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KÄHLER MANIFOLDS WITH SMALL EIGENVALUES OF THE DIRAC OPERATOR AND A CONJECTURE OF LICHNEROWICZ

by Andrei MOROIANU

1. Introduction.

The problem of finding optimal lower bounds for the eigenvalues of the Dirac operator on compact manifolds was for the first time considered in 1980 by Th. Friedrich [3]. Using the Lichnerowicz formula [19] and a modified spin connection, he proved that the first eigenvalue λ of the Dirac operator on a compact spin manifold (M^n, g) of positive scalar curvature S satisfies

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M S. \tag{1}$$

The limiting case of this inequality is characterized by the existence of *real Killing spinors*. After several partial results of Th. Friedrich, I. Kath, R. Grunewald, and O. Hijazi ([12], [6], [7], [8], [12]), the geometrical description of simply connected manifolds carrying Killing spinors was obtained in 1991 by C. Bär [1], who made an ingenious use of the cone construction. On the other hand, already in 1984, O. Hijazi remarked ([11], [13]) that Kähler manifolds never carry Killing spinors (except in complex dimension 1), and thus raised the question of improving Friedrich's inequality for Kähler manifolds. This was done in 1986 by K.-D. Kirchberg

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[16], [17], who showed that every eigenvalue λ of the Dirac operator on a compact Kähler manifold (M^{2m}, g, J) of positive scalar curvature S satisfies

$$\lambda^2 \geq \frac{m+1}{4m} \inf_M S, \quad \text{if } m \text{ is odd,} \tag{2}$$

and

$$\lambda^2 \geq \frac{m}{4(m-1)} \inf_M S, \quad \text{if } m \text{ is even.} \tag{3}$$

The manifolds which satisfy the limiting case of these inequalities (called *limiting manifolds* for the remaining of this paper), are characterized by the existence of *Kählerian Killing spinors* (see [17], [14]) for m odd and by the existence of spinors satisfying some more complicated equations ((8)–(11) below), for $m \geq 4$ even (cf. [17], [10]). In complex dimension $m = 2$, they were classified in 1993 by Th. Friedrich [5]. Limiting manifolds of odd complex dimension were geometrically described by the author in 1994 [21], whereas the problem in even complex dimensions $m \geq 4$ has remained open until now.

It was remarked by K.-D. Kirchberg [17] that a product $N \times T^2$, where N is a limiting manifold of odd complex dimension and T^2 is a flat torus, is a limiting manifold of even complex dimension. More generally, this holds for suitable twisted products, *i.e.* for suspensions of commuting pairs of isometries of N preserving a Kählerian Killing spinor, over parallelograms in $\mathbb{R}^2$ (Section 7).

The main goal of this paper is to show that, conversely, every limiting manifold of even complex dimension can be obtained in this way (see Thm. 7.4 for a precise statement). A similar result (omitting, however, the twisted case) appears in [20], but the argument is incomplete, as observed in [22]. The assertion that each limiting manifold of even complex dimension is locally isometric to $N \times \mathbb{R}^2$, where N is a limiting manifold of odd complex dimension will therefore be referred to as Lichnerowicz' conjecture.

The most important part of this paper (Section 3 to 6) is devoted to proving that the Ricci tensor of a limiting manifold is parallel, a missing point in [20]. The main ideas are the following: in [22] we showed that the Ricci tensor of a limiting manifold M has only two eigenvalues, and that M is foliated by the integral manifolds of the two corresponding eigenspaces (see Theorem 3.1 and Corollary 3.2 below). Using this, we consider (Section 4) a suitable 1-parameter family of metrics on M and show that they are all limiting metrics, thus obtaining information about the curvature of M. In Section 5, ideas from [21] and the classification

of (simply connected) Spin^c manifolds with parallel spinors [23] allow us to show that every Spin^c structure carrying Kählerian Killing spinors (of special algebraic form) has to be a *spin* structure. Next comes the key step of the proof: we consider (Section 6) the restriction of a limiting spinor to the maximal leaves of one of the above distributions (corresponding to the non-zero eigenvalue of the Ricci tensor), and show that it is a Spin^c Kählerian Killing spinor on each such leaf. Together with the results obtained in the two previous sections, this implies that the Ricci tensor of M is parallel and that the leaves are limiting manifolds of odd complex dimension. This proves the above mentioned (local) conjecture of Lichnerowicz.

Finally, the complete classification of limiting manifolds of even complex dimension is obtained in Section 7, after a careful analysis of the action of the fundamental group of M on its universal cover.

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2. Preliminaries.

We follow here the presentation and notations from [10]. For basic definitions concerning spin and Spin^c structures see [23]. Let (M^{2m}, g, J) be a spin Kähler manifold and let ΣM be the spinor bundle of M. We denote by S the scalar curvature of M and by Ω the Kähler form, defined by $\Omega(X,Y) = g(JX,Y)$.

A k-form ω acts by Clifford multiplication on ΣM by

$$\omega \cdot \Psi := \sum_{i_1<\cdots<i_k} \omega(e_{i_1}, \cdots, e_{i_k})\, e_{i_1} \cdot \ldots \cdot e_{i_k} \cdot \Psi,$$

where $\{e_1, \cdots, e_n\}$ is an arbitrary local orthonormal frame on M. The Clifford action of the Kähler form on a spinor Ψ may then be written as

$$\Omega \cdot \Psi = \frac{1}{2} \sum_{i=1}^{2m} e_i \cdot Je_i \cdot \Psi = -\frac{1}{2} \sum_{i=1}^{2m} Je_i \cdot e_i \cdot \Psi. \tag{4}$$

For every vector $X \in TM$ and every spinor Ψ we have

$$\Omega \cdot X \cdot \Psi = X \cdot \Omega \cdot \Psi + 2JX \cdot \Psi. \tag{5}$$

It is well-known (see [15]) that ΣM splits with respect to the Clifford action of Ω into

$$\Sigma M = \bigoplus_{q=0}^{m} \Sigma^q M, \tag{6}$$

where $\Sigma^q M$ is the eigenbundle of rank $\binom{m}{q}$ associated to the eigenvalue $i\mu^q = i(2q-m)$ of Ω.

On the other hand, on every even-dimensional spin manifold M^{2m}, the Clifford action of the complex volume element $\omega^{\mathbb{C}} := i^m e_1 \cdot \ldots \cdot e_{2m}$ (where $\{e_1, \cdots, e_{2m}\}$ is an oriented local orthonormal frame) yields a decomposition $\Sigma M = \Sigma_+ M \oplus \Sigma_- M$, where $\Sigma_\pm M$ is the eigenbundle of ΣM corresponding to the eigenvalue ± 1 of $\omega^{\mathbb{C}}$. If $\Psi = \Psi_+ + \Psi_-$ with respect to this decomposition, we define its conjugate $\bar{\Psi} := \Psi_+ - \Psi_- = \omega^{\mathbb{C}} \cdot \Psi$. It is easy to check that for M Kähler, this decomposition of ΣM is related to (6) by

$$\Sigma_+ M = \bigoplus_{q \text{ even}} \Sigma^q M, \quad \text{and} \quad \Sigma_- M = \bigoplus_{q \text{ odd}} \Sigma^q M. \tag{7}$$

We also recall that ΣM carries a parallel $\mathbb{C}$-anti-linear automorphism $\mathfrak{j}$ commuting with the Clifford multiplication and satisfying $\mathfrak{j}^2 = (-1)^{\frac{m(m+1)}{2}}$. The $\mathbb{C}$-anti-linearity of $\mathfrak{j}$ easily shows that $\mathfrak{j}(\mathfrak{t}^q \mathfrak{M}) = \mathfrak{t}^{m-q} \mathfrak{M}$.

We now turn our attention to limiting manifolds of complex dimension $m = 2\ell \geq 4$ and recall their characterization in terms of special spinors.

