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CONSTRUCTION AND ANALYSIS OF SOME CONVOLUTION ALGEBRAS

by Arne BEURLING (1)

1. A construction method of convolution algebras.

Let $\mathcal{G}$ be a locally compact Abelian group with the invariant measure dx. In order to avoid trivial cases we assume that $\mathcal{G}$ is not compact. We consider a normed family Ω of strictly positive functions $\omega(x)$ on $\mathcal{G}$, which are measurable, summable with respect to dx, and furthermore, together with the norm $N(\omega)$, satisfy the following conditions:

1. For each $\omega \in \Omega$, $N(\omega)$ takes a finite value, such that

$$0 < \int \omega \, dx \leqslant N(\omega). \tag{1.1}$$

II. If λ is a positive number and $\omega \in \Omega$, then $\lambda\omega \in \Omega$ and

$$N(\lambda\omega) = \lambda N(\omega). \tag{1.2}$$

III. If $\omega_1, \omega_2 \in \Omega$, the sum $\omega_1 + \omega_2$ as well as the convolution $\omega_1 * \omega_2$ are also in Ω and

$$N(\omega_1 + \omega_2) \leqslant N(\omega_1) + N(\omega_2) \tag{1.3}$$

$$N(\omega_1 * \omega_2) \leqslant N(\omega_1)N(\omega_2). \tag{1.4}$$

(1) This paper was presented at a Conference in harmonic analysis held at Cornell University 1955. Except for the conference record no reprints were prepared. The paper is now made available to a larger public on the request by analysts interested in the subject.

IV. Ω is complete under the norm N in the sense that for any sequence $\{\omega_n\}_1^\infty \subset \Omega$ such that $\sum_1^\infty N(\omega_n) < \infty$, it holds that $\omega = \sum_1^\infty \omega_n$ is in Ω and

$$(1.5) \qquad N(\omega) \leqslant \sum_1^\infty N(\omega_n).$$

The set of measures $\{\omega\, dx;\ \omega \in \Omega\}$ constitutes our starting point for the following constructions of Banach algebras and shall be referred to as *a normed semi-ring of positive finite measures.* By Ω_0 we shall denote the subset of Ω defined by the condition

$$N(\omega) = 1$$

and we shall call such an ω normalized. In the sequel we shall assume that $1 < p < \infty$ and set $q = p/p - 1$. For a fixed p, we shall use the notation

$$(1.6) \qquad \omega' = \frac{1}{\omega^{p-1}},$$

and write Hölder's inequality in the form

$$(1.7) \qquad \int |\Phi F|\, dx \leqslant \left\{\int |\Phi|^q \omega\, dx\right\}^{\frac{1}{q}} \left\{\int |F|^p \omega'\, dx\right\}^{\frac{1}{p}}.$$

We associate with each normalized ω the Banach spaces $L^p_{\omega'}$ and L^q_ω of functions, measurable on $\mathcal{G}$ and having the norms

$$\|F\|_{L^p_{\omega'}} = \left\{\int |F|^p \omega'\, dx\right\}^{\frac{1}{p}},$$

$$\|\Phi\|_{L^q_\omega} = \left\{\int |\Phi|^q \omega\, dx\right\}^{\frac{1}{q}},$$

respectively. From these spaces, we obtain two sets of functions

$$A^p(\mathcal{G}, \Omega) = A^p \qquad \text{and} \qquad B^q(\mathcal{G}, \Omega) = B^q$$

by setting

$$(1.8) \qquad A^p = \bigcup_{\omega \in \Omega_0} L^p_{\omega'},$$

$$(1.9) \qquad B^q = \bigcap_{\omega \in \Omega_0} L^q_\omega$$

and we define

$$(1.10) \qquad ||F|| = ||F||_{A^p} = \inf_{\omega \in \Omega_0} ||F||_{L^p_\omega},$$

$$(1.11) \qquad ||\Phi|| = ||\Phi||_{B^q} = \sup_{\omega \in \Omega_0} ||\Phi||_{L^q_\omega}.$$

We shall now prove

THEOREM I. — *In the norm* (1.10) A^p *is a Banach algebra under addition and convolution, and*

$$(1.12) \qquad ||F_1 * F_2|| \leqslant ||F_1|| \, ||F_2||.$$

In the norm (1.11) B^q *is a Banach space, which is the dual of* A^p *in the sense that each linear functional* $\varphi(F)$ *on* A^p *has the form*

$$(1.13) \qquad \varphi(F) = \int \Phi F \, dx$$

where Φ *is a unique element of* B^q *and*

$$(1.14) \qquad \sup_{F \in A^p} \frac{\left|\int \Phi F \, dx\right|}{||F||} = ||\Phi||.$$

An immediate consequence of the definitions is that B^q is a Banach space under the norm (1.11). The same is true of A^p under the norm (1.10), but the proof is not trivial. However, we may infer at once by the Hölder's inequality (1.7) applied to $\Phi \in B^q$, $F \in A^p$, that

$$(1.15) \qquad \int |\Phi F| \, dx \leqslant \inf_{\omega \in \Omega_0} ||\Phi||_{L^q_\omega} ||F||_{L^p_\omega} \leqslant ||\Phi|| \, ||F||.$$

For $\Phi \equiv 1$ we have by (1.1),

$$(1.16) \qquad ||\Phi|| = \sup_{\omega \in \Omega_0} \left\{\int \omega \, dx\right\}^{\frac{1}{q}} \leqslant \sup_{\omega \in \Omega_0} N^{\frac{1}{q}}(\omega) = 1,$$

which together with (1.15) yields

$$(1.17) \qquad \int |F| \, dx \leqslant ||F||.$$

Thus, $||F||$ is a majorant of the L^1-norm and A^p is a subset of L^1, while L^∞ is a subset of B^q and the L^∞-norm is a minorant of $||\Phi||$. That A^p is actually a Banach space will be proved by a new definition of the norm $||F||$. We set

$$(1.18)\quad W_p(F;\ \omega) = W(F;\ \omega) = \frac{1}{p}\int \frac{|F|^p}{\omega^{p-1}}\,dx + \frac{1}{q}\,N(\omega),$$

and we call $\omega \in \Omega$ relatively extremal for an $F \in A^p$ if (1.18) is finite and if furthermore the minimum of $W(\lambda) = W(F;\ \lambda\omega)$ for $\lambda > 0$ is attained for $\lambda = 1$. From the relation

$$\frac{dW}{d\lambda} = \frac{1}{q}\left\{-\frac{1}{\lambda^p}\int \frac{|F|^p}{\omega^{p-1}}\,dx + N(\omega)\right\}$$

it follows that ω has this property if, and only if

$$(1.19)\qquad \int \frac{|F|^p}{\omega^{p-1}}\,dx = N(\omega) = W(F,\ \omega).$$

If ω_0 is normalized and λ is determined such that $\lambda\omega_0 = \omega$ is relatively extremal for F, we find that

$$\left\{\int \frac{|F|^p}{\omega_0^{\,p-1}}\,dx\right\}^{\frac{1}{p}} = \operatorname*{Min}_{\lambda>0} W(F;\ \lambda\omega_0) = W(F;\ \omega).$$

Therefore

$$(1.20)\qquad ||F|| = \inf_{\omega\in\Omega} W(F;\ \omega),$$

which is our new definition of the norm.

Let us now establish the triangle inequality

$$(1.21)\qquad ||F_1 + F_2|| \leqslant ||F_1|| + ||F_2||.$$

Let ω_1, $\omega_2 \in \Omega$ be such that $W(F_\nu,\ \omega_\nu)$ are finite ($\nu = 1,\ 2$), and set

$$(1.22)\quad W(\theta) = W(\theta|F_1| + (1-\theta)|F_2|;\ \theta\omega_1 + (1-\theta)\omega_2)$$
$$= \frac{1}{p}\int \frac{(\theta|F_1| + (1-\theta)|F_2|)^p}{(\theta\omega_1 + (1-\theta)\,\omega_2)^{p-1}}\,dx + \frac{1}{q}\,N(\theta\omega_1 + (1-\theta)\omega_2.)$$

This function $W(\theta)$ is convex in the interval $[0,\ 1]$. In fact, according to (1.2) and (1.3) we conclude that the last term is a convex function of θ. Furthermore, we observe that the integrand is of the form $h = l_1^p/l_2^{p-1}$ where l_1 and l_2 are linear and non-negative functions of θ. For the second derivative of h we obtain

$$h'' = p(p-1)h\left(\frac{l_1'}{l_1} - \frac{l_2'}{l_2}\right)^2 \geqslant 0.$$

Hence, the integrand of (1.22), as well as $W(\theta)$ is convex in $[0, 1]$, and consequently

$$2W\left(\frac{1}{2}\right) \leqslant W(1) + W(0).$$

Since $2W\left(\frac{1}{2}\right) = W(|F_1| + |F|_2;\ \omega_1 + \omega_2)$ we shall have

$$(1.23) \quad W(F_1 + F_2;\ \omega_1 + \omega_2) \leqslant W(|F_1| + |F_2|;\ \omega_1 + \omega_2) \leqslant W(F_1;\ \omega_1) + W(F_2;\ \omega_2),$$

which, combined with (1.20) implies (1.21).

