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A NON-ABELIAN TENSOR PRODUCT OF LEIBNIZ ALGEBRAS

by Allahtan V. GNEDBAYE

Introduction.

Let \mathfrak{g} be a Lie algebra and let M be a representation of \mathfrak{g} , seen as a right \mathfrak{g} -module. Given a \mathfrak{g} -equivariant map $\mu : M \rightarrow \mathfrak{g}$, one can endow the \mathbb{K} -module M with a bracket $([m, m'] := m^{\mu(m')})$ which is not skew-symmetric but satisfies the *Leibniz rule of derivations*:

$$[m, [m', m'']] = [[m, m'], m''] - [[m, m''], m'].$$

Such objects were baptized *Leibniz algebras* by Jean-Louis Loday and are studied as a non-commutative variation of Lie algebras (see [8]). One of the main examples of Lie algebras comes from the notion of *derivations*. For the Leibniz algebras, there is an analogue notion of *biderivations* (see [7]).

The aim of this article is to “integrate” the Leibniz algebra of biderivations by means of a non-abelian tensor product of Leibniz algebras as it is done for Lie algebras.

In the classical case, D. Guin (see [5]) has shown that, given crossed Lie \mathfrak{g} -algebras \mathfrak{M} and \mathfrak{N} , the set of derivations $\text{Der}_{\mathfrak{g}}(\mathfrak{M}, \mathfrak{N})$ has a structure of pre-crossed Lie \mathfrak{g} -algebra. Moreover the functor $\text{Der}_{\mathfrak{g}}(\mathfrak{N}, -)$ is right adjoint to the functor $-\otimes_{\mathfrak{g}}\mathfrak{N}$ where $-\otimes_{\mathfrak{g}}-$ is the non-abelian tensor product of Lie algebras defined by G. J. Ellis (see [3]). D. Guin uses these objects

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to construct a non-abelian (co)homology theory for Lie algebras, which enables him to compare the \mathbb{K} -modules $\mathrm{HC}_1(A)$ and $\mathrm{K}_2^{M\mathrm{add}}(A)$ where A is an arbitrary associative algebra. We give a non-commutative version of his results, in the sense that Leibniz algebras play the role of Lie algebras, the additive Milnor K -theory $\mathrm{K}_*^{M\mathrm{add}}(A)$ (resp. the cyclic homology $\mathrm{HC}_*(A)$) being replaced by the Milnor-type Hochschild homology $\mathrm{HH}_*^M(A)$ (resp. the classical Hochschild homology $\mathrm{HH}_*(A)$).

To this end, we introduce the notion of (pre)crossed Leibniz \mathfrak{g} -algebra as a simultaneous generalization of notions of representation and two-sided ideal of the Leibniz algebra \mathfrak{g} . Given crossed Leibniz \mathfrak{g} -algebras \mathfrak{M} and \mathfrak{N} , we equip the set $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{M}, \mathfrak{N})$ of biderivations with a structure of pre-crossed Leibniz \mathfrak{g} -algebra. On the other hand, we construct a non-abelian tensor product $\mathfrak{M} \star \mathfrak{N}$ of Leibniz algebras with mutual actions on one another. When \mathfrak{M} and \mathfrak{N} are crossed Leibniz \mathfrak{g} -algebras, this tensor product has also a structure of crossed Leibniz \mathfrak{g} -algebra. It turns out that the functor $- \star_{\mathfrak{g}} \mathfrak{N}$ is left adjoint to the functor $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{N}, -)$. Another characterization of this tensor product is the following. If the Leibniz algebra \mathfrak{g} is perfect (and free as a \mathbb{K} -module), then the Leibniz algebra $\mathfrak{g} \star \mathfrak{g}$ is the universal central extension of \mathfrak{g} (see [4]). We give also low-degrees (co)homological interpretations of these objects, which yield an exact sequence of \mathbb{K} -modules

$$\begin{aligned} A/[A, A] \otimes \mathrm{HH}_1(A) \oplus \mathrm{HH}_1(A) \otimes A/[A, A] &\rightarrow \mathfrak{H}\mathcal{L}_1(\mathfrak{A}, L(\mathfrak{A})) \\ &\rightarrow \mathfrak{H}\mathcal{L}_1(\mathfrak{A}, [\mathfrak{A}, \mathfrak{A}]) \rightarrow \mathrm{HH}_1(\mathfrak{A}) \rightarrow \mathrm{HH}_1^{\mathrm{M}}(\mathfrak{A}) \rightarrow [\mathfrak{A}, \mathfrak{A}]/[\mathfrak{A}, [\mathfrak{A}, \mathfrak{A}]] \rightarrow 0 \end{aligned}$$

where $L(A)$ is the \mathbb{K} -module $A \otimes A / \mathrm{im}(b_3)$ equipped with a suitable Leibniz bracket (see section 1.2).

Throughout this paper the symbol \mathbb{K} denotes a commutative ring with a unit element and \otimes stands $\otimes_{\mathbb{K}}$.

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1. Prerequisites on Leibniz algebras.

1.1. Leibniz algebras.

A *Leibniz algebra* is a \mathbb{K} -module \mathfrak{g} equipped with a bilinear map $[-, -] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, called *bracket* and satisfying only the *Leibniz identity*

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

for any $x, y, z \in \mathfrak{g}$. In the presence of the condition $[x, x] = 0$, the Leibniz identity is equivalent to the so-called *Jacobi identity*. Therefore Lie algebras are examples of Leibniz algebras.

A *morphism* of Leibniz algebras is a linear map $f : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ such that

$$f([x, y]) = [f(x), f(y)]$$

for any $x, y \in \mathfrak{g}_1$. It is clear that Leibniz algebras and their morphisms form a category that we denote by **(Leib)**.

A *two-sided ideal* of a Leibniz algebra \mathfrak{g} is a submodule \mathfrak{h} such that $[x, y] \in \mathfrak{h}$ and $[y, x] \in \mathfrak{h}$ for any $x \in \mathfrak{h}$ and any $y \in \mathfrak{g}$. For any two-sided ideal \mathfrak{h} in \mathfrak{g} , the quotient module $\mathfrak{g}/\mathfrak{h}$ inherits a structure of Leibniz algebra induced by the bracket of \mathfrak{g} . In particular, let $([x, x])$ be the two-sided ideal in \mathfrak{g} generated by all brackets $[x, x]$. The Leibniz algebra $\mathfrak{g}/([x, x])$ is in fact a Lie algebra, said *canonically associated* to \mathfrak{g} and is denoted by $\mathfrak{g}_{\text{Lie}}$.

Let \mathfrak{g} be a Leibniz algebra. Denote by $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ the submodule generated by all brackets $[x, y]$. The Leibniz algebra \mathfrak{g} is said to be *perfect* if $\mathfrak{g}' = \mathfrak{g}$. It is clear that any submodule of \mathfrak{g} containing \mathfrak{g}' is a two-sided ideal in \mathfrak{g} .

1.2. Examples.

Let M be a representation of a Lie algebra \mathfrak{g} (the action of \mathfrak{g} on M being denoted by m^g for $m \in M$ and $g \in \mathfrak{g}$). For any \mathfrak{g} -equivariant map $\mu : M \rightarrow \mathfrak{g}$, the bracket given by $[m, m'] := m^{\mu(m')}$ induces a structure of Leibniz (non-Lie) algebra on M . Observe that any Leibniz algebra \mathfrak{g} can be obtained in such a way by taking the canonical projection $\mathfrak{g} \rightarrow \mathfrak{g}_{\text{Lie}}$ (which is obviously $\mathfrak{g}_{\text{Lie}}$ -equivariant).

Let A be an associative algebra and let $b_3 : A^{\otimes 3} \rightarrow A^{\otimes 2}$ be the Hochschild boundary that is, the linear map defined by

$$b_3(a \otimes b \otimes c) := ab \otimes c - a \otimes bc + ca \otimes b, \quad a, b, c \in A.$$

Then the bracket given by

$$[a \otimes b, c \otimes d] := (ab - ba) \otimes (cd - dc), \quad a, b, c, d \in A,$$

defines a structure of Leibniz algebra on the \mathbb{K} -module $L(A) := A^{\otimes 2} / \text{im}(b_3)$. Moreover, we have an exact sequence of \mathbb{K} -modules

$$0 \rightarrow \text{HH}_1(A) \rightarrow L(A) \xrightarrow{b_2} A \rightarrow \text{HH}_0(A)$$

where $\text{HH}_*(A)$ denotes the Hochschild homology groups and $b_2(x, y) = [x, y] := xy - yx$ for any $x, y \in A$.

1.3. Free Leibniz algebra.

Let V be a \mathbb{K} -module and let $\overline{T}(V) := \bigoplus_{n \geq 1} V^{\otimes n}$ be the reduced tensor module. The bracket defined inductively by

$$\begin{aligned} [x, v] &= x \otimes v, \quad \text{if } x \in \overline{T}(V) \text{ and } v \in V \\ [x, y \otimes v] &= [x, y] \otimes v - [x \otimes v, y], \quad \text{if } x, y \in \overline{T}(V) \text{ and } v \in V, \end{aligned}$$

satisfies the Leibniz identity. The Leibniz algebra so defined is the *free Leibniz algebra* over V and is denoted by $\mathcal{F}(V)$ (see [8]). Observe that one has

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n = [\cdots [[v_1, v_2], v_3] \cdots v_n], \quad \forall v_1, \dots, v_n \in V.$$

Moreover, the *free Lie algebra* over V is nothing but the Lie algebra $\mathcal{F}(V)_{\text{Lie}}$.