THEOREM 2.1 (cf. [17], see also [10]). — *Let (M, g, J) be a spin compact Kähler manifold of complex dimension $m = 2\ell$. Then every eigenvalue λ of the Dirac operator of M satisfies the inequality (3). Moreover, for $m \geq 4$, equality holds in (3) if and only if the scalar curvature S of M is a positive constant and there exists a spinor $\Psi \in \Gamma(\Sigma^{\ell+1} M)$ such that*

$$\nabla_X \Psi = -\frac{1}{n}(X - iJX) \cdot D\Psi, \ \forall X, \tag{8}$$

$$\nabla_X D\Psi = -\frac{1}{4}(\mathrm{Ric}(X) + iJ\,\mathrm{Ric}(X)) \cdot \Psi, \ \forall X, \tag{9}$$

$$\kappa(X - iJX) \cdot \Psi = (\mathrm{Ric}(X) - iJ\,\mathrm{Ric}(X)) \cdot \Psi, \ \forall X, \tag{10}$$

$$\kappa(X - iJX) \cdot D\Psi = (\mathrm{Ric}(X) - iJ\,\mathrm{Ric}(X)) \cdot D\Psi, \ \forall X, \tag{11}$$

where $\kappa = \dfrac{S}{n-2}$. In particular, (8) implies (after a Clifford contraction) that $D\Psi \in \Gamma(\Sigma^\ell M)$.

These relations correspond to formulas (58), (59), (60) and (74) from [17], with the remark that Ψ above and $\psi^{\ell-1}$ of [17] are related by $\Psi = j\psi^{\ell-1}$. They are also obtained in [10], where one can moreover find a very elegant proof of inequalities (1)–(3) by means of elementary linear algebra.

In the next section we will also need the following stronger version of the "if" part of the above theorem

THEOREM 2.2 (cf. [10], Proposition 2). — *Let (M, g, J) be a spin compact Kähler manifold of complex dimension $m = 2\ell$. Suppose that there exists a spinor $\Psi \in \Gamma(\Sigma^{\ell+1}M)$ and a real number λ such that Ψ satisfies (8), $D\Psi \in \Gamma(\Sigma^{\ell}M)$ and $D^2\Psi = \lambda^2\Psi$. Then λ satisfies the equality in (3).*

¿From now on, when speaking about limiting manifolds without specifying their dimension, we always understand that they have even complex dimension $m = 2\ell \geq 4$.

3. Eigenvalues of the Ricci tensor of limiting manifolds.

In [22] we obtained the following

THEOREM 3.1 (cf. [22], Thm. 3.1). — *The Ricci tensor of a limiting manifold of even complex dimension has two eigenvalues, κ and 0, the first one with multiplicity $n-2$ and the second one with multiplicity 2.*

COROLLARY 3.2. — *The tangent bundle of M splits into a J-invariant orthogonal direct sum $TM = \mathcal{E} \oplus \mathcal{F}$ (where $\mathcal{E}$ and $\mathcal{F}$ are the eigenbundles of TM corresponding to the eigenvalues 0 and κ of* Ric *respectively). Moreover, the distributions $\mathcal{E}$ and $\mathcal{F}$ are integrable.*

Proof. — All but the last statement are clear from Theorem 3.1, so we only prove the integrability of $\mathcal{E}$ and $\mathcal{F}$. Let ρ denote the Ricci form of M, defined by $\rho(X,Y) = \mathrm{Ric}(JX,Y)$, which, of course satisfies $d\rho = 0$. Remark that $X \lrcorner \rho = 0$ for $X \in \mathcal{E}$ and $X \lrcorner \rho = -\kappa JX$ for $X \in \mathcal{F}$. We consider arbitrary vector fields $X, Y \in \mathcal{E}$ and $Z \in \mathcal{F}$ and obtain (σ stands for the cyclic sum)

$$\begin{aligned} 0 = d\rho(X,Y,Z) &= \sigma(X(\rho(Y,Z) - \rho([X,Y],Z)) \\ &= -\rho([X,Y],Z), \end{aligned}$$

so $\mathcal{E}$ is integrable. Similarly, for $X, Y \in \mathcal{F}$ and $Z \in \mathcal{E}$ we have

$$\begin{aligned} 0 &= d\rho(X,Y,Z) = \sigma(X(\rho(Y,Z) - \rho([X,Y],Z)) \\ &= Z(\rho(X,Y)) - \rho([Y,Z],X) - \rho([Z,X],Y) \\ &= \kappa(Z(g(JX,Y)) - g(J[Y,Z],X) - g(J[Z,X],Y)) \\ &= \kappa(g(\nabla_Y Z, JX) - g(\nabla_X Z, JY) \\ &= -\kappa g([X,Y],JZ), \end{aligned}$$

which proves the integrability of $\mathcal{F}$. □

¿From (8)–(11) follows that for every section X of $\mathcal{E}$ we have

$$(X - iJX) \cdot \Psi = (X - iJX) \cdot D\Psi = 0 \tag{12}$$

$$\nabla_X \Psi = \nabla_X D\Psi = 0. \tag{13}$$

Conversely, we have

LEMMA 3.3. — *The equations (12) and (13) characterize the kernel of the Ricci tensor; in other words, if X satisfies these equations, then $X \in \mathcal{E}$.*

Proof. — Immediate consequence of (9), (10). □

For later use, we remark that taking the covariant derivative in (12) with respect to some arbitrary vector field Y on M yields

$$\nabla_Y(X - iJX) \cdot \Psi = \nabla_Y(X - iJX) \cdot D\Psi = 0. \tag{14}$$

4. The curvature tensor of limiting manifolds.

In this section we collect information on the curvature tensor of limiting manifolds by using deformations of the metric tensor in the $\mathcal{E}$-directions. More precisely, we show that such a deformation by a constant factor does not affect the property of M to be a limiting manifold, and using the results from the previous sections we interpret this in terms of the curvature tensor.

DEFINITION 4.1. — *An* adapted frame $\{e_i, f_j\}$ $(i \in \{1,2\},\ j \in \{1,\dots,n-2\})$ *is a local orthonormal oriented frame on M such that $\mathcal{E} = \operatorname{span}(e_i)$ and $\mathcal{F} = \operatorname{span}(f_j)$.*

¿From now on, we shall use such adapted frames for several computations, without explicitly stating it at each time.

THEOREM 4.2. — *Let M be a limiting manifold and $TM = \mathcal{E} \oplus \mathcal{F}$ the decomposition given by Theorem 3.1. Denote by $g^{\mathcal{E}}$ and $g^{\mathcal{F}}$ the restrictions of the metric tensor to the two distributions $\mathcal{E}$ and $\mathcal{F}$ and define a family of Riemannian metrics on M by $g^t = t^2 g^{\mathcal{E}} + g^{\mathcal{F}}$ (so, of course, $g^1 = g$). Then, (M, g^t) is a limiting manifold for each $t > 0$. Moreover, the Ricci tensor of g^t does not depend on t.*

Proof. — Let us denote by $X \mapsto X^t$ the canonical isometry between (T_xM, g) and (T_xM, g^t) given by $X^t = X$ for $X \in \mathcal{F}_x$ and $X^t = X/t$ for $X \in \mathcal{E}_x$. We choose a (local) adapted frame $u = \{e_i, f_j\}$ and let X, Y, Z belong to this frame. Using the Koszul formula and Corollary 3.2 we compute

$$g^t(\nabla^t_{X^t} Y^t, Z^t) = \begin{cases} g(\nabla_X Y, Z)\,, & \text{if } N \text{ is even} \\ (1/t) g(\nabla_X Y, Z)\,, & \text{if } N \text{ is odd} \end{cases} \tag{15}$$

where N is the number of X, Y, Z belonging to $\mathcal{E}$. Using this, we compute the spin covariant derivative in the new metric (for the sake of simplicity, we set $(M, g^t) = M^t$). The isometry $X \mapsto X^t$ constructed above yields a bundle isomorphism $P_{SO_n} M \to P_{SO_n} M^t$, $u \mapsto u^t$, satisfying $(ua)^t = u^t a$ for all $a \in SO_n$, hence the composition

