G is cyclic of order p, and we have the cup-product

$$H^1(N, \mathbb{Z}/p\mathbb{Z}(i)) \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \to H^2(N, \mathbb{Z}/p\mathbb{Z}(i))$$

whose kernel is equal to

$$N_{M/N}(H^1(M, \mathbb{Z}/p\mathbb{Z}(i))) \otimes H^1(G, \mathbb{Z}/pZ).$$

To see this, we may assume without loss of generality that N contains μ_p, in which case this product is just a twisted version of the standard cup-product into the Brauer group of F. Restricting the last morphism to the subgroup $H^1(N, \mathbb{Z}_p(i))/p \otimes H^1(G, \mathbb{Z}/p\mathbb{Z})$ yields the result. □

Let us return now to the situation considered before: p is odd and L/F is a cyclic extension of number fields of degree p with Galois group G. We are going to compare the global and local maps constructed in Proposition 2.9. Let $C_v := \operatorname{coker}(H^2(F_v, \mathbb{Z}_p(i)) \to (\oplus_{w|v} H^2(L_w, \mathbb{Z}_p(i)))^G$. Then by definition

$$C_v = 0 \Leftrightarrow v \notin T^{(i)}_{L/F}$$

and $C_v = H^2(G, H^1(L_w, \mathbb{Z}_p(i)) \cong H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i))$ if $v \in T^{(i)}_{L/F}$. The following commutative diagram:

$$\begin{array}{ccccc} & & H^2(G, K^{\text{ét}}_{2i-1}(L)) & \to & \prod_v C_v \\ & & \downarrow & & \downarrow \\ 0 & \to & H^2(F, \mathbb{Z}/p\mathbb{Z}(i)) & \to & \prod_v H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i)) \end{array}$$

then shows that the image of $H^2(G, K^{\text{ét}}_{2i-1}(L))$ in $\prod_v H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i))$ is in fact contained in $\oplus_{v \in T^{(i)}_{L/F}} H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i))$. The injectivity of the localization map $H^2(F, \mathbb{Z}/p\mathbb{Z}(i)) \to \prod_v H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i))$ is proved for instance in [33, Section 2, Lemma 7]. We can now conclude:

LEMMA 2.10.— *The canonical map $H^2(G, K^{\text{ét}}_{2i-1}(L)) \to H^2(F, \mathbb{Z}/p\mathbb{Z}(i))$ induces the map*

$$\alpha' : H^2(G, K^{\text{ét}}_{2i-1}(L)) \to \bigoplus_{v \in T^{(i)}_{L/F}} H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i)).$$

Furthermore:

$$\ker \alpha' \cong [K^{\text{ét}}_{2i-1}(F)/p \cap N_{L/F}(H^1(L, \mathbb{Z}/p\mathbb{Z}(i))]/N_{L/F}(K^{\text{ét}}_{2i-1}(L)/p)$$

and

$$|\operatorname{im} \alpha'| = [K^{\text{ét}}_{2i-1}(F)/p : K^{\text{ét}}_{2i-1}(F)/p \cap N_{L/F}(H^1(L, \mathbb{Z}/p\mathbb{Z}(i)))].$$

Combining this with the calculation of ker β' provides the main result of this section, a "genus formula" for the étale wild kernels for cyclic extensions of degree p:

THEOREM 2.11.— *Let L/F be a cyclic extension of number fields of degree p, p odd, with Galois group G. Assume that $T^{(i)}_{L/F} \neq \varnothing$ and that some $v \in T^{(i)}_{L/F}$ is undecomposed in F_1 if $i \equiv 1 \bmod d$. Then the natural map $WK^{\text{ét}}_{2i-2}(L)_G \to WK^{\text{ét}}_{2i-2}(F)$ is surjective and its kernel has order*

$$\frac{p^{|T^{(i)}_{L/F}|}}{|H^0(F, \mathbb{Z}/p\mathbb{Z}(1-i))| \cdot [K^{\text{ét}}_{2i-1}(F)/p : K^{\text{ét}}_{2i-1}(F)/p \cap N_{L/F}(H^1(L, \mathbb{Z}/p\mathbb{Z}(i)))]}.$$

Remark 2.12. — Let us consider the special case that $i \equiv 1 \bmod d$, and that all p-adic primes of L are undecomposed in L_∞. Then $T^{(i)}_{L/F}$ contains all undecomposed p-adic primes as well as all tamely ramified primes. Hence we can rewrite the formula in the preceding theorem as

$$\frac{|WK^{\text{ét}}_{2i-2}(L)^G|}{|WK^{\text{ét}}_{2i-2}(F)|}$$

$$= \frac{\prod_{\mathfrak{p}|p} d_{\mathfrak{p}}(L/F)\cdot\prod_{\mathfrak{p}\nmid p} e_{\mathfrak{p}}(L/F)}{[L:F]\cdot[K^{\text{ét}}_{2i-1}(F)/p : K^{\text{ét}}_{2i-1}(F)/p\cap N_{L/F}(H^1(L,\mathbb{Z}/p\mathbb{Z}(i)))]}.$$

Here $d_{\mathfrak{p}}(L/F)$ and $e_{\mathfrak{p}}(L/F)$ denote the local degrees and the ramification indices, respectively. If we replace the étale K-theory index by the index $[U'_F : U'_F \cap N_{L/F}(L^*)]$ for the p-units U'_F, then this becomes precisely the genus formula for the p-class groups. We will return to this peculiarity later on.