To prove that A^p is complete it is sufficient to show that for any sequence $\{F_n\}_1^\infty \subset A^p$ such that $\Sigma\|F_n\| < \infty$, there exists an $F \in A^p$ with the property that

$$(1.24) \qquad \lim_{n=\infty} \left\| F - \sum_1^n F_\nu \right\| = 0.$$

Let k be any number > 1; let $\omega_n \in \Omega$ be relatively extremal for F_n; and suppose that $W(F_n;\ \omega_n) \leqslant k\|F\|$. According to (1.19)

$$(1.25) \quad \int \frac{|F_n|^p}{\omega_n^{p-1}}\, dx = N(\omega_n) = W(F_n;\ \omega_n) \leqslant k\|F_n\|.$$

Hence, the series $\Sigma N(\omega_n)$ converges and $\omega = \Sigma\omega_n$ therefore belongs to Ω. On applying (1.23) repeatedly, we obtain

$$W\left(\sum_1^n |F_\nu|;\ \sum_1^n \omega_\nu\right) \leqslant k \sum_1^n \|F_\nu\|,$$

and from (1.25) and (1.18) we find

$$(1.26) \quad W\left(\sum_1^n |F_\nu|;\ \omega\right) \leqslant k \sum_1^n \|F_\nu\| + \frac{k}{q} \sum_{n+1}^\infty \|F_\nu\| \leqslant k \sum_1^\infty \|F_\nu\|.$$

Setting

$$H = \sum_1^\infty |F_\nu|, \qquad F = \sum_1^\infty F_\nu$$

we conclude from (1.26) that $W(H;\ \omega)$ is finite, and consequently that H and F are defined as elements of the space $L^p_{\omega'}$. Furthermore, we have

$$W\left(F - \sum_1^n F_\nu; \omega\right) \leqslant W\left(H - \sum_1^n |F_\nu|; \omega\right),$$

where Lebesgue's theorem of dominated convergence can be applied to the last integral. This prove that

$$\lim_{n=\infty} W\left(F - \sum_1^n F_\nu; \omega\right) = 0,$$

from which (1.24) follows.

We have shown at this point that A^p is a Banach space. We shall now verify that A^p is also an algebra under convolution. Let F_1, $F_2 \in A^p$, and let ω_1, $\omega_2 \in \Omega_0$ be such that

$$\left[\int \frac{|F_\nu|^p}{\omega_\nu^{p-1}} dx\right]^{\frac{1}{p}} \leqslant k\|F_\nu\|, \quad (\nu = 1,\ 2),$$

where k is a given constant > 1. Set [2]

$$F = F_1 * F_2 = \int F_1(x - y) F_2(y)\, dy,$$

$$\omega = \omega_1 * \omega_2 = \int \omega_1(x - y)\omega_2(y)\, dy.$$

By Hölder's inequality, we have

$$|F(x)| \leqslant \left\{\int \frac{|F_1(x-y)|^p}{\omega_1(x-y)^{p-1}} \cdot \frac{|F_2(y)|^p}{\omega_2(y)^{p-1}} dy\right\}^{\frac{1}{p}} \left\{\int \omega_1(x-y)\omega_2(y)\, dy\right\}^{\frac{p-1}{p}}.$$

Hence

$$\int \frac{|F(x)|^p}{\omega(x)^{p-1}} dx \leqslant \iint \frac{|F_1(x-y)|^p}{\omega_1(x-y)^{p-1}} \cdot \frac{|F_2(y)|^p}{|\omega_2(y)|^{p-1}} dy\, dx$$
$$= \int \frac{|F_1|^p}{\omega_1^{p-1}} dx \int \frac{|F_2|^p}{\omega_2^{p-1}} dy \leqslant k^{2p} \|F_1\|^p \|F_2\|^p.$$

Because of the inequality $N(\omega) \leqslant N(\omega_1).N(\omega_2) \leqslant 1$, the first integral is by definition $\geqslant \|F\|^p$, and consequently

$$\|F\| \leqslant k^2 \|F_1\|\, \|F_2\|$$

which proves (1.12), since k is arbitrarily close to 1.

(2) We denote by $+$ the group operation in $\mathcal{G}$ and by $-y$ the inverse of y.

As to the second part of the theorem we start by recalling that $L^p_{\omega'}$ and L^q_ω are conjugate spaces. We also know that for any $\omega \in \Omega_0$, $L^p_{\omega'}$ is a subset of A^p while the norm in $L^p_{\omega'}$ is a majorant of $||F||$. Therefore a linear functional $\varphi(F)$ on A^p is also a linear functional on $L^p_{\omega'}$, where it has the representation

$$\varphi(F) = \int \Phi F \, dx,$$

Φ being a uniquely determined element of L^q_ω. We will prove that this Φ is independent of ω and belongs to all of the spaces L^q_ω for $\omega \in \Omega_0$. In fact, for any ω_1, $\omega_2 \in \Omega_0$,

$$\omega_3 = \frac{\omega_1 + \omega_2}{N(\omega_1 + \omega_2)} \in \Omega_0$$

and $\varphi(F)$ has the representations

$$\varphi(F) = \int \Phi_\nu \, F \, dx, \quad (\Phi_\nu \in L^q_{\omega_\nu})$$

in the spaces $L^p_{\omega'}$, $\nu = 1, 2, 3$. However, it is obvious that $L^p_{\omega'_3}$ contains the union of $L^p_{\omega'_1}$ and $L^p_{\omega'_2}$. This implies $\Phi_1 = \Phi_3$ and $\Phi_2 = \Phi_3$ and our assertion follows.

It now remains only to prove that for each number $k < 1$ and for every fixed $\Phi \in B^q$, there is an $F \in A^p$ such that

$$\left| \int \Phi F \, dx \right| \geqslant k ||\Phi|| \, ||F||.$$

By choosing $\omega \in \Omega_0$ such that

$$\left\{ \int |\Phi|^q \omega \, dx \right\}^{\frac{1}{q}} \geqslant k ||\Phi||$$

and taking

$$F = \frac{|\Phi|^q}{\Phi} \omega$$

we find

$$||F|| \leqslant \left\{ \int \frac{|F|^p}{\omega^{p-1}} dx \right\}^{\frac{1}{p}} = \left\{ \int |\Phi|^q \omega \, dx \right\}^{\frac{1}{p}},$$

$$\int \Phi F \, dx = \int |\Phi|^q \omega \, dx \geqslant k ||\Phi|| \, ||F||,$$

which completes the proof of Theorem I.

From the definition we conclude that our spaces A^p and B^q satisfy the following inclusion relations. Assume $1 < p_1 < p$

and let $\Omega_1 \subset \Omega$ be two families which satisfy our four conditions under the norms N_1 and N respectively. If $N_1(\omega) \geqslant N(\omega)$ in Ω_1 we obtain

$$A^p(\mathcal{G}, \Omega_1) \subset A^p(\mathcal{G}, \Omega) \subset A^{p_1}(\mathcal{G}, \Omega),$$
$$B^{\frac{p}{p-1}}(\mathcal{G}, \Omega_1) \supset B^{\frac{p}{p-1}}(\mathcal{G}, \Omega) \supset B^{\frac{p_1}{p_1-1}}(\mathcal{G}, \Omega).$$

Furthermore, we find that the norm in $A^p(\mathcal{G}, \Omega)$ is a non-decreasing function of p, while the converse situation holds for the norm in $B^{\frac{p}{p-1}}(\mathcal{G}, \Omega)$. We also observe that if $|F(x)|$ is almost everywhere equal to a function $\omega(x) \in \Omega$, then $F(x)$ belongs to all of the spaces A^p, $(1 < p < \infty)$ and $||F||_{A^p} \leqslant N(\omega)$.

Up till now we have assumed $1 < p < \infty$. However, for any choice of Ω satisfying our conditions, we will have

$$\lim_{p=1+0} ||F||_{A^p} = \int |F| \, dx,$$
$$\lim_{q=\infty} ||\Phi||_{B^q} = \text{true max } |\Phi(x)|$$

where both sides are finite or infinite at the same time. The spaces $A^1(\mathcal{G}, \Omega)$ and $B^\infty(\mathcal{G}, \Omega)$ are therefore to be identified with $L^1(\mathcal{G})$ and $L^\infty(\mathcal{G})$ respectively. In the other limit case $p = \infty$ we will have

$$||F||_{A^\infty} = \inf_\omega N(\omega)$$

where the inf is taken for those $\omega \in \Omega$ which are essential majorants of $|F(x)|$; while

$$||\Phi||_{B^1} = \inf_{\omega \in \Omega_0} \int |\Phi| \omega \, dx.$$

From this we conclude that theorem 1 holds also in the case $p = 1$, and that the first part of the theorem is true even for $p = \infty$.