$$P_{\mathrm{Spin}_n} M \to P_{SO_n} M \to P_{SO_n} M^t$$

defines a spin structure on M^t. Nevertheless, in order to avoid confusion, we shall not identify the spin structures on M and M^t, but only denote the canonical isomorphism between them by $\gamma \mapsto \gamma^t$. We thus obtain an isomorphism of the spin bundles of M and M^t, $\Phi \mapsto \Phi^t$ which satisfies $(X \cdot \Phi)^t = X^t \cdot \Phi^t$. Consider our spinor Ψ and write it as $\Psi = [\gamma, \xi]$, where γ is a local section of $P_{\mathrm{Spin}_n} M$ projecting over the adapted frame u. Then, classical formulas for the (spin) covariant derivative together with (15) yield, for $X \in \mathcal{F}$,

$$\begin{aligned} \nabla^t_{X^t} \Psi^t = \; & [\gamma^t, X^t(\xi)] + \frac{1}{2} \sum_{i<j} g^t(\nabla^t_{X^t} f_i^t, f_j^t) f_i^t \cdot f_j^t \cdot \Psi^t \\ & + \frac{1}{2} \sum_{i \le 2, j \le n-2} g^t(\nabla^t_{X^t} e_i^t, f_j^t) e_i^t \cdot f_j^t \cdot \Psi^t \\ & + \frac{1}{2} \sum_{i<j} g^t(\nabla^t_{X^t} e_i^t, e_j^t) e_i^t \cdot e_j^t \cdot \Psi^t \end{aligned}$$

$$= (\nabla_X \Psi)^t - (1 - \frac{1}{t})\frac{1}{2} \sum_{i\leq 2, j\leq n-2} g(\nabla_X e_i, f_j)(e_i \cdot f_j \cdot \Psi)^t$$
$$= (\nabla_X \Psi)^t + (1 - \frac{1}{t})\frac{1}{2} \sum_{i\leq 2} (\mathrm{pr}_{\mathcal{F}}(\nabla_X e_i) \cdot e_i \cdot \Psi)^t.$$

On the other hand, a simple use of (12) and (14) shows that the last sum vanishes:

$$\begin{aligned}\sum_{i\leq 2} \mathrm{pr}_{\mathcal{F}}(\nabla_X e_i) \cdot e_i \cdot \Psi &= \mathrm{pr}_{\mathcal{F}}(\nabla_X e_1) \cdot e_1 \cdot \Psi + \mathrm{pr}_{\mathcal{F}}(\nabla_X Je_1) \cdot Je_1 \cdot \Psi \\ &= i\,\mathrm{pr}_{\mathcal{F}}(\nabla_X e_1) \cdot Je_1 \cdot \Psi + \mathrm{pr}_{\mathcal{F}}(\nabla_X Je_1) \cdot Je_1 \cdot \Psi \\ &= -iJe_1 \cdot \mathrm{pr}_{\mathcal{F}}(\nabla_X e_1) \cdot \Psi + \mathrm{pr}_{\mathcal{F}}(\nabla_X Je_1) \cdot Je_1 \cdot \Psi \\ &= Je_1 \cdot \mathrm{pr}_{\mathcal{F}}(\nabla_X Je_1) \cdot \Psi + \mathrm{pr}_{\mathcal{F}}(\nabla_X Je_1) \cdot Je_1 \cdot \Psi \\ &= 0,\end{aligned}$$

so finally

$$\nabla^t_X \Psi^t = (\nabla_X \Psi)^t, \tag{16}$$

for every section X of $\mathcal{F}$. A similar argument shows that this equation is also satisfied if X is a section of $\mathcal{E}$. Finally, the equations that we used to prove this formula (*i.e.* (12) and (14)) also hold for $D\Psi$, so we obtain in the same way

$$\nabla^t_X (D\Psi)^t = (\nabla_X D\Psi)^t, \ \forall X. \tag{17}$$

Then, using (13) and (16) yields $D^t\Psi^t = \sum f_i^t \cdot \nabla^t_{f_i^t} \Psi^t = (D\Psi)^t$, and similarly $D^t(D\Psi)^t = (D^2\Psi)^t = \lambda^2 \Psi^t$, hence $(D^t)^2\Psi^t = \lambda^2 \Psi^t$. We have thus shown that Ψ^t is an eigenspinor for the square of the Dirac operator D^t on M^t with the eigenvalue λ^2. Theorem 2.2 then implies that M^t is a limiting manifold for all t. Consequently, by Theorem 3.1, the Ricci tensor of M^t has the same eigenvalues as that of M. In order to show that $\mathrm{Ric} = \mathrm{Ric}^t$, it then suffices to show that the kernels of Ric and Ric^t are the same. This follows from Lemma 3.3 together with (16) and (17). □

LEMMA 4.3.

1. *Let X, Y be sections of $\mathcal{E}$ and $\mathcal{F}$ respectively and $\{e_i, f_j\}$ an adapted frame. Then*

$$\sum_{i=1}^{2} g(\nabla_{e_i} Y, e_i) = \sum_{j=1}^{n-2} g(\nabla_{f_j} X, f_j) = 0, \tag{18}$$

or equivalently $\sum \nabla_{e_i} e_i \in \mathcal{E}$ and $\sum \nabla_{f_j} f_j \in \mathcal{F}$ for every adapted frame.

2. *If X_1, X_2 are sections of $\mathcal{E}$ and Y_1, Y_2 are sections of $\mathcal{F}$, then*

$$g(\nabla_{X_1} Y_i, X_2) = g(\nabla_{X_2} Y_i, X_1)\ ,\ g(\nabla_{Y_1} X_i, Y_2) = g(\nabla_{Y_2} X_i, Y_1). \tag{19}$$

Proof.

1. Since S is constant, we have $0 = dS/2 = \delta \operatorname{Ric}$. Thus

$$\begin{aligned} 0 &= \sum (\nabla_{e_i} \operatorname{Ric})(e_i) + \sum (\nabla_{f_j} \operatorname{Ric})(f_j) \\ &= -\operatorname{Ric}(\nabla_{e_i} e_i) + \kappa \nabla_{f_j} f_j - \operatorname{Ric}(\nabla_{f_j} f_j) \\ &= -\kappa \operatorname{pr}_{\mathcal{F}}(\nabla_{e_i} e_i) + \kappa \operatorname{pr}_{\mathcal{E}}(\nabla_{f_j} f_j), \end{aligned}$$

and the first assertion follows.

2. This is a direct consequence of the fact that $\mathcal{E}$ and $\mathcal{F}$ are orthogonal and integrable. □

We use now Theorem 4.2 and Lemma 4.3 in order to compare the Ricci tensor of M with the Ricci tensor of a maximal leaf, say N, of the distribution $\mathcal{F}$. The covariant derivative on N is of course obtained by projection on N of the covariant derivative on M : $\nabla^N_X Y = \operatorname{pr}_{\mathcal{F}}(\nabla_X Y)$ for X, Y tangent to N.

PROPOSITION 4.4. — *Let $X \in TN$. Then*

$$\operatorname{Ric}(X, X) = \operatorname{Ric}^N(X, X) - \sum_{i,j} g(\nabla_{e_i} X, e_j)^2 \tag{20}$$

and

$$\sum_i e_i(g(\nabla_X X, e_i)) - \sum_{i,k} g(\nabla_X X, e_k)\, g(e_k, \nabla_{e_i} e_i) + 2 \sum_{i,j} g(\nabla_{e_i} X, f_j)\, g(f_j, \nabla_X e_i) = 0. \tag{21}$$

Proof. — We adopt henceforth the summation convention on repeated subscripts and compute (using Lemma 4.3 and an adapted frame)

$$\begin{aligned} R(f_j, X, X, f_j) &= f_j(g(\nabla_X X, f_j)) - g(\nabla_X X, \nabla_{f_j} f_j) - X(g(\nabla_{f_j} X, f_j)) \\ &\quad + g(\nabla_{f_j} X, \nabla_X f_j) - g(\nabla_{[f_j, X]} X, f_j) \\ &= f_j(g(\nabla_X X, f_j)) - g(\nabla_X X, f_k) g(f_k, \nabla_{f_j} f_j) \\ &\quad - X(g(\nabla_{f_j} X, f_j)) + g(\nabla_{f_j} X, e_i) g(e_i, \nabla_X f_j) \\ &\quad + g(\nabla_{f_j} X, f_k) g(f_k, \nabla_X f_j) - g(\nabla_{[f_j, X]} X, f_j) \\ &= \operatorname{Ric}^N(X, X) + g(\nabla_{f_j} X, e_i) g(e_i, \nabla_X f_j), \end{aligned}$$

hence

$$R(f_j, X, X, f_j) = \mathrm{Ric}^N(X, X) + g(\nabla_{f_j} X, e_i)^2. \tag{22}$$