2. Crossed Leibniz algebras.

2.1. Leibniz action.

Let \mathfrak{g} and \mathfrak{M} be Leibniz algebras. A *Leibniz action* of \mathfrak{g} on \mathfrak{M} is a couple of bilinear maps

$$\mathfrak{g} \times \mathfrak{M} \rightarrow \mathfrak{M}, \quad (\mathfrak{g}, m) \mapsto {}^{\mathfrak{g}}m \quad \text{and} \quad \mathfrak{M} \times \mathfrak{g} \rightarrow \mathfrak{M}, \quad (m, \mathfrak{g}) \mapsto m^{\mathfrak{g}}$$

satisfying the axioms

- i) $m^{[g, g']} = (m^g)^{g'} - (m^{g'})^g,$
- ii) $[g, g']m = ({}^g m)^{g'} - g(m^{g'}),$

- iii) $g(g' m) = -g(mg')$,
- iv) $g[m, m'] = [g m, m'] - [g m', m]$,
- v) $[m, m']^g = [m^g, m'] + [m, m'^g]$,
- vi) $[m, {}^g m'] = -[m, m'^g]$

for any $m, m' \in \mathfrak{M}$ and $g, g' \in \mathfrak{g}$. We say that \mathfrak{M} is a *Leibniz \mathfrak{g} -algebra*. Observe that the axiom i) applied to the triples $(m; g, g')$ and $(m; g', g)$ yields the relation

$$m^{[g, g']} = -m^{[g', g]}.$$

2.2. Examples.

Any two-sided ideal of a Leibniz algebra \mathfrak{g} is a Leibniz \mathfrak{g} -algebra, the action being given by the initial bracket.

A \mathbb{K} -module M equipped with two operations of a Leibniz algebra \mathfrak{g} satisfying the axioms i), ii) and iii) is called a *representation of \mathfrak{g}* (see [8]). Therefore representations of a Leibniz algebra \mathfrak{g} are abelian Leibniz \mathfrak{g} -algebras.

2.3. Crossed Leibniz algebras.

Let \mathfrak{g} be a Leibniz algebra. A *pre-crossed Leibniz \mathfrak{g} -algebra* is a Leibniz \mathfrak{g} -algebra \mathfrak{M} equipped with a morphism of Leibniz algebras $\mu : \mathfrak{M} \rightarrow \mathfrak{g}$ such that

$$\mu({}^g m) = [g, \mu(m)] \quad \text{and} \quad \mu(m^g) = [\mu(m), g]$$

for any $g \in \mathfrak{g}$ and $m \in \mathfrak{M}$. Moreover if the relations

$$\mu^{(m)} m' = [m, m'] \quad \text{and} \quad m^{\mu(m')} = [m, m'], \quad \forall m, m' \in \mathfrak{M},$$

hold, then (\mathfrak{M}, μ) is called a *crossed Leibniz \mathfrak{g} -algebra*.

2.4. Examples.

Any Leibniz algebra \mathfrak{g} , equipped with the identity map $\text{id}_{\mathfrak{g}}$, is a crossed Leibniz \mathfrak{g} -algebra.

Any two-sided ideal \mathfrak{h} of a Leibniz algebra \mathfrak{g} , equipped with the inclusion map $\mathfrak{h} \hookrightarrow \mathfrak{g}$, is a crossed Leibniz \mathfrak{g} -algebra.

Let $\alpha : \mathfrak{c} \twoheadrightarrow \mathfrak{g}$ be a central extension of Leibniz algebras (i.e., a surjective morphism whose kernel is contained in the centre of \mathfrak{c} , see [4]). Define operations of \mathfrak{g} on \mathfrak{c} by

$${}^g\mathfrak{c} := [\alpha^{-1}(g), c] \quad \text{and} \quad \mathfrak{c}^g := [c, \alpha^{-1}(g)]$$

where $\alpha^{-1}(g)$ is any pre-image of g in \mathfrak{c} . Then (\mathfrak{c}, α) is a crossed Leibniz \mathfrak{g} -algebra.

PROPOSITION 2.1. — *For any pre-crossed Leibniz \mathfrak{g} -algebra (\mathfrak{M}, μ) , the image $\text{im}(\mu)$ (resp. the kernel $\ker(\mu)$) is a two-sided ideal in \mathfrak{g} (resp. \mathfrak{M}). Moreover, if (\mathfrak{M}, μ) is crossed, then $\ker(\mu)$ is contained in the centre of \mathfrak{M} .*

Proof. — Let m be an element of \mathfrak{M} . For any $g \in \mathfrak{g}$, we have

$$[\mu(m), g] = \mu(m^g) \in \text{im}(\mu) \quad \text{and} \quad [g, \mu(m)] = \mu({}^g m) \in \text{im}(\mu).$$

Thus, $\text{im}(\mu)$ is a two-sided ideal in \mathfrak{g} . Assume that $m \in \ker(\mu)$; then for any $m' \in \mathfrak{M}$, we have

$$\mu([m, m']) = [\mu(m), \mu(m')] = 0 = [\mu(m'), \mu(m)] = \mu([m', m]).$$

Therefore $\ker(\mu)$ is a two-sided ideal in \mathfrak{M} . Moreover if the Leibniz action of \mathfrak{g} on \mathfrak{M} is crossed, then we have

$$[m, m'] = \mu^{(m)} m' = 0 = m' \mu^{(m)} = [m', m]$$

for any $m \in \ker(\mu)$ and $m' \in \mathfrak{M}$. Thus $\ker(\mu)$ is contained in the centre of \mathfrak{M} . □

2.5. Morphism of pre-crossed Leibniz algebras.

Let \mathfrak{g} be a Leibniz algebra and let (\mathfrak{M}, μ) and (\mathfrak{N}, ν) be pre-crossed Leibniz \mathfrak{g} -algebras. A *morphism* from (\mathfrak{M}, μ) to (\mathfrak{N}, ν) is a Leibniz algebra morphism $f : \mathfrak{M} \rightarrow \mathfrak{N}$ such that

$$f({}^g m) = {}^g(f(m)), \quad f(m^g) = (f(m))^g \quad \text{and} \quad \mu = \nu f$$

for any $m \in \mathfrak{M}$ and $g \in \mathfrak{g}$. A morphism of crossed Leibniz \mathfrak{g} -algebras is the same as a morphism of pre-crossed Leibniz \mathfrak{g} -algebras. It is clear that pre-crossed (resp. crossed) Leibniz \mathfrak{g} -algebras and their morphisms form a category that we denote by $(\mathbf{pc}\text{-Leib}(\mathfrak{g}))$ (resp. $(\mathbf{c}\text{-Leib}(\mathfrak{g}))$).

PROPOSITION 2.2. — *Let $f : (\mathfrak{M}, \mu) \rightarrow (\mathfrak{N}, \nu)$ be a crossed Leibniz \mathfrak{g} -algebra morphism. Then (\mathfrak{M}, f) is a crossed Leibniz \mathfrak{N} -algebra via the Leibniz action of \mathfrak{N} on \mathfrak{M} given by*

$${}^n m := \nu^{(n)} m \quad \text{and} \quad m^n := m^{\nu^{(n)}}, \quad \forall m \in \mathfrak{M}, n \in \mathfrak{N}.$$

Proof. — One easily checks that \mathfrak{M} is a Leibniz \mathfrak{N} -algebra. For any $m, m' \in \mathfrak{M}$ and $n \in \mathfrak{N}$, we have

$$f({}^n m) = f(\nu^{(n)} m) = \nu^{(n)} f(m) = [n, f(m)],$$

$$f(m^n) = f(m^{\nu^{(n)}}) = f(m)^{\nu^{(n)}} = [f(m), n];$$

thus (\mathfrak{M}, f) is a pre-crossed Leibniz \mathfrak{N} -algebra. Moreover we have

$$f^{(m)} m' = \nu^{(f(m))} m' = \mu^{(m)} m' = [m, m'],$$

$$m^{f(m')} = m^{\nu^{(f(m'))}} = m^{\mu^{(m')}} = [m, m'];$$

thus (\mathfrak{M}, f) is a crossed Leibniz \mathfrak{N} -algebra. □

2.6. Exact sequences.

We say that a sequence

$$(\mathfrak{L}, \lambda) \xrightarrow{\alpha} (\mathfrak{M}, \mu) \xrightarrow{\beta} (\mathfrak{N}, \nu)$$

is exact in the category $(\mathbf{pc}\text{-Leib}(\mathfrak{g}))$ (resp. $(\mathbf{c}\text{-Leib}(\mathfrak{g}))$) if the sequence

$$\mathfrak{L} \xrightarrow{\alpha} \mathfrak{M} \xrightarrow{\beta} \mathfrak{N}$$

is exact as sequence of Leibniz algebras.

PROPOSITION 2.3. — *If the sequence*

$$(\mathfrak{L}, \lambda) \xrightarrow{\alpha} (\mathfrak{M}, \mu) \xrightarrow{\beta} (\mathfrak{N}, \nu)$$

is exact in the category $(\mathbf{pc}\text{-Leib}(\mathfrak{g}))$ (resp. $(\mathbf{c}\text{-Leib}(\mathfrak{g}))$), then the map λ is zero. Moreover if the Leibniz \mathfrak{g} -algebra (\mathfrak{L}, λ) is crossed, then the Leibniz algebra \mathfrak{L} is abelian.

