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ON SOLUTIONS OF THE SCHRÖDINGER EQUATION WITH RADIATION CONDITIONS AT INFINITY: THE LONG-RANGE CASE

by Y. GÂTEL & D. YAFAEV

1. Introduction.

In this paper we obtain a complete description of solutions $u(x)$ of the homogeneous Schrödinger equation with a long-range potential V

$$-\Delta u(x) + V(x)u(x) = \lambda u(x), \quad x \in \mathbb{R}^d, \ \lambda > 0, \ V = \overline{V}, \tag{1.1}$$

obeying the natural condition

$$\int_{|x|\leq\rho} |u(x)|^2 dx \leq C\rho \quad \text{as} \quad \rho \to \infty. \tag{1.2}$$

Our aim is to show that every function $u(x)$ satisfying (1.1) and (1.2) is asymptotically a sum of incoming and outgoing distorted spherical waves

$$w_\pm(x, \lambda) = |x|^{-(d-1)/2} e^{\pm i\varphi(x,\lambda)}, \tag{1.3}$$

where the phase φ is a suitably chosen approximate solution of the eikonal equation

$$|\nabla_x \varphi(x, \lambda)|^2 + V(x) = \lambda. \tag{1.4}$$

Clearly, for functions $w_\pm$, estimate (1.2) is fulfilled. Remainders $\epsilon(x)$ in our asymptotic formulas will satisfy the condition

$$\lim_{\rho\to\infty} \rho^{-1} \int_{|x|\leq\rho} |\epsilon(x)|^2 dx = 0, \tag{1.5}$$

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which will be denoted as $\epsilon(x) = o_{av}(|x|^{-(d-1)/2})$. Of course, condition (1.5) is satisfied if $\epsilon(x) = o(|x|^{-(d-1)/2})$ in the usual sense.

The main result of our paper can be formulated as follows:

A) For any function $a_- \in L^2(\mathbb{S}^{d-1})$, there exist a function $a_+ \in L^2(\mathbb{S}^{d-1})$ and a solution u of equation (1.1) with asymptotics (uniform in λ from any compact subinterval of $(0, \infty)$)

$$u(x) = a_+(\hat{x}) w_+(x, \lambda) - a_-(\hat{x}) w_-(x, \lambda) + o_{av}(|x|^{-(d-1)/2}), \quad \hat{x} = x/|x| \tag{1.6}$$

$$\partial_r u(x) = i\sqrt{\lambda}\Big(a_+(\hat{x}) w_+(x, \lambda) + a_-(\hat{x}) w_-(x, \lambda)\Big) + o_{av}(|x|^{-(d-1)/2}), \quad \partial_r = \partial/\partial|x|, \tag{1.7}$$

as $|x| \to \infty$. Moreover one has $||a_+|| = ||a_-||$.

B) Every u satisfying (1.1) and (1.2) has the asymptotics (1.6), (1.7) with some functions $a_\pm \in L^2(\mathbb{S}^{d-1})$.

We emphasize that in the part A) the function a_+ and the solution u are determined uniquely by a_-. Of course, the roles of a_+ and a_- can be interchanged here. This implies that the operator $\Sigma(\lambda) : a_- \mapsto a_+$ is unitary in $L^2(\mathbb{S}^{d-1})$. Thus, we establish the one-to-one correspondence between functions $a_\pm \in L^2(\mathbb{S}^{d-1})$ and solutions of (1.1) satisfying condition (1.2).

Usually the scattering matrix $\mathcal{S}(\lambda) : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ for the Schrödinger operator $H = -\Delta + V$ is defined in terms of solutions of the time-dependent equation. Let $\Omega(x,t) = -\Omega(x,-t)$ be a solution (perhaps approximate) of the time-dependent eikonal equation

$$\partial\Omega/\partial t + |\nabla\Omega|^2 + V = 0. \tag{1.8}$$

It can be shown that for any f orthogonal to eigenvectors of H there exist elements $f_\pm$ such that

$$(e^{-iHt} f)(x) = e^{i\Omega(x,t)}(2it)^{-d/2} f_\pm((2t)^{-1}x) + \epsilon_\pm(x,t), \tag{1.9}$$

where $||\epsilon_\pm(\cdot, t)|| = o(1)$ as $t \to \pm\infty$. Let us write $f_\pm(x)$ as $f_\pm(\hat{x}, |x|)$. The operator $\mathcal{S}(\lambda) : f_-(\cdot, \sqrt{\lambda}) \mapsto f_+(\cdot, \sqrt{\lambda})$ is known as the scattering matrix. It turns out that the operators $\mathcal{S}(\lambda)$ and $\Sigma(\lambda)$ essentially coincide. Namely, $\mathcal{S}(\lambda) = \Sigma(\lambda)\mathcal{T}$, where $\mathcal{T}$, $(\mathcal{T}a)(\omega) = a(-\omega)$, is the reflection operator on the sphere. Thus, in the long-range case, our approach provides a way to construct the scattering matrix in terms of solutions of the stationary equation (1.1).

We emphasize that neither relation (1.6) nor (1.9) define the scattering matrix $\mathcal{S}(\lambda)$ uniquely. Indeed, an arbitrary function $\Theta(\hat{x},\lambda)$ can be added to $\varphi(x,\lambda)$ in definition (1.3) or the function $-\operatorname{sgn} t\,\Theta(\hat{x},|x|^2/(4t^2))$ can be added to $\Omega(x,t)$ in (1.9). This corresponds to the replacement of $\mathcal{S}(\lambda)$ by $e^{i\Theta_+(\lambda)}\mathcal{S}(\lambda)e^{i\Theta_-(\lambda)}$ where $\Theta_\pm(\lambda)$ is multiplication by $\Theta(\pm\hat{x},\lambda)$.

In the short-range case (when $\varphi(x,\lambda) = \sqrt{\lambda}|x| - (d-3)\pi/4$), the results formulated above were proven in [15], where the techniques of [2], [3] were extensively used.

Our proofs rely heavily on the well-elaborated machinery of long-range scattering theory (see [5], [12]). More precisely, we need the sharp form of the limiting absorption principle [1], [8], [10] and the asymptotics [6], [9] of the function $((H-\lambda\pm i0)^{-1}f)(x)$ as $|x|\to\infty$.

Let us mention also the recent paper [11] where a result of type A) (but not of type B)) was obtained in the general context of asymptotically Euclidean manifolds. In [11] only perturbations of the metrics which correspond to very short-range potentials and only functions $a_\pm \in C^\infty(\mathbb{S}^{d-1})$ were considered.

2. Preliminaries.

1. Let $\mathfrak{S} = \mathfrak{S}(\mathbb{R}^d)$ be the Schwartz space of test functions and $L^2_\gamma = L^2_\gamma(\mathbb{R}^d)$ be the Hilbert space of functions f such that

$$||f||^2_\gamma = \int_{\mathbb{R}^d}(1+|x|^2)^\gamma |f(x)|^2dx < \infty.$$

Recall definitions of spaces B, B^*, B^*_0 introduced in the context of scattering theory in [2]. The space B consists of functions f such that the norm

$$||f||_B = \left(\int_{|x|\le 1}|f(x)|^2dx\right)^{1/2} + \sum_{n=0}^{\infty}\left(2^n\int_{2^n\le|x|\le 2^{n+1}}|f(x)|^2dx\right)^{1/2} < \infty.$$

The space B^*, dual to B with respect to $L^2 = L^2(\mathbb{R}^d)$, is a Banach space with one of its equivalent norms given by

$$||f||_{B^*} = \sup_{\rho\ge 1}\left(\rho^{-1}\int_{|x|\le\rho}|f(x)|^2dx\right)^{1/2}.$$

We denote by $\langle\cdot,\cdot\rangle$ the corresponding duality symbol. Clearly,

$$L^2_\gamma \subset B \subset L^2_{1/2} \subset L^2 \subset L^2_{-1/2} \subset B^* \subset L^2_{-\gamma}, \quad \forall\gamma > 1/2.$$

The closure B_0^* of L^2 in the norm of B^* consists of functions $\epsilon(x)$ satisfying (1.5).