Similarly,

$$\begin{aligned} R(e_i, X, X, e_i) &= e_i(g(\nabla_X X, e_i)) - g(\nabla_X X, \nabla_{e_i} e_i) - X(g(\nabla_{e_i} X, e_i)) \\ &\quad + g(\nabla_{e_i} X, \nabla_X e_i) - g(\nabla_{[e_i, X]} X, e_i) \\ &= e_i(g(\nabla_X X, e_i)) - g(\nabla_X X, e_k) g(e_k, \nabla_{e_i} e_i) \\ &\quad - X(g(\nabla_{e_i} X, e_i)) + g(\nabla_{e_i} X, e_k) g(e_k, \nabla_X e_i) \\ &\quad + g(\nabla_{e_i} X, f_j) g(f_j, \nabla_X e_i) - g([e_i, X], e_k) g(\nabla_{e_k} X, e_i) \\ &\quad - g([e_i, X], f_j) g(\nabla_{f_j} X, e_i). \end{aligned}$$

In the last equality, the third term vanishes by Lemma 4.3, and the fourth term vanishes too, since it is of the form $a_{ij} b_{ij}$ with a_{ij} symmetric and b_{ij} skew-symmetric, so we are left with

$$\begin{aligned} \mathrm{Ric}(X, X) &= R(f_j, X, X, f_j) + R(e_i, X, X, e_i) \\ &= \mathrm{Ric}^N(X, X) + g(\nabla_{f_j} X, e_i)^2 + e_i(g(\nabla_X X, e_i)) \\ &\quad - g(\nabla_X X, e_k) g(e_k, \nabla_{e_i} e_i) + g(\nabla_{e_i} X, f_j) g(f_j, \nabla_X e_i) \\ &\quad - g(\nabla_{e_i} X, e_k) g(\nabla_{e_k} X, e_i) - g(\nabla_{e_i} X, f_j) g(\nabla_{f_j} X, e_i) \\ &\quad + g(\nabla_X e_i, f_j) g(\nabla_{f_j} X, e_i) \\ &= \mathrm{Ric}^N(X, X) - g(\nabla_{e_i} X, e_k) g(\nabla_{e_k} X, e_i) + e_i(g(\nabla_X X, e_i)) \\ &\quad - g(\nabla_X X, e_k) g(e_k, \nabla_{e_i} e_i) + 2 g(\nabla_{e_i} X, f_j) g(f_j, \nabla_X e_i). \end{aligned}$$

This formula holds for any of the metrics g^t, by Theorem 4.2. Using (15) we then find

$$\mathrm{Ric}^t(X, X) = \mathrm{Ric}_0(X, X) + t^{-2} \mathrm{Ric}_1(X, X), \tag{23}$$

where $\mathrm{Ric}_0(X, X) = \mathrm{Ric}^N(X, X) - g(\nabla_{e_i} X, e_k) g(\nabla_{e_k} X, e_i)$ and $\mathrm{Ric}_1(X, X)$ is the sum of the remaining terms in the above relation. By Theorem 4.2 again, we then obtain $\mathrm{Ric}_1(X, X) = 0$ (which is equivalent to (21)), so finally $\mathrm{Ric}(X, X) = \mathrm{Ric}_0(X, X) = \mathrm{Ric}^N(X, X) - g(\nabla_{e_i} X, e_j)^2$. □

COROLLARY 4.5. — *The following relation holds:*

$$\sum_i R(e_i, X, X, e_i) = -\sum_{i,j} (g(\nabla_X e_i, f_j)^2 + g(\nabla_{e_i} X, e_j)^2). \tag{24}$$

Proof. — Immediate consequence of (20) and (22):

$$R(e_i, X, X, e_i) = \mathrm{Ric}(X, X) - R(f_j, X, X, f_j)$$

$$
\begin{aligned}
&= \mathrm{Ric}^N(X,X) - g(\nabla_{e_i}X, e_j)^2 - (\mathrm{Ric}^N(X,X) \\
&\quad + g(\nabla_{f_j}X, e_i)^2) \\
&= -g(\nabla_X e_i, f_j)^2 - g(\nabla_{e_i}X, e_j)^2. \qquad \square
\end{aligned}
$$

5. Kählerian Killing spinors on Spin^c manifolds.

In this section we give a classification of Spin^c Hodge manifolds of odd complex dimension carrying Kählerian Killing spinors lying in the "middle" of the spectrum of the Kähler form. Such a classification has, of course, some interest independently of other considerations, but its real importance to our problem will only become clear in the next section.

DEFINITION 5.1. — *A Kählerian Killing spinor on a* Spin^c *Kähler manifold* $(M^{4\ell-2}, g, J)$ *is a spinor* Ψ *satisfying*

$$\nabla^A_X \Psi + \alpha X \cdot \Psi + i\alpha(-1)^\ell JX \cdot \bar{\Psi} = 0, \ \forall X, \tag{25}$$

for some real constant $\alpha \neq 0$.

THEOREM 5.2. — *Let* (M^n, g, J), $n = 4\ell - 2$ *be a simply connected compact Hodge manifold endowed with a* Spin^c *structure carrying a Kählerian Killing spinor* $\Psi \in \Gamma(\Sigma^{\ell-1}M \oplus \Sigma^\ell M)$. *Then this* Spin^c *structure on* M *is actually a spin structure, and* M *is a limiting manifold of odd complex dimension.*

Proof. — The proof is in two steps. We first show, as in [21] that the Kählerian Killing spinor on M induces a Killing spinor on some S^1 bundle over M, and then use the classification of Spin^c manifolds carrying Killing spinors to conclude.

By rescaling the metric of M and taking the conjugate of Ψ if necessary, we may suppose that the Killing constant satisfies $\alpha = (-1)^\ell/2$, so (25) becomes

$$\nabla^A_X \Psi + \frac{(-1)^\ell}{2} X \cdot \Psi + \frac{i}{2} JX \cdot \bar{\Psi} = 0, \ \forall X. \tag{26}$$

The Hodge condition just means that $\dfrac{r}{2\pi}[\Omega] \in H^2(M, \mathbb{Z})$ for some $r \in \mathbb{R}^*$, and we will fix some $r < 0$ with this property. The isomorphism $H^2(M, \mathbb{Z}) \simeq H^1(M, S^1)$ guarantees the existence of some principal $U(1)$

bundle $\pi : S \to M$ whose first Chern class satisfies $c_1(S) = \frac{r}{2\pi}[\Omega]$. Furthermore, the Thom-Gysin exact sequence shows that S is simply connected if r is chosen in such a way that $\frac{r}{2\pi}[\Omega]$ is not a multiple of some integral class in $H^2(M,\mathbb{Z})$ (cf. [2], p. 85).