Proof. — Indeed, since $\beta\alpha = 0$, we have $\lambda = \nu\beta\alpha = 0$. From whence $\ker(\lambda) = \mathfrak{L}$, and by Proposition 2.1, it is clear that the Leibniz algebra \mathfrak{L} is abelian. □

3. Biderivations of Leibniz algebras.

In this section, we fix a Leibniz algebra \mathfrak{g} .

3.1. Derivations and anti-derivations.

Let (\mathfrak{M}, μ) and (\mathfrak{N}, ν) be pre-crossed Leibniz \mathfrak{g} -algebras. A *derivation* from (\mathfrak{M}, μ) to (\mathfrak{N}, ν) is a linear map $d : \mathfrak{M} \rightarrow \mathfrak{N}$ such that

$$d([m, m']) = d(m)\mu(m') + \mu(m)d(m'), \quad \forall m, m' \in \mathfrak{M}.$$

An *anti-derivation* from (\mathfrak{M}, μ) to (\mathfrak{N}, ν) is a linear map $D : \mathfrak{M} \rightarrow \mathfrak{N}$ such that

$$D([m, m']) = D(m)\mu(m') - D(m')\mu(m), \quad \forall m, m' \in \mathfrak{M}.$$

3.2. Examples.

Let (\mathfrak{N}, ν) be a crossed Leibniz \mathfrak{g} -algebra and let n be any element of \mathfrak{N} . By the axiom iii) (resp. i)) of 2.1, the linear map

$$\mathfrak{g} \rightarrow \mathfrak{N}, \quad \mathfrak{g} \mapsto n \quad (\text{resp. } \mathfrak{g} \rightarrow \mathfrak{N}, \quad \mathfrak{g} \mapsto -n^{\mathfrak{g}})$$

is a derivation (resp. an anti-derivation) from $(\mathfrak{g}, \text{id}_{\mathfrak{g}})$ to (\mathfrak{N}, ν) .

3.3. Biderivations.

Let (\mathfrak{M}, μ) and (\mathfrak{N}, ν) be pre-crossed Leibniz \mathfrak{g} -algebras. We denote by $\text{Bider}_{\mathfrak{g}}(\mathfrak{M}, \mathfrak{N})$ the free \mathbb{K} -module generated by the triples (d, D, g) , where d (resp. D) is a derivation (resp. an anti-derivation) from (\mathfrak{M}, μ) to (\mathfrak{N}, ν) and g is an element of \mathfrak{g} such that

$$\begin{aligned} \nu(d(m)) &= \mu(m^g), \quad \nu(D(m)) = -\mu({}^g m), \\ {}^h d(m) &= {}^h D(m), \quad D(m^h) = -D({}^h m) \end{aligned}$$

for any $h \in \mathfrak{g}$ and $m \in \mathfrak{M}$.

PROPOSITION 3.1. — *If the Leibniz \mathfrak{g} -algebra (\mathfrak{M}, ν) is crossed, then there is a Leibniz algebra structure on the \mathbb{K} -module $\text{Bider}_{\mathfrak{g}}(\mathfrak{M}, \mathfrak{M})$ for the bracket defined by*

$$[(d, D, g), (d', D', g')] := (\delta, \Delta, [g, g'])$$

where

$$\delta(m) := d'(m^g) - d(m^{g'}) \quad \text{and} \quad \Delta(m) = -D(m^{g'}) - d'(g_m), \quad \forall m \in \mathfrak{M}.$$

Proof. — Let us show that the maps δ and Δ are respectively a derivation and an anti-derivation. Indeed, for any $m, m' \in \mathfrak{M}$, we have

$$\begin{aligned} \delta([m, m']) &= d'([m, m']^g) - d([m, m']^{g'}) \\ &= d'([m^g, m']) + d'([m, m'^g]) - d([m^{g'}, m']) - d([m, m'^{g'}]) \\ &= d'(m^g)^{\mu(m')} + \mu(m^g) d'(m') + d'(m)^{\mu(m'^g)} + \mu(m) d'(m'^g) \\ &\quad - d(m^{g'})^{\mu(m')} - \mu(m^{g'}) d(m') - d(m)^{\mu(m'^{g'})} \\ &\quad - \mu(m) d(m'^{g'}) \\ &= (d'(m^g) - d(m^{g'}))^{\mu(m')} + \mu(m) (d'(m'^g) - d(m'^{g'})) \\ &\quad + \nu(d(m)) d'(m') + d'(m)^{\nu(d(m'))} - \nu(d'(m)) d(m') \\ &\quad - d(m)^{\nu(d'(m'))} \\ &= \delta(m)^{\mu(m')} + \mu(m) \delta(m') + [d(m), d'(m')] \\ &\quad + [d'(m), d(m')] - [d'(m), d(m')] - [d(m), d'(m')] \\ &= \delta(m)^{\mu(m')} + \mu(m) \delta(m') \end{aligned}$$

and

$$\begin{aligned} \Delta([m, m']) &= -D([m, m']^{g'}) - d'(g[m, m']) \\ &= -D([m^{g'}, m']) - D([m, m'^g]) - d'([g_m, m']) + d'([g_m', m]) \\ &= -D(m^{g'})^{\mu(m')} + D(m')^{\mu(m^{g'})} - D(m)^{\mu(m'^{g'})} + D(m'^g)^{\mu(m)} \\ &\quad - d'(g_m)^{\mu(m')} - \mu(g_m) d'(m') + d'(g_m')^{\mu(m)} + \mu(g_m') d'(m) \\ &= (-D(m^{g'}) - d'(g_m))^{\mu(m')} - (-D(m'^g) - d'(g_m'))^{\mu(m)} \\ &\quad + D(m')^{\nu(d'(m))} - D(m)^{\nu(d'(m'))} + \nu(D(m)) d'(m') \\ &\quad - \nu(D(m')) d'(m) \\ &= \Delta(m)^{\mu(m')} - \Delta(m')^{\mu(m)} + [D(m'), d'(m)] \\ &\quad - [D(m), d'(m')] + [D(m), d'(m')] - [D(m'), d'(m)] \\ &= \Delta(m)^{\mu(m')} - \Delta(m')^{\mu(m)}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 \nu(\delta(m)) &= \nu(d'(m^g)) - \nu(d(m^{g'})) = \mu((m^g)^{g'}) - \mu((m^{g'})^g) = \mu(m^{[g,g']}), \\
 \nu(\Delta(m)) &= -\nu(D(m^{g'})) - \nu(d'({}^g m)) = \mu({}^g(m^{g'})) - \mu(({}^g m)^{g'}) = -\mu({}^{[g,g']}m), \\
 {}^h\delta(m) &= {}^h d'(m^g) - {}^h d(m^{g'}) = {}^h D'(m^g) - {}^h D(m^{g'}) \\
 &= -{}^h D'({}^g m) - {}^h D(m^{g'}) = -{}^h d'({}^g m) - {}^h D(m^{g'}) \\
 &= {}^h \Delta(m), \\
 \Delta({}^h m) &= -D(({}^h m)^{g'}) - d'({}^g({}^h m)) \\
 &= -D({}^{[h,g']}m) - D({}^h(m^{g'})) + d'({}^g(m^h)) \\
 &= D((m^h)^{g'}) + d'({}^g(m^h)) = -\Delta(m^h).
 \end{aligned}$$

Therefore the triple $(\delta, \Delta, [g, g'])$ is a biderivation from (\mathfrak{M}, μ) to (\mathfrak{N}, ν) . Moreover, let (d, D, g) , (d', D', g') and (d'', D'', g'') be biderivations from (\mathfrak{M}, μ) to (\mathfrak{N}, ν) . We set

$$\begin{aligned}
 (\delta, \Delta, [g', g'']) &:= [(d', D', g'), (d'', D'', g'')], \\
 (\delta_0, \Delta_0, g_0) &:= [(d, D, g), (\delta, \Delta, [g', g''])], \\
 (\delta', \Delta', [g, g']) &:= [(d, D, g), (d', D', g')], \\
 (\delta_1, \Delta_1, g_1) &:= [(\delta', \Delta', [g, g']), (d'', D'', g'')], \\
 (\delta'', \Delta'', [g, g'']) &:= [(d, D, g), (d'', D'', g'')], \\
 (\delta_2, \Delta_2, g_2) &:= [(\delta'', \Delta'', [g, g'']), (d', D', g')].
 \end{aligned}$$

It is clear that $g_0 = g_1 - g_2$. For any $m \in \mathfrak{M}$, we have

$$\begin{aligned}
 (\delta_1 - \delta_2)(m) &= d''(m^{[g,g']}) - \delta'(m^{g''}) - d'(m^{[g,g']}) + \delta''(m^{g'}) \\
 &= d''((m^g)^{g'}) - d''((m^{g'})^g) - d'((m^{g''})^g) + d((m^{g''})^{g'}) \\
 &\quad - d'((m^g)^{g''}) + d'((m^{g''})^g) + d''((m^{g'})^g) - d((m^{g'})^{g''}) \\
 &= d''((m^g)^{g'}) - d'((m^g)^{g''}) - d(m^{[g',g'']}) \\
 &= \delta(m^g) - d(m^{[g',g'']}) = \delta_0(m)
 \end{aligned}$$

and

$$\begin{aligned}
 (\Delta_1 - \Delta_2)(m) &= -\Delta'(m^{g''}) - d''({}^{[g,g']}m) + \Delta''(m^{g'}) + d'({}^{[g,g'']}m) \\
 &= D((m^{g''})^{g'}) + d'({}^g(m^{g''})) - d''(({}^g m)^{g'}) + d''({}^g(m^{g'})) \\
 &\quad - D((m^{g'})^{g''}) - d''({}^g(m^{g'})) + d'(({}^g m)^{g''}) - d'({}^g(m^{g''})) \\
 &= -D(m^{[g',g'']}) - d''(({}^g m)^{g'}) + d'(({}^g m)^{g''}) \\
 &= -D(m^{[g',g'']}) - \delta({}^g m) = \Delta_0(m).
 \end{aligned}$$

Therefore the \mathbb{K} -module $\text{Bider}_{\mathfrak{g}}(\mathfrak{M}, \mathfrak{N})$ is a Leibniz algebra. \square

Let us equip the set $\text{Bider}_{\mathfrak{g}}(\mathfrak{M}, \mathfrak{N})$ with a Leibniz action of \mathfrak{g} .