Denote by $\mathbf{H}^2 = \mathbf{H}^2(\mathbb{R}^d)$ the Sobolev space and by $\mathbf{H}^2_{\mathrm{loc}}$ the space of locally $\mathbf{H}^2$ functions; C and c are different positive constants whose precise values are of no importance. We need the following elementary property of the space B^*.

LEMMA 2.1. — *If $f \in \mathbf{H}^2_{\mathrm{loc}}$, then*

$$||\nabla f||_{B^*} \leq C(||f||_{B^*} + ||\Delta f||_{B^*}).$$

Proof. — For any real $\zeta \in C_0^\infty(\mathbb{R}^d)$, one has the identity

$$\int_{\mathbb{R}^d} |\nabla f|^2 \zeta dx = 2^{-1} \int_{\mathbb{R}^d} |f|^2 \Delta \zeta dx - \mathrm{Re} \int_{\mathbb{R}^d} \Delta f \bar{f} \zeta dx. \tag{2.1}$$

Indeed, integrating by parts we see that

$$-\int_{\mathbb{R}^d} \Delta f \bar{f} \zeta dx = \int_{\mathbb{R}^d} |\nabla f|^2 \zeta dx + \int_{\mathbb{R}^d} \nabla f \bar{f} \nabla \zeta dx, \tag{2.2}$$

and

$$\int_{\mathbb{R}^d} \nabla f \bar{f} \nabla \zeta dx = -\int_{\mathbb{R}^d} f \nabla \bar{f} \nabla \zeta dx - \int_{\mathbb{R}^d} |f|^2 \Delta \zeta dx.$$

The last equality is equivalent to

$$2\,\mathrm{Re} \int_{\mathbb{R}^d} \nabla f \bar{f} \nabla \zeta dx = -\int_{\mathbb{R}^d} |f|^2 \Delta \zeta dx. \tag{2.3}$$

Combining (2.2) and (2.3), we arrive at (2.1).

Suppose now that $\zeta(x) = 1$ for $|x| \leq 1$, $\zeta(x) = 0$ for $|x| \geq 2$ and set $\zeta_\rho(x) = \zeta(x/\rho)$. Then (2.1) yields

$$\int_{\mathbb{R}^d} |\nabla f|^2 \zeta(x/\rho) dx = 2^{-1} \rho^{-2} \int_{\mathbb{R}^d} |f|^2 (\Delta \zeta)(x/\rho) dx - \mathrm{Re} \int_{\mathbb{R}^d} \Delta f \bar{f} \zeta(x/\rho) dx$$

and hence

$$\int_{|x| \leq \rho} |\nabla f|^2 dx \leq C \int_{|x| \leq 2\rho} (|\Delta f|^2 + |f|^2) dx \tag{2.4}$$

with C independent of $\rho \geq 1$. □

COROLLARY 2.2. — *Let $u \in \mathbf{H}^2_{\mathrm{loc}} \cap B^*$ be a solution of equation (1.1) with a bounded potential V. Then*

$$||\nabla u||_{B^*} \leq C||u||_{B^*}.$$

In the following assertion the functions $w_\pm$ are defined by formula (1.3) but the function φ may be arbitrary.

LEMMA 2.3. — *Let $u \in \mathbf{H}^2_{\mathrm{loc}}$ be a solution of equation (1.1) satisfying, for some $a_\pm \in L^2(\mathbb{S}^{d-1})$, conditions (1.6) and (1.7). Then $||a_+|| = ||a_-||$.*

Proof. — Since $\operatorname{Im} \Delta u\overline{u} = 0$ according to (1.1), integration by parts yields

$$\operatorname{Im} \int_{|x|=r} u_r \overline{u}\, dS_r = \operatorname{Im} \int_{|x|\le r} \Delta u \overline{u}\, dx = 0. \tag{2.5}$$

Using (1.6), (1.7) and the identity

$$\operatorname{Im} i(a_-\overline{a}_+ w_- \overline{w}_+ - a_+\overline{a}_- w_+ \overline{w}_-) = 0,$$

we see that

$$\begin{aligned} \operatorname{Im} u_r\overline{u} &= \operatorname{Im} i\sqrt{\lambda}(a_+w_+ + a_-w_-)(\overline{a}_+\overline{w}_+ - \overline{a}_-\overline{w}_-) + \epsilon(x) \\ &= \sqrt{\lambda} r^{1-d}(|a_+|^2 - |a_-|^2) + \epsilon(x), \end{aligned} \tag{2.6}$$

where

$$\int_{|x|\le\rho} |\epsilon(x)| dx = o(\rho), \ \ \rho \to \infty.$$

Inserting (2.6) into (2.5) and integrating over $r \in (0,\rho)$, we obtain that $||a_+||^2 - ||a_-||^2 = o(1)$ and hence $||a_+|| = ||a_-||$. □

We need also an abstract theorem of H. Hahn (see e.g. [16], chapter 7 §5) which we formulate in terms adapted to our case.

THEOREM 2.4. — *Let $\mathcal{F} : B \to L^2(\mathbb{S}^{d-1})$ be a bounded operator. Suppose that the adjoint operator $\mathcal{F}^* : L^2(\mathbb{S}^{d-1}) \to B^*$ has a closed range $R(\mathcal{F}^*)$. Then an element $u \in B^*$ belongs to $R(\mathcal{F}^*)$ if (and only if) $\langle u, f\rangle = 0$ for all f from the kernel $N(\mathcal{F})$ of the operator $\mathcal{F}$. Moreover, $R(\mathcal{F}) = L^2(\mathbb{S}^{d-1})$ if $N(\mathcal{F}^*) = \{0\}$.*

2. Let us now give precise conditions on the potential V. We always suppose that

$$V(x) = V_L(x) + V_S(x) \tag{2.7}$$

where the long-range part $V_L(x)$ is a C^3-function such that

$$|\partial^\alpha V_L(x)| \le C(1+|x|)^{-\delta-|\alpha|}, \ \delta \in (0,1], \ 0 \le |\alpha| \le 3, \tag{2.8}$$

while the short-range part $V_S(x)$ satisfies

$$|V_S(x)| \leq C(1+|x|)^{-1-\delta}. \tag{2.9}$$

Note however that many intermediary results require (2.8) for $|\alpha| \leq 1$ or $|\alpha| \leq 2$ only.

The Schrödinger operator $H = -\Delta + V(x)$ is of course self-adjoint in $L^2(\mathbb{R}^d)$ on the domain $\mathbf{H}^2$.

We start with a reformulation of a standard uniqueness result (see e.g. [12], for a similar assertion) in terms convenient for us.

PROPOSITION 2.5. — *Suppose that $u \in \mathbf{H}^2_{\mathrm{loc}}$ satisfies (1.1) and $u \in B_0^*$. Then $u = 0$. In particular, the operator H does not have positive eigenvalues.*

Omitting the proof, we only note that inequality (2.4) implies the inclusion $\partial_r u \in B_0^*$. Hence there exists a sequence $r_n \to \infty$ such that

$$\int_{|x|=r_n} (|\partial_r u|^2 + |u|^2) dS_{r_n} \to 0.$$

The equality $u = 0$ is deduced from this relation.

Combining Proposition 2.5 with Lemma 2.3 we also obtain

PROPOSITION 2.6. — *Suppose that $u \in \mathbf{H}^2_{\mathrm{loc}}$ satisfies (1.1) and for some $a \in L^2(\mathbb{S}^{d-1})$ (and one of the signs " + " or " − ")*

$$u(x) = a(\hat{x}) w_\pm(x,\lambda) + o_{av}(|x|^{-(d-1)/2}), \tag{2.10}$$

$$\partial_r u(x) = \pm i\sqrt{\lambda}\, a(\hat{x}) w_\pm(x,\lambda) + o_{av}(|x|^{-(d-1)/2}) \tag{2.11}$$

as $|x| \to \infty$. Then $u = 0$.

This assertion shows that a solution u of equation (1.1) satisfying (1.6) and (1.7) is determined uniquely either by a_+ or by a_-.

3. Construction of scattering theory relies on the following fundamental result on the resolvent $R(z) = (H-z)^{-1}$ of the operator H known as the limiting absorption principle. Denote by $||R(z)||_{B,B^*}$ the norm of the operator $R(z) : B \to B^*$.