The condition above on the first Chern class of S shows that there is a connection on S whose curvature form G satisfies $G = -ir\pi^*\Omega$. This connection induces a 1-parameter family of metrics on S which turn the bundle projection $\pi : S \to M$ into a Riemannian submersion with totally geodesic fibers. These metrics are given by

$$g_S^t(X,Y) = g(\pi_*(X),\pi_*(Y)) - t^2\,\omega(X)\,\omega(Y) \quad (t>0),$$

where ω denotes the (imaginary valued) connection form on S. Let V^t denote the unit vertical vector field on S defined by $\omega(V^t) = i/t$ and for $X \in TM$, let X^* denote its horizontal lift to TS. We now compute the O'Neill tensors [25] $\mathcal{A}$ and $\mathcal{T}$ of the submersion $S \to M$. For every vector fields X, Y on M,

$$G(X^*,Y^*) = d\omega(X^*,Y^*) = -\frac{1}{2}\omega([X^*,Y^*]) = \frac{1}{2ti}g_S^t([X^*,Y^*],V^t),$$

so

$$\begin{aligned}\Omega(X,Y) = \pi^*\Omega(X^*,Y^*) &= \frac{1}{2rt}g_S^t([X^*,Y^*],V^t) = \frac{1}{rt}g_S^t(\nabla_{X^*}Y^*,V^t)\\ &= -\frac{1}{rt}g_S^t(Y^*,\nabla_{X^*}V^t) = -\frac{1}{rt}g(Y,\pi(\mathcal{A}_{X^*}V^t)).\end{aligned}$$

For the remaining of this section, we fix $t = -1/r$ and denote $V := V^t$ and $g_S := g_S^t$. We thus have obtained

$$\mathcal{A}_{X^*}V = J(X)^*. \tag{27}$$

Now,

$$0 = G(V,X^*) = d\omega(V,X^*) = -\frac{1}{2}\,\omega([V,X^*]),$$

so $[V,X^*]$ is horizontal for every vector field X on M. On the other hand, V projects to 0 and X^* to X, so $[V,X^*]$ projects to 0, *i.e.* it is vertical. We have shown that $[V,X^*] = 0$ for every vector field X on M. Or, $g_S(V,V) = 1$ implies that $g_S(V,\nabla_{X^*}V) = 0$ and thus

$$0 = g_S(V,\nabla_V X^*) = g_S(\nabla_V V, X^*),\ \forall X \in TM,$$

so $\nabla_V V = 0$, which shows that the submersion π has totally geodesic fibers (equivalently, $\mathcal{T} = 0$).

By pull-back from M, on S we obtain a Spin^c structure whose spinor bundle is just $\pi^*\Sigma M$ (for M spin, this was shown in [21], Section 3; the Spin^c case is similar). Clifford multiplication is given by

$$X^* \cdot \pi^*\Psi = \pi^*(X \cdot \Psi), \tag{28}$$

$$V \cdot \pi^*\Psi = \pi^*(i\bar{\Psi}). \tag{29}$$

We now relate covariant derivatives of spinors on S and M.

PROPOSITION 5.3. — *Let $S \to M$ be a Riemannian submersion with totally geodesic one-dimensional fibers. Suppose that M is endowed with a* Spin^c *structure with covariant derivative ∇^A on the corresponding spinor bundle, and let $\nabla^{S,A}$ denote the covariant derivative on the spinor bundle of S corresponding to the* Spin^c *structure on S obtained by pull-back from M. Then for every spinor Ψ on M we have*

$$\nabla^{S,A}_{X^*}\pi^*\Psi = \pi^*\Big(\nabla^A_X\Psi - \frac{1}{2}i\pi_*(\mathcal{A}_{X^*}V)\cdot\bar{\Psi}\Big), \tag{30}$$

and

$$\nabla^{S,A}_{V}\pi^*\Psi = -\frac{1}{4}\pi^*\Big(\sum_{j=1}^{n}\pi_*(\mathcal{A}_{X_j^*}V)\cdot X_j\cdot\Psi\Big). \tag{31}$$

We will skip the proof, which is similar to that of [21], Prop. 2.

Applying this result, together with (4), (26) and (27), to our Kählerian Killing spinor Ψ yields

$$\nabla^{S,A}_{X^*}\pi^*\Psi = \frac{1}{2}\pi^*((-1)^\ell X\cdot\Psi) = \frac{(-1)^\ell}{2}X^*\cdot\pi^*\Psi$$

and

$$\nabla^{S,A}_{V}\pi^*\Psi = \frac{1}{4}\pi^*(2\Omega\cdot\Psi) = \frac{(-1)^\ell}{2}\pi^*(i\bar{\Psi}) = \frac{(-1)^\ell}{2}V\cdot\pi^*\Psi.$$

Here we have used the fact that $\Psi \in \Sigma^{\ell-1}M \oplus \Sigma^\ell M$ and thus $\Omega\cdot\Psi = (-1)^\ell i\bar{\Psi}$ (recall that $\Sigma^\ell M \subset \Sigma_+ M$ exactly when ℓ is even).

These two equations just mean that $\pi^*\Psi$ is a Killing spinor of the pull-back Spin^c structure on S.

Remark 5.4. — At this point, the reader might be slightly confused by the fact that Kählerian Killing spinors on Spin^c manifolds (inducing Killing spinors on suitable S^1 bundles) also appear in [24]. But, in contrast to our present situation, they do not live in the "middle" of the spectrum

of Ω, and the Spin^c structures of the S^1 bundles considered there are not the same as here (see [24], Prop. 3.2).

Now, a standard argument shows that $\pi^*\Psi$ induces a parallel spinor Φ on the cone $\bar{S}$ over S, endowed with the pull-back Spin^c structure (see [23]). Since S is compact, a theorem of Gallot ([9], Prop. 3.1) shows that $\bar{S}$ is an irreducible Riemannian manifold. From ([23], Thm.3.1) we then deduce that either the Spin^c structure of $\bar{S}$ is actually a spin structure, or there exists a Kähler structure I on $\bar{S}$ such that

$$X \cdot \Phi = iI(X) \cdot \Phi, \ \forall X \in T\bar{S}, \tag{32}$$

and the Spin^c structure of $\bar{S}$ is the canonical Spin^c structure induced by I (these two cases do not exclude each other). In the first case we are done since then the Spin^c structure on M has to be a spin structure, and Ψ has to be a usual Kählerian Killing spinor, so M is a limiting manifold.

In the second case, we first remark that $\bar{S}$ carries another Kähler structure, say J, which comes from that of M, and such that Φ lies in the kernel of the Kähler form Ω_J of J (see [21]).

Taking the Clifford product with Ω_J in (32) and using (5) yields

$$JX \cdot \Phi = iJI(X) \cdot \Phi, \ \forall X \in T\bar{S}, \tag{33}$$

so replacing X by JX in (32) and using (33) shows that $IJ = JI$. Now it is clear that $I \neq \pm J$ since $\Omega_J \cdot \Phi = 0$ and, by (32), $\Omega_I \cdot \Psi = 2i\ell\Phi$, where Ω_I denotes the Kähler form of I. On the other hand, IJ is a symmetric parallel involution of $T\bar{S}$, so the decomposition $T\bar{S} = T^+ \oplus T^-$, where $T^\pm = \{X \mid IJX = \pm X\}$ gives a holonomy reduction of $\bar{S}$, which contradicts the above mentioned result of Gallot. □

6. Restrictions of spinors to the leaves of $\mathcal{F}$.

We are now ready to complete the proof of our main result:

THEOREM 6.1. — *The Ricci tensor of a limiting manifold of even complex dimension is parallel.*

Proof. — Proposition 4.4 shows that the Ricci curvature of any maximal leaf N of $\mathcal{F}$ is greater than κ. Since N is complete, Myers' Theorem implies that N is compact. Moreover, N being Kähler with positive defined

Ricci tensor, a theorem of Kobayashi ([18], Thm. A) shows that N is simply connected. We shall now consider the restriction Φ^N of $\Phi := \Psi + \frac{2}{\sqrt{n\kappa}} D\Psi$ to N. First of all, what kind of object is Φ^N? To answer this question, we recall the classical representation of spinor bundles on Kähler manifolds (*e.g.* [15]):

$$\Sigma M \simeq (\mathcal{K}^M)^{\frac{1}{2}} \otimes \Lambda^{0,*} M.$$

where, as usual, $(\mathcal{K}^M)^{\frac{1}{2}}$ denotes a square root of the canonical bundle of M. Through this identification, the spin covariant derivative on the left hand side corresponds to the Levi-Civita covariant derivative on the right hand side. In our particular situation we have the following isomorphisms of complex vector bundles:

$$\mathcal{K}^M \simeq \Lambda^m(T^{0,1}M) \simeq \Lambda^m(\mathcal{E}^{0,1} \oplus \mathcal{F}^{0,1}) \simeq \mathcal{E}^{0,1} \otimes \Lambda^{m-1}(\mathcal{F}^{0,1}),$$

so

$$\mathcal{K}^M|_N \simeq \mathcal{E}^{0,1}|_N \otimes \mathcal{K}^N. \tag{34}$$

Similarly,

$$\Lambda^{0,*} M|_N \simeq \Lambda^{0,*} N \oplus (\mathcal{E}^{1,0}|_N \otimes \Lambda^{0,*} N),$$

and thus

$$\Sigma M|_N \simeq ((\mathcal{K}^M)^{\frac{1}{2}}|_N \otimes \Lambda^{0,*} N) \oplus ((\mathcal{K}^M)^{\frac{1}{2}}|_N \otimes (\mathcal{E}^{1,0}|_N \otimes \Lambda^{0,*} N)). \tag{35}$$