2. The relation between certain algebras A^p and the space of functions bounded in the mean of order q.

In this section we shall focus our interest on algebras A^p over the euclidean space R^n, $(n \geqslant 1)$, which are generated by some particularly simple families Ω. Consider first the set

$\Omega = \Omega(R^n)$ of positive and summable $\omega(x)$ which are non-increasing functions of $|x|$ and have the norm

$$N(\omega) = \int_{R^n} \omega \, dx = \int_0^\infty \omega(r) \, dV_n(r), \tag{2.1}$$

where $V_n(r)$ denotes the volume of the sphere $|x| \leqslant r$ in R^n. To prove that this set Ω satisfies our conditions I-IV it is sufficient to show that the convolution of ω_1, $\omega_2 \in \Omega$ is in Ω *i.e.*, that

$$\omega(|x|) = \int_{R^n} \omega_1(|x-y|)\omega_2(|y|) \, dy$$

is a non-increasing function of $|x|$. Without loss of generality we may assume that $\omega_1(r)$ is differentiable for $r > 0$. Denoting by y^* the symmetric point of y with respect to the hyperplane $(y - x, x) = 0$, we obtain for y situated in the halfspace H_x where $(y - x, x) < 0$,

$$\frac{d\omega_1(|x-y|)}{d|x|} = -\frac{d\omega_1(|x-y^*|)}{d|x|} \leqslant 0.$$

Hence,

$$\frac{d\omega(|x|)}{d|x|} = \int_{H_x} \frac{d\omega_1(|x-y|)}{d|x|} (\omega_2(|y|) - \omega_2(|y^*|)dy$$

which is $\leqslant 0$ since $|y| \leqslant |y^*|$.

We shall also consider the family Ω_1 defined as the subset of Ω consisting of functions with the property

$$\omega(0) = \lim_{x=0} \omega(x) < \infty.$$

The norm in Ω_1 will be defined as

$$N(\omega) = \omega(0) + \int_{R^n} \omega \, dx. \tag{2.2}$$

Among the conditions I-IV, the inequality

$$N(\omega_1 * \omega_2) \leqslant N(\omega_1)N(\omega_2)$$

is the only one that is not obvious. For the proof we may restrict ourselves to the normalized case $N(\omega_1) = N(\omega_2) = 1$. Under this assumption we obtain

$$(\omega_1 * \omega_2)_{x=0} \leqslant \omega_1(0) \, (1 - \omega_2(0)),$$

and the requested inequality follows,

$$N(\omega_1 * \omega_2) \leqslant \omega_1(0)\,(1-\omega_2(0)) + (1-\omega_1\,(0))(1-\omega_2(0)) < 1.$$

In this and the following sections we shall denote by $B^q = B^q(R^n)$ and by $\mathcal{B}^q = \mathcal{B}^q(R^n)$ the spaces of measurable functions on R^n which are L^q over compact sets and have the norms

$$(2.3)\qquad ||\Phi||_{B^q} = \sup_{r>0}\left\{\frac{\int_{|x|\leqslant r}|\Phi|^q\,dx}{V_n(r)}\right\}^{\frac{1}{q}},$$

$$(2.4)\qquad ||\Phi||_{\mathcal{B}^q} = \sup_{r>0}\left\{\frac{\int_{|x|\leqslant r}|\Phi|^q\,dx}{1+V_n(r)}\right\}^{\frac{1}{q}}.$$

Theorem II. — *B^q is the conjugate space of the algebra $A^p = A^p(R^n, \Omega)$. Similarly, $\mathcal{B}^q$ is the conjugate space of the algebra $\mathcal{A}^p = A^p(R^n, \Omega_1)$, where Ω and Ω_1 are the families defined in this section.*

Let us first show that $\mathcal{B}^q$ is identical with $B^q(R^n, \Omega_1)$ defined as in § 1. Assume $\Phi \in B^q(R^n, \Omega_1)$ and let $\{\omega_\nu\}_1^\infty \subset \Omega_1$ be a decreasing sequence such that for a given $r > 0$,

$$\lim_{\nu=\infty} \omega_\nu(x) = \begin{cases} 1, & |x| \leqslant r \\ 0, & |x| > r \end{cases}$$
$$\lim_{\nu=\infty} N(\omega_\nu) = 1 + V_n(r).$$

Thus if $||\Phi||$ denotes the norm in $B^q(R^n, \Omega_1)$,

$$\int_{|x|\leqslant r} |\Phi|^q\,dx \leqslant \int |\Phi|^q\omega_\nu\,dx \leqslant ||\Phi||^q N(\omega_\nu).$$

For $\nu \to \infty$ we obtain

$$(2.5)\qquad \int_{|x|\leqslant r} |\Phi|^q\,dx \leqslant ||\Phi||^q(1 + V_n(r)),\quad (r>0)$$

which implies that

$$(2.6)\qquad ||\Phi||_{\mathcal{B}^q} \leqslant ||\Phi||.$$

On the other hand, the assumption $\Phi \in \mathcal{B}^q$ implies the truth of (2.5) with $||\Phi||$ replaced by $||\Phi||_{\mathcal{B}^q}$, and if $\varphi(r)$ denotes the left hand member of (2.5) we shall have

$$\int |\Phi|^q\omega\, dx = \int_0^\infty \omega(r)\, d\varphi(r) = \int_0^\infty \varphi(r) d[-\omega(r)]$$
$$\leqslant \|\Phi\|^q_{\mathfrak{B}^q} \int_0^\infty (1 + V_n(r) d[-\omega(r)] = \|\Phi\|^q_{\mathfrak{B}^q} N(\omega).$$

Hence,

$$\|\Phi\| \leqslant \|\Phi\|_{\mathfrak{B}^{q'}}, \tag{2.7}$$

which together with (2.6) proves our statement $\mathfrak{B}^q = B^q(R^n, \Omega_1)$. In the same way we find that $B^q = B^q(R^n, \Omega)$ and theorem II follows on applying Theorem I.

Finally, we give some examples that will not be considered further in this paper, but are apt to illustrate the relation between the algebras $L^1(\mathcal{G})$ and $A^p(\mathcal{G}, \Omega)$.

Example 2*:* Ω_2 is the set of all positive summable functions on $\mathcal{G}$ with $N(\omega)$ determined as in (2.1). Then $A^p = L^1$ and $B^q = L^\infty$ for $1 \leqslant p \leqslant \infty$.

Example 3*:* Ω_3 is the subset of Ω_2 determined by the condition

$$\sup_{x\in g} \omega(x) \leqslant \int \omega\, dx = N(\omega).$$

As is easily seen, this property is conserved under addition and convolution, so that conditions I-IV are all satisfied. For the norm in B^q we find

$$\|\Phi\| = \sup_E \left\{ \int_E |\Phi|^q\, dx \right\}^{\frac{1}{q}},$$

where E is a variable set of measure $\leqslant 1$. The norm in A^p will be

$$\|F\| = \inf \sum_1^\infty \left\{ \int_{E_n} |F|^p\, dx \right\}^{\frac{1}{p}},$$

where $\{E_n\}_1^\infty$ is a variable sequence of sets of measures $\leqslant 1$ covering each compact part of $\mathcal{G}$. It is to be expected that for all p the properties of A^p are very closely related to those of L^1.

Example 4*:* Ω_4 is the subset of Ω_2 consisting of all continuous functions with real and positive Fourier transforms; $N(\omega)$

is defined as in (2.2). This algebra seems to be of considerable interest in the case that the dual group of $\mathcal{G}$ is not compact.

Example 5*:* Ω_5 is defined over a euclidien space R^n and possesses the property that the Fourier transform of an $\omega \in \Omega_5$ is also in Ω_5; $N(\omega)$ defined as in (2.2).

3. The algebra A^2 over the real line.

The algebras A^2 and $\mathcal{A}^2$ appearing in theorem II and defined over the real line are among the most interesting cases of convolution algebras derived by the method of § 1. We shall find that the main problems of these two algebras permit a complete solution, and furthermore, that the analysis can be carried through without introduction of the conjugate spaces B^2 and $\mathcal{B}^2$.

We begin with A^2 and recall that

$$||F||^2 = \inf_{\omega \in \Omega} \int \omega \, dx \int \frac{|F|^2}{\omega} \, dx,$$

where Ω consists of summable positive ω which are non-increasing functions of $|x|$. By capitals $F(x)$, $G(x)$, ... we shall denote elements of A^2, while $f(t)$, $g(t)$, ... will be their Fourier transforms in the definition

$$f(t) = \int e^{-itx} F(x) \, dx.$$

The ring of Fourier transforms f, $(F \in A^2)$, will be denoted by $\tilde{A}^2$ and will have the norm $||f|| = ||F||$.

Other notations that will be used throughout the paper are

$$\eta(\alpha) = \eta(\alpha, f) = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(t+\alpha) - f(t)|^2 \, dt},$$

$$\Delta_\alpha f(t) = f(t+\alpha) - f(t),$$

$$A(f) = \int_0^{\infty} \eta(\alpha, f) \frac{d\alpha}{\alpha^{3/2}}.$$

This integral $A(f)$ will be an important tool for the analysis of A^2 as shown by

THEOREM III. — *A function f belongs to the ring $\tilde{A}^2$ if and only if:*

a) *f is continuous,*

b) $\lim_{t=\pm\infty} f(t) = 0$,

c) $A(f) < \infty$.

Under these conditions, f is the Fourier transform of an $F \in A^2$, *and the following inequalities hold:*

$$\|f\| < A(f) < 5\|f\| \tag{3.1}$$

(*provided that* $f \neq 0$).

We prove first that our conditions are sufficient and imply $\|F\| < A(f)$. From *c*) and the definition of $\eta(\alpha, f)$ and $A(f)$, it follows that $\Delta_\alpha f(t) \in L^2$ for α belonging to a certain set E whose complement is of measure zero. We point out also, that $\eta(\alpha) > 0$ for $\alpha \neq 0$ except in the trivial case $f \equiv 0$ which we exclude.