PROPOSITION 3.2. — *Let (\mathfrak{M}, μ) (resp. (\mathfrak{N}, ν)) be a pre-crossed (resp. crossed) Leibniz \mathfrak{g} -algebra. The set $\text{Bider}_{\mathfrak{g}}(\mathfrak{M}, \mathfrak{N})$ is a pre-crossed Leibniz \mathfrak{g} -algebra for the operations defined by*

$${}^h(d, D, g) := ({}^h d, {}^h D, [h, g]) \quad \text{and} \quad (d, D, g)^h := (d^h, D^h, [g, h])$$

where

$$\begin{aligned} ({}^h d)(m) &= d(m^h) - d(m)^h, & ({}^h D)(m) &:= {}^h d(m) - d({}^h m), \\ (d^h)(m) &:= d(m)^h - d(m^h), & (D^h)(m) &:= D(m)^h - D(m^h). \end{aligned}$$

Proof. — Everything can be smoothly checked and we merely give an example of these verifications. By definition we have

$$\begin{aligned} {}^h[(d, D, g), (d', D', g')] &= ({}^h \delta, {}^h \Delta, [h, [g, g']]), \\ [{}^h(d, D, g), (d', D', g')] &= (\delta_1, \Delta_1, [[h, g], g']), \\ [{}^h(d', D', g'), (d, D, g)] &= (\delta_2, \Delta_2, [[h, g'], g]). \end{aligned}$$

For any $m \in \mathfrak{M}$ we have

$$\begin{aligned} (\delta_1 - \delta_2)(m) &= d'(m^{[h, g]}) - ({}^h d)(m^{g'}) - d(m^{[h, g']}) + ({}^h d')(m^g) \\ &= d'((m^h)^g) - d'((m^g)^h) - d((m^{g'})^h) + d(m^{g'})^h \\ &\quad - d((m^h)^{g'}) + d((m^{g'})^h) + d'((m^g)^h) - d'(m^g)^h \\ &= (d'((m^h)^g) - d((m^h)^{g'})) - (d'(m^g) - d(m^{g'}))^h \\ &= \delta(m^h) - \delta(m)^h = ({}^h \delta)(m) \end{aligned}$$

and

$$\begin{aligned} (\Delta_1 - \Delta_2)(m) &= - ({}^h D)(m^{g'}) - d'({}^{[h, g]} m) + ({}^h D')(m^g) + d({}^{[h, g']} m) \\ &= - {}^h D(m^{g'}) + d({}^h(m^{g'})) - d'(({}^h m)^g) + d'({}^h(m^g)) \\ &\quad + {}^h D'(m^g) - d'({}^h(m^g)) + d(({}^h m)^{g'}) - d({}^h(m^{g'})) \\ &= {}^h(D'(m^g) - D(m^{g'})) - (d'(({}^h m)^g) - d(({}^h m)^{g'})) \\ &= {}^h \delta(m) - \delta({}^h m) = ({}^h \Delta)(m). \end{aligned}$$

Thus we get

$${}^h[(d, D, g), (d', D', g')] = [{}^h(d, D, g), (d', D', g')] - [{}^h(d', D', g'), (d, D, g)].$$

□

Now we can state the fundamental result which is a consequence of Propositions 3.1 and 3.2.

THEOREM 3.3. — For any pre-crossed (resp. crossed) Leibniz \mathfrak{g} -algebra (\mathfrak{M}, μ) (resp. (\mathfrak{N}, ν)), the Leibniz \mathfrak{g} -algebra $\text{Bider}_{\mathfrak{g}}(\mathfrak{M}, \mathfrak{N})$ is pre-crossed for the morphism

$$\rho : \text{Bider}_{\mathfrak{g}}(\mathfrak{M}, \mathfrak{N}) \rightarrow \mathfrak{g}, (\mathfrak{d}, \mathfrak{D}, \mathfrak{g}) \mapsto \mathfrak{g}. \quad \square$$

3.4. Remarks.

For any element g of \mathfrak{g} , the linear map $\text{ad}_g : h \mapsto [h, g]$ (resp. $\text{Ad}_g : h \mapsto -[g, h]$) is a derivation (resp. an anti-derivation) of the Leibniz algebra \mathfrak{g} . In the classical sense (i.e., without “crossing”, see [7]) the couple $(\text{ad}_g, \text{Ad}_g)$ is called *inner biderivation* of \mathfrak{g} . Therefore the pre-crossed Leibniz \mathfrak{g} -algebra $\text{Bider}_{\mathfrak{g}}(\mathfrak{M}, \mathfrak{N})$ can be seen as the set of biderivations from (\mathfrak{M}, μ) to (\mathfrak{N}, ν) over inner biderivations of \mathfrak{g} .

On the other hand, given a pre-crossed Leibniz \mathfrak{g} -algebra (\mathfrak{M}, μ) , one easily checks that the map $\text{Bider}_{\mathfrak{g}}(\mathfrak{M}, -)$ is a functor from the category of crossed Leibniz \mathfrak{g} -algebras to the category of pre-crossed Leibniz \mathfrak{g} -algebras.

4. Non-abelian tensor product of Leibniz algebras.

4.1. Leibniz pairings.

Let \mathfrak{M} and \mathfrak{N} be Leibniz algebras with mutual Leibniz actions on one another. A *Leibniz pairing* of \mathfrak{M} and \mathfrak{N} is a triple (\mathfrak{P}, h_1, h_2) where \mathfrak{P} is a Leibniz algebra and $h_1 : \mathfrak{M} \times \mathfrak{N} \rightarrow \mathfrak{P}$ (resp. $h_2 : \mathfrak{N} \times \mathfrak{M} \rightarrow \mathfrak{P}$) is a bilinear map such that

$$\begin{aligned} h_1(m, [n, n']) &= h_1(m^n, n') - h_1(m^{n'}, n), \\ h_2(n, [m, m']) &= h_2(n^m, m') - h_2(n^{m'}, m), \\ h_1([m, m'], n) &= h_2({}^m n, m') - h_1(m, n^{m'}), \\ h_2([n, n'], m) &= h_1({}^n m, n') - h_2(n, m^{n'}), \\ h_1(m, {}^{m'} n) &= -h_1(m, n^{m'}), \quad h_2(n, {}^{n'} m) = -h_2(n, m^{n'}), \\ h_1(m^n, {}^{m'} n') &= [h_1(m, n), h_1(m', n')] = h_2({}^m n, {}^{m'} m'), \\ h_1({}^n m, {}^{n'} m') &= [h_2(n, m), h_2(n', m')] = h_2(n^m, {}^{n'} m'), \\ h_1(m^n, n'^{m'}) &= [h_1(m, n), h_2(n', m')] = h_2({}^m n, n'^{m'}), \\ h_1({}^n m, m'^{n'}) &= [h_2(n, m), h_1(m', n')] = h_2(n^m, m'^{n'}) \end{aligned}$$

for any $m, m' \in \mathfrak{M}$ and $n, n' \in \mathfrak{N}$.

4.2. Example.

Let \mathfrak{M} and \mathfrak{N} be two-sided ideals of a same Leibniz algebra \mathfrak{g} . Take $\mathfrak{P} := \mathfrak{M} \cap \mathfrak{N}$ and define

$$h_1(m, n) := [m, n] \quad \text{and} \quad h_2(n, m) := [n, m].$$

Then the triple (\mathfrak{P}, h_1, h_2) is a Leibniz pairing of \mathfrak{M} and \mathfrak{N} .

4.3. Non-abelian tensor product.

A Leibniz pairing (\mathfrak{P}, h_1, h_2) of \mathfrak{M} and \mathfrak{N} is said to be *universal* if for any other Leibniz pairing $(\mathfrak{P}', h'_1, h'_2)$ of \mathfrak{M} and \mathfrak{N} there exists a unique Leibniz algebra morphism $\theta : \mathfrak{P} \rightarrow \mathfrak{P}'$ such that

$$\theta h_1 = h'_1 \quad \text{and} \quad \theta h_2 = h'_2.$$

It is clear that a universal pairing, when it exists, is unique up to a unique isomorphism. Here is a construction of the universal pairing as a *non-abelian tensor product*.