Now formula (12) just means that the $(\mathcal{K}^M)^{\frac{1}{2}}|_N \otimes \Lambda^{0,*} N$-part of Φ^N vanishes, hence Φ^N is a section of

$$(\mathcal{K}^M)^{\frac{1}{2}}|_N \otimes (\mathcal{E}^{1,0}|_N \otimes \Lambda^{0,*} N),$$

and by the above, this is just the spinor bundle of some Spin^c structure on N with associated line bundle $\mathcal{E}^{1,0}|_N$. In fact we may write locally

$$\begin{aligned}(\mathcal{K}^M)^{\frac{1}{2}}|_N \otimes (\mathcal{E}^{1,0}|_N \otimes \Lambda^{0,*} N) &\simeq ((\mathcal{E}^{1,0})|_N)^{\frac{1}{2}} \otimes ((\mathcal{K}^N)^{\frac{1}{2}} \otimes \Lambda^{0,*} N) \\ &\simeq ((\mathcal{E}^{1,0})|_N)^{\frac{1}{2}} \otimes \Sigma N,\end{aligned}$$

but, of course, neither $((\mathcal{E}^{1,0})|_N)^{\frac{1}{2}}$ nor ΣN need not exist globally on N.

We now want to compute the covariant derivative of Φ^N as Spin^c spinor on N. Note, first, that each of the above vector bundles inherit a covariant derivative (that we shall call *natural*), coming from the Levi-Civita covariant derivative on M. Indeed, all these bundles are exterior and tensor products of sub-bundles of $TM^{\mathbb{C}}$. On each sub-bundle of $TM^{\mathbb{C}}$ we have a covariant derivative obtained from the usual covariant derivative on

M, followed by the projection to the considered sub-bundle. But, in general, the above isomorphisms do not preserve the covariant derivatives obtained in this way (because $\mathcal{E}$ and $\mathcal{F}$ are not parallel distributions — at least, we do not know this yet!). Nevertheless, the next lemma shows that we may compute the covariant derivative of Φ^N using the above isomorphisms.

Let us denote by A the natural connection induced on $(\mathcal{E}^{1,0})|_N$ by the Levi-Civita covariant derivative of M, and by $\nabla^{N,A}$ the corresponding Spin^c covariant derivative on $(\mathcal{K}^M)^{\frac{1}{2}}|_N \otimes (\mathcal{E}^{1,0}|_N \otimes \Lambda^{0,*}N)$. It should be noted that the natural covariant derivatives on $\Lambda^{m-1}(\mathcal{F}^{0,1}) \simeq \mathcal{K}^N$ and $\Lambda^*(\mathcal{F}^{0,1}) \simeq \Lambda^{0,*}N$ coincide with those coming from the Levi-Civita connection on N. With these notations we have

LEMMA 6.2.

$$\nabla^{N,A}_Y \Phi^N = (\nabla_Y \Phi)|_N, \ \forall Y \in TN. \tag{36}$$

Proof. — By (8), (9) and (12) easily follows

$$(X - iJX) \cdot \nabla_Y \Psi = (X - iJX) \cdot \nabla_Y D\Psi = 0, \ \forall Y \in TM, \ X \in \mathcal{E}. \tag{37}$$

This implies, as before, that the $(\mathcal{K}^M)^{\frac{1}{2}}|_N \otimes \Lambda^{0,*}N$-part of $(\nabla_Y \Phi)|_N$ vanishes for all Y, hence $(\nabla_Y \Phi)|_N$ is also a section of $(\mathcal{K}^M)^{\frac{1}{2}}|_N \otimes (\mathcal{E}^{1,0}|_N \otimes \Lambda^{0,*}N)$. We then remark that the natural covariant derivative on the bundle $B := (\mathcal{K}^M)^{\frac{1}{2}}|_N \otimes (\mathcal{E}^{1,0}|_N \otimes \Lambda^{0,*}N)$ is obtained from the spin covariant derivative on B (identified *via* (35) to a sub-bundle of $\Sigma M|_N$) followed by the projection back to B. But, as the isomorphism (34) preserves the covariant derivatives, we deduce that the natural covariant derivative on B is just the Spin^c covariant derivative $\nabla^{N,A}$. We have thus obtained

$$\nabla^{N,A}_Y \Phi^N = pr_B((\nabla_Y \Phi)|_N) = (\nabla_Y \Phi)|_N, \ \forall Y \in TN. \qquad \square$$

For later use, we compute the curvature form iF of the complex line bundle $L = (\mathcal{E}^{1,0})|_N$. Let $e \in \mathcal{E}$ be a local unit vector field defining a local section $\sigma := e - iJe$ of L. Then

$$\begin{aligned}
\nabla^A_X \sigma &= \mathrm{pr}_{\mathcal{E}^{1,0}}(\nabla_X \sigma) = \mathrm{pr}_{\mathcal{E}}(\nabla_X \sigma) \\
&= g(\nabla_X e, Je)Je - ig(\nabla_X Je, e)e = ig(\nabla_X e, Je)(e - iJe) \\
&= ig(\nabla_X e, Je)\sigma,
\end{aligned}$$

which yields

$$\begin{aligned}
iF(X,Y)\sigma &= i(Xg(\nabla_Y e, Je) - Yg(\nabla_X e, Je) - g(\nabla_{[X,Y]}e, Je))\sigma \\
&= i(g(\nabla_X \nabla_Y e, Je) - g(\nabla_Y e, \nabla_X Je) - g(\nabla_Y \nabla_X e, Je) \\
&\quad + g(\nabla_X e, \nabla_Y Je) - g(\nabla_{[X,Y]}e, Je))\sigma \\
&= i(R(X,Y,e,Je) + 2g(\nabla_X e, \nabla_Y Je))\sigma,
\end{aligned}$$

thus showing that

$$F(X,Y) = R(e,Je,X,Y) + 2g(\nabla_X e, \nabla_Y Je). \tag{38}$$

On the other hand, (12) implies that $e \cdot Je \cdot \Psi = i\Psi$ and $e \cdot Je \cdot D\Psi = iD\Psi$ for all unit vectors e in $\mathcal{E}$. We fix such a vector e for a moment, and remark that, as elements of the Clifford bundle, the Kähler form of M, Ω, and that of N, Ω^N are related by $\Omega = \Omega^N + e \cdot Je$. Recall now (Section 2) that $\Psi \in \Sigma^{\ell+1}M$ and $D\Psi \in \Sigma^{\ell}M$, *i.e.* $\Omega \cdot \Psi = i(2(\ell+1) - m)\Psi$ and $\Omega \cdot D\Psi = i(2\ell - m)D\Psi$. This shows that $\Omega^N \cdot \Psi|_N = i(2\ell - (m-1))\Psi|_N$ and $\Omega^N \cdot D\Psi|_N = i(2(\ell-1) - (m-1))D\Psi|_N$, *i.e.* $\Psi|_N \in \Sigma^{\ell}N$ and $D\Psi|_N \in \Sigma^{\ell-1}N$. Using (36), (8), (9) and (7) we then obtain

$$\nabla^{N,A}_X \Phi^N + \alpha(X \cdot \Phi^N + i\varepsilon JX \cdot \overline{\Phi}^N) = 0, \ \forall X \in TN, \tag{39}$$

where $\varepsilon = (-1)^{\ell}$ and $\alpha = \sqrt{\frac{\kappa}{4n}}$. Thus Φ^N is a *Kählerian Killing* Spin^c *spinor* on N (with Killing constant α). Moreover, N is a Hodge manifold: if we denote by i the inclusion $N \to M$ and by ρ the Ricci form of M, then $\kappa\Omega^N = i^*\rho$, which implies $\kappa[\Omega^N] = i^*(2\pi c_1(M))$, and thus $[\Omega^N]$ is a real multiple of $i^*(c_1(M)) \in H^2(N,\mathbb{Z})$.