Example 2.13. — 1) Take $p = i = 3$ and $F = \mathbb{Q}$ the field of rationals. Since $K_4(\mathbb{Z})$ is trivial (cp. [30], [31], [32]), so is $WK^{\text{ét}}_4(\mathbb{Z})$. We are going to give an infinite family of cubic fields L such that $WK^{\text{ét}}_4(L) = 0$. For this, consider the set of primes (see also [37, Remarks page 182])

$$\begin{aligned} P &= \{\ell\ ;\ \ell \equiv 1 \bmod 3 \text{ and } 3^{\frac{\ell-1}{3}} \equiv 1 \bmod \ell\} \\ &= \{\ell\ ;\ \ell \equiv 1 \bmod 3 \text{ and } \sqrt[3]{3} \in \mathbb{Z}/\ell\mathbb{Z}\}. \end{aligned}$$

Obviously, by Hensel's lemma, we have

$$\begin{aligned} P &= \{\ell\ ;\ \mu_3 \subset \mathbb{Q}_\ell \text{ and } \sqrt[3]{3} \in \mathbb{Q}_\ell\} \\ &= \{\ell\ ;\ \ell \text{ splits in } \mathbb{Q}(\mu_3, \sqrt[3]{3})\}. \end{aligned}$$

We are interested in the infinite family (of density $\frac{1}{6} - \frac{1}{18}$) of the primes ℓ in P which do not split in $\mathbb{Q}(\mu_9, \sqrt[3]{3})$. Now let L be the cubic extension of $\mathbb{Q}$ contained in $\mathbb{Q}(\mu_\ell)$ and $G = G(L/\mathbb{Q})$. Then $T^{(3)}_{L/\mathbb{Q}} = \{\ell\}$ and, according to Theorem 2.11, the wild kernel $WK^{\text{ét}}_4(L) = 0$.

2) In this example, we are going to determine the Galois p-extensions M of $\mathbb{Q}$, for which the p-part of the classical wild kernel is trivial. The two cases $p = 3$ and $p \geqslant 5$ are completely different due to the fact that in the latter case the considered fields do not contain the maximal real subfield $\mathbb{Q}(\mu_p)^+$of the cyclotomic field $\mathbb{Q}(\mu_p)$. For $p \geqslant 5$, the Galois p-extensions M of $\mathbb{Q}$ for which $WK_2(M)\{p\} = 0$ are exactly the layers $\mathbb{Q}_n$ of the $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty/\mathbb{Q}$. Indeed, since $\mathbb{Q}_\infty$ is the maximal p-ramified

pro-p-extension of $\mathbb{Q}$, we see that the maximal p-ramified extension of $\mathbb{Q}$ contained in M is a layer $\mathbb{Q}_n$ of $\mathbb{Q}_\infty$. If $M = \mathbb{Q}_n$ then, by Corollary 2.7, $WK_2(M)\{p\} = 0$. Otherwise, choose a tower of degree p cyclic extensions

$$\mathbb{Q}_n = M_0 \subset M_1 \subset \cdots \subset M_r = M.$$

Since $T^{(2)}_{M_1/\mathbb{Q}_n} \neq \varnothing$, we have $WK_2(M_1)\{p\} \neq 0$ (Theorem 2.11). Moreover, for each intermediate extension $M_{\nu+1}/M_\nu$, the canonical map

$$WK_2(M_{\nu+1})\{p\}_{G(M_{\nu+1}/M_\nu)} \to WK_2(M_\nu)\{p\}$$

is surjective (Proposition 2.3), which shows that $WK_2(M)\{p\} \neq 0$. A number field M for which $H^2_{\text{ét}}(o_M, \mathbb{Z}/p\mathbb{Z}) = 0$, is called p-rational [25], [24]. Moreover, if M contains $\mathbb{Q}(\mu_p)^+$, then it is also called p-regular [10]. The p-regularity of M is simply expressed by the triviality of the p-part of the tame kernel $K_2(o_M)$. As the $\mathbb{Q}_n$ are not the only p-extensions of $\mathbb{Q}$ which are p-rational, we notice that, for $p \geqslant 5$, among the p-extensions M of $\mathbb{Q}$, some are p-rational but have a non-trivial $WK_2(M)\{p\}$. Now take $p = 3$. Then by Moore's exact sequence $WK_2(M)\{3\} = 0$ if and only if the tame kernel $K_2(o_M)$ has no 3-torsion. Hence the number field M is 3-rational or 3-regular. In this case, $WK_2(M)\{3\} = 0$ if and only if outside the prime 3, the 3-extension $M/\mathbb{Q}$ is at most ramified at one prime l, which is inert in the $\mathbb{Z}_3$-extension $\mathbb{Q}_\infty/\mathbb{Q}$ (cp. [10], [25], [24]).

Let us have a closer look at the cup-product

$$K^{\text{ét}}_{2i-1}(F)/p \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \to H^2(F, \mathbb{Z}/p\mathbb{Z}(i)),$$

which occurred in the proof of Proposition 2.9, and describe the maps α' and β'. Let $E = F(\mu_p)$ and $\Delta = \mathrm{Gal}(E/F)$. Over E we have

$$H^2(E, \mathbb{Z}/p\mathbb{Z}(i)) \cong H^2(E, \mathbb{Z}/p\mathbb{Z}(1))(i-1) \cong {}_p\mathrm{Br}(E)(i-1),$$

where $\mathrm{Br}(E)$ stands for the Brauer group of E. The set $T^{(i)}_{LE/E}$ is independent of i, and we simply denote it by $T_{LE/E}$. Obviously, every prime in E which lies above a prime in $T^{(i)}_{L/F}$ belongs to $T_{LE/E}$. Conversely, let v_E be a prime in $T_{LE/E}$, and let v denote the prime of F below v_E. Then $v \in T^{(i)}_{L/F}$ if and only if $H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i)) \neq 0$. Let $\mathrm{Br}^T(E)$ denote the subgroup of $\mathrm{Br}(E)$ of all isomorphism classes of central simple E-algebras split outside $T_{LE/E}$. It is now easy to see that

$$\ker \beta' \cong ({}_p\mathrm{Br}^T(E)(i-1))^\Delta \cong ({}_p\mathrm{Br}^T(E))^{[1-i]},$$

where ω denotes the Teichmüller character of Δ, and $A^{[j]}$ denotes the j-th eigenspace of ω acting on a Δ-module A.