DEFINITION-THEOREM 4.1. — *Let \mathfrak{M} and \mathfrak{N} be Leibniz algebras with mutual Leibniz actions on one another. Let V be the free \mathbb{K} -module generated by the symbols $m * n$ and $n * m$ where $m \in \mathfrak{M}$ and $n \in \mathfrak{N}$. Let $\mathfrak{M} * \mathfrak{N}$ be the Leibniz algebra quotient of the free Leibniz algebra generated by V by the two-sided ideal defined by the relations*

i) $\lambda(m * n) = \lambda m * n = m * \lambda n, \lambda(n * m) = \lambda n * m = n * \lambda m,$

ii) $(m + m') * n = m * n + m' * n, (n + n') * m = n * m + n' * m,$
 $m * (n + n') = m * n + m * n', n * (m + m') = n * m + n * m',$

iii) $m * [n, n'] = m^n * n' - m^{n'} * n, n * [m, m'] = n^m * m' - n^{m'} * m,$
 $[m, m'] * n = {}^m n * m' - m * n^{m'}, [n, n'] * m = {}^n m * n' - n * m^{n'},$

iv) $m * {}^{m'} n = -m * n^{m'}, n * {}^{n'} m = -n * m^{n'},$

v) $m^n * {}^{m'} n' = [m * n, m' * n'] = {}^m n * m'^{n'},$
 $m^n * n'^{m'} = [m * n, n' * m'] = {}^m n * n'^{m'},$
 ${}^n m * n'^{m'} = [n * m, n' * m'] = n^m * n'^{m'}$
 ${}^n m * m'^{n'} = [n * m, m' * n'] = n^m * m'^{n'}$

for any $\lambda \in \mathbb{K}, m, m' \in \mathfrak{M}, n, n' \in \mathfrak{N}$. Define maps

$$h_1 : \mathfrak{M} \times \mathfrak{N} \rightarrow \mathfrak{M} * \mathfrak{N}, h_1(m, n) := m * n$$

and

$$h_2 : \mathfrak{N} \times \mathfrak{M} \rightarrow \mathfrak{M} \star \mathfrak{N}, \quad h_2(n, m) := n \star m.$$

Then the triple $(\mathfrak{M} \star \mathfrak{N}, h_1, h_2)$ is the universal Leibniz pairing of \mathfrak{M} and \mathfrak{N} and called the non-abelian tensor product (or tensor product for short) of \mathfrak{M} and \mathfrak{N} .

Proof. — It is straightforward to see that the triple $(\mathfrak{M} \star \mathfrak{N}, h_1, h_2)$ so-defined is a Leibniz pairing of \mathfrak{M} and \mathfrak{N} . For the universality, notice that if $(\mathfrak{P}, h'_1, h'_2)$ is another Leibniz pairing of \mathfrak{M} and \mathfrak{N} , then the map θ is necessarily given on generators by

$$\theta(m \star n) = h'_1(m, n) \quad \text{and} \quad \theta(n \star m) = h'_2(n, m)$$

for any $m \in \mathfrak{M}$ and $n \in \mathfrak{N}$. □

As an illustration of this construction, we give now a description of the non-abelian tensor product when the actions are trivial.

PROPOSITION 4.2. — *If the Leibniz algebras \mathfrak{M} and \mathfrak{N} act trivially on each other, then there is an isomorphism of abelian Leibniz algebras*

$$\mathfrak{M} \star \mathfrak{N} \cong \mathfrak{M}_{\text{ab}} \otimes \mathfrak{N}_{\text{ab}} \oplus \mathfrak{N}_{\text{ab}} \otimes \mathfrak{M}_{\text{ab}}$$

where $\mathfrak{M}_{\text{ab}} := \mathfrak{M}/[\mathfrak{M}, \mathfrak{M}]$ and $\mathfrak{N}_{\text{ab}} := \mathfrak{N}/[\mathfrak{N}, \mathfrak{N}]$.

Proof. — Recall that the underlying \mathbb{K} -module of the free Leibniz algebra generated by V is

$$\overline{\mathbb{T}}(V) = V \oplus V^{\otimes 2} \oplus \dots \oplus V^{\otimes n} \oplus \dots$$

Since the actions are trivial, the definition of the bracket on $\overline{\mathbb{T}}(V)$ and the relations v) enable us to see that $\mathfrak{M} \star \mathfrak{N}$ is an abelian Leibniz algebra and that the summands $V^{\otimes n}$ (for $n \geq 2$) are killed. Relations i) and ii) of 4.1 say that the \mathbb{K} -module $\mathfrak{M} \star \mathfrak{N}$ is the quotient of $\mathfrak{M} \otimes \mathfrak{N} \oplus \mathfrak{N} \otimes \mathfrak{M}$ by the relations iii). These later imply that $\mathfrak{M} \star \mathfrak{N}$ is the abelian Leibniz algebra $\mathfrak{M}_{\text{ab}} \otimes \mathfrak{N}_{\text{ab}} \oplus \mathfrak{N}_{\text{ab}} \otimes \mathfrak{M}_{\text{ab}}$. □

4.4. Compatible Leibniz actions.

Let \mathfrak{M} and \mathfrak{N} be Leibniz algebras with mutual Leibniz actions on one another. We say that these actions are *compatible* if we have

$$\begin{aligned} \binom{m}{n} m' &= [m^n, m'], & \binom{m}{n} n' &= [n^m, n'], \\ \binom{n}{m} m' &= [n^m, m'], & \binom{n}{m} n' &= [m^n, n'], \\ m \binom{m'}{n} &= [m, m'^n], & n \binom{n'}{m} &= [n, n'^m], \\ m \binom{n}{m'} &= [m, n^m'], & n \binom{m'}{n'} &= [n, m'n'] \end{aligned}$$

for any $m, m' \in \mathfrak{M}$ and $n, n' \in \mathfrak{N}$.

4.5. Examples.

If \mathfrak{M} and \mathfrak{N} are two-sided ideals of a same Leibniz algebra, then the actions (given by the initial bracket) are compatible.

Let (\mathfrak{M}, μ) and (\mathfrak{N}, ν) be pre-crossed Leibniz \mathfrak{g} -algebras. Then one can define a Leibniz action of \mathfrak{M} on \mathfrak{N} (resp. of \mathfrak{N} on \mathfrak{M}) by setting

$$\begin{aligned} {}^m n &:= \mu(m)n \quad \text{and} \quad n^m := n^{\mu(m)} \\ (\text{resp. } {}^n m &:= \nu(n)m \quad \text{and} \quad m^n := m^{\nu(n)}). \end{aligned}$$

If the Leibniz \mathfrak{g} -algebras (\mathfrak{M}, μ) and (\mathfrak{N}, ν) are crossed, then these Leibniz actions are compatible.

4.6. First crossed structure.

Let \mathfrak{M} and \mathfrak{N} be Leibniz algebras with mutual compatible actions on one another. Consider the operations of \mathfrak{M} on $\mathfrak{M} \star \mathfrak{N}$ given by

$$\begin{aligned} {}^m(m' \star n') &:= [m, m'] \star n' - {}^m n' \star m', \quad {}^m(n' \star m') := {}^m n' \star m' - [m, m'] \star n', \\ (m \star n)^{m'} &:= [m, m'] \star n + m \star n^{m'}, \quad (n \star m)^{m'} := n^{m'} \star m + n \star [m, m'] \end{aligned}$$

and those of \mathfrak{N} on $\mathfrak{M} \star \mathfrak{N}$ given by

$$\begin{aligned} {}^n(m' \star n') &:= {}^n m' \star n' - [n, n'] \star m', \quad {}^n(n' \star m') := [n, n'] \star m' - {}^n m' \star n', \\ (m \star n)^{n'} &:= m^{n'} \star n + m \star [n, n'], \quad (n \star m)^{n'} := [n, n'] \star m + n \star m^{n'} \end{aligned}$$

for any $m, m' \in \mathfrak{M}$ and $n, n' \in \mathfrak{N}$. Then we have

PROPOSITION 4.3. — *With the above operations, the map*

$$\begin{aligned} \mu : \mathfrak{M} \star \mathfrak{N} &\rightarrow \mathfrak{M}, \quad m \star n \mapsto m^n, \quad n \star m \mapsto {}^n m \\ (\text{resp. } \nu : \mathfrak{M} \star \mathfrak{N} &\rightarrow \mathfrak{N}, \quad m \star n \mapsto {}^m n, \quad n \star m \mapsto n^m) \end{aligned}$$

induces on $\mathfrak{M} \star \mathfrak{N}$ a structure of crossed Leibniz \mathfrak{M} -algebra (resp. \mathfrak{N} -algebra).

Proof. — Once again everything can be readily checked thanks to the compatibility conditions. For example we have

$$\begin{aligned} \mu^{(m \star n)}(m' \star n') &= m^n(m' \star n') = [m^n, m'] \star n' - (m^n) n' \star m' \\ &= (m^n) n' \star m' - m^n \star n' m' - (m^n) n' \star m' \\ &= m^n \star m' n = [m \star n, m' \star n'] \end{aligned}$$

for any $m, m' \in \mathfrak{M}$ and $n, n' \in \mathfrak{N}$. □

4.7. Second crossed structure.

Let (\mathfrak{M}, μ) and (\mathfrak{N}, ν) be pre-crossed Leibniz \mathfrak{g} -algebras, equipped with the mutual Leibniz actions given in Examples 4.5. One easily checks that the operations given by

$$\begin{aligned} {}^g(m * n) &:= {}^g m * n - {}^g n * m, & {}^g(n * m) &:= {}^g n * m - {}^g m * n, \\ (m * n)^g &:= m^g * n + m * n^g, & (n * m)^g &:= n^g * m + n * m^g, \end{aligned}$$

define a Leibniz action of \mathfrak{g} on $\mathfrak{M} * \mathfrak{N}$.