We then apply Theorem 5.2 and deduce that the Spin^c structure on N has actually to be a *spin* structure (*i.e.* $\mathcal{E}^{1,0}|_N$ is a flat bundle on N, or, equivalently, $F = 0$). We shall now see that the vanishing of F implies that $\mathcal{E}$ and $\mathcal{F}$ are parallel, and this will complete the proof.

For an arbitrary vector field X on N we compute, using the first Bianchi identity, Lemma 4.3, (38) and (24)

$$\begin{aligned}
0 = F(X,JX) &= R(e,Je,X,JX) + 2g(\nabla_X e, \nabla_{JX} Je) \\
&= -R(Je,X,e,JX) - R(X,e,Je,JX) + 2g(\nabla_X e, f_i)g(f_i, \nabla_{JX} Je) \\
&= -R(Je,X,X,Je) - R(e,X,X,e) + 2g(\nabla_X e, f_i)g(JX, \nabla_{f_i} Je) \\
&= -R(e_i,X,X,e_i) + 2g(\nabla_X e, f_i)g(X, \nabla_{f_i} e) \\
&= g(\nabla_X e_i, f_j)^2 + g(\nabla_{e_i} X, e_j)^2 + 2g(\nabla_X e, f_i)^2 \\
&= 2g(\nabla_X e_i, f_j)^2 + g(\nabla_{e_i} X, e_j)^2.
\end{aligned}$$

This clearly shows that $\mathcal{E}$ and $\mathcal{F}$ are parallel distributions at each point of N, so they are parallel on M because the N's foliate M. □

As an immediate corollary of Theorem 6.1 we obtain

THEOREM 6.3. — *The universal cover $\widetilde{M}$ of a limiting manifold M of even complex dimension is isometric to the Riemannian product $N \times \mathbb{R}^2$, where N is a limiting manifold of odd complex dimension.*

Proof. — Denote by π the covering projection $\widetilde{M} \to M$ and take an arbitrary point $x \in M$. It is clear from the above proof that the maximal leaf N of $\mathcal{F}$ containing x is a limiting manifold of odd complex dimension. We have seen moreover that N is simply connected, so each connected component of $\pi^{-1}(N)$ is isometric to N. Take $y \in \pi^{-1}(x)$ and let $\widetilde{N}$ be the maximal leaf of the pull-back of $\mathcal{F}$ to $\widetilde{M}$ containing y. The decomposition theorem of de Rham implies that $\widetilde{M} \simeq \widetilde{N} \times \mathbb{R}^2$. Finally, it is easy to see that $\widetilde{N}$ is just the connected component of $\pi^{-1}(N)$ containing y, and thus $\widetilde{N}$ is a limiting manifold of odd complex dimension. □

By taking into account the classification of limiting manifolds of odd complex dimension [21] we can refine this result as follows

COROLLARY 6.4. — *Let M^{2m} be a limiting manifold of even complex dimension $m = 2\ell$, $\ell \geq 2$ and $\widetilde{M}$ its universal cover. Then*

- *if ℓ is odd, $\widetilde{M}$ is isometric to the Riemannian product $CP^{2\ell-1} \times \mathbb{R}^2$, where $CP^{2\ell-1}$ is the complex projective space endowed with the Fubini-Study metric.*
- *if ℓ is even, $\widetilde{M}$ is isometric to the Riemannian product $N \times \mathbb{R}^2$, where N is the twistor space of some quaternionic Kähler manifold of positive scalar curvature.*

7. The classification of limiting manifolds.

Let $M^{4\ell}$ be a limiting manifold of complex dimension 2ℓ, $\pi : \widetilde{M} \to M$ its universal cover and Γ the fundamental group of M. Obviously, Γ can be seen as a discrete group of isometries acting freely on $\widetilde{M}$, and M is isomorphic to $\widetilde{M}/\Gamma$. Theorem 6.3 says that $\widetilde{M}$ is isometric to a Riemannian product $N \times \mathbb{R}^2$, where N is a limiting manifold of odd complex dimension. We first recall the following (probably well-known) general result

LEMMA 7.1. — *Let M', M'' be Riemannian manifolds. Then the group $\mathcal{I}_0(M' \times M'')$ of isometries of $M' \times M''$ preserving the horizontal and vertical distributions is canonically isomorphic to the product $\mathcal{I}(M') \oplus \mathcal{I}(M'')$ of the isometry groups of M' and M''.*

Proof. — Let $\gamma \in \mathcal{I}_0(M' \times M'')$. It is clear that γ maps each submanifold $M' \times \{m''\}$ isometrically onto $M' \times \{\widetilde{m}''\}$ for some $\widetilde{m}''$ (depending

only on m''), and thus $\gamma(m',m'') = (\gamma'_{m''}(m'),\gamma''(m''))$, where $\gamma'_{m''}$ are isometries of M' depending (*a priori*) on m'' and γ'' is a transformation of M'' not depending on m'. As the situation is symmetric with respect to M' and M'', we deduce that $\gamma(m',m'') = (\gamma'(m'),\gamma''(m''))$, where γ', γ'' are isometries of M', M'' respectively. □

In our case, $\mathcal{I}(N \times \mathbb{R}^2) = \mathcal{I}_0(N \times \mathbb{R}^2)$ because every isometry of $\widetilde{M}$ preserves the kernel of the Ricci tensor and its orthogonal complement. Moreover, as M is Kähler, Γ consists of *holomorphic* isometries of $\widetilde{M}$. Hence $\Gamma \subset \mathcal{I}^h(N) \times \mathcal{I}^h(\mathbb{R}^2)$, where $\mathcal{I}^h(X)$ denotes the group of holomorphic isometries of the Kähler manifold X. Let us denote by Γ', resp. Γ'' the projections of Γ on $\mathcal{I}^h(N)$, resp. $\mathcal{I}^h(\mathbb{R}^2)$.

LEMMA 7.2. — *The group Γ'' consists of translations only.*

Proof. — We use again the theorem of Kobayashi, which implies that there is no group of holomorphic isometries acting freely on N. Let $\gamma = (\gamma',\gamma'') \in \Gamma$ and suppose that γ'' is not a translation. Since γ'' is holomorphic, it is of the form $v \mapsto \alpha v + \beta$, $\alpha,\beta \in \mathbb{C}$, $\alpha \neq 1$, so it has a fixed point, say v_0. This implies that for every n, either γ'^n has no fixed point, or $\gamma^n = 1_{N\times\mathbb{R}^2}$. Consider the subgroup $\langle\gamma\rangle$ of Γ generated by γ. If $\langle\gamma\rangle$ is finite (of order $n > 1$), then by the above γ'^m has no fixed point for $m < n$, hence $\langle\gamma'\rangle$ acts freely on N, which is impossible. Hence $\langle\gamma\rangle$ has infinite order, and thus γ'^n has no fixed point for all $n \geq 1$. Again by the theorem of Kobayashi, it follows that $\langle\gamma'\rangle$ does not act freely on N. As N is compact, we can then find $x \in N$ and a sequence $n_i \to \infty$ such that $\gamma'^{n_i}(x) \to x$. This implies that $\gamma^{n_i}(x,v_0) \to (x,v_0)$, so the action of $\langle\gamma\rangle$ on $N \times \mathbb{R}^2$ is not free, a contradiction. This shows that γ'' is a translation. □

The above argument actually proves slightly more, namely that if some $\gamma = (\gamma',\gamma'') \in \Gamma$ satisfies $\gamma'' = 1_{\mathbb{R}^2}$, then $\gamma' = 1_N$. In particular, this implies, firstly, that Γ has no element of finite order, and secondly, that Γ is Abelian, because the Γ''-part of any commutator is the identity, by the above lemma. Hence $\Gamma \simeq \mathbb{Z}^k$ and the compactness of M easily implies that $k = 2$. Let $\gamma_i = (\gamma'_i,\gamma''_i)$, $i = 1,2$ be two elements generating Γ, where γ'_i are commuting holomorphic isometries of N and γ''_i are translations of $\mathbb{R}^2$.