Using the Plancherel theorem, we define for all $\alpha \in E$ a function $F_\alpha(x)$ by setting

$$(e^{-i\alpha x} - 1)F_\alpha(x) = \lim_{n=\infty} \frac{1}{2\pi} \int_{-n}^{n} e^{itx} \Delta_\alpha f(t)\, dt.$$

If β is another number $\in E$, all four functions appearing in the identity

$$\Delta_\alpha f(t+\beta) - \Delta_\alpha f(t) = \Delta_\beta f(t+\alpha) - \Delta_\beta f(t)$$

belong to L^2, and we obtain, on taking the Fourier transforms of each member,

$$(e^{-i\beta x} - 1)\,(e^{-i\alpha x} - 1)F_\alpha(x) = (e^{-i\alpha x} - 1)\,(e^{-i\beta x} - 1)F_\beta(x).$$

Hence, $F_\alpha(x) = F(x)$ is independent of α, and the Parseval relation yields

$$\eta^2(\alpha) = 4 \int_{-\infty}^{\infty} |F(x)|^2 \sin^2 \frac{\alpha x}{2}\, dx, \quad (\alpha \in E). \tag{3.2}$$

Our aim is now prove to the existence of an $\omega \in \Omega$ such that $|F|^2/\omega$ is summable. For this purpose, we divide both members of (3.2) by $\eta(\alpha)\alpha^{3/2}$ and integrate with respect to α over $(0, \infty)$, obtaining

$$(3.3) \qquad A(f) = \int_{-\infty}^{\infty} |F(x)|^2 \int_0^{\infty} \frac{4 \sin^2 \frac{\alpha x}{2}}{\eta(\alpha)\alpha^{3/2}} \, d\alpha \, dx.$$

By defining

$$\frac{1}{\mu(x)} = 4 \int_0^{\infty} \frac{\sin^2 \frac{\alpha x}{2}}{\eta(\alpha)\alpha^{3/2}} \, d\alpha,$$

$$\omega(x) = \frac{1}{8} \int_0^{\frac{4}{|x|}} \frac{\eta(\alpha)}{\alpha^{1/2}} \, d\alpha,$$

and using Schwarz' inequality, we get for $x > 0$,

$$\frac{\omega(x)}{\mu(x)} > \frac{1}{2} \left\{ \int_0^{\frac{4}{x}} \frac{\sin \frac{\alpha x}{2}}{\alpha} \, d\alpha \right\}^2 = \frac{1}{2} \left\{ \int_0^{2} \frac{\sin \beta}{\beta} \, d\beta \right\}^2 > 1.$$

Hence

$$A(f) = \int_{-\infty}^{\infty} \frac{|F(x)|^2}{\mu(x)} \, dx > \int_{-\infty}^{\infty} \frac{|F(x)|^2}{\omega(x)} \, dx,$$

$$2 \int_0^{\infty} \omega(x) \, dx = \frac{1}{4} \int_0^{\infty} dx \int_0^{\frac{4}{x}} \frac{\eta(\alpha)}{\alpha^{\frac{1}{2}}} \, d\alpha = \int_0^{\infty} \frac{\eta(\alpha)}{\alpha^{3/2}} \, d\alpha = A(f).$$

Consequently $\omega \in \Omega$, and

$$||F||^2 \leqslant \int_{-\infty}^{\infty} \omega(x) \, dx \int_{-\infty}^{\infty} \frac{|F(x)|^2}{\omega(x)} \, dx < A^2(f)$$

which verifies the inequality $||F|| < A(f)$, provided that f is actually the Fourier transform of F. In accordance with the definition of F and the fact that f is continuous, we have, for all α and t,

$$f(t + \alpha) - f(t) = \int_{-\infty}^{\infty} e^{-i(t+\alpha)x} F(x) \, dx - \int_{-\infty}^{\infty} e^{-itx} F(x) \, dx.$$

The Riemann-Lebesgue theorem implies that the first integral tends to 0 as $\alpha \to \pm \infty$, t being fixed. This shows that the limit

$$f(\infty) = \lim_{\alpha = \pm \infty} f(+ \alpha)$$

exists, and consequently, that

$$f(t) = f(\infty) + \int_{-\infty}^{\infty} e^{-itx} F(x)\, dx, \tag{3.4}$$

which is the desired relation, since by hypothesis, $f(\infty) = 0$.

Since the conditions a) and b) are obviously fulfilled for every $f \in \tilde{A}^2$, it remains only to prove that $f \in \tilde{A}^2$ implies $A(f) < 5\|F\|$. The proof of this inequality depends essentially on the following lemma, which is proved elsewhere [3].

Let $\omega(x)$ be non-increasing and summable over $0 < x < \infty$, and let a and b be constants such that $0 < a < 1 < b$. Then $\omega(x)$ admits a majorant $\omega^*(x)$ with the following properties: $x^a \omega^*(x)$ is non-increasing, $x^b \omega^*(x)$ is non-decreasing and

$$\int_0^{\infty} \omega^*(x)\, dx \leqslant \frac{b}{(1-a)\,(b-1)} \int_0^{\infty} \omega(x)\, dx. \tag{3.5}$$

Let $\omega \in \Omega$ be normalized and have the property that $|F|^2/\omega$ is summable, and let ω^* be the majorant described above and corresponding to the case $a = 1/2$, $b = 3/2$. Thus

$$\int_0^{\infty} \omega^*(x)\, dx \leqslant 6 \int_0^{\infty} \omega(x)\, dx = 3.$$

By Schwarz' inequality,

$$A^2(f) = \left\{ \int_0^{\infty} \frac{\eta(\alpha)}{\alpha^{3/2}}\, d\alpha \right\}^2 \leqslant \int_0^{\infty} \frac{\eta^2(\alpha)\, d\alpha}{\omega^*\left(\dfrac{2}{\alpha}\right)\alpha} \int_0^{\infty} \omega^*\left(\frac{2}{\alpha}\right) \frac{d\alpha}{\alpha^2},$$

where the last integral is $\leqslant 3/2$. Hence, on substituting the right member of (3.2) for $\eta^2(\alpha)$, we find

$$A^2(f) \leqslant 6 \int_{-\infty}^{\infty} |F(x)|^2 \int_0^{\infty} \frac{\sin^2 \dfrac{\alpha x}{2}}{\omega^*\left(\dfrac{2}{\alpha}\right)\alpha}\, d\alpha\, dx. \tag{3.6}$$

From the inequalities

$$\frac{\omega^*(x)}{\omega^*\left(\dfrac{x}{\beta}\right)} \leqslant \begin{cases} \dfrac{1}{\beta^{3/2}}, & 0 < \beta \leqslant 1 \\ \dfrac{1}{\beta^{3/2}}, & \beta \geqslant 1 \end{cases}$$

[3] A. BEURLING, On the spectral synthesis at bounded functions, *Acta Math.*, t. 81, 1949.

we infer

$$\int_0^\infty \frac{\sin^2 \frac{\alpha x}{2}}{\omega^*\left(\frac{2}{\alpha}\right)\alpha} d\alpha = \int_0^1 + \int_1^\infty \frac{\sin^2 \beta}{\omega^*\left(\frac{x}{\beta}\right)\beta} d\beta$$

$$< \frac{1}{\omega^*(x)} \int_0^1 \frac{d\beta}{\beta^{1/2}} + \frac{1}{\omega^*(x)} \int_1^\infty \frac{d\beta}{\beta^{3/2}} = \frac{4}{\omega^*(x)}.$$

Thus

$$A^2(f) < 24 \int_{-\infty}^\infty \frac{|F|^2}{\omega^*} dx \leqslant 24 \int_{-\infty}^\infty \frac{|F|^2}{\omega} dx,$$

and this completes the proof of theorem III.

On the continuity theorem in $\tilde{A}^2$ *and the principles of contraction.*

A function $K(z)$ defined in the whole complex plane and having the properties

$$|K(z') - K(z)| \leqslant |z' - z|$$
$$K(0) = 0$$

will in the sequel be called a contractor. We shall, in particular, be interested in the circular contractor K_ε defined by the formula

$$(3.7) \qquad K_\varepsilon(z) = \begin{cases} z & |z| \leqslant \varepsilon, \\ \varepsilon \dfrac{z}{|z|}, & |z| > \varepsilon. \end{cases}$$

Furthermore, a function $g(t)$ shall be called a contraction of $f(t)$ if for all t and t'

$$(3.8) \qquad \begin{cases} |g(t)| \leqslant |f(t)|, \\ |g(t') - g(t)| \leqslant |f(t') - f(t)|. \end{cases}$$

We shall also consider conditions such as

$$(3.9) \qquad \begin{cases} |g(t)| \leqslant \sum_1^N |f_\nu(t)|, \\ |g(t') - g(t)| \leqslant \sum_1^N |f_\nu(t') - f_\nu(t)|, \end{cases}$$

and in that case we shall say that g is a contraction of the series $\sum_1^N f_\nu$.

One of the most striking consequences of theorem III is that each contractor K is a bounded operator on $\tilde{A}^2$. In fact, if f is continuous and tends to 0 for $t \to \pm \infty$, the same is true of $g = K(f)$. Clearly, $A(g) \leqslant A(f)$ and theorem III yields

$$(3.10) \qquad ||K(f)|| < 5||f||.$$

The continuity theorem for $\tilde{A}^2$ is now a consequence of the following stronger.