PROPOSITION 2.7. — *Under the assumptions (2.7) – (2.9)*

$$||R(z)||_{B,B^*} \leq C, \quad \mathrm{Im}\, z \neq 0, \tag{2.12}$$

where the constant C does not depend on z as long as $0 < \lambda_0 \leq \mathrm{Re}\, z \leq \lambda_1 < \infty$. Moreover, for any $f, g \in B$, the function $(R(z)f, g)$ (resp. the operator-function $R(z) : L^2_\gamma \to L^2_{-\gamma}, \gamma > 1/2$) is continuous (resp. continuous in norm) with respect to z up to the cut along $[0, \infty)$.

The proof of this result, based on the Mourre estimate, can be found in [10] (see also [5], for an alternative proof).

COROLLARY 2.8. — *Under the assumptions of Proposition 2.7*

$$||\nabla R(z)||_{B,B^*} \leq C, \quad \mathrm{Im}\, z \neq 0. \tag{2.13}$$

Proof. — Let $f \in B$ and $v = R(z)f \in B^*$. Since $\Delta v = (V - z)v - f$, Proposition 2.7 implies that

$$||\Delta v||_{B^*} \leq c(||v||_{B^*} + ||f||_{B^*}) \leq C||f||_B.$$

Now (2.13) follows from Lemma 2.1. □

3. Generalized Fourier transforms.

1. We need an approximate solution of the eikonal equation (1.4) (see e.g. [9]).

LEMMA 3.1. — *For any $\lambda > 0$ there exists a real-valued function $\Phi(x, \lambda)$ satisfying the equation*

$$2\sqrt{\lambda}\,\frac{\partial \Phi}{\partial |x|}(x, \lambda) = |\nabla_x \Phi(x, \lambda)|^2 + V_L(x) \tag{3.1}$$

for $|x|$ large enough and the estimates

$$|\partial_x^\alpha \partial_\lambda^\beta \Phi(x, \lambda)| \leq C(1 + |x|)^{1-|\alpha|-\delta}, \quad |\alpha| \leq 3, \ \beta \leq 2. \tag{3.2}$$

Note that all our results, in particular estimate (3.2) as well as all estimates below, are uniform in λ from any compact subinterval of $(0, \infty)$. Estimates (3.2) for $\beta > 0$ are used in Section 5 for construction of time-dependent wave operators. Otherwise we need only continuity of $\Phi(x, \lambda)$ with respect to λ. Relation (3.1) ensures that

$$\varphi(x, \lambda) = \sqrt{\lambda}|x| - \Phi(x, \lambda) - \pi(d - 3)/4 \tag{3.3}$$

is a solution of equation (1.4) with V replaced by V_L for sufficiently large $|x|$. In what follows the functions $w_\pm(x, \lambda)$ are always given by (1.3) where

$\Phi(x,\lambda)$ is any function satisfying (3.1) up to a term $O(|x|^{-1-\varepsilon})$, $\varepsilon > 0$, as $|x| \to \infty$ and estimates (3.2). In fact, one can construct by iterations (see for example [12]) an explicit function $\Phi(x,\lambda)$ satisfying these conditions but, in general, this construction requires assumption (2.8) for $|\alpha| > 3$.

Let us denote $R_L(z) = (-\Delta + V_L - z)^{-1}$. Our paper relies essentially on the following result.

THEOREM 3.2. — *Let V_L satisfy (2.8) and $f \in \mathfrak{S}$. Then*

$$(3.4) \quad (R_L(\lambda \pm i0)f)(x) = \pi^{1/2}\lambda^{-1/4} a_\pm^L(\pm\hat{x}) w_\pm(x,\lambda) + o\big(|x|^{-(d-1)/2}\big),$$

$$(3.5) \quad (\partial_r R_L(\lambda \pm i0)f)(x) = \pm i\pi^{1/2}\lambda^{1/4} a_\pm^L(\pm\hat{x}) w_\pm(x,\lambda) + o\big(|x|^{-(d-1)/2}\big),$$

for some $a_\pm^L \in L^2(\mathbb{S}^{d-1})$ as $|x| \to \infty$.

This result was first proven in [6] for $\delta > 1/2$ and for a suitably chosen sequence $|x_n| \to \infty$. These technical restrictions have been independently overcome in [9] and [12].

Relation (3.4) allows us to define the mapping $\mathcal{F}_\pm^L(\lambda) : \mathfrak{S} \to L^2(\mathbb{S}^{d-1})$ by the equality $(\mathcal{F}_\pm^L(\lambda)f)(\hat{x}) = a_\pm^L(\hat{x})$. Set $v_\pm = R_L(\lambda \pm i0)f$ for $f \in \mathfrak{S}$ so that

$$(-\Delta + V_L - \lambda)v_\pm = f.$$

Integrating by parts and using relations (3.4), (3.5), we obtain that

$$\begin{aligned}\int_{|x|\le r} (v_\pm \overline{f} - f\overline{v}_\pm)\,dx &= \int_{|x|\le r} (\overline{v}_\pm \Delta v_\pm - v_\pm \Delta \overline{v}_\pm)\,dx \\ &= \int_{|x|=r} (\overline{v}_\pm \partial_r v_\pm - v_\pm \partial_r \overline{v}_\pm)\,dx \\ &= \pm 2\pi i \int_{\mathbb{S}^{d-1}} |a_\pm^L(\hat{x})|^2\,d\hat{x} + o(1) \qquad (3.6)\end{aligned}$$

and hence, passing to the limit $r \to \infty$,

$$(3.7) \quad \langle R_L(\lambda + i0)f, f\rangle - \langle R_L(\lambda - i0)f, f\rangle = 2\pi i \int_{\mathbb{S}^{d-1}} |(\mathcal{F}_\pm^L(\lambda)f)(\hat{x})|^2 d\hat{x}.$$

Proposition 2.7 and relation (3.7) yield

LEMMA 3.3. — *The operator $\mathcal{F}_\pm^L(\lambda)$ extends by continuity to a bounded operator from B into $L^2(\mathbb{S}^{d-1})$.*

The asymptotics (3.4) and (3.5) can now be extended to an arbitrary $f \in B$. At the same time, the short-range part of the potential will be included.

THEOREM 3.4. — *Let V satisfy (2.7) – (2.9) and $f \in B$. Then*

$$(3.8)\quad (R(\lambda \pm i0)f)(x) = \pi^{1/2}\lambda^{-1/4}a_\pm(\pm\hat{x})w_\pm(x,\lambda) + o_{av}(|x|^{-(d-1)/2}),$$

$$(3.9)\quad (\partial_r R(\lambda \pm i0)f)(x) = \pm i\pi^{1/2}\lambda^{1/4}a_\pm(\pm\hat{x})w_\pm(x,\lambda) + o_{av}(|x|^{-(d-1)/2})$$

for some $a_\pm \in L^2(\mathbb{S}^{d-1})$ as $|x| \to \infty$.

Proof. — Choose, for example, the sign "+". Consider first (3.8) for $R_L(\lambda + i0)f$. Let $f_n \in \mathfrak{S}$ be such that $||f_n - f||_B \to 0$ as $n \to \infty$. Then

$$\begin{aligned} &\rho^{-1/2}||R_L(\lambda + i0)f - \alpha(\lambda)w_+\mathcal{F}_+^L(\lambda)f||_{L^2(|x|\le\rho)} \\ &\quad \le \rho^{-1/2}||R_L(\lambda + i0)(f - f_n)||_{L^2(|x|\le\rho)} \\ &\quad +\rho^{-1/2}\alpha(\lambda)||w_+\mathcal{F}_+^L(\lambda)(f - f_n)||_{L^2(|x|\le\rho)} \\ (3.10)\qquad &\quad +\rho^{-1/2}||R_L(\lambda + i0)f_n - \alpha(\lambda)w_+\mathcal{F}_+^L(\lambda)f_n||_{L^2(|x|\le\rho)}, \end{aligned}$$

where $\alpha(\lambda) = \pi^{1/2}\lambda^{-1/4}$. Clearly, the first term in the right-hand side is bounded by $C||R_L(\lambda + i0)(f - f_n)||_{B^*}$ and the second equals $c||\mathcal{F}_+^L(\lambda)(f - f_n)||_{L^2(\mathbb{S}^{d-1})}$. Thus, both of them are estimated by $||f - f_n||_B$ according to Proposition 2.7 and Lemma 3.3. The last term in (3.10) tends to zero for fixed n and $\rho \to \infty$ according to (3.4). This yields asymptotics (3.8) for $R_L(\lambda + i0)f$ and any $f \in B$.