We now show that every isometry γ' of N such that $\gamma' \in \Gamma'$ lifts to an isomorphism of the spin bundle of N preserving a Kählerian Killing spinor Φ^N on N (not depending on γ'). For this we first need the following well-known classical result.

LEMMA 7.3.

a) *The universal cover $\widetilde{M}$ of a spin manifold $M = \widetilde{M}/\Gamma$ carries a unique spin structure.*

b) *The spin structures on M are in one-to-one correspondence with lifts to $P_{\mathrm{Spin}_n}\widetilde{M}$ of the tangent action Γ_* of Γ on $P_{SO_n}\widetilde{M}$.*

Proof.

a) Uniqueness is obvious. To prove existence, we denote the covering projection by π and remark that $\pi^* P_{SO_n} M$ is isomorphic to $P_{SO_n}\widetilde{M}$. It follows that the pull-back by π of the spin structure on M defines a spin structure on $\widetilde{M}$.

b) Using a), for every spin structure on M we may view the spin structure on $\widetilde{M}$ as a pull-back. We then define $\widetilde{\gamma}[\widetilde{m}, u_m] = [\gamma(\widetilde{m}), u_m]$, where $\widetilde{m} \in \widetilde{M}$, $m = \pi(\widetilde{m})$ and u_m is an element of $P_{\mathrm{Spin}_n} M$. It is easy to check that this is a lift to $P_{\mathrm{Spin}_n}\widetilde{M}$ of the tangent action of γ on $P_{SO_n}\widetilde{M}$. Conversely, if $\widetilde{\Gamma}$ is a lift to $P_{\mathrm{Spin}_n}\widetilde{M}$ of the tangent action Γ_* on $P_{SO_n}\widetilde{M}$, then we simply define $P_{\mathrm{Spin}_n} M = (P_{\mathrm{Spin}_n}\widetilde{M})/\widetilde{\Gamma}$, and it is clear that the two constructions are inverse to each other. □

Let Φ be the eigenspinor of the Dirac operator on M defined in Section 5 and $\widetilde{\Phi}$ the spinor induced on $\widetilde{M}$ by pull-back. From (13) follows that $\nabla_X \Phi = 0$ for all vectors $X \in \mathcal{E}$ so obviously $\nabla_X \widetilde{\Phi} = 0$ for all vectors X on $\widetilde{M} = N \times \mathbb{R}^2$ tangent to $\mathbb{R}^2$. This shows that the restriction of $\widetilde{\Phi}$ to $N \times \{v\}$ is a Kählerian Killing spinor Φ^N on N which does not depend on $v \in \mathbb{R}^2$. Now, by Lemma 7.3, b) the spin structure of M corresponds to a lift of Γ to $P_{\mathrm{Spin}_n}\widetilde{M}$, which preserves $\widetilde{\Phi}$. Take an element $\gamma = (\gamma', \gamma'')$ in Γ. The fact that $\gamma_* \widetilde{\Phi} = \widetilde{\Phi}$ shows that $\gamma'_* \Phi^N = \Phi^N$, where γ'_* is the lift of the action of γ' to $P_{\mathrm{Spin}_n} N$ given by the restriction of γ_*. Thus every isometry $\gamma' \in \Gamma'$ lifts to an isomorphism of $P_{\mathrm{Spin}_n} N$ preserving Φ^N.

Conversely, let N be an odd dimensional limiting manifold, γ'_1, γ'_2 be two commuting holomorphic isometries of N with the above property and γ''_1, γ''_2 two (linearly independent) translations of $\mathbb{R}^2$. Then $M := (N \times \mathbb{R}^2)/\Gamma$ is an even dimensional limiting manifold, where $\Gamma = \langle (\gamma'_i, \gamma''_i) \rangle$. To see this, remark first that the spin structure of $N \times \mathbb{R}^2$ is obtained from that of N by pull-back on $N \times \mathbb{R}^2$ and enlargement of the structure group. Hence the Kählerian Killing spinor Φ^N preserved by f_i induces a spinor $\widetilde{\Phi}$ on $N \times \mathbb{R}^2$ satisfying (8)–(11), and the action of γ'_i on $P_{\mathrm{Spin}_n} N$ induces an action of Γ on $P_{\mathrm{Spin}_n}(N \times \mathbb{R}^2)$ preserving $\widetilde{\Phi}$. Consequently, we obtain

a spinor Φ on $M := (N \times \mathbb{R}^2)/\Gamma$ satisfying (8)–(11), so M is a limiting manifold. We have obtained

THEOREM 7.4. — *A Kähler manifold M of even complex dimension $m \geq 4$ is a limiting manifold if and only if it is isometric to $(N \times \mathbb{R}^2)/\Gamma$, where N is a limiting manifold of odd complex dimension $m-1$, and $\Gamma = \langle(\gamma_1', \gamma_1''), (\gamma_2', \gamma_2'')\rangle$, where γ_i'' are independent translations of $\mathbb{R}^2$ and γ_i' are commuting holomorphic isometries of N which lift to commuting isomorphisms γ_{i*}' of $P_{\mathrm{Spin}_n} N$ preserving a Kählerian Killing spinor of N.*

Remark 7.5. — Let P be the parallelogram in $\mathbb{R}^2$ with vertices 0, $\gamma_1''(0)$, $\gamma_2''(0)$, $\gamma_1''(0) + \gamma_2''(0)$. Then the quotient $(N \times \mathbb{R}^2)/\Gamma$ can be seen as $N \times P/\sim$, where $\sim$ is the equivalence relation

$$(n, t\gamma_1''(0)) \sim (\gamma_1'(n), t\gamma_1''(0) + \gamma_2''(0))$$

and

$$(n, s\gamma_2''(0)) \sim (\gamma_2'(n), \gamma_1''(0) + s\gamma_2''(0)),$$

for all $n \in N$ and $s, t \in [0,1]$. This is just the *suspension* of the commuting pair of isometries γ_1', γ_2' of N over the parallelogram P.

In order to complete the classification, we have to decide when two limiting manifolds obtained in this way are isomorphic (*i.e.* holomorphically isometric). This is achieved by the following

LEMMA 7.6. — *Let $M_1 = (N_1 \times \mathbb{R}^2)/\Gamma_1$ and $M_2 = (N_2 \times \mathbb{R}^2)/\Gamma_2$ be two limiting manifolds, where N_1, N_2, Γ_1, Γ_2 are as in Theorem 7.4. Then M_1 is holomorphically isometric to M_2 if and only if there exist a holomorphic isometry $\varphi : N_1 \to N_2$ and $\alpha \in S^1$, such that $\Gamma_1' = \varphi^{-1} \circ \Gamma_2' \circ \varphi$ and $\Gamma_1'' = \alpha\Gamma_2''$. In particular, a limiting manifold $M = (N \times \mathbb{R}^2)/\Gamma$ is decomposable (i.e. isometric to a product $N \times T^2$) if and only if $\Gamma = \{1_N\}$.*

Proof. — Any holomorphic isometry $M_1 \to M_2$ obviously lifts to a holomorphic isometry $\Phi : \widetilde{M_1} = N_1 \times \mathbb{R}^2 \to N_2 \times \mathbb{R}^2 = \widetilde{M_2}$ of the universal covers, which, by Lemma 7.1, can be written $\Phi = (\varphi, A)$, where φ is a holomorphic isometry $N_1 \to N_2$ and $A : \mathbb{R}^2 \to \mathbb{R}^2$ is of the form $Av = \alpha v + \beta$, $\alpha \in S^1$, $\beta \in \mathbb{C}$. Now, such a holomorphic isometry Φ descends to the quotients of $\widetilde{M_1}$, $\widetilde{M_2}$ through Γ_1, Γ_2 if and only if $\Gamma_1 = \Phi^{-1} \circ \Gamma_2 \circ \Phi$, which is equivalent to our statement. □

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