Since $K_{2i-1}^{\text{ét}}(E)/p$ is contained in $H^1(E, \mathbb{Z}/p\mathbb{Z}(i)) \cong (E^*/E^{*p})(i-1)$, there exists a subgroup $D_E^{(i)}$ of E^* containing E^{*p} - the analog of the Tate-kernel in case $i = 2$ - such that

$$K_{2i-1}^{\text{ét}}(E)/p \cong (D_E^{(i)}/E^{*p})(i-1),$$

and hence

$$K_{2i-1}^{\text{ét}}(F)/p \cong ((D_E^{(i)}/E^{*p})(i-1))^{\Delta} \cong (D_E^{(i)}/E^{*p})^{[1-i]}.$$

Note that for $i \equiv 1 \bmod d$ we can similarly define $D_F^{(i)}$, and clearly in this case $(D_E^{(i)}/p)^{\Delta} \cong D_F^{(i)}/p$. The considerations after Proposition 2.9 now show that the cup-product over E is explicitly given as

$$(D_E^{(i)}/E^{*p})(i-1) \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \to {}_p\mathrm{Br}^T(E)(i-1),$$

where

$$D_E^{(i)}/E^{*p} \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \to {}_p\mathrm{Br}^T(E),$$

is the classical cup-product $x \otimes \chi \mapsto (\chi, x)$ (cp. [34, Chap. XIV]). Descending to F, we see that the image of α' is precisely the image of the cup-product

$$(D_E^{(i)}/E^{*p})^{[1-i]} \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \to ({}_p\mathrm{Br}^T(E))^{[1-i]}.$$

We can therefore reformulate the condition for Galois co-descent of the wild kernel as follows:

THEOREM 2.14. — *Let L/F be a cyclic extension of number fields of degree p, p odd, with Galois group G. Assume that $T_{L/F}^{(i)} \neq \varnothing$ and that some $v \in T_{L/F}^{(i)}$ is undecomposed in F_1 if $i \equiv 1 \bmod d$. Then the étale wild kernel $WK_{2i-2}^{\text{ét}}(L)$ satisfies Galois co-descent if and only if the cup-product*

$$(D_E^{(i)}/E^{*p})^{[1-i]} \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \to ({}_p\mathrm{Br}^T(E))^{[1-i]}$$

is surjective.

In the special case where $i \equiv 1 \bmod d$, the condition can be reformulated as: The cup-product

$$D_F^{(i)}/F^{*p} \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \to {}_p\mathrm{Br}^{T_{L/F}^{(i)}}(F)$$

is surjective. Moreover, in this special case the genus-formula simplifies to:

$$\frac{|WK^{\text{ét}}_{2i-2}(L)^G|}{|WK^{\text{ét}}_{2i-2}(F)|} = \frac{p^{|T^{(i)}_{L/F}|-1}}{[D^{(i)}_F : D^{(i)}_F \cap N_{L/F}(L^*)]}.$$

Since $i \geqslant 2$, we can reinterpret the cup-product as a Galois symbol: Let $EL = E(\sqrt[p]{\delta})$. Then

$$(D^{(i)}_E/E^{*p})(i-1) \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) = (D^{(i)}_E/E^{*p}) \otimes H^1(G, \mu_p)(i-2),$$

and therefore the cup-product over E is the $(i-2)$-th twist of the Galois symbol

$$D^{(i)}_E/E^{*p} \otimes H^1(G, \mu_p) \to \mu_p \otimes {}_p\mathrm{Br}^T(E).$$

The Kummer radical $H^1(G, \mu_p)$ is generated by δ, and the map is given by (cp. [23])

$$x \otimes \delta \mapsto \zeta_p \otimes \left[\left(\frac{x, \delta}{E}\right)\right],$$

where ζ_p is a primitive pth root of unity and $[(\frac{x,\delta}{E})]$ denotes the isomorphism class of the cyclic algebra $(\frac{x,\delta}{E})$, with generators u, v and relations: $u^p = x, v^p = \delta, vu = \zeta_p uv$.

In general, not much is known about the higher "Tate-kernels" $D^{(i)}_E$ defined by $K^{\text{ét}}_{2i-1}(E)/p \cong (D^{(i)}_E/E^{*p})$. However, for n large, the groups $D^{(i)}_{E_n}/E^{*p}_n$ can be characterized in terms of local conditions, if we assume the Gross conjecture, which we describe next: Let F be any number field. For a finite prime v in F let $\hat{F}_v = \varprojlim F^*_v/F^{*p^k}_v$, let $\hat{U}_v = \mathbb{Z}_p \otimes U_v$ and let $\mathcal{N}_v \subset \hat{F}_v$ denote the group of norms from the cyclotomic $\mathbb{Z}_p$-extension of F_v. Thus $\mathcal{N}_v = \hat{U}_v$ if $v \notin S_p$, and for $v \in S_p$ we have the following characterization:

$$a \in \mathcal{N}_v \Leftrightarrow \log_p(N_{F_v/\mathbb{Q}_p}(a)) = 0,$$

where $\log_p$ denotes the p-adic logarithm normalized by $\log_p(p) = 0$ (cp. [9], [19]). There is a natural homomorphism

$$g_F : \mathbb{Z}_p \otimes U'_F \to \bigoplus_{v|p} \hat{F}^*_v/\mathcal{N}_v$$

and the Gross kernel $GK(F) := \ker g_F$ has $\mathbb{Z}_p$-rank $r_1(F) + r_2(F) + \delta_F$, where $\delta_F \geqslant 0$ is the Gross defect. $GK(F)$ is therefore characterized by the following local conditions:

$$\epsilon \in GK(F) \Leftrightarrow \epsilon \in \mathbb{Z}_p \otimes U'_F \quad \text{and} \quad \log_p(N_{F_v/\mathbb{Q}_p}(\epsilon)) = 0 \quad \forall v \in S_p.$$

The Gross Conjecture postulates that $\delta_F = 0$, which is true for instance for abelian fields F. Let - as before - $E = F(\mu_p)$ and $\Delta = \mathrm{Gal}(E/F)$. The following result was proved for $i = 2$ in [19, Theorem 2.5]. The method was extended to higher étale K-theory in [5].

THEOREM 2.15.— *For n large there is an exact sequence*

$$0 \to K_{2i-1}^{\text{ét}}(E_n)/p \to (\ker g_{E_n}/p)(i-1) \to (\mathbb{Z}/p\mathbb{Z})^{\delta_n} \to 0,$$

where δ_n denotes the Gross defect for the field E_n.

COROLLARY 2.16.— *Assume that the Gross Conjecture holds for E_n, n large. Then*

$$D_{E_n}^{(i)}/E_n^{*p} \cong GK(E_n)/p \qquad \text{for } n \text{ large.}$$

In particular, for n large, the groups $D_{E_n}^{(i)}/E_n^{*p}$ are independent of i.

So far in this section we have ignored the prime 2. Let us briefly discuss the case $p = 2$ in the classical situation $i = 2$, where special attention has to be paid to real infinite primes in F. Let $L = F(\sqrt{\delta})$ be a quadratic extension of number fields with Galois group G. Denote by $T_{L/F}$ the set of finite primes in F which consists of all ramified non-dyadic primes and of all undecomposed dyadic primes v of F, for which either $\mu(L_w)\{2\} = \mu(F_v)\{2\}$ or L_w is not contained in the cyclotomic $\mathbb{Z}_2$-extension of F_v, where w is the prime above v in L. Also, denote by D_F the subgroup of F^* of all elements x, such that $\{-1, x\} = 1$ in $K_2(F)$. This is the classical Tate-kernel. Then the following results can be proved along the same lines as for odd p:

PROPOSITION 2.17.— *The canonical map $WK_2(L)\{2\}_G \to WK_2(F)\{2\}$ is surjective precisely in the following situations, and has cokernel of order 2 otherwise:*

i) $|\mu(L)\{2\}| > |\mu(F)\{2\}|$ *and* $L \subset F_\infty$.

ii) $|\mu(L)\{2\}| > |\mu(F)\{2\}|$, $L \not\subset F_\infty$ *and* $\mu(L_w)\{2\} = \mu(L)\{2\}$ *for some* $w \mid v$, $v \in T_{L/F}$.

iii) $\mu(L)\{2\} = \mu(F)\{2\}$ *and* $\mu(L_w)\{2\} = \mu(F_v)\{2\}$ *for some* $v \in T_{L/F}$.

We note in particular that the map $WK_2(L)\{2\}_G \to WK_2(F)\{2\}$ is always surjective if a non-dyadic prime of F is ramified in L.

THEOREM 2.18. — *Let L/F be a relative quadratic extension with Galois group G.*

a) *If $|\mu(L)\{2\}| > |\mu(F)\{2\}|$ and $L \subset F_\infty$, then $WK_2(L)\{2\}_G \cong WK_2(F)\{2\}$.*

b) *If either $|\mu(L)\{2\}| = |\mu(F)\{2\}|$ or $L \not\subset F_\infty$, and if either a real infinite prime of F ramifies in L or if $|\mu(F_v)\{2\}| = |\mu(F)\{2\}|$ for some prime $v \in T_{L/F}$, then*

$$\frac{|WK_2(L)\{2\}_G|}{|WK_2(F)\{2\}|} = \frac{2^{|T_{L/F}|-1}}{[D_F : D_F \cap N_{L/F}(L^*)]}.$$

In [6], Browkin and Schinzel computed the 2-rank of the wild kernel of a quadratic number field and obtained a complete list of quadratic number fields with trivial 2-primary wild kernels. A combination of their results with the genus formula in Theorem 2.18 and methods of [12] yield a complete list of bi-quadratic fields with trivial 2-primary wild kernels. Details will appear elsewhere.

3. Capitulation kernels.

Let p be an odd prime and let F_∞/F be an arbitrary $\mathbb{Z}_p$-extension of F with finite layers F_n. Let $A'_n = A'(F_n)$ denote the p-part of the p-class group of F_n and $A'_\infty = \varinjlim A'_n$. We define the capitulation kernel $\mathrm{Cap}_0(F_\infty/F_n) = \ker(A'_n \to A'_\infty)$. As is well-known (cp. [13]) these kernels stabilize, more precisely, the norm $N_{F_m/F_n} : \mathrm{Cap}_0(F_\infty/F_m) \to \mathrm{Cap}_0(F_\infty/F_n)$ is an isomorphism for n large and $m \geqslant n$ and we set $\mathrm{Cap}_0(F_\infty) = \varprojlim \mathrm{Cap}_0(F_\infty/F_n)$.