THEOREM IV. — *Let g be a contraction of the series $\sum_1^N f_\nu$ where each f_ν belongs to $\tilde{A}^2$. Then*

$$(3.11) \qquad g \in \tilde{A}^2, \; ||g|| \leqslant k \sum_1^N ||f_\nu||,$$

where $k(< 5)$ is a constant depending only on the space.

If, in a sequence $\{g_n\}_1^\infty$, each function is a contraction of $\sum_1^N f_\nu$, then the assumption

$$\lim_{n=\infty} M(g_n) = 0$$

implies

$$(3.12) \qquad \lim_{n=\infty} ||g_n|| = 0.$$

This property of a space will be refered to as *the principle of uniform contraction*, whereas the implication (3.11) alone will be called *the principle of contraction.*

Since (3.11) is obvious we start by proving (3.12). We have

$$|\Delta_\alpha g_n(t)| \leqslant \sum_1^N |\Delta_\alpha f_\nu(t)| \in L^2,$$

while

$$\lim_{n=\infty} \Delta_\alpha g_n(t) = 0$$

for fixed α and t. Hence, by the Lebesgue theorem of dominated convergence,

$$\lim_{n=\infty} \eta(\alpha, g_n) = 0.$$

Similarly,

$$\frac{\eta(\alpha, g_n)}{\alpha^{3/2}} \leqslant \sum_1^N \frac{\eta(\alpha, f_\nu)}{\alpha^{3/2}} \in L^1,$$

and we may apply the same theorem to the integral $A(g_n)$ obtaining

$$\lim_{n=\infty} A(g_n) = 0,$$

and (3.12) follows from theorem III.

In the later part of this paper we will meet several rings satisfying the principle of uniform contraction and it is therefore convenient to give at this instance a short account of their basic properties. We shall denote by Γ a normed ring of continuous numeric functions $f(t)$ defined on any space S and with the property that at each point $t \in S$,

$$(3.13) \qquad \sup_{||g|| \leqslant 1} \frac{|g(t)|}{||g||} = 1. \text{ (}^4\text{)}$$

We shall also assume that the principle of uniform contraction is valid in Γ with a certain constant k.

To the ring Γ we may adjoin each function h with the property that $g \in \Gamma$ implies $gh \in \Gamma$. By the closed graph theorem

$$(3.14) \qquad ||gh|| \leqslant m||g||, \quad (m < \infty)$$

and we define the norm of h as the least number m satisfiyng (3.14) By this completion of Γ we obtain a new ring which will be denoted by ex Γ. In order to avoid ambiguities, the norm in ex Γ will sometimes be denoted by $||h||_{ex}$; the notation $M(h)$ shall stand for the supremum norm of h. We observe at once that (3.13) implies that

$$(3.15) \qquad M(h) \leqslant ||h||_{ex}.$$

By a closed ideal $\mathfrak{J}$ in Γ we shall mean a closed linear subset of Γ such that $f \in \mathfrak{J}$, $g \in \Gamma$ imply $fg \in \mathfrak{J}$. The closure of $\mathfrak{J}$ in the uniform topology will be denoted by $\bar{\mathfrak{J}}$. The ideal generated by an $f \in \Gamma$ will be denoted by $\mathfrak{J}_f$ and defined as the closure of the set $\{gf;\ g \in \Gamma\}$.

(4) In this paper we don't make any distinction between Banach algebras and normed rings, but we prefer the latter notation for Banach spaces of numeric functions which are algebras under pointwise addition and multiplication.

Using these notations we shall now prove the following four elementary lemmas.

LEMMA I. — *Let f, g belong to Γ and let there exist a positive number m such that*

$$(|f(t)| - m)|g(t)| \geqslant 0. \tag{3.16}$$

Then the equation $hf - g = 0$ possesses a solution $h \in \Gamma$ and

$$||h|| \leqslant k \frac{||g||}{m}\left(1 + \frac{||f||}{m}\right). \tag{3.17}$$

The condition (3.16) implies that $g(t)$ vanishes on the set E where $|f(t)| < m$. If h is defined as $= 0$ on E, and $= g/f$ on the complement of E, we find whether this set is empty or not,

$$|h(t') - h(t)| \leqslant \frac{1}{m}|g(t') - g(t)| + \frac{|g(t')|}{m^2}|f(t') - f(t)|.$$

Hence, h is a contraction of the series

$$m^{-1}g(t) + m^{-2}||g||f(t)$$

and this proves our statement.

LEMMA II. — *A function $g \in \Gamma$ belongs to the closed ideal $\mathfrak{J}$ if and only if, for each given $\varepsilon > 0$, the inequality*

$$\mathrm{M}\left(\frac{g}{1 + |f|}\right) < \varepsilon \tag{3.18}$$

has a solution $f \in \mathfrak{J}$. In particular, we shall have

$$\mathfrak{J} = \bar{\mathfrak{J}} \cap \Gamma, \tag{3.19}$$

and the $\mathfrak{J}_f$ consists of those $g \in \Gamma$ for which

$$\lim_{n=\infty} \mathrm{M}\left(\frac{g}{1 + n|f|}\right) = 0. \tag{3.20}$$

Assume that f, g verify (3.18). If $\varepsilon < 1$, then $|g(t)| < 2\varepsilon$ on the set where $|g(t)| < \varepsilon$. Setting $g_1 = g - \mathrm{K}_{2\varepsilon}(g)$, where $\mathrm{K}_{2\varepsilon}$ is the circular contractor defined by (3.7), we see that

$$(|f(t)| - \varepsilon)|g_1(t)| \geqslant 0.$$

Thus by lemma I there exists an $h \in \Gamma$ which satisfies the equation $hf - g_1 = 0$. Consequently

$$||hf - g|| \leqslant ||K_{2\varepsilon}(g)|| = \delta(\varepsilon, g),$$

where $\delta(\varepsilon, g)$ tends to 0 for $\varepsilon \to 0$. This proves lemma II.

LEMMA III. — *If Γ is separable then each closed ideal $\mathfrak{J}$ contains an element f such that $\mathfrak{J} = \mathfrak{J}_f$.*

Our assumption implies that $\mathfrak{J}$ is likewise separable and there exists then a sequence $\{f_n\}_1^\infty \subset \mathfrak{J}$ which is dense in $\mathfrak{J}$. Observing that the functions $|f_n|^2$ belong to $\mathfrak{J}$ we set

$$f = \sum_1^\infty a_n |f_n|^2,$$

and choose $a_n > 0$ and so small that the series converges in norm. If n and p are given integers we can always find an integer q so large that

$$M\left(\frac{1 + p|f_n|}{1 + qf}\right) < 2.$$

Thus the assumption

$$\lim_{p=\infty} M\left(\frac{g}{1 + p|f_n|}\right) = 0$$

implies

$$\lim_{q=\infty} M\left(\frac{g}{1 + qf}\right) = 0.$$

Hence, by lemma II, $\mathfrak{J}_{f_n} \subset \mathfrak{J}_f$, $(n = 1, 2, \ldots)$. From this we conclude that

$$\mathfrak{J}_f \subset \mathfrak{J} = \text{closure} \bigcup_{n \geqslant 1} \mathfrak{J}_{f_n} \subset \mathfrak{J}_f \tag{3.21}$$

and lemma III is established.

LEMMA IV. — *If Γ satisfies the principle of contraction with the constant k then the extended ring ex Γ satisfies the same principle with the constant $3k$.*

Let the function h be a contraction a of the series $\sum_1^N h_\nu$ where each term belongs to ex Γ, and let g belong to Γ and have a norm $\leqslant 1$. From the relations

$$g(t')h(t') - g(t)\,h(t) = g(t)\,(h(t') - h(t)) + h(t')\,(g(t') - g(t)),$$
$$|g(t')h(t') - g(t)h(t)| \leqslant |g(t)||h(t') - h(t)| + M(h)|g(t') - g(t)|,$$
$$|g(t)||h_\nu(t') - h_\nu(t)| \leqslant |g(t')h_\nu(t') - g(t)h_\nu(t)| + M(h_\nu)|g(t') - g(t)|,$$

it follows that

$$|g(t')h(t') - g(t)h(t)| \leqslant b|g(t') - g(t)| + \sum_1^N |g(t')h_\nu(t') - g(t)h_\nu(t')|,$$

with

$$b = \mathrm{M}(h) + \sum_1^N \mathrm{M}(h_\nu) \leqslant 2 \sum_1^N ||h_\nu||_{ex}.$$

Thus gh is a contraction of the series

$$bg(t) + \sum_1^N g(t)h_\nu(t),$$

where by assumption each term gelongs to Γ. Therefore $gh \in \Gamma$ and

$$||gh|| \leqslant k\left(b + \sum_1^N ||h_\nu||_{ex}\right) \leqslant 3k \sum_1^N ||h_\nu||_{ex},$$

which proves our statement.

Let us now return to the ring $\tilde{\mathrm{A}}^2$ and consider its closed ideals. Taking the Fourier transform $g(t)$ of $e^{it_0x}\omega(x)$, where ω is a normalized function $\in \Omega$, we see that $|g(t_0)| = ||g|| = 1$. Thus $\tilde{\mathrm{A}}^2$ satisfies the conditions stipulated for Γ, and even lemma III applies to $\tilde{\mathrm{A}}^2$ since A^2 is obviously separable.