Taking into account the resolvent identity and Proposition 2.7, we see that

$$R(\lambda + i0)f = R_L(\lambda + i0)g \quad \text{where} \quad g = f - V_S R(\lambda + i0)f \in B.$$

Therefore (3.8) is an immediate consequence of the same relation for $R_L(\lambda + i0)g$.

The proof of (3.9) can be obtained quite similarly on the basis of (3.5) and the boundedness of the operator $\partial_r R_L(\lambda + i0) : B \to B^*$. □

We define now the mapping $\mathcal{F}_\pm(\lambda) : B \to L^2(\mathbb{S}^{d-1})$ by the equality

$$(3.11)\qquad (\mathcal{F}_\pm(\lambda)f)(\hat{x}) = a_\pm(\hat{x}).$$

It follows from (3.8) and (3.9) that relation (3.6) holds for some sequence $r_n \to \infty$ which implies the equality

$$(3.12)\quad \langle R(\lambda + i0)f, f\rangle - \langle R(\lambda - i0)f, f\rangle = 2\pi i \int_{\mathbb{S}^{d-1}} |(\mathcal{F}_\pm(\lambda)f)(\hat{x})|^2 d\hat{x}.$$

This gives us

PROPOSITION 3.5. — *The operator $\mathcal{F}_\pm(\lambda)$ is bounded from B into $L^2(\mathbb{S}^{d-1})$, and its adjoint $\mathcal{F}_\pm^*(\lambda)$ is a bounded operator from $L^2(\mathbb{S}^{d-1})$ into B^*.*

Note that in the free case $V = 0$, the operators $\mathcal{F}_+(\lambda) = \mathcal{F}_-(\lambda) = \mathcal{F}_0(\lambda)$ where

$$(\mathcal{F}_0(\lambda)f)(\omega) = 2^{-1/2}\lambda^{(d-2)/4}\hat{f}(\lambda^{1/2}\omega) \tag{3.13}$$

and $\hat{f}$ is the Fourier transform of f.

2. Our aim is now to derive a convenient representation for the operator $\mathcal{F}_\pm^*(\lambda)$ on functions $a \in C^\infty(\mathbb{S}^{d-1})$. Let $\eta \in C^\infty(\mathbb{R}^d)$ be such that $\eta(x) = 0$ in a neighbourhood of $x = 0$ and $\eta(x) = 1$ for large $|x|$. Obviously, the functions

$$u_\pm(x,\lambda) = \eta(x)a(\pm\hat{x})w_\pm(x,\lambda) \tag{3.14}$$

belong to the space B^*. It follows from (3.2) that

$$\partial_r u_\pm(x,\lambda) = \pm i\lambda^{1/2}u_\pm(x,\lambda) + o(|x|^{-(d-1)/2-\delta}). \tag{3.15}$$

Set also

$$g_\pm(\lambda) = (-\Delta + V - \lambda)u_\pm(\lambda) =: G_\pm(\lambda)a. \tag{3.16}$$

Straightforward calculations (which can be found for example in [9]) show that

$$g_\pm(\lambda) \in L^2_\gamma(\mathbb{R}^d) \quad \text{for } \gamma < 1/2 + \delta. \tag{3.17}$$

Of course, the definition of the operator $G_\pm(\lambda)$ depends on η, but as we will see below, this dependence is inessential.

PROPOSITION 3.6. — *Let $a \in C^\infty(\mathbb{S}^{d-1})$ and let $u_\pm$, $g_\pm$ be given by (3.14), (3.16). Then (as elements of B^*)*

$$\pm 2i\pi^{1/2}\lambda^{1/4}\mathcal{F}_\pm^*(\lambda)a = u_\pm(\lambda) - R(\lambda \mp i0)g_\pm(\lambda). \tag{3.18}$$

Proof. — We consider only the " + " case, proofs being similar. It suffices to check that for any $f \in B$

$$\langle u_+, f\rangle - \langle g_+, R(\lambda + i0)f\rangle = 2i\pi^{1/2}\lambda^{1/4}(a, \mathcal{F}_+(\lambda)f)_{L^2(\mathbb{S}^{d-1})}. \tag{3.19}$$

Applying the Green formula to u_+ and $v = R(\lambda + i0)f$, we see that

$$\begin{aligned}\int_{|x|\le r}(u_+\overline{f} - g_+\overline{R(\lambda+i0)f})\,dx &= \int_{|x|\le r}(\overline{v}\Delta u_+ - u_+\Delta\overline{v})dx \\ &= \int_{|x|=r}(\overline{v}\partial_r u_+ - u_+\partial_r\overline{v})\,dS_r.\end{aligned} \tag{3.20}$$

It follows from Theorem 3.4 that

$$\int_{|x|=r_n}(|v - bw_+|^2 + |\partial_r v - i\lambda^{1/2}bw_+|^2)dS_{r_n} = o(1), \quad b = \pi^{1/2}\lambda^{-1/4}\mathcal{F}_+(\lambda)f,$$

for some sequence $r_n \to \infty$. Therefore, taking into account (3.14), (3.15), we obtain, by the Schwarz inequality, that

$$\int_{|x|=r_n} ((\overline{v} - \overline{bw_+})\partial_r u_+ - u_+(\partial_r \overline{v} + i\lambda^{1/2}\overline{bw_+}))dS_{r_n} = o(1).$$

Using again (3.14), (3.15), we see now that

$$\begin{aligned}\int_{|x|=r_n} (\overline{v}\partial_r u_+ - u_+\partial_r \overline{v})\, dS_{r_n} &= \int_{|x|=r_n} (\partial_r u_+ + i\lambda^{1/2}u_+)\overline{bw_+}\, dS_{r_n} + o(1)\\ &= 2i\lambda^{1/2}(a,b)_{L^2(\mathbb{S}^{d-1})} + o(1).\end{aligned}$$

Thus, the right-hand side of (3.20) tends to the right-hand side of (3.19) as $r_n \to \infty$. Obviously, the left-hand side of (3.20) tends to the left-hand side of (3.19) as $r \to \infty$. □

Corollary 3.7. — *For any $a \in C^\infty(\mathbb{S}^{d-1})$, the function $\mathcal{F}^*_\pm(\lambda)a \in \mathbf{H}^2_{\mathrm{loc}} \cap B^*$ and*

$$(-\Delta + V - \lambda)\mathcal{F}^*_\pm(\lambda)a = 0. \tag{3.21}$$

Taking also into account Proposition 3.5 and Corollary 2.2, we deduce the following result

Proposition 3.8. — *For any $a \in L^2(\mathbb{S}^{d-1})$, the function $\mathcal{F}^*_\pm(\lambda)a \in \mathbf{H}^2_{\mathrm{loc}} \cap B^*$ and the equation (3.21) is satisfied. The operators $\Delta\mathcal{F}^*_\pm(\lambda)$ and $\partial_j\mathcal{F}^*_\pm(\lambda)$, $j = 1, \dots, d$, defined by (3.18) on the set $C^\infty(\mathbb{S}^{d-1})$ extend by continuity to bounded operators from $L^2(\mathbb{S}^{d-1})$ into B^*.*

Finally, we note that, according again to Proposition 2.7, all objects introduced here are continuous (in a weak sense) with respect to λ. For example, the function $(\mathcal{F}_\pm(\lambda)f, a)$ is continuous in λ for all $f \in B$ and $a \in L^2(\mathbb{S}^{d-1})$.

Representation (3.18) is not new. It occurred already, e.g., in [9] where it was used for the proof of Theorem 3.2. We have given the proof of Proposition 3.6 only for the sake of completeness of our exposition.