PROPOSITION 4.4. — *Let (\mathfrak{M}, μ) and (\mathfrak{N}, ν) be pre-crossed Leibniz \mathfrak{g} -algebras. Then the map $\eta : \mathfrak{M} * \mathfrak{N} \rightarrow \mathfrak{g}$ defined on generators by*

$$\eta(m * n) := [\mu(m), \nu(n)] \quad \text{and} \quad \eta(n * m) := [\nu(n), \mu(m)],$$

*confers to $\mathfrak{M} * \mathfrak{N}$ a structure of pre-crossed Leibniz \mathfrak{g} -algebra. Moreover, if one of the Leibniz \mathfrak{g} -algebras \mathfrak{M} or \mathfrak{N} is crossed, then the Leibniz \mathfrak{g} -algebra $\mathfrak{M} * \mathfrak{N}$ is crossed.*

Proof. — It is immediate to check that the map η passes to the quotient and defines a Leibniz algebra morphism. Moreover we have

$$\begin{aligned} \eta({}^g(m * n)) &= [\mu({}^g m), \nu(n)] - [\nu({}^g n), \mu(m)] \\ &= [[g, \mu(m)], \nu(n)] - [[g, \nu(n)], \mu(m)] \\ &= [g, [\mu(m), \nu(n)]] = [g, \eta(m * n)]; \\ \eta({}^g(n * m)) &= -\eta({}^g(m * n)) = -[g, \eta(m * n)] \\ &= -[g, [\mu(m), \nu(n)]] = [g, [\nu(n), \mu(m)]] = [g, \eta(n * m)]; \\ \eta((m * n)^g) &= [\mu(m^g), \nu(n)] + [\mu(m), \nu(n^g)] \\ &= [[\mu(m), g], \nu(n)] + [\mu(m), [\nu(n), g]] \\ &= [[\mu(m), \nu(n)], g] = [\eta(m * n), g]; \\ \eta((n * m)^g) &= [\nu(n^g), \mu(m)] + [\nu(n), \mu(m^g)] \\ &= [[\nu(n), g], \mu(m)] + [\nu(n), [\mu(m), g]] \\ &= [[\nu(n), \mu(m)], g] = [\eta(n * m), g]; \end{aligned}$$

thus $(\mathfrak{M} * \mathfrak{N}, \eta)$ is a pre-crossed Leibniz \mathfrak{g} -algebra. Assume that, for instance,

the Leibniz \mathfrak{g} -algebra \mathfrak{M} is crossed. Then we have

$$\begin{aligned} \eta^{(m \star n)}(m' \star n') &= [\mu^{(m), \nu^{(n)}}](m' \star n') = \mu^{(m^{\nu^{(n)}})}(m' \star n') \\ &= \mu^{(m^{\nu^{(n)}})}m' \star n' - \mu^{(m^{\nu^{(n)}})}n' \star m' \\ &= [m^{\nu^{(n)}}, m'] \star n' - \mu^{(m^{\nu^{(n)}})}n' \star m' \\ &= \mu^{(m^{\nu^{(n)}})}n' \star m' - m^{\nu^{(n)}} \star n' \mu^{(m')} - \mu^{(m^{\nu^{(n)}})}n' \star m' \\ &= m^{\nu^{(n)}} \star \mu^{(m')}n' = [m \star n, m' \star n'] \end{aligned}$$

and

$$\begin{aligned} (m \star n)^{\eta(m' \star n')} &= (m \star n)^{[\mu(m'), \nu(n')]} = (m \star n)^{\mu(m'^{\nu(n')})} \\ &= m \mu^{(m'^{\nu(n')})} \star n + m \star n \mu^{(m'^{\nu(n')})} \\ &= [m, m'^{\nu(n')}] \star n + m \star n \mu^{(m'^{\nu(n')})} \\ &= \mu^{(m)}n \star m'^{\nu(n')} - m \star n \mu^{(m'^{\nu(n')})} + m \star n \mu^{(m'^{\nu(n')})} \\ &= [m \star n, m' \star n']. \end{aligned}$$

By the same way, one easily gets

$$\begin{aligned} \eta^{(m \star n)}(n' \star m') &= [m \star n, n' \star m'], \quad (m \star n)^{\eta(n' \star m')} = [m \star n, n' \star m'], \\ \eta^{(n \star m)}(n' \star m') &= [n \star m, n' \star m'], \quad (n \star m)^{\eta(n' \star m')} = [n \star m, n' \star m'], \\ \eta^{(n \star m)}(m' \star n') &= [n \star m, m' \star n'], \quad (n \star m)^{\eta(m' \star n')} = [n \star m, m' \star n']. \end{aligned}$$

So we have proved that the Leibniz \mathfrak{g} -algebra $\mathfrak{M} \star \mathfrak{N}$ is crossed. □

4.8. Remark.

It is clear that if (\mathfrak{M}, μ) (resp. (\mathfrak{N}, ν)) is a crossed Leibniz \mathfrak{g} -algebra, then the map $\mathfrak{M} \star -$ (resp. $- \star \mathfrak{N}$) is a functor from the category of pre-crossed Leibniz \mathfrak{g} -algebras to the category of crossed Leibniz \mathfrak{g} -algebras.

PROPOSITION 4.5. — *Let (\mathfrak{N}, ν) be a crossed Leibniz \mathfrak{g} -algebra. The functor $F(-) := - \star \mathfrak{N}$ is a right exact functor from the category of pre-crossed Leibniz \mathfrak{g} -algebras to the category of crossed Leibniz \mathfrak{g} -algebras.*

Proof. — Taking into account Proposition 2.3, let

$$0 \rightarrow (\mathfrak{P}, \circ) \xrightarrow{f} (\Omega, \lambda) \xrightarrow{g} (\mathfrak{X}, \gamma) \rightarrow \circ$$

be an exact sequence of pre-crossed Leibniz \mathfrak{g} -algebras. Consider the sequence of Leibniz algebras

$$F(\mathfrak{P}) \xrightarrow{\mathfrak{F}(f)} \mathfrak{F}(\Omega) \xrightarrow{\mathfrak{F}(g)} \mathfrak{F}(\mathfrak{X}) \rightarrow \circ.$$

It is clear that the morphism $F(g)$ is surjective. Since the map $F(f)$ is a morphism of crossed Leibniz \mathfrak{g} -algebras, by Proposition 2.2, $(F(\mathfrak{P}), \mathfrak{F}(f))$ is a crossed Leibniz $F(\Omega)$ -algebra; and by Proposition 2.1, the image $\text{im } F(f)$ is a two-sided ideal in $F(\Omega)$. By composition we have $F(g)F(f) = F(gf) = 0$, which yields a factorisation

$$\overline{F(g)} : F(\Omega) / \text{im } \mathfrak{F}(f) \rightarrow \mathfrak{F}(\mathfrak{A}).$$

In fact, the morphism $\overline{F(g)}$ is an isomorphism. To see it, let us consider the map

$$\Gamma : F(\mathfrak{A}) \rightarrow \mathfrak{F}(\Omega) / \text{im } \mathfrak{F}(f)$$

given on generators by

$$\Gamma(r * n) := g^{-1}(r) * n \text{ mod } \text{im } F(f) \text{ and } \Gamma(n * r) := n * g^{-1}(r) \text{ mod } \text{im } F(f)$$

where $g^{-1}(r)$ is any pre-image of r in Ω . Indeed, if q and q' are two pre-images of r , then $q - q' = f(p)$ for some p in \mathfrak{P} . Therefore we have

$$\begin{aligned} q * m - q' * n &= (q - q') * n = f(p) * n = F(f)(p * n) \in \text{im } F(f), \\ n * q - n * q' &= n * (q - q') = n * f(p) = F(f)(n * p) \in \text{im } F(f); \end{aligned}$$

thus the map Γ is well-defined. One easily checks that Γ is a morphism of Leibniz algebras and inverse to $\overline{F(g)}$. □

5. Adjunction theorem.

In this section we show that, for any crossed Leibniz \mathfrak{g} -algebra (\mathfrak{N}, ν) , the functor $- * \mathfrak{N}$ is left adjoint to the functor $\text{Bider}_{\mathfrak{g}}(\mathfrak{N}, -)$. For technical reasons, we assume that the relations

$$iv) \quad m * \mu(m')n = -m * n\mu(m'), \quad n * \nu(n')m = -n * m\nu(n')$$

defining the tensor product $\mathfrak{M} * \mathfrak{N}$ are extended to the relations

$$iv)' \quad m * {}^g n = -m * n^g, \quad n * {}^g m = -n * m^g$$

for any $m, m' \in \mathfrak{M}$, $n, n' \in \mathfrak{N}$ and $g \in \mathfrak{g}$. To avoid confusion, we denote this later tensor product by $\mathfrak{M} *_{\mathfrak{g}} \mathfrak{N}$. For instance, the Leibniz \mathfrak{g} -algebras $\mathfrak{M} * \mathfrak{N}$ and $\mathfrak{M} *_{\mathfrak{g}} \mathfrak{N}$ coincide if the maps μ and ν are surjective.