Remark 3.1. — Let A_n denote the p-part of the (usual) class group of F_n and let $A_\infty = \varinjlim A_n$. Once again, the capitulation kernels $\ker(A_n \to A_\infty)$ stabilize, and we can consider $\widetilde{\mathrm{Cap}}(F_\infty) = \varprojlim \ker(A_n \to A_\infty)$. We note that in general $\widetilde{\mathrm{Cap}}(F_\infty) \neq \mathrm{Cap}_0(F_\infty)$. Indeed, from the explicit examples elaborated by Greenberg in [11, section 8], it is not hard to see that if we take $F = \mathbb{Q}(\sqrt{142})$, $p = 3$, and let F_∞ be the cyclotomic $\mathbb{Z}_3$-extension of F, then $\tilde{\mathrm{Cap}}(F_\infty) \cong \mathbb{Z}/3\mathbb{Z}$, whereas $\mathrm{Cap}_0(F_\infty)$ is trivial. From a K-theoretic point of view, $\mathrm{Cap}_0(F_\infty)$ is the appropriate object to study.

We want to consider the analog of these kernels in higher étale K-theory.

Let again S be a finite set of primes in F containing S_p. To simplify notation, we put

$$\tilde{K}^{\text{ét}}_{2i-1}(F_\infty) = \varinjlim K^{\text{ét}}_{2i-1}(F_n)$$

and

$$\tilde{K}^{\text{ét}}_{2i-2}(o^S_\infty) = \varinjlim K^{\text{ét}}_{2i-2}(o^S_n),$$

where o^S_n denotes the ring of S-integers in F_n, i.e. the integral closure of o^S_F in F_n. We now define for $i \geqslant 2$:

$$\mathrm{Cap}_{i-1}(F_\infty/F_n) = \ker(K^{\text{ét}}_{2i-2}(o^S_n) \to \tilde{K}^{\text{ét}}_{2i-2}(o^S_\infty)).$$

The following result implies in particular that the definition is independent of the choice of the finite set S containing S_p. Let Γ_n denote the Galois group of F_∞/F_n with the usual convention $\Gamma_0 = \Gamma$.

PROPOSITION 3.2.— *For $i \geqslant 2$ there is a short exact sequence*

$$0 \to H^1(\Gamma_n, \tilde{K}^{\text{ét}}_{2i-1}(F_\infty)) \to K^{\text{ét}}_{2i-2}(o^S_n) \to \tilde{K}^{\text{ét}}_{2i-2}(o^S_\infty)^{\Gamma_n} \to 0.$$

Proof.— For each $m \geqslant n$, Theorem 1.2 gives an exact sequence

$$0 \to H^1(\Gamma_n/\Gamma_m, K^{\text{ét}}_{2i-1}(F_m)) \to K^{\text{ét}}_{2i-2}(o^S_n) \to K^{\text{ét}}_{2i-2}(o^S_m)^{\Gamma_n/\Gamma_m} \to H^2(\Gamma_n/\Gamma_m, K^{\text{ét}}_{2i-1}(F_m)) \to 0.$$

From Corollary 1.4 we see that the orders of the groups $H^2(\Gamma_n/\Gamma_m, K^{\text{ét}}_{2i-1}(F_m))$ are bounded independently of m by the order of $K^{\text{ét}}_{2i-2}(o^S_n)$, and therefore the limit

$$H^2(\Gamma_n, \tilde{K}^{\text{ét}}_{2i-1}(F_\infty)) = \varinjlim H^2(\Gamma_n/\Gamma_m, K^{\text{ét}}_{2i-1}(F_m))$$

is finite. On the other hand, this group is divisible, since $cd_p(\Gamma_n) = 1$, hence trivial. □

In the classical case $i = 1$, it was shown by Iwasawa (cp. [13, Theorem 12]) that

$$\mathrm{Cap}_0(F_\infty/F_n) \cong H^1(\Gamma_n, U'_\infty),$$

where $U'_\infty = \varinjlim U'_n$ and U'_n denotes the group of p-units of F_n. Therefore Proposition 3.2 gives, in particular, the following higher-dimensional analog of this result:

COROLLARY 3.3.— *For $i \geqslant 2$*

$$\mathrm{Cap}_{i-1}(F_\infty/F_n) \cong H^1(\Gamma_n, \tilde{K}^{\text{ét}}_{2i-1}(F_\infty)).$$

To go further, we quote the following general result of Kahn (cp. [16, Proposition 6.2]), which he attributes to Nguyen Quang Do:

LEMMA 3.4.— *Let A be a discrete torsion free Γ-module. Assume that for all integers $n \geqslant 0$:*

i) *$H^0(\Gamma_n, A)$ is finitely generated;*

ii) *$H^1(\Gamma_n, A)$ is finite;*

iii) *$H^2(\Gamma_n, A) = 0$.*

Then the groups $H^1(\Gamma_n, A)$ stabilize, in particular $\varprojlim H^1(\Gamma_n, A)$ is finite.