4. The main result.

1. Now we are able to obtain the spatial asymptotics of functions $(\mathcal{F}^*_\pm(\lambda)a)(x)$. For $a \in C^\infty(\mathbb{S}^{d-1})$, let us set

$$\begin{aligned}\Sigma_+(\lambda)a &= \pi^{1/2}\lambda^{-1/4}\mathcal{F}_+(\lambda)G_-(\lambda)\mathcal{T}a,\\ \Sigma_-(\lambda)a &= \pi^{1/2}\lambda^{-1/4}\mathcal{T}\mathcal{F}_-(\lambda)G_+(\lambda)a,\end{aligned} \tag{4.1}$$

where $\mathcal{T}$, $(\mathcal{T}a)(\omega) = a(-\omega)$, is the reflection operator on the sphere and $G_\pm(\lambda)$ is defined by (3.14), (3.16). Recall that, according to (3.17), $G_\pm(\lambda) : C^\infty(\mathbb{S}^{d-1}) \to L^2_\gamma(\mathbb{R}^d)$ for some $\gamma > 1/2$ while $\mathcal{F}_\pm(\lambda) : L^2_{\gamma'}(\mathbb{R}^d) \to L^2(\mathbb{S}^{d-1})$ for any $\gamma' > 1/2$. Hence $\Sigma_\pm(\lambda) : C^\infty(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ is well defined. Put $\beta(\lambda) = -i2^{-1}\pi^{-1/2}\lambda^{-1/4}$.

PROPOSITION 4.1. — *The operators $\Sigma_\pm(\lambda)$ extend by continuity to unitary operators on $L^2(\mathbb{S}^{d-1})$ and*

$$\Sigma_+(\lambda) = \Sigma_-^*(\lambda), \quad \Sigma_-(\lambda) = \Sigma_+^*(\lambda). \tag{4.2}$$

For any $a_\pm \in L^2(\mathbb{S}^{d-1})$, the functions

$$u_+ = \mathcal{F}_+^*(\lambda)a_+ \quad \text{and} \quad u_- = \mathcal{F}_-^*(\lambda)\mathcal{T}a_-$$

have the following asymptotics as $|x| \to \infty$:

$$u_+(x) = \beta(\lambda)\Big(a_+(\hat{x})w_+(x,\lambda) - (\Sigma_-(\lambda)a_+)(\hat{x})w_-(x,\lambda)\Big) + o_{av}(|x|^{-(d-1)/2}), \tag{4.3}$$

$$(\partial_r u_+)(x) = i\lambda^{1/2}\beta(\lambda)\Big(a_+(\hat{x})w_+(x,\lambda) + (\Sigma_-(\lambda)a_+)(\hat{x})w_-(x,\lambda)\Big) + o_{av}(|x|^{-(d-1)/2}), \tag{4.4}$$

and

$$u_-(x) = \beta(\lambda)\Big((\Sigma_+(\lambda)a_-)(\hat{x})w_+(x,\lambda) - a_-(\hat{x})w_-(x,\lambda)\Big) + o_{av}(|x|^{-(d-1)/2}), \tag{4.5}$$

$$(\partial_r u_-)(x) = i\lambda^{1/2}\beta(\lambda)\Big((\Sigma_+(\lambda)a_-)(\hat{x})w_+(x,\lambda) + a_-(\hat{x})w_-(x,\lambda)\Big) + o_{av}(|x|^{-(d-1)/2}). \tag{4.6}$$

Proof. — If $a_\pm \in C^\infty(\mathbb{S}^{d-1})$, then (4.3) – (4.6) follow directly from Theorem 3.4 and Proposition 3.6. By Proposition 3.8, the functions u_+ and u_- satisfy equation (1.1). Moreover, relations (4.3), (4.4) and (4.5), (4.6) play the role of (1.6), (1.7). Therefore isometricity of $\Sigma_-(\lambda)$ and $\Sigma_+(\lambda)$ on the set $C^\infty(\mathbb{S}^{d-1})$ is a direct consequence of Lemma 2.3 applied to u_+ and u_-. Then the boundedness of the operators $\mathcal{F}_\pm^*(\lambda)$, $\partial_r\mathcal{F}_\pm^*(\lambda)$ and $\Sigma_\pm(\lambda)$ allows us (*cf.* Theorem 3.4) to extend (4.3) – (4.6) to arbitrary $a_\pm \in L^2(\mathbb{S}^{d-1})$.

It remains to check relations (4.2) or, in view of the isometricity of $\Sigma_\pm(\lambda)$, the relations

$$\Sigma_+(\lambda)\Sigma_-(\lambda) = \Sigma_-(\lambda)\Sigma_+(\lambda) = I. \tag{4.7}$$

Recall that by Proposition 2.6 a solution of equation (1.1) satisfying (1.6), (1.7) is determined uniquely either by a coefficient at w_+ or by a coefficient at w_-. Therefore comparing (4.3), (4.4) with (4.5), (4.6) we see that equality $\Sigma_-(\lambda)a_+ = a_-$ (equality $a_+ = \Sigma_+(\lambda)a_-$) implies that $a_+ = \Sigma_+(\lambda)a_-$ (that $\Sigma_-(\lambda)a_+ = a_-$). This yields the first (the second) equality (4.7). □

Combined together Propositions 3.8 and 4.1 conclude the proof of the part A) of the main result formulated in Section 1.

2. For the proof of the part B) we need the following properties of the operator $\mathcal{F}^*_\pm(\lambda)$.

PROPOSITION 4.2. — *For any $a \in L^2(\mathbb{S}^{d-1})$*

$$\lim_{\rho\to\infty} \rho^{-1} \int_{|x|\le\rho} |(\mathcal{F}^*_\pm(\lambda)a)(x)|^2\,dx = (2\pi)^{-1}\lambda^{-1/2}||a||^2_{L^2(\mathbb{S}^{d-1})}. \tag{4.8}$$

Proof. — According to (4.3) or (4.5),

$$4\pi\lambda^{1/2}\rho^{-1}\int_{|x|\le\rho} |(\mathcal{F}^*_\pm(\lambda)a)(x)|^2\,dx = ||a||^2_{L^2(\mathbb{S}^{d-1})} + ||\Sigma_\mp(\lambda)a||^2_{L^2(\mathbb{S}^{d-1})}$$
$$-2\rho^{-1}\,\mathrm{Re}\left(\int_{\mathbb{S}^{d-1}}(\Sigma_\mp(\lambda)a)(\omega)\bar{a}(\omega)d\omega \int_0^\rho \exp(\mp 2i\varphi(r\omega,\lambda))\,dr\right) + o(1).$$

Integrating with the help of (3.2) by parts, we see that

$$\lim_{\rho\to 0}\rho^{-1}\int_0^\rho \exp(\mp 2i\varphi(r\omega,\lambda))\,dr = 0.$$

Therefore (4.8) follows from isometricity of the operator $\Sigma_\mp(\lambda)$. □

Combining Propositions 3.5 and 4.2, we obtain

PROPOSITION 4.3. — *One has the two-sided estimate*

$$(2\pi)^{-1/2}\lambda^{-1/4}||a||_{L^2(\mathbb{S}^{d-1})} \le ||\mathcal{F}^*_\pm(\lambda)a||_{B^*} \le C||a||_{L^2(\mathbb{S}^{d-1})}.$$

Hence the kernel $N(\mathcal{F}^*_\pm(\lambda))$ of the operator $\mathcal{F}^*_\pm(\lambda) : L^2(\mathbb{S}^{d-1}) \to B^*$ is trivial and its range $R(\mathcal{F}^*_\pm(\lambda))$ is closed.

Let us return to equation (1.1).