THEOREM 5.1. — *Let (\mathfrak{M}, μ) be a pre-crossed Leibniz \mathfrak{g} -algebra and let (\mathfrak{N}, ν) and (\mathfrak{P}, λ) be crossed Leibniz \mathfrak{g} -algebras. There is an isomorphism of \mathbb{K} -modules*

$$\text{Hom}_{(\text{pc-Leib}(\mathfrak{g}))}(\mathfrak{M}, \text{Bider}_{\mathfrak{g}}(\mathfrak{N}, \mathfrak{P})) \cong \text{Hom}_{(\text{c-Leib}(\mathfrak{g}))}(\mathfrak{M} *_{\mathfrak{g}} \mathfrak{N}, \mathfrak{P}).$$

Proof. — Let $\phi \in \text{Hom}_{(\mathbf{pc}\text{-Leib}(\mathfrak{g}))}(\mathfrak{M}, \text{Bider}_{\mathfrak{g}}(\mathfrak{N}, \mathfrak{P}))$ and put $(d_m, D_m, g_m) := \phi(m)$ for $m \in \mathfrak{M}$. Notice that we have $g_m = \mu(m)$ thanks to the relation $\rho\phi = \mu$, where $\rho : \text{Bider}_{\mathfrak{g}}(\mathfrak{N}, \mathfrak{P}) \rightarrow \mathfrak{g}$ is the crossing morphism. We associate to ϕ the map $\Phi : \mathfrak{M} \star_{\mathfrak{g}} \mathfrak{N} \rightarrow \mathfrak{P}$ defined on generators by

$$\Phi(m * n) := -D_m(n) \quad \text{and} \quad \Phi(n * m) := d_m(n), \quad \forall m \in \mathfrak{M}, n \in \mathfrak{N}.$$

LEMMA 5.2. — *The map Φ is a morphism of crossed Leibniz \mathfrak{g} -algebras.*

Conversely, given an element $\sigma \in \text{Hom}_{(\mathbf{c}\text{-Leib}(\mathfrak{g}))}(\mathfrak{M} \star_{\mathfrak{g}} \mathfrak{N}, \mathfrak{P})$, we associate the map $\Sigma : \mathfrak{M} \rightarrow \text{Bider}_{\mathfrak{g}}(\mathfrak{N}, \mathfrak{P})$ defined by

$$\Sigma(m) := (\delta_m, \Delta_m, \mu(m)), \quad \forall m \in \mathfrak{M},$$

where

$$\delta_m(n) := \sigma(n * m) \quad \text{and} \quad \Delta_m(n) := -\sigma(m * n), \quad \forall n \in \mathfrak{N}.$$

LEMMA 5.3. — *The map Σ is a morphism of pre-crossed Leibniz \mathfrak{g} -algebras.*

It is clear that the maps $\phi \mapsto \Phi$ and $\sigma \mapsto \Sigma$ are inverse to each other, which proves the adjunction theorem. □

Proof of Lemma 5.2. — There is a lot of things to check in order to show that the map Φ is well-defined. Let us give some examples of these verifications. For any $m, m' \in \mathfrak{M}$, $n, n' \in \mathfrak{N}$ and $h \in \mathfrak{g}$, we have

$$\begin{aligned} \Phi({}^n m * n' - n * m^{n'}) &= -D_{\nu(n)m}(n') - d_{m\nu(n')}(n) \\ &= -({}^{\nu(n)}D_m)(n') - ((d_m)^{\nu(n')})(n) \\ &= -{}^{\nu(n)}D_m(n') + d_m({}^{\nu(n)}n') - d_m(n)^{\nu(n')} + d_m(n^{\nu(n')}) \\ &= -{}^{\nu(n)}d_m(n') + d_m([n, n']) - d_m(n)^{\nu(n')} + d_m([n, n']) \\ &= d_m([n, n']) = \Phi([n, n'] * m). \end{aligned}$$

We also compute

$$\begin{aligned} \Phi(m * {}^h n) &= -D_m({}^h n) = D_m(n^h) = -\Phi(m * n^h), \\ \Phi(n * {}^h m) &= d_{hm}(n) = ({}^h d_m)(n) = -((d_m)^h)(n) = -d_{m^h}(n) = -\Phi(n * m^h) \end{aligned}$$

and

$$\begin{aligned}
 \Phi(m^n * m' n') &= -D_{m^{\nu(n)}}(\mu(m') n') = -((D_m)^{\nu(n)})(\mu(m') n') \\
 &= -D_m(\mu(m') n')^{\nu(n)} + D_m((\mu(m') n')^{\nu(n)}) \\
 &= -D_m(\mu(m') n')^{\nu(n)} + D_m([\mu(m') n', n]) \\
 &= -D_m(n)^{\nu(\mu(m') n')} = D_m(n)^{\lambda(D_{m'}(n'))} \\
 &= [D_m(n), D_{m'}(n')] = [\Phi(m * n), \Phi(m' * n')] \\
 &= \Phi([m * n, m' * n']).
 \end{aligned}$$

Now let $m \in \mathfrak{M}$, $n \in \mathfrak{N}$ and $g \in \mathfrak{g}$. One has successively

$$\begin{aligned}
 \Phi({}^g(m * n)) &= \Phi({}^g m * n) - \Phi({}^g n * m) = -D_{g_m}(n) - d_m({}^g n) \\
 &= ({}^g D_m)(n) - d_m({}^g n) = -{}^g D_m(n) = {}^g \Phi(m * n), \\
 \Phi({}^g(n * m)) &= -\Phi({}^g(m * n)) = -{}^g \Phi(m * n) = {}^g D_m(n) = {}^g d_m(n) = {}^g \Phi(n * m), \\
 \Phi((m * n)^g) &= \Phi(m^g * n) + \Phi(m * n^g) = -D_{m^g}(n) - D_m(n^g) \\
 &= -((D_m)^g)(n) - D_m(n^g) = -D_m(n)^g = \Phi(m * n)^g, \\
 \Phi((n * m)^g) &= \Phi(n^g * m) + \Phi(n * m^g) = d_m(n^g) + d_{m^g}(n) \\
 &= d_m(n^g) + ((d_m)^g)(n) = d_m(n)^g = \Phi(n * m)^g; \\
 \lambda \Phi(m * n) &= -\lambda(D_m(n)) = \nu(\mu(m) n) = [\mu(m), \nu(n)] = \eta(m * n), \\
 \lambda \Phi(n * m) &= \lambda(d_m(n)) = \nu(n^{\mu(m)}) = [\nu(n), \mu(m)] = \eta(n * m).
 \end{aligned}$$

Therefore the map Φ is a morphism of crossed Leibniz \mathfrak{g} -algebras. □

Proof of Lemma 5.3. — Let us first show that $\Sigma(m)$ is a well-defined biderivation. For any $n, n' \in \mathfrak{N}$, we have

$$\begin{aligned}
 &\delta_m(n)^{\nu(n')} + \nu(n) \delta_m(n') \\
 &= \sigma(n * m)^{\nu(n')} + \nu(n) \sigma(n' * m) = \sigma((n * m)^{\nu(n')}) + \sigma(\nu(n)(n' * m)) \\
 &= \sigma(n^{\nu(n')} * m) + \sigma(n * m^{\nu(n')}) + \sigma(\nu(n) n' * m) - \sigma(\nu(n') m * n') \\
 &= 2\sigma([n, n'] * m) - \sigma(\nu(n) m * n' - n * m^{\nu(n')}) \\
 &= 2\sigma([n, n'] * m) - \sigma([n, n'] * m) = \sigma([n, n'] * m) = \delta_m([n, n']),
 \end{aligned}$$

thus δ_m is a derivation. Moreover, we have

$$\begin{aligned}
 &\Delta_m(n)^{\nu(n')} - \Delta_m(n')^{\nu(n)} \\
 &= -\sigma(m * n)^{\nu(n')} + \sigma(m * n')^{\nu(n)} = \sigma((m * n')^{\nu(n)}) - \sigma((m * n)^{\nu(n')}) \\
 &= \sigma(m^{\nu(n)} * n') + \sigma(m * n'^{\nu(n)}) - \sigma(m^{\nu(n')} * n) - \sigma(m * n^{\nu(n')}) \\
 &= \sigma(m^{\nu(n)} * n' - m^{\nu(n')} * n) - \sigma(m * \nu(n) n') - \sigma(m * n^{\nu(n')}) \\
 &= \sigma(m * [n, n']) - \sigma(m * [n, n']) - \sigma(m * [n, n']) \\
 &= -\sigma(m * [n, n']) = \Delta_m([n, n']),
 \end{aligned}$$

thus Δ_m is an anti-derivation. We have also

$$\begin{aligned} \lambda(\delta_m(n)) &= \lambda(\sigma(n * m)) = \eta(n * m) = [\nu(n), \mu(m)] = \nu(n^{\mu(m)}), \\ \lambda(\Delta_m(n)) &= -\lambda(\sigma(m * n)) = -\eta(m * n) = -[\mu(m), \nu(n)] = -\nu(\mu(m)n), \\ {}^h\delta_m(n) &= {}^h\sigma(n * m) = \sigma({}^h(n * m)) = -\sigma({}^h(m * n)) = -{}^h\sigma(m * n) = -{}^h\Delta_m(n), \\ \Delta_m({}^hn) &= -\sigma(m * {}^hn) = \sigma(m * n^h) = -\Delta_m(n^h). \end{aligned}$$

Therefore $\Sigma(m) = (\delta_m, \Delta_m, \mu(m))$ is a biderivation from (\mathfrak{N}, ν) to (\mathfrak{P}, λ) .