If E is a closed set of points t we shall denote by $\mathfrak{J}_\mathrm{E}$ the set of functions $g \in \tilde{\mathrm{A}}^2$ which vanish on E.

Consider now the ideal $\mathfrak{J}_f$ generated by f and let E be the zeros of f. By the criterion (3.18) of lemma II, $g \in \tilde{\mathrm{A}}^2$ belongs to $\mathfrak{J}_f$ if, for each $\varepsilon > 0$, there exists an integer n such that for all t

$$|g(t)| < \varepsilon + n|f(t)|.$$

However, for continuous functions which tend to 0 for $t \to \pm\infty$ this condition is satisfied provided that g vanishes on E. Thus

$$\mathfrak{J}_\mathrm{E} \subset \mathfrak{J}_f \subset \mathfrak{J}_\mathrm{E}, \quad \mathfrak{J}_f \subset \mathfrak{J}_\mathrm{E} \subset \mathfrak{J}_f, \tag{3.22}$$

and we have proved

THEOREM V. — *For an ideal* $\mathfrak{J}_f$, *we have* $\mathfrak{J}_f = \mathfrak{J}_\mathrm{E}$, *where* E *is the set of zeros of* f.

This property obviously remains true if E is empty in which case $\mathfrak{J}_f = \mathfrak{J}_E = \tilde{A}^2$. For arbitrary closed ideals we obtain

THEOREM VI. — *Let* E *be the set of common zeros of functions belonging to a closed ideal* $\mathfrak{J}$. *Then* $\mathfrak{J} = \mathfrak{J}_E$.

By lemma III $\mathfrak{J}$ contains a generating element f, i.e., with the property $\mathfrak{J}_f = \mathfrak{J}$. From this we conclude that f can't vanish outside E and our statement follows on applying theorem V.

The extended ring ex $\tilde{A}^2$.

Following the procedure described in a previous section we form the extended ring ex $\tilde{A}^2$. By Lemma IV we know already that the principle of contraction must be valid in the new ring with the constant $k = 15$. In the actual case we shall find, however, that a much stronger result is true. This depends mainly on a property of $\tilde{A}^2$ which has not been relevant earlier, viz. that $\tilde{A}^2$ contains a sequence $\{g_n\}_1^\infty$ with norms $\leqslant 1$ such that $g_n(t)$ converges uniformly to 1 over each compact set as $n \to \infty$. If, in fact, $\omega(x)$ is a normalized function belonging to Ω the same is true of $n\omega(nx)$ for $n = 1,2 \ldots$ and the latter function has a Fourier transform $g_n(t)$ with the desired properties. Using this fact we shall prove

THEOREM VII. — *The ring ex* $\tilde{A}^2$ *consists of all functions of the form*

$$h = c + f \tag{3.23}$$

where c *is a constant and* $f \in \tilde{A}^2$.

The non-trivial part of this theorem expresses that ex $\tilde{A}^2$ does not contain any other functions than (3.23). In the proof of this, and of similar theorems later on, we shall use the notation

$$\xi(\alpha, g, h) = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |g(t)|^2|h(t+\alpha) - h(t)|^2\, dt},$$

and the identity

$$\Delta_\alpha g(t)h(t) = g(t)\Delta_\alpha h(t) + h(t+\alpha)\Delta_\alpha g(t),$$

which by Minkowsky's inequality yields

$$|\xi(\alpha, g, h) - \eta(\alpha, gh)| \leqslant M(h)\eta(\alpha, g).$$

On dividing both members by $\alpha^{3/2}$ and on integrating with respect to α over $(0, \infty)$,

$$\left|\int_0^\infty \xi(\alpha, g, h)\frac{d\alpha}{\alpha^{3/2}} - \mathrm{A}(gh)\right| \leqslant \mathrm{M}(h)\mathrm{A}(g).$$

Assuming $h \in \operatorname{ex} \tilde{\mathrm{A}}^2$ we insert in this formula the function g_n defined above. By theorem III,

$$\mathrm{A}(g_n h) + \mathrm{M}(h)\mathrm{A}(g_n) < 5\|g_n h\| + 5\mathrm{M}(h)\|g_n\| \leqslant 10\|h\|_{\mathrm{ex}}$$

Thus for $n \to \infty$,

$$\mathrm{A}(h) = \lim_{n=\infty} \int_0^\infty \xi(\alpha, g_n, h)\frac{d\alpha}{\alpha^{3/2}} \leqslant 10\|h\|_{\mathrm{ex}}.$$

Being necessarily continuous h thus satisfies conditions $a)$ and $c)$ of theorem III. However, by the proof of that theorem we showed that $a)$ and $c)$ imply the representation (3.4) which is the desired result.

In the theory of ex $\tilde{\mathrm{A}}^2$ it is convenient to set

$$h(\infty) = \lim_{t=\pm\infty} h(t),$$

and to adjoin $t = \infty$ as an ideal point of the t-axis. Under these conventions theorem VI remains true: Each closed ideal $\mathcal{J}$ in ex $\tilde{\mathrm{A}}^2$ has the form $\mathcal{J}_{\mathrm{E}}$ where E may or may not contain the point ∞. For the rest we only point out that the previous results on $\tilde{\mathrm{A}}^2$ remain valid for the extended ring. We finally observe the interesting inequality

$$\|h^{-1}\|_{\mathrm{ex}} < \frac{5}{m^2}\|h\|_{\mathrm{ex}}$$

which holds for functions $h \in \operatorname{ex} \tilde{\mathrm{A}}^2$ with $|h(t)| \geqslant m > 0$.

4. The algebra $\mathcal{A}^2$ over the real line.

For the norm in $\mathcal{A}^2$ we have

$$(4.1) \qquad \|\mathrm{F}\|^2 = \inf_{\omega\in\Omega_1}\left(\omega(0) + \int_{-\infty}^{\infty}\omega\, dx\right)\int_{-\infty}^{\infty}\frac{|\mathrm{F}|^2}{\omega}dx,$$

ω being here a positive and non-increasing function of $|x|$ with the property that

$$\omega(0) = \lim_{x=0} \omega(x) < \infty.$$

We first prove

THEOREM VIII. — *The space* $\mathcal{A}^2$ *is the intersection of* A^2 *and* L^2, *and the norms in these spaces satisfy the inequalities*

$$(4.2) \qquad ||F||_{\mathcal{A}^2} > ||F||_{A^2},$$
$$(4.3) \qquad ||F||_{\mathcal{A}^2} > ||F||_{L^2},$$
$$(4.4) \qquad ||F||_{\mathcal{A}^2} < ||F||_{A^2} + ||F||_{L^2}.$$

The relation (4.2) is a consequence of the fact that Ω_1 is subset of Ω in which $N_1(\omega) > N(\omega)$. As to the proof of (4.3) we recall that

$$||\Phi||_{\mathcal{B}^2} = \sup_{r>0}\left\{\frac{1}{1+2r}\int_{-r}^{r} |\Phi|^2\, dx\right\}^{\frac{1}{2}} < ||\Phi||_{L^2}.$$

Hence, on taking $\Phi = \overline{F}$ in (1.14),

$$||F||^2_{L^2} = \int F\Phi\, dx \leqslant ||F||_{\mathcal{A}^2}||\Phi||_{\mathcal{B}^2} < ||F||_{\mathcal{A}^2}||F||_{L^2},$$

and (4.3) follows. For the purpose of proving (4.4) we set

$$||F||_{L^2} = a, \quad \left\{\int_{-\infty}^{\infty} \frac{|F|^2}{\omega} dx\right\}^{\frac{1}{2}} = b,$$

where $\omega \in \Omega$ is normalized and b is a number close to $||F||_{A^2}$. Clearly

$$\omega_1(x) = \frac{ab\omega(x)}{a + b\omega(x)}$$

belongs to Ω_1 and

$$N(\omega_1) = \omega_1(0) + \int_{-\infty}^{\infty} \omega_1\, dx < (a + b).$$

Thus, on inserting ω_1 in (4.1) we obtain

$$||F||^2 < (a + b)\int_{-\infty}^{\infty} |F|^2\left(\frac{1}{b\omega} + \frac{1}{a}\right) dx = (a + b)^2,$$

which proves (4.4) with the sign $\leqq$. That the strict inequality actually holds for $F \neq 0$ is easily established.

By $\tilde{\mathcal{A}}^2$ we shall denote the ring of Fourier transform f, ($F \in \mathcal{A}^2$), with the norm $||f|| = ||F||$. Combining theorem III and VIII we obtain

Theorem IX. — *A function f belongs to $\tilde{\mathcal{A}}^2$ if and only if:*

a) *f is continuous,*

b) $f \in L^2$,

c) $A(f) < \infty$.

Under these conditions the following inequalities hold:

$$||f|| < A(f) + \frac{1}{\sqrt{2\pi}}||f||_{L^2} < 6||f||. \tag{4.5}$$

(*provided that* $f \neq 0$).