LEMMA 4.4. — *Let $u \in \mathbf{H}^2_{\mathrm{loc}} \cap B^*$ be a solution of (1.1). Then $\langle u, f\rangle = 0$ for any $f \in B$ such that $\mathcal{F}_\pm(\lambda)f = 0$.*

Proof. — By Corollary 2.2, $\partial_r u \in B^*$ and hence

$$\int_{|x|\le\rho} (|u(x)|^2 + |\partial_r u(x)|^2)\, dx \le C\rho, \quad \rho \ge 1. \tag{4.9}$$

If $f \in B$ and $\mathcal{F}_\pm(\lambda) f = 0$ then, according to Theorem 3.4, the function $v_\pm = R(\lambda \pm i0) f$ satisfies the condition

$$\int_{|x|\le\rho} (|v_\pm(x)|^2 + |\partial_r v_\pm(x)|^2)\, dx = o(\rho) \tag{4.10}$$

as $\rho \to \infty$. By the Schwarz inequality, it follows from (4.9) and (4.10) that

$$\int_{|x|\le\rho} (\overline{v}_\pm \partial_r u - u \partial_r \overline{v}_\pm) dx = o(\rho).$$

Consequently, there exists a sequence $r_n \to \infty$ such that

$$\lim_{r_n\to\infty} \int_{|x|=r_n} (\overline{v}_\pm \partial_r u - u \partial_r \overline{v}_\pm) dS_{r_n} = 0.$$

Now integrating by parts and taking into account equation (1.1) for u, we find that

$$\begin{aligned}\langle u, f\rangle &= \lim_{r_n\to\infty} \int_{|x|\le r_n} u(-\Delta + V - \lambda)\overline{v}_\pm\, dx \\ &= \lim_{r_n\to\infty} \int_{|x|=r_n} (\overline{v}_\pm \partial_r u - u \partial_r \overline{v}_\pm) dS_{r_n} = 0.\end{aligned}$$

□

It follows from Proposition 4.3 that Theorem 2.4 can be applied to the operator $\mathcal{F}_\pm(\lambda)$. Therefore, by Lemma 4.4, every solution $u \in \mathbf{H}^2_{\text{loc}} \cap B^*$ of equation (1.1) belongs to the range of $\mathcal{F}^*_\pm(\lambda)$. Taking also into account Proposition 3.8, we obtain a description of solutions of equation (1.1) in terms of the operator $\mathcal{F}^*_\pm(\lambda)$.

PROPOSITION 4.5. — *For $u \in \mathbf{H}^2_{\text{loc}} \cap B^*$ to be a solution of equation (1.1), it is necessary and sufficient that $u \in R(\mathcal{F}^*_\pm(\lambda))$.*

According to Proposition 4.1, Proposition 4.5 leads immediately to the part B) of our main result. Let us reformulate both parts adding informations contained in Propositions 2.6, 4.2 and 4.3. Note that functions $a_\pm$ and u in this formulation are related by the equalities

$$u(x) = 2i\pi^{1/2}\lambda^{1/4}(\mathcal{F}^*_+(\lambda)a_+)(x), \quad u(x) = 2i\pi^{1/2}\lambda^{1/4}(\mathcal{F}^*_-(\lambda)\mathcal{T}a_-)(x).$$

THEOREM 4.6. — *Let assumptions (2.7) – (2.9) hold and let $w_\pm(x,\lambda)$ be given by (1.3) where $\Phi(x,\lambda)$ is some function satisfying (3.1)* *up to a term $O(|x|^{-1-\varepsilon})$, $\varepsilon > 0$, as $|x| \to \infty$ and estimates (3.2). For any $a_- \in L^2(\mathbb{S}^{d-1})$, there exist a function $a_+ \in L^2(\mathbb{S}^{d-1})$ and a solution $u \in \mathbf{H}^2_{\mathrm{loc}} \cap B^*$ of equation (1.1) with asymptotics (1.6), (1.7) as $|x| \to \infty$. The function a_+ and the solution u are determined uniquely by a_-. Moreover, $||a_-|| = ||a_+||$ and the roles of a_+ and a_- in this formulation can be interchanged.*

Conversely, any $u \in \mathbf{H}^2_{\mathrm{loc}} \cap B^*$ satisfying equation (1.1) has the asymptotics (1.6), (1.7) as $|x| \to \infty$ for some functions $a_\pm \in L^2(\mathbb{S}^{d-1})$. Furthermore,

$$||u||_{B^*} \leq C||a_\pm||_{L^2(\mathbb{S}^{d-1})} \text{ and } \lim_{\rho\to\infty} \rho^{-1} \int_{|x|\leq\rho} |u(x)|^2 dx = 2||a_\pm||^2_{L^2(\mathbb{S}^{d-1})}.$$

Set $\mathcal{S}(\lambda) = \Sigma_+(\lambda)\mathcal{T}$. By Proposition 4.1, this operator known as the scattering matrix is unitary on $L^2(\mathbb{S}^{d-1})$ and $\mathcal{S}^*(\lambda) = \mathcal{T}\Sigma_-(\lambda)$. As an easy consequence of Theorem 4.6, we obtain

PROPOSITION 4.7. — *In order that the equation (1.1) have a solution u with asymptotics (1.1) where $a_\pm \in L^2(\mathbb{S}^{d-1})$, it is necessary and sufficient that $a_+ = \mathcal{S}(\lambda)\mathcal{T}a_-$ or $a_- = \mathcal{T}\mathcal{S}^*(\lambda)a_+$.*

COROLLARY 4.8. — *If u satisfies equation (1.1) and condition (2.10) for some $a \in L^2(\mathbb{S}^{d-1})$ (and one of the signs " + " or " − "), then $u = 0$.*

Since condition (2.11) is not assumed here, Corollary 4.8 improves Proposition 2.6.

The following result shows that the scattering matrix "relates" the operators $\mathcal{F}_+(\lambda)$ and $\mathcal{F}_-(\lambda)$.

PROPOSITION 4.9. — *For any $f \in B$,*

$$\mathcal{F}_+(\lambda)f = \mathcal{S}(\lambda)\mathcal{F}_-(\lambda)f. \tag{4.11}$$

Proof. — By Proposition 3.8 for any $a \in L^2(\mathbb{S}^{d-1})$, the function

$$u(x) = (\mathcal{F}_+^*(\lambda)a)(x) - (\mathcal{F}_-^*(\lambda)\mathcal{S}^*(\lambda)a)(x)$$

satisfies equation (1.1). Moreover, by Proposition 4.1,

$$u(x) = o_{av}(|x|^{-(d-1)/2}) \text{ as } |x| \to \infty.$$

Therefore Corollary 4.8 implies that $u = 0$ or, equivalently, that $\mathcal{F}_+^*(\lambda) = \mathcal{F}_-^*(\lambda)\mathcal{S}^*(\lambda)$. □

On the set $C^\infty(\mathbb{S}^{d-1})$, the scattering matrix can be expressed in terms of the operators $G_\pm(\lambda)$ defined by (3.14), (3.16). Indeed, it follows from (4.1) that

$$\mathcal{S}(\lambda)a = \pi^{1/2}\lambda^{-1/4}\mathcal{F}_+(\lambda)G_-(\lambda)a, \quad \mathcal{S}^*(\lambda)a = \pi^{1/2}\lambda^{-1/4}\mathcal{F}_-(\lambda)G_+(\lambda)a,$$

which coincides with the representation for the scattering matrix obtained in [13]. The first of these equalities shows that, for any $a, b \in C^\infty(\mathbb{S}^{d-1})$,

$$\begin{aligned}(\mathcal{S}(\lambda)a, b) &= \pi^{1/2}\lambda^{-1/4}\langle G_-(\lambda)a, \mathcal{F}_+^*(\lambda)b\rangle \\ &= -i2^{-1}\lambda^{-1/2}\langle G_-(\lambda)a, u_+(\lambda) - R(\lambda - i0)G_+(\lambda)b\rangle.\end{aligned}$$

Since $G_-(\lambda)a$ and $G_+(\lambda)b$ are continuous functions of λ in $L^2_\gamma(\mathbb{R}^d)$ for some $\gamma > 1/2$ and $u_+(\lambda)$ is continuous in $L^2_{-\gamma'}$ for any $\gamma' > 1/2$, the continuity of $(\mathcal{S}(\lambda)a, b)$ with respect to λ follows from Proposition 2.7. Taking into account that the operators $\mathcal{S}(\lambda)$ and $\mathcal{S}^*(\lambda)$ are isometric, we obtain

PROPOSITION 4.10. — *The operators $\mathcal{S}(\lambda)$ and $\mathcal{S}^*(\lambda)$ are strongly continuous functions of λ.*

3. Here we formulate a corollary of our considerations for the non-homogeneous Schrödinger equation.

PROPOSITION 4.11. — *For any $a \in L^2(\mathbb{S}^{d-1})$, there exist (but of course not unique) functions $f_\pm \in B$ and $u_\pm \in \mathbf{H}^2_{\mathrm{loc}}$ with asymptotics (2.10), (2.11) satisfying the equation*

$$-\Delta u_\pm + V u_\pm = \lambda u_\pm + f_\pm. \tag{4.12}$$

Proof. — According to the last statement of Theorem 2.4, it follows from Proposition 4.3 that $R(\mathcal{F}_\pm(\lambda)) = L^2(\mathbb{S}^{d-1})$. Therefore there exist functions $f_\pm \in B$ such that $\mathcal{F}_+(\lambda)f_+ = a$ or $\mathcal{F}_-(\lambda)f_- = \mathcal{T}a$. Clearly, functions $u_\pm = R(\lambda \pm i0)f_\pm$ satisfy (4.12) and have asymptotics (2.10), (2.11) by virtue of Theorem 3.4. □

5. Time-dependent scattering matrix.

Our goal here is to show that the scattering matrix $\mathcal{S}(\lambda)$ constructed in Section 4 coincides with the time-dependent one. To that end we need to introduce modified time-dependent wave operators and to express them in terms of the operators $\mathcal{F}_\pm(\lambda)$.