For any $h \in \mathfrak{g}$, $m \in \mathfrak{M}$ and $n \in \mathfrak{N}$, we have

$$\begin{aligned} ({}^h(\delta_m))(n) &= \delta_m(n^h) - \delta_m(n)^h = \sigma(n^h * m) - \sigma(n * m)^h \\ &= -\sigma(n * m^h) = \sigma(n * {}^hm) = \delta_{hm}(n), \end{aligned}$$

$$\begin{aligned} ({}^h(\Delta_m))(n) &= {}^h\Delta_m(n) - \Delta_m({}^hn) = {}^h\sigma(m * n) - \sigma({}^hn * m) \\ &= \sigma({}^hm * n) = \Delta_{hm}(n); \end{aligned}$$

and obviously $[h, \mu(m)] = \mu({}^hm)$, thus we have $\Sigma({}^hm) = {}^h\Sigma(m)$. On the other side, we have

$$\begin{aligned} ((\delta_m)^h)(n) &= \delta_m(n)^h - \delta_m(n^h) = \sigma(n * m)^h - \sigma(n^h * m) \\ &= \sigma(n * m^h) = \delta_{m^h}(n) \end{aligned}$$

and

$$\begin{aligned} ((\Delta_m)^h)(n) &= \Delta_m(n)^h - \Delta_m(n^h) = -\sigma(m * n)^h + \sigma(m * n^h) \\ &= -\sigma(m^h * n) = \Delta_{m^h}(n). \end{aligned}$$

Since $[\mu(m), h] = \mu(m^h)$, we get $\Sigma(m^h) = \Sigma(m)^h$. By definition of the map Σ , we have $\rho\Sigma(m) = \mu(m)$. Therefore the map Σ is a morphism of pre-crossed Leibniz \mathfrak{g} -algebras. \square

6. Cohomological characterizations.

6.1. Non-abelian Leibniz cohomology.

Let \mathfrak{g} be a Leibniz algebra viewed as the crossed Leibniz \mathfrak{g} -algebra $(\mathfrak{g}, \text{id}_{\mathfrak{g}})$, and let (\mathfrak{M}, μ) be a crossed Leibniz \mathfrak{g} -algebra. Given an element $m \in \mathfrak{M}$, we denote by d_m (resp. D_m) the derivation (resp. anti-derivation) $g \mapsto {}^g m$ (resp. $g \mapsto -m^g$) from $(\mathfrak{g}, \text{id}_{\mathfrak{g}})$ to (\mathfrak{M}, μ) , and by $\overline{\mu(m)} := \mu(m) \text{ mod } Z(\mathfrak{g})$, where $Z(\mathfrak{g})$ is the centre of \mathfrak{g} . One easily checks that the triple $(d_m, D_m, \overline{\mu(m)})$ is a well-defined element of $\text{Bider}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{M})$.

DEFINITION-PROPOSITION 6.1. — *Let \mathfrak{J} be the \mathbb{K} -module freely generated by the biderivations $(d_m, D_m, \overline{\mu(m)})$, $m \in \mathfrak{M}$. Then \mathfrak{J} is a two-sided ideal of $\text{Bider}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{M})$. The Leibniz algebra $\text{Bider}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{M})/\mathfrak{J}$ is denoted by $\mathfrak{L}^1(\mathfrak{g}, \mathfrak{M})$.*

Proof. — For any $m \in \mathfrak{M}$ and $(d, D, g) \in \text{Bider}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{M})$, we have

$$[(d, D, g), (d_m, D_m, \overline{\mu(m)})] = (\delta_m, \Delta_m, [g, \overline{\mu(m)}])$$

with

$$\begin{aligned} \delta_m(x) &= d_m([x, g]) - d([x, \overline{\mu(m)}]) = [x, g]m - d([x, \mu(m)]) \\ &= \mu(d(x))m - d(x)^{\mu(m)} - {}^x d(\mu(m)) \\ &= [d(x), m] - [d(x), m] - {}^x D(\mu(m)) \\ &= d_{m_1}(x) \end{aligned}$$

where $m_1 := -D(\mu(m))$,

$$\begin{aligned} \Delta_m(x) &= -D([x, \overline{\mu(m)}]) - d_m([g, x]) = -D([x, \mu(m)]) - [g, x]m \\ &= -D(x)^{\mu(m)} - D(\mu(m))^x + \mu(D(x))m \\ &= -[D(x), m] + D(\mu(m))^x + [D(x), m] \\ &= D_{m_1}(x), \end{aligned}$$

$$\mu(m_1) = -\mu(D(\mu(m))) = [g, \mu(m)] = [g, \overline{\mu(m)}];$$

thus we have $[(d, D, g), (d_m, D_m, \overline{\mu(m)})] \in \mathfrak{J}$. On the other side, we have

$$[(d_m, D_m, \overline{\mu(m)}), (d, D, g)] = (\delta'_m, \Delta'_m, [\overline{\mu(m)}, g])$$

with

$$\begin{aligned} \delta'_m(x) &= d([x, \overline{\mu(m)}]) - d_m([x, g]) = d([x, \mu(m)]) - [x, g]m \\ &= d(x)^{\mu(m)} + {}^x d(\mu(m)) - \mu(d(x))m \\ &= [d(x), m] + {}^x d(\mu(m)) - [d(x), m] \\ &= d_{m_2}(x) \end{aligned}$$

where $m_2 := d(\mu(m))$,

$$\begin{aligned} \Delta'_m(x) &= -D_m([x, g]) - d([\overline{\mu(m)}, x]) = m^{[x, g]} - d([\mu(m), x]) \\ &= m^{\mu(d(x))} - d(\mu(m))^x - \mu(m)d(x) \\ &= [m, d(x)] - d(\mu(m))^x - [m, d(x)] \\ &= D_{m_2}(x), \end{aligned}$$

$$\mu(m_2) = \mu(d(\mu(m))) = [\mu(m), g] = [\overline{\mu(m)}, g];$$

thus we have $[(d_m, D_m, \overline{\mu(m)}), (d, D, g)] \in \mathfrak{J}$. Therefore the set \mathfrak{J} is a two-sided ideal of $\text{Bider}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{M})$. □

Similarly, given a crossed Leibniz \mathfrak{g} -algebra (\mathfrak{M}, μ) , one defines

$$\mathfrak{H}\mathcal{L}^0(\mathfrak{g}, \mathfrak{M}) := \{m \in \mathfrak{M} : {}^{\mathfrak{g}}m = m^{\mathfrak{g}} = 0, \forall \mathfrak{g} \in \mathfrak{g}\}$$

that is, the set of invariant elements of \mathfrak{M} . From the relations

$$[m, m'] = m\mu(m') = 0 = \mu(m')m = [m', m], \quad m \in \mathfrak{H}\mathcal{L}^0(\mathfrak{g}, \mathfrak{M}), \quad m' \in \mathfrak{M},$$

it is clear that $\mathfrak{H}\mathcal{L}^0(\mathfrak{g}, \mathfrak{M})$ is contained in the centre of the Leibniz algebra \mathfrak{M} .

PROPOSITION 6.2. — *For any exact sequence of crossed Leibniz \mathfrak{g} -algebras*

$$0 \rightarrow (\mathfrak{A}, \circ) \xrightarrow{\alpha} (\mathfrak{B}, \lambda) \xrightarrow{\beta} (\mathfrak{C}, \mu) \rightarrow 0,$$

there exists an exact sequence of \mathbb{K} -modules

$$\begin{aligned} 0 \rightarrow \mathfrak{H}\mathcal{L}^0(\mathfrak{g}, \mathfrak{A}) \rightarrow \mathfrak{H}\mathcal{L}^0(\mathfrak{g}, \mathfrak{B}) \rightarrow \mathfrak{H}\mathcal{L}^0(\mathfrak{g}, \mathfrak{C}) \xrightarrow{\partial} \mathfrak{H}\mathcal{L}^1(\mathfrak{g}, \mathfrak{A}) \\ \rightarrow \mathfrak{H}\mathcal{L}^1(\mathfrak{g}, \mathfrak{B}) \xrightarrow{\beta^1} \mathfrak{H}\mathcal{L}^1(\mathfrak{g}, \mathfrak{C}) \end{aligned}$$

where β^1 is a Leibniz algebra morphism.

Proof. — Everything goes smoothly except the definition of the connecting homomorphism ∂ . Given an element $c \in \mathfrak{H}\mathcal{L}^0(\mathfrak{g}, \mathfrak{C})$, let $b \in \mathfrak{B}$ be any pre-image of c in \mathfrak{B} . For any $x \in \mathfrak{g}$, we have

$$\beta({}^x b) = {}^x c = 0 = c^x = \beta(b^x).$$

Thus the element ${}^x b$ (resp. b^x) is in $\ker(\beta) = \text{im}(\alpha)$. Since the morphism α is injective, the map $d^c : x \mapsto \alpha^{-1}({}^x b)$ (resp. $D^c : x \mapsto \alpha^{-1}(b^x)$) is a derivation (resp. an anti-derivation) from $(\mathfrak{g}, \text{id}_{\mathfrak{g}})$ to (\mathfrak{A}, \circ) . One easily checks that the triple $(d^c, D^c, 0)$ is a well-defined element of $\text{Bider}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{A})$ whose class in $\mathfrak{H}\mathcal{L}^1(\mathfrak{g}, \mathfrak{A})$ does not depend on the choice of the pre-image b . We put

$$\partial(c) := \text{class}(d^c, D^c, 0). \quad \square$$

6.2. Non-abelian Leibniz homology.

Let \mathfrak{g} be a Leibniz algebra