From this we see that the principle of uniform contraction is valid in $\tilde{\mathcal{A}}^2$ with the constant $k = 6$. Forming the Fourier transform $g(t)$ of $e^{it_0x}\omega(x)$, where $\omega \in \Omega_1$ is normalized and $\omega(0) < \varepsilon$, we find that $|g(t_0)| > 1 - \varepsilon$, $||g|| < 1$. Thus $\tilde{\mathcal{A}}^2$ verifies the conditions stipulated for the rings Γ. Therefore theorems V and VI remain true, and nothing new is to be found with regard to the ideal theory. On the other hand, the completion of $\tilde{\mathcal{A}}^2$ turns out to be quite a fascinating problem showing new and interesting aspects.

Each function $f \in \tilde{\mathcal{A}}^2$ being of summable square, it follows that

$$\eta(\alpha, f) \leqslant \frac{2}{\sqrt{2\pi}}||f||_{L^2}. \tag{4.6}$$

The amount of the integral $A(f)$ which derives from the range $1 < \alpha < \infty$, is therefore no longer significant. For this reason we introduce

$$A_1(f) = \int_0^1 \frac{\eta(\alpha, f)}{\alpha^{3/2}}\, d\alpha, \tag{4.7}$$

and observe that

$$A_1(f) < A(f) < A_1(f) + \frac{4}{\sqrt{2\pi}}||f||_{L^2}. \tag{4.8}$$

On combining this with (4.5),

$$(4.9) \qquad \|f\| < A_1(f) + \frac{5}{\sqrt{2\pi}}\|f\|_{L^2}.$$

Similarly,

$$(4.10) \qquad \|f\| > \frac{1}{6}\left(A_1(f) + \frac{1}{\sqrt{2\pi}}\|f\|_{L^2}\right).$$

The completion of $\tilde{\mathcal{A}}^2$.

By lemma IV we already know that ex $\tilde{\mathcal{A}}^2$ satisfies the principle of contraction with a constant $k < 18$. We also know that continuity and boundedness are necessary conditions for functions belonging to ex $\tilde{\mathcal{A}}^2$. If h has these properties we find that the boundedness of $A_1(gh)$ for $g \in \tilde{\mathcal{A}}^2$, $\|g\| \leqslant 1$, is a both necessary and sufficient condition. In fact, by our previous inequalities (3.1), (4.3) and (4.9),

$$(4.11) \qquad \frac{1}{5}A_1(gh) < \|gh\| < A_1(gh) + 5M(h).$$

Assuming still $\|g\| \leqslant 1$ we obtain by (3.29),

$$(4.12) \qquad \left|\int_0^1 \xi(\alpha, g, h)\frac{d\alpha}{\alpha^{3/2}} - A_1(gh)\right| \leqslant M(h)A_1(g) < 5M(h).$$

Setting

$$a(h) = \sup_{\|g\|\leqslant 1} A_1(gh), \qquad \xi(h) = \sup_{\|g\|\leqslant 1}\int_0^1 \xi(\alpha, g, h)\frac{d\alpha}{\alpha^{3/2}},$$

it follows from (4.11) that

$$\frac{1}{5}a(h) < \|h\|_{ex} < a(h) + 5M(h).$$

Combining this result with the inequalities

$$|\xi(h) - a(h)| < 5M(h), \qquad M(h) \leqslant \|h\|_{ex}$$

we find that

$$(4.13) \qquad \frac{1}{10}\xi(h) < \|h\|_{ex} < \xi(h) + 10M(h).$$

Consider first the case that h possesses a modulus of continuity

$$\theta(\alpha) = \sup_{-\infty<t<\infty} |h(t+\alpha) - h(t)|$$

which is summable on [0, 1] with respect to the measure $\alpha^{-3/2}\, d\alpha$. Then it follows that

$$\xi(\alpha, g, h) \leqslant \frac{\theta(\alpha)\|g\|_{L^2}}{\sqrt{2\pi}} \leqslant \theta(\alpha),$$

$$\|h\|_{ex} \leqslant \int_0^1 \frac{\theta(\alpha)\, d\alpha}{\alpha^{3/2}} + 10M(h). \tag{4.14}$$

After these preliminaries we have reached the main problem of this chapter: to obtain a complete characterization of functions belonging to the extended ring ex $\tilde{\mathcal{A}}^2$. Some kind of uniform continuity is obviously required, but the stated property of $\theta(\alpha)$ is too strong and the problem of finding the adequate notion of uniformity is not quite easy and needs a series of lemmas.

LEMMA V. — *Let $\{E_n\}_1^\infty$ be a sequence of closed sets on the t-axis such that the distance between E_m and E_n is $\geqslant 1$ for $m \neq n$. Let f belong to $\mathcal{A}^2$ and have the expansion $\sum_1^\infty f_n$ where f_n vanishes outsides E_n. Then $f_n \in \tilde{\mathcal{A}}^2$ and*

$$\sqrt{\sum_1^\infty \|f_n\|^2} < 10\|f\|. \tag{4.15}$$

For $0 < \alpha < 1$ we have by assumption

$$\Delta_\alpha f_m(t)\Delta_\alpha f_n(t) = 0 \quad (m \neq n)$$

and it follows that

$$\sum_1^\infty \eta^2(\alpha, f_n) = \eta^2(\alpha, f),$$

$$\sum_1^\infty \int_{-\infty}^{\infty} |f_n|^2\, dt = \int_{-\infty}^{\infty} |f|^2\, dt.$$

By Schwarz' inequality

$$A_1^2(f_n) = \left\{\int_0^1 \eta(\alpha, f_n)\frac{d\alpha}{\alpha^{3/2}}\right\}^2 \leqslant A_1(f)\int_0^1 \frac{\eta^2(\alpha, f_n)}{\eta(\alpha, f)}\frac{d\alpha}{\alpha^{3/2}}.$$

Hence, on summation,

$$\sum_1^\infty A_1^2(f_n) \leqslant A_1^2(f).$$

By (4.9),

$$||f_n||^2 < 2A_1^2(f_n) + \frac{50}{2\pi}||f_n||^2_{L^2}.$$

Since $A_1(f) < 5||f||$ and $||f||_{L^2} < 2\pi||f||$, we finally get,

$$\sum_1^\infty ||f_n||^2 < 2A_1^2(f) + \frac{50}{2\pi}||f||^2_{L^2} < 100||f||^2.$$

LEMMA VI. — *There is a finite constant k_1 such that for any $g \in \tilde{\mathcal{A}}^2$*

$$\sqrt{\sum_{-\infty}^{\infty} b_n^2} \leqslant k_1||g||, \tag{4.14}$$

where b_n is the maximum of $|g(t)|$ on the interval $[n, n+1]$.

Let $\gamma_n(t)$, $(n = 0, \pm 1, \ldots)$ denote the continuous function which is $= 1$ on $[n, n+1]$, $= 0$ outside $[n-1, n+2]$ and linear on the two remaining intervals. Clearly

$$\gamma(t) = \sum_{-\infty}^{\infty} \gamma_{4n}(t) \in \text{ex}\, \tilde{\mathcal{A}}^2.$$

Thus γg belongs to $\tilde{\mathcal{A}}^2$ and the series

$$\gamma g = \sum_{-\infty}^{\infty} \gamma_{4n} g$$

verifies the conditions of lemma V. Hence,

$$\sum_{-\infty}^{\infty} b_{4n}^2 \leqslant \sum_{-\infty}^{\infty} ||\gamma_{4n} g||^2 \leqslant 100||\gamma||^2_{\text{ex}}||g||^2.$$

From this we conclude that (4.14) must be true if we take $k_1 = 20||\gamma||_{\text{ex}}$.

LEMMA VII. — *There is a finite constant k_2 such that for any sequence $\{a_n\}_{-\infty}^{\infty}$ of non-negative numbers with finite square sum, we can find a $g \in \tilde{\mathcal{A}}^2$ satisfying the following conditions:*

$$\min_{[n, n+1]} |g(t)| \geqslant a_n, \qquad ||g|| \leqslant k_2 \sqrt{\sum_{-\infty}^{\infty} a_n^2}.$$

Setting

$$g_i = \sum_{n=-\infty}^{n=\infty} a_{4n+i}\gamma_{4n+i}, \quad (i = 0, 1, 2, 3).$$

we obtain for each g_i,

$$\eta^2(\alpha, g_i) = \sum_{-\infty}^{\infty} a_{4n+i}^2 \eta^2(\alpha, \gamma_{4n+i}) < \frac{\alpha^2}{\pi} \sum_{-\infty}^{\infty} a_{4n+i}^2,$$

$$A_1(g_i) < \sqrt{\frac{4}{\pi} \sum_{-\infty}^{\infty} a_{4n+i}^2},$$

$$\int_{-\infty}^{\infty} |g_i|^2 = \frac{5}{3} \sum_{-\infty}^{\infty} a_{4n+i}^2$$

Thus by (4.9)

$$||g_i|| < \left(\sqrt{\frac{4}{\pi}} + \sqrt{\frac{125}{6\pi}}\right) \sqrt{\sum_{-\infty}^{\infty} a_{4n+i}^2},$$

and we find by Schwarz' inequality that

$$g = \sum_{0}^{3} g_i$$

verifies our condition if we take

$$k_2 = 2\left(\sqrt{\frac{4}{\pi}} + \sqrt{\frac{125}{6\pi}}\right) < 8.$$

Let us now return to our main problem and recall the definition

$$\xi^2(\alpha, g, h) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |g(t)|^2 |h(t + \alpha) - h(t)|^2 \, dt.$$