1. We start with a standard expansion theorem in eigenfunctions of the operator H. Let $E(\cdot)$ be the spectral projection of H and let $\tilde{\mathcal{H}} = L^2(\mathbb{R}_+, L^2(\mathbb{S}^{d-1}))$ be the Hilbert space of $L^2(\mathbb{S}^{d-1})$-valued square integrable functions on $\mathbb{R}_+$ with the Lebesgue measure. Recall that the operator $\mathcal{F}_\pm(\lambda) : B \to L^2(\mathbb{S}^{d-1})$ was defined by equalities (3.8) and (3.11). Now we introduce the mapping $F_\pm : B \to \tilde{\mathcal{H}}$ by the formula

$$(F_\pm f)(\lambda) = \mathcal{F}_\pm(\lambda) f. \tag{5.1}$$

Integrating (3.12) over $\mathbb{R}_+$ and using the spectral theorem, we find that

$$\begin{aligned} ||F_\pm f||^2 &= \int_0^\infty ||\mathcal{F}_\pm(\lambda) f||^2_{L^2(\mathbb{S}^{d-1})} \, d\lambda = \int_0^\infty \frac{d(E(\lambda)f, f)}{d\lambda} d\lambda \\ &= ||Pf||^2, \quad P = E(0,\infty), \end{aligned}$$

for any $f \in B$. Hence the operator $F_\pm$ extends by continuity to a bounded operator on the entire space $\mathcal{H}$. This operator is isometric on the absolutely continuous subspace $\mathcal{H}^{(a)} = P\mathcal{H}$ of the operator H and vanishes on its orthogonal complement, i.e.,

$$F_\pm^* F_\pm = P. \tag{5.2}$$

Denote by Λ multiplication by the independent variable λ in the space $\tilde{\mathcal{H}}$ and by $E_\Lambda(X)$ its spectral projection acting as multiplication by the characteristic function of the Borel set $X \cap \mathbb{R}_+$. To check the intertwining property

$$F_\pm E(X) = E_\Lambda(X) F_\pm, \tag{5.3}$$

it suffices to show that $(F_\pm H f, \tilde{g}) = (F_\pm f, \Lambda \tilde{g})$ for $f \in \mathfrak{S}$ and $\tilde{g}(\lambda) = v(\lambda) a$, where $v \in C_0^\infty(\mathbb{R}_+)$ and $a \in L^2(\mathbb{S}^{d-1})$. By definition (5.1), this is equivalent to equality

$$\int_0^\infty \langle Hf, \mathcal{F}_\pm^*(\lambda) a\rangle v(\lambda)\, d\lambda = \int_0^\infty \lambda \langle f, \mathcal{F}_\pm^*(\lambda) a\rangle v(\lambda)\, d\lambda,$$

which is a direct consequence of Proposition 3.8.

Let us finally show that $R(F_\pm) = \mathcal{H}$.

LEMMA 5.1. — *The operator (5.1) satisfies*

$$F_\pm F_\pm^* = I. \tag{5.4}$$

Proof. — According to (5.2), (5.3), for any $f \in \mathcal{H}$, $\tilde{g} \in \tilde{\mathcal{H}}$ and any $X \subset \mathbb{R}_+$,

$$(E_\Lambda(X) F_\pm F_\pm^* \tilde{g}, F_\pm f) = (E_\Lambda(X) \tilde{g}, F_\pm f). \tag{5.5}$$

If $f \in B$ and $\tilde{h} = F_\pm F_\pm^* \tilde{g}$, then relation (5.5) can be rewritten as

$$\int_X (\tilde{h}(\lambda), \mathcal{F}_\pm(\lambda) f)_{L^2(\mathbb{S}^{d-1})}\, d\lambda = \int_X (\tilde{g}(\lambda), \mathcal{F}_\pm(\lambda) f)_{L^2(\mathbb{S}^{d-1})}\, d\lambda.$$

Since $f \in B$ and $X \subset \mathbb{R}$ are arbitrary, this implies that

$$\mathcal{F}_\pm^*(\lambda)(\tilde{h}(\lambda) - \tilde{g}(\lambda)) = 0$$

for almost all $\lambda \in \mathbb{R}_+$. By Proposition 4.3, the kernel of $\mathcal{F}_\pm^*(\lambda)$ is trivial and hence $\tilde{h}(\lambda) = \tilde{g}(\lambda)$ for almost all $\lambda \in \mathbb{R}_+$. Therefore $F_\pm F_\pm^* \tilde{g} = \tilde{g}$. □

Let us summarize the results obtained.

THEOREM 5.2. — *For every $f \in B$ the function $\mathcal{F}_\pm(\lambda) f$ belongs to the space $\tilde{\mathcal{H}}$. The operator $F_\pm : \mathcal{H} \to \tilde{\mathcal{H}}$ defined by (5.1) extends by continuity to a bounded operator on the entire space $\mathcal{H}$ and satisfies relations (5.2), (5.3) and (5.4).*

Similarly to the short-range case [15], Theorem 5.2 can be reformulated in terms of solutions of equation (1.1) with asymptotics (1.6), (1.7) at infinity but we shall not dwell upon it here.

2. To find the asymptotics of $e^{-iHt} f$ as $t \to \pm\infty$, we proceed from Theorem 5.2. Choose $\tilde{f}(\lambda) = v(\lambda) a$ where $a \in C^\infty(\mathbb{S}^{d-1})$ and $v \in C_0^\infty(\mathbb{R}^+)$. The set of linear combinations of such $\tilde{f}$ is dense in $\tilde{\mathcal{H}}$, so that the set of linear combinations of $f = F_\pm^* \tilde{f}$ is dense in $\mathcal{H}^{(a)}$. It follows from (5.3) that

$$e^{-iHt} f = \int_0^\infty e^{-i\lambda t} \mathcal{F}_\pm^*(\lambda) a v(\lambda) d\lambda. \tag{5.6}$$

Recall that the function $\mathcal{F}_\pm^*(\lambda) a$ satisfies (3.18) and as shown in [7] the contribution to (5.6) of the term $R(\lambda \mp i0) g_\pm(\lambda)$ disappears in the limit $t \to \pm\infty$. Taking into account that the function $u_\pm(x, \lambda)$ is given by (3.14) we obtain

LEMMA 5.3. — *Set $w(\lambda) = \lambda^{-1/4} v(\lambda)$ and*

$$I_\pm(x, t) = \int_0^\infty e^{-i\lambda t \pm i\varphi(x,\lambda)} w(\lambda) d\lambda. \tag{5.7}$$

Then

$$(e^{-iHt} f)(x) = \mp i 2^{-1} \pi^{-1/2} |x|^{-(d-1)/2} \eta(x) a(\pm \hat{x}) I_\pm(x, t) + \epsilon_\pm(x, t), \tag{5.8}$$

where $||\epsilon_\pm(\cdot, t)|| = o(1)$ as $t \to \pm\infty$.