Let

$$\bar{K}^{\text{ét}}_{2i-1}(F_n) = K^{\text{ét}}_{2i-1}(F_n)/\mathit{torsion}$$

and

$$\bar{K}^{\text{ét}}_{2i-1}(F_\infty) = \tilde{K}^{\text{ét}}_{2i-1}(F_\infty)/\mathit{torsion}.$$

We want to apply the previous lemma with $A = \bar{K}^{\text{ét}}_{2i-1}(F_\infty)$. From the exact sequence

$$0 \to H^0(F_\infty, \mathbb{Q}_p/\mathbb{Z}_p(i)) \to \tilde{K}^{\text{ét}}_{2i-1}(F_\infty) \to \bar{K}^{\text{ét}}_{2i-1}(F_\infty) \to 0,$$

we deduce the exact sequence

$$0 \to \bar{K}^{\text{ét}}_{2i-1}(F_n) \to \bar{K}^{\text{ét}}_{2i-1}(F_\infty)^{\Gamma_n} \to H^1(\Gamma_n, H^0(F_\infty, \mathbb{Q}_p/\mathbb{Z}_p(i)))$$
$$\to H^1(\Gamma_n, \tilde{K}^{\text{ét}}_{2i-1}(F_\infty)) \to H^1(\Gamma_n, \bar{K}^{\text{ét}}_{2i-1}(F_\infty)) \to 0,$$

as well as an isomorphism

$$H^2(\Gamma_n, \tilde{K}^{\text{ét}}_{2i-1}(F_\infty)) \cong H^2(\Gamma_n, \bar{K}^{\text{ét}}_{2i-1}(F_\infty)).$$

The proof of Proposition 2.2 showed that $H^2(\Gamma_n, \bar{K}^{\text{ét}}_{2i-1}(F_\infty)) = 0$, and hence we see that $\bar{K}^{\text{ét}}_{2i-1}(F_\infty)$ satisfies the assumptions of the previous lemma. We obtain the fact that the groups $H^1(\Gamma_n, \bar{K}^{\text{ét}}_{2i-1}(F_\infty))$ stabilize and therefore that $\varprojlim H^1(\Gamma_n, \bar{K}^{\text{ét}}_{2i-1}(F_\infty))$ is finite. To obtain the same result for the groups $H^1(\Gamma_n, \tilde{K}^{\text{ét}}_{2i-1}(F_\infty))$ and their limit, we look at the term $H^1(\Gamma_n, H^0(F_\infty, \mathbb{Q}_p/\mathbb{Z}_p(i)))$ in the above exact sequence: The group $H^0(F_\infty, \mathbb{Q}_p/\mathbb{Z}_p(i))$ is either $\mathbb{Q}_p/\mathbb{Z}_p(i)$ or finite. In the first case, Tate's Lemma implies that $H^1(\Gamma_n, H^0(F_\infty, \mathbb{Q}_p/\mathbb{Z}_p(i))) = 0$, hence

$$H^1(\Gamma_n, \tilde{K}^{\text{ét}}_{2i-1}(F_\infty)) \cong H^1(\Gamma_n, \bar{K}^{\text{ét}}_{2i-1}(F_\infty)).$$

In the second case, $H^1(\Gamma_n, H^0(F_\infty, \mathbb{Q}_p/\mathbb{Z}_p(i)))$ stabilizes for n large, and hence in any case we obtain:

PROPOSITION 3.5. — *The groups* $\mathrm{Cap}_{i-1}(F_\infty/F_n)$ *stabilize; more precisely, the corestriction maps*

$$\mathrm{Cap}_{i-1}(F_\infty/F_{n+1}) \to \mathrm{Cap}_{i-1}(F_\infty/F_n)$$

are surjective for all n *and* $\varprojlim \mathrm{Cap}_{i-1}(F_\infty/F_n)$ *is finite.*

We now define

$$\mathrm{Cap}_{i-1}(F_\infty) = \varprojlim \mathrm{Cap}_{i-1}(F_\infty/F_n).$$

Now let us specialize and take F_∞/F to be the *cyclotomic* $\mathbb{Z}_p$-extension. As in the case $i = 1$, the finite groups $\mathrm{Cap}_{i-1}(F_\infty)$ then have various characterizations in terms of Iwasawa-theory. Let $E = F(\mu_p)$, let $E_\infty = F(\mu_{p^\infty})$ be the cyclotomic $\mathbb{Z}_p$-extension of E and identify Γ_n with the Galois group of E_∞/E_n. We first describe $\mathrm{Cap}_{i-1}(E_\infty)$. Let $\mathcal{X}_\infty$ denote the standard Iwasawa-module for E_∞, i.e. the Galois group over E_∞ of the maximal abelian p-ramified pro-p-extension of E_∞. Denote by $\mathrm{tor}_\Lambda \mathcal{X}_\infty$ the torsion part of $\mathcal{X}_\infty$ as a module over $\Lambda = \mathbb{Z}_p[[\Gamma]]$. As is well-known, there exists an injective homomorphism ([11, Theorem 3])

$$\mathcal{X}_\infty/\mathrm{tor}_\Lambda \mathcal{X}_\infty \to \Lambda^{r_2(E)}$$

with finite cokernel H. The following result is due to Iwasawa ([13]) for $i = 1$, to Coates ([7]) for $i = 2$ and to Nguyen Quang Do([27, section 4]) in general:

THEOREM 3.6. — *For all* $i \geqslant 1$ *and all* $n \geqslant 0$, *there are canonical isomorphisms*

$$\mathrm{Cap}_{i-1}(E_\infty/E_n) \cong H^*(i)_{\Gamma_n}.$$

Since H is finite, the group Γ_n acts trivially on $H^*(i)$ for all i provided n is large enough. Therefore, as abstract groups, all capitulation kernels $\mathrm{Cap}_{i-1}(E_\infty)$ are isomorphic to H.