We set

$$\eta_n^2 = \eta_n^2(\alpha, h) = \frac{1}{2\pi} \int_{n}^{n+1} |h(t + \alpha) - h(t)|^2 \, dt, \quad (n = 0, \pm 1, \ldots)$$

and call $\eta_n(\alpha)$ the local modulus of the L^2-continuity of h. For later use we point out the inequality

(4.15)
$$|\eta_n(\alpha + \beta) - \eta_n(\alpha)| \leqslant \eta_n(\beta) + \eta_{n+1}(\beta), \quad (\alpha, \beta \geqslant 0, \alpha \leqslant 1).$$

Denoting by b_n the maximum, and by a_n the minimum of $|g(t)|$ on the interval $[n, n + 1]$, we obtain

$$(4.16) \quad \int_0^1 \sqrt{\sum_{-\infty}^{\infty} a_n^2 \eta_n^2} \frac{d\alpha}{\alpha^{3/2}} \leqslant \int_0^1 \xi(\alpha, g, h) \frac{d\alpha}{\alpha^{3/2}}$$
$$\leqslant \int_0^1 \sqrt{\sum_{-\infty}^{\infty} b_n^2 \eta_n^2} \frac{d\alpha}{\alpha^{3/2}},$$

which combined with lemma VI and VII yields the following result :

THEOREM X. — *A continuous bounded function h belongs to ex* $\tilde{\mathfrak{A}}^2$ *if and only if*

$$(4.18) \qquad K(h) = \sup_{\psi \in \mathcal{C}} \int_0^1 \sqrt{\psi(\alpha)} \frac{d\alpha}{\alpha^{3/2}} < \infty,$$

when ψ *varies in the convex set*

$$\mathcal{C} = \left\{ \psi(\alpha) = \sum_{-\infty}^{\infty} \tau_n \eta_n^2(\alpha, h); \quad \tau_n \geqslant 0, \quad \sum_{-\infty}^{\infty} \tau_n \leqslant 1 \right\}.$$

We shall now see that K(h) also admits another and more significant definition. If the functions $\{\eta_n\}_{-\infty}^{\infty}$ possess a common majorant $\varphi(\alpha)$, summable over [0, 1] with respect to the measure $\alpha^{-3/2}d\alpha$, then clearly

$$K(h) \leqslant \int_0^1 \varphi \frac{d\alpha}{\alpha^{3/2}}.$$

However, this inequality remains true if we choose φ in a wider class of functions. We shall say that $\varphi \geqslant 0$ is a *mean-majorant* of $\{\eta_n\}_{-\infty}^{\infty}$ on [0, 1] with respect to the measure $\alpha^{-3/2}d\alpha$ if

$$(4.20) \qquad \int_0^1 \frac{\eta_n^2}{\varphi} \frac{d\alpha}{\alpha^{3/2}} \leqslant \int_0^1 \varphi \frac{d\alpha}{\alpha^{3/2}}, \quad (n = 0, \pm 1, \pm \cdots)$$

If such a function is summable for $\alpha^{-3/2}d\alpha$ we obtain by Schwarz' inequality for each $\psi \in \mathcal{C}$,

$$(4.21) \quad \int_0^1 \sqrt{\psi} \frac{d\alpha}{\alpha^{3/2}} \leqslant \left\{ \int_0^1 \frac{\psi}{\varphi} \frac{d\alpha}{\alpha^{3/2}} \int_0^1 \varphi \frac{d\alpha}{\alpha^{3/2}} \right\}^{\frac{1}{2}} \leqslant \int_0^1 \varphi \frac{d\alpha}{\alpha^{3/2}}$$

LEMMA VIII. — *If* K(h) $< \infty$, *then the functions* $\psi \in \mathcal{C}$ *are equicontinuous and the uniform closure* $\overline{\mathcal{C}}$ *of* $\mathcal{C}$ *contains a unique element* ψ^* *such that* $\varphi = \sqrt{\psi^*}$ *is a mean-majorant with the property*

$$(4.22) \qquad K(h) = \int_0^1 \varphi \frac{d\alpha}{\alpha^{3/2}}.$$

Let us first show that $K(h) < \infty$ implies the existence of a bounded function $\lambda(\varepsilon)$, $(0 < \varepsilon \leqslant 1)$, tending to 0 at the origin and such that for each $\psi \in \mathcal{C}$

$$(4.23) \qquad \int_0^\varepsilon \sqrt{\psi}\,\frac{d\alpha}{\alpha^{3/2}} \leqslant \lambda(\varepsilon), \quad (0 < \varepsilon \leqslant 1).$$

If this statement were false there would exist a number $\lambda_0 > 0$ such that for each given $\varepsilon > 0$ the inequality

$$\int_0^\varepsilon \sqrt{\psi}\,\frac{d\alpha}{\alpha^{3/2}} > \lambda_0$$

would hold for some $\psi \in \mathcal{C}$. If p is a given integer we could also find non-overlapping intervals $\{\gamma_\nu\}_1^p \subset (0, 1]$ and functions $\{\psi_\nu\}_1^p \in \mathcal{C}$ such that

$$\int_{\gamma_\nu} \sqrt{\psi_\nu}\,\frac{d\alpha}{\alpha^{3/2}} > \lambda_0.$$

Since $\mathcal{C}$ is convex, $\psi = \dfrac{1}{p}\sum\limits_1^p \psi_\nu$ would belong to $\mathcal{C}$ and

$$\int_0^1 \sqrt{\psi}\,\frac{d\alpha}{\alpha^{3/2}} > p^{-1/2} \sum_1^p \int_{\gamma_\nu} \sqrt{\psi_\nu}\,\frac{d\alpha}{\alpha^{3/2}} > p^{\frac{1}{2}}\lambda_0.$$

This contradiction proves our statement.

From (4.23) we may now derive some important conclusions. For $0 \leqslant 2\beta \leqslant 2\alpha \leqslant 1$, we shall have by (4.15),

$$\eta_n(2\alpha) \leqslant \eta_n(\alpha + \beta) + \eta_n(\alpha - \beta) + \eta_{n+1}(\alpha - \beta).$$

An integration with respect to β over $(0, \alpha)$ yield,

$$\alpha\eta_n(2\alpha) \leqslant \int_0^{2\alpha} \eta_n(\beta)\,d\beta + \int_0^\alpha \eta_{n+1}(\beta)\,d\beta.$$

Here we may apply (4.23) to $\psi = \eta_n^2,\ \eta_{n+1}^2$. Thus

$$\alpha\eta_n(2\alpha) \leqslant (2\alpha)^{3/2}\lambda(2\alpha) + \alpha^{3/2}\lambda(\alpha).$$

Since we may assume $\lambda(\varepsilon)$ nondecreasing this inequality follows on setting $2\alpha = \varepsilon$,

$$(4.24) \quad \eta_n(\varepsilon) < 3\sqrt{\varepsilon}\,\lambda(\varepsilon), \quad (0 < \varepsilon \leqslant 1, \quad n = 0, \pm 1, \ldots)$$

By virtue of (4.15) we obtain for $\psi \in \mathcal{C}$,

$$(4.25)\quad |\psi(\alpha+\varepsilon)-\psi(\alpha)| \leqslant \sum_{-\infty}^{\infty} \tau_n|\eta_n(\alpha+\varepsilon)-\eta_n(\alpha)|(\eta_n(\alpha+\varepsilon)+\eta_n(\alpha)) \leqslant 36\lambda(1)\sqrt{\varepsilon}\lambda(\varepsilon),\quad (0<\varepsilon\leqslant 1).$$

By this equicontinuity and by (4.23) it follows that the mapping

$$\bar{\mathcal{C}} \ni \psi \to \int_0^1 \sqrt{\psi}\,\frac{d\alpha}{\alpha^{3/2}}$$

is continuous. The integral therefore assumes its least upper bound $K(h)$ for some $\psi^* \in \bar{\mathcal{C}}$. Since $\bar{\mathcal{C}}$ is convex,

$$(1-\theta)\psi^* + \theta\eta_n^2 \in \bar{\mathcal{C}}$$

for $0 \leqslant \theta \leqslant 1$ and for each n. Hence

$$\int_0^1 \{\psi^* + \theta(\eta_n^2 - \psi^*)\}^{\frac{1}{2}} \frac{d\alpha}{\alpha^{3/2}} \leqslant \int_0^1 \psi^* \frac{d\alpha}{\alpha^{3/2}}.$$

This implies

$$\int_0^1 \frac{\eta_n^2 - \psi^*}{\sqrt{\psi^*}} \frac{d\alpha}{\alpha^{3/2}} \leqslant 0,$$

and proves that $\sqrt{\psi^*}$ is a minimal mean-majorant of $\{\eta_n\}_{-\infty}^{\infty}$. If $\bar{\mathcal{C}}$ contained another function, say ψ, with the same properties, then the sign of equality would hold in (4.21) with $\varphi = \sqrt{\psi^*}$, and the two functions ψ^* and ψ would have to be proportional and therefore identic.

According to the previous results the quantity $M(h) + K(h)$ is a norm equivalent with $||h||_\alpha$ and the extension problem for $\tilde{\mathcal{A}}^2$ is thus completely solved.

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