It remains to find the asymptotics of integral (5.7). Suppose that $v(\lambda) = 0$ for $\lambda \le \mu$ and $\lambda \ge \nu$. If $|x| \le \alpha |t|$ or $|x| \ge \beta |t|$ for sufficiently small $\alpha = \alpha(\mu,\nu)$ and sufficiently large $\beta = \beta(\mu,\nu)$, then integrating by parts we find that

$$I_\pm(x,t) = -i\int_0^\infty e^{-i\lambda t \pm i\varphi(x,\lambda)}\partial_\lambda\Big((t\mp 2^{-1}\lambda^{-1/2}|x| \pm \Phi_\lambda(x,\lambda))^{-1}w(\lambda)\Big)d\lambda.$$

In view of (3.2), this yields the estimate

$$|I_\pm(x,t)| \le C(|x|+|t|)^{-1}.$$

Therefore function (5.8) satisfies

$$(5.9)\quad \lim_{|t|\to\infty}\left(\int_{|x|\le\alpha|t|}|(e^{-iHt}f)(x)|^2dx + \int_{|x|\ge\beta|t|}|(e^{-iHt}f)(x)|^2dx\right) = 0.$$

In the region

$$(5.10)\qquad \alpha|t| \le |x| \le \beta|t|$$

the asymptotics of integral (5.7) is determined by stationary points $\lambda_s = \lambda_s(x,t)$ where λ_s is the solution of equation

$$(5.11)\qquad -t \pm \varphi_\lambda(x,\lambda_s) = 0,$$

or according to (3.3)

$$(5.12)\qquad \lambda_s^{1/2} = (2|t|)^{-1}|x| - |t|^{-1}\Phi_\lambda(x,\lambda_s)\lambda_s^{1/2}.$$

Using again estimate (3.2) and the method of successive approximations we obtain

LEMMA 5.4. — *Let x satisfy (5.10) for some positive α and β. Then for sufficiently large $|t|$, equation (5.12) has a unique solution $\lambda_s = \lambda_s(x,t)$ and*

$$(5.13)\qquad \lambda_s^{1/2}(x,t) = (2|t|)^{-1}|x| + O(t^{-\delta}).$$

Set

$$(5.14)\qquad \psi(x,\lambda,t) = -\lambda + |t|^{-1}\varphi(x,\lambda).$$

Applying now to integral (5.7) the stationary phase method (see e.g. [4]) we obtain, under the assumptions of Lemma 5.4, that

$$(5.15)\quad I_\pm(x,t) = (2\pi)^{1/2}|t|^{-1/2}|\psi_{\lambda\lambda}|^{-1/2}e^{it\psi\pm\frac{\pi i}{4}\operatorname{sgn}\psi_{\lambda\lambda}}w(\lambda_s(x,t)) + O(|t|^{-1}),$$

where

$$\psi = \psi(x,\lambda_s(x,t),t),\quad \psi_{\lambda\lambda} = \psi_{\lambda\lambda}(x,\lambda_s(x,t),t).$$

Note that, by virtue of (5.13), $w(\lambda_s(x,t))$ can be replaced in (5.15) by $w((2t)^{-2}|x|^2)$ with an error term tending to zero in L_2 over the spherical layer (5.10). Furthermore, differentiating (5.14) and taking into account (5.13), we find that

$$\psi_{\lambda\lambda}(x,\lambda_s,t) = -(4|t|)^{-1}|x|\lambda_s^{-3/2} - |t|^{-1}\Phi_{\lambda\lambda}(x,\lambda_s) = -2|x|^{-2}|t|^2 + O(|t|^{-\delta}).$$

In particular, $\operatorname{sgn}\psi_{\lambda\lambda} = -1$. Set

$$\Omega(x,t) = t\psi(x,\lambda_s(x,t)) \pm \pi(d-3)/4 = -\lambda_s t \pm \lambda_s^{1/2}|x| \mp \Phi(x,\lambda_s), \ \ \pm t > 0. \tag{5.16}$$

Then, according to (5.13),

$$\Omega(x,t) = (4t)^{-1}|x|^2 + O(|t|^{1-\delta}). \tag{5.17}$$

Plugging these expressions into (5.15) and then into (5.8) and taking into account (5.9), we obtain

LEMMA 5.5. — *Let $\tilde{f}(\lambda) = v(\lambda)a$ where $a \in C^\infty(\mathbb{S}^{d-1})$ and $v \in C_0^\infty(\mathbb{R}^+)$, let the function $\Omega(x,t)$ be defined by equations (5.12), (5.16) and let $f = F_\pm^* \tilde{f}$. Then*

$$(e^{-iHt}f)(x) = 2^{1/2}(2it)^{-d/2}e^{i\Omega(x,t)}|(2t)^{-1}x|^{-(d-2)/2}v((4t^2)^{-1}|x|^2)a(\pm\hat{x}) + \epsilon_\pm(x,t) \tag{5.18}$$

where $||\epsilon_\pm(\cdot,t)|| = o(1)$ as $t \to \pm\infty$.

3. Recall that the operators $\mathcal{F}_0(\lambda)$ were defined by (3.13) and set $(F_0 f)(\lambda) = \mathcal{F}_0(\lambda)f$ (cf. (5.1)). The operator $F_0 : \mathcal{H} \to \tilde{\mathcal{H}}$ is of course unitary and

$$\widehat{(F_0^* \tilde{f})}(\xi) = 2^{1/2}|\xi|^{-(d-2)/2}v(|\xi|^2)a(\hat{\xi}) \quad \text{if} \quad \tilde{f}(\lambda) = v(\lambda)a.$$

Let us introduce a family of unitary operators by the equality

$$(U_0(t)f)(x) = e^{i\Omega(x,t)}(2it)^{-d/2}\hat{f}((2t)^{-1}x). \tag{5.19}$$

Then (5.18) can be rewritten as

$$\lim_{t\to\pm\infty} ||e^{-iHt}f - U_0(t)F_0^*F_\pm f|| = 0. \tag{5.20}$$

Since both operators e^{-iHt} and $U_0(t)$ are unitary, this relation extends to all $f \in \mathcal{H}^{(a)}$. Taking additionally into account Theorem 5.2, we obtain

THEOREM 5.6. — *Let $\Omega(x,t)$ be defined by equations (5.12), (5.16) and $U_0(t)$ – by (5.19). Then the strong limits*

$$W_\pm = s - \lim_{t\to\pm\infty} e^{iHt}U_0(t) \tag{5.21}$$

exist and $W_\pm = F_\pm^* F_0$. In particular, the wave operators $W_\pm$ are complete, that is $R(W_\pm) = \mathcal{H}^{(a)}$.

Wave operators (5.21) with $U_0(t)$ defined by (5.19) were introduced in [14]. However the phase function $\Omega(x,t)$ was defined in [14] as an approximate solution of eikonal equation (1.8) satisfying condition (5.17) in any region (5.10). Of course both functions Ω coincide. To check it, we shall show that function (5.16) satisfies equation (1.8) (with V replaced by V_L). Indeed, it follows from (5.11) that

$$\Omega_t = -\lambda_s - t\partial\lambda_s/\partial t \pm \varphi_\lambda(x,\lambda_s)\partial\lambda_s/\partial t = -\lambda_s$$

and

$$\Omega_{x_i} = -t\partial\lambda_s/\partial x_i \pm \varphi_{x_i}(x,\lambda_s) \pm \varphi_\lambda(x,\lambda_s)\partial\lambda_s/\partial x_i = \pm\varphi_{x_i}(x,\lambda_s)$$

so that

$$\Omega_t(x,t) + |\nabla_x\Omega(x,t)|^2 = -\lambda_s(x,t) + |(\nabla_x\varphi)(x,\lambda_s(x,t))|^2.$$

Therefore (1.8) is an immediate consequence of equation (1.4) (with V replaced by V_L).

4. It follows from Theorem 5.6 that the time-dependent scattering operator $\mathbf{S} = W_+^* W_-$ satisfies the equality

$$\mathbf{S} = F_0^* F_+ F_-^* F_0.$$

Of course it commutes with the operator H_0. Taking into account equality (4.11) we obtain

PROPOSITION 5.7. — *The operator $F_0\mathbf{S}F_0^*$ acts in the space $\tilde{\mathcal{H}}$ as multiplication by $\mathcal{S}(\lambda) : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$.*

Thus, the time-dependent scattering matrix coincides with the stationary one, constructed in Section 4.

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