Let $\Delta = \mathrm{Gal}(E/F)$ and let d denote the order of Δ. Now clearly

$$\mathrm{Cap}_{i-1}(F_\infty) = \mathrm{Cap}_{i-1}(E_\infty)^\Delta.$$

Theorem 3.6 shows that $\mathrm{Cap}_{i-1}(E_\infty)$ and $\mathrm{Cap}_{j-1}(E_\infty)$ are isomorphic as Δ-modules for $i \equiv j \bmod d$. Therefore we obtain the following periodicity result:

COROLLARY 3.7.— *Let p be odd and let F_∞/F be the cyclotomic $\mathbb{Z}_p$-extension of F. Then*

$$\mathrm{Cap}_{i-1}(F_\infty) \cong \mathrm{Cap}_{j-1}(F_\infty)$$

for all $i, j \geqslant 1$, $i \equiv j \bmod d$.

Next we would like to discuss another well-known relation between capitulation kernels and Iwasawa-theory: We continue to assume that $E_\infty = F(\mu_{p^\infty})$ is the cyclotomic $\mathbb{Z}_p$-extension of $E = F(\mu_p)$, and that p is odd. As usual, let X'_∞ denote the Galois group over E_∞ of the maximal abelian unramified pro-p-extension of E_∞, in which all primes above p are completely decomposed. Thus $X'_\infty \cong \varprojlim A'_n(E)$. The co-invariants $(X'_\infty)_\Gamma$ have been described by Jaulent as a group of logarithmic classes $\tilde{cl}(E)$ which can be interpreted as the class field theory analog of the wild kernels corresponding to the case $i = 1$. The Galois co-descent for these modules $\tilde{cl}(E)$ has been studied in [14]. Now, let $(X'_\infty)^0$ denote the maximal finite submodule of X'_∞. It is well-known (cp. [21]) that

$$\mathrm{Cap}_0(E_\infty) \cong (X'_\infty)^0.$$

On the other hand, we have for all $n \geqslant 0$ and all $i \geqslant 2$, an isomorphism

$$(X'_\infty(i-1))_{\Gamma_n} \cong WK^{\text{ét}}_{2i-2}(E_n)$$

(cp. 33, section 6, Lemma 1]), and therefore

$$\begin{aligned}\ker\,(WK^{\text{ét}}_{2i-2}(E_n) &\to WK^{\text{ét}}_{2i-2}(E_m))\\ &\cong \ker\,((X'_\infty(i-1))_{\Gamma_n} \to (X'_\infty(i-1))_{\Gamma_m})\\ &\cong (X'_\infty)^0(i-1)\end{aligned}$$

for n large and m sufficiently larger than n. If we define

$$\tilde{W}K_{2i-2}^{\text{ét}}(E_\infty) = \varinjlim WK_{2i-2}^{\text{ét}}(E_n),$$

then we obtain

PROPOSITION 3.8.— *For $i \geqslant 2$ and n sufficiently large we have:*

$$\text{Cap}_{i-1}(E_\infty) \cong \ker(WK_{2i-2}^{\text{ét}}(E_n) \to \tilde{W}K_{2i-2}^{\text{ét}}(E_\infty)) \cong (X'_\infty)^0(i-1)$$

as Δ-modules.

For the original field F and the cyclotomic $\mathbb{Z}_p$-extension F_∞/F this implies:

$$\text{Cap}_{i-1}(F_\infty) = \ker(WK_{2i-2}^{\text{ét}}(F_n) \to \tilde{W}K_{2i-2}^{\text{ét}}(F_\infty)) \cong ((X'_\infty)^0(i-1))^\Delta.$$

Again let ω denote the Teichmüller character on Δ. We have

$$((X'_\infty)^0(i-1))^\Delta \cong ((X'_\infty)^0)^{[1-i]} \cong (X'_\infty{}^{[1-i]})^0,$$

and hence

$$\text{Cap}_{i-1}(F_\infty) \cong (X'_\infty{}^{[1-i]})^0 \cong \text{Cap}_0(E_\infty)^{[1-i]}$$

for all $i \geqslant 1$. We therefore obtain a decomposition of $\text{Cap}_0(E_\infty)$ into eigenspaces:

$$\underset{0}{\text{Cap}}(E_\infty) \cong \bigoplus_{j=0}^{d-1} \underset{j}{\text{Cap}}(F_\infty)$$

with $\text{Cap}_j(F_\infty)$ being isomorphic to the $(d-j)$-th eigenspace of $\text{Cap}_0(E_\infty)$. The following result gives the connection with Section 2:

PROPOSITION 3.9.— *For $i \geqslant 2$, the following statements are equivalent:*

i) $\text{Cap}_{i-1}(F_\infty) \cong WK_{2i-2}^{\text{ét}}(F_n)$ *for large n.*

ii) $X'_\infty{}^{[1-i]}$ *is finite.*

Proof. — As already mentioned we have for $i \geqslant 2$:

$$(X'_\infty(i-1))_{\Gamma_n} \cong WK_{2i-2}^{\text{ét}}(E_n),$$

hence

$$X'_\infty(i-1) \cong \varprojlim\, WK_{2i-2}^{\text{ét}}(E_n),$$

and therefore

$$X'^{[1-i]}_{\infty} \cong \varprojlim WK^{\text{ét}}_{2i-2}(F_n).$$

The equivalence of i) and ii) is now obvious. □

Let us assume now that the base field F is totally real. Then E is a CM-field with maximal real subfield E^+. Since obviously the plus-part of the group H is trivial in this situation, Theorem 3.6 implies that $\mathrm{Cap}_{i-1}(F_\infty) = 0$ for all even $i \geqslant 2$, hence that the minus-part of $\mathrm{Cap}_0(E_\infty)$ vanishes: $\mathrm{Cap}_0(E_\infty)^- = 0$. Let X_∞ denote the Galois group of the maximal abelian unramified pro-p-extension of E_∞. Greenberg's Conjecture (cp. [11]) for the cyclotomic $\mathbb{Z}_p$-extension F_∞ of the totally real field F is equivalent to the fact that X^{Δ}_{∞} is finite. Clearly this implies that $(X'_\infty)^\Delta$ is also finite, and the converse implication is true if one assumes for example that Leopoldt's Conjecture holds for the layers F_n of F_∞/F. We will refer to Greenberg's Conjecture in the form: $(X'_\infty)^\Delta$ *is finite*. In fact we will consider Greenberg's Conjecture for the field E^+. Using Proposition 3.9, we can summarize:

PROPOSITION 3.10. — *Let F be a totally real number field, p an odd prime, $E = F(\mu_p)$ and E^+ the maximal real subfield of E. Furthermore, let F_∞ denote the cyclotomic $\mathbb{Z}_p$-extension of F and E_∞ the cyclotomic $\mathbb{Z}_p$-extension of E. Then:*

i) $\mathrm{Cap}_0(E_\infty)^- = 0$, *i.e.* $\mathrm{Cap}_{i-1}(F_\infty) = 0$ *for all even* $i \geqslant 2$.

ii) $\mathrm{Cap}_{i-1}(F_\infty) \cong WK^{\text{ét}}_{2i-2}(F_n)$ *for large n and all odd $i \geqslant 3$, if and only if Greenberg's Conjecture holds for E^+.*

As an immediate consequence of part ii), we obtain that under Greenberg's Conjecture the étale wild kernels $WK^{\text{ét}}_{2i-2}(F_n)$ show the same periodic behaviour as the capitulation kernels for n large and $i \geqslant 3$ odd. On the other hand, under Greenberg's Conjecture for E^+, we also have $\mathrm{Cap}_0(E^+_\infty) = \mathrm{Cap}_0(E_\infty)^+ = A'_n(E)^+$ for n large; hence for all $i \geqslant 3$ odd:

$$\mathrm{Cap}_{i-1}(F_\infty) \cong A'_n(E)^{[1-i]} \cong WK^{\text{ét}}_{2i-2}(F_n) \qquad \text{for } n \text{ large.}$$

Therefore, the Galois co-descent results of Section 2 also apply to both $\mathrm{Cap}_{i-1}(F_\infty)$ and the eigenspaces $A'_n(E)^{[1-i]}$ of $A'_n(E^+)$ for n large. In particular:

THEOREM 3.11. — *Let L/F be a cyclic extension of totally real number fields of degree p, p odd, with Galois group G and let $E = F(\mu_p)$. Assume Greenberg's conjecture holds for E^+, LE^+ and the Gross* *conjecture holds for* E_n*, n large. Then for* $i \geqslant 3$ *odd, n large and* $T_{L_n/F_n} \neq \varnothing$*, Galois co-descent holds for* $\mathrm{Cap}_{i-1}(L_\infty)$ *and* $A'_n(LE)^{[1-i]}$ *if and only if the cup-product*

$$(GK(E_n)/p)^{[1-i]} \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \to {}_p\mathrm{Br}^T(E_n)^{[1-i]}$$

is surjective.

Remark 3.12. — If $i \equiv 1 \bmod d$, then, under the assumptions of Theorem 3.11, we can compare the genus formulae for $WK^{\text{ét}}_{2i-2}(L_n)$ and $A'_n(L)$ to obtain for large n:

$$[U'_n : U'_n \cap N_{L_n/F_n}(L_n^*)] = [GK(F_n) : GK(F_n) \cap N_{L_n/F_n}(L_n^*)],$$

a result which one can also prove directly.

4. Galois co-descent for the étale tame kernel.

In this final section we briefly discuss how the methods of Section 2 can be used to study the much easier problem of Galois co-descent for the étale tame kernels again for cyclic extensions L/F of degree p, p odd. Results for arbitrary finite Galois p-extensions have been obtained by Assim (cp. [1], [2]) in terms of primitive ramification, however under the assumption that Leopoldt's Conjecture holds for the fields $L(\mu_{p^n})$ for all n. Let S be the finite set of primes of F, consisting of the set S_p and the tamely ramified primes in L/F. We have the following exact sequence:

$$0 \to K^{\text{ét}}_{2i-2}(o_F) \to K^{\text{ét}}_{2i-2}(o_F^S) \to \bigoplus_{v \in S \setminus S_p} H^2(F_v, \mathbb{Z}_p(i)) \to 0,$$

which, combined with Proposition 1.3, shows that the canonical map

$$K^{\text{ét}}_{2i-2}(o_L)_G \to K^{\text{ét}}_{2i-2}(o_F)$$

is always surjective and that the kernel of this map is isomorphic to the cokernel of the map

$$K^{\text{ét}}_{2i-2}(o_L^S)^G \to \left(\bigoplus_{w \in S'_L} H^2(L_w, \mathbb{Z}_p(i))\right)^G,$$

where S'_L consists of the primes in L above $S \setminus S_p$. We recall that $S \setminus S_p$ is always contained in $T^{(i)}_{L/F}$. The following is now clear from the results in Section 2:

THEOREM 4.1. — *The kernel of the surjective map* $K^{\text{ét}}_{2i-2}(o_L)_G \to K^{\text{ét}}_{2i-2}(o_F)$ *is isomorphic to the cokernel of the map*

$$K^{\text{ét}}_{2i-1}(F)/p \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \to \bigoplus_{v \in S \setminus S_p} H^2(F_v, \mathbb{Z}/p\mathbb{Z}(i)).$$

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Manuscrit reçu le 8 avril 1999,
accepté le 6 juillet 1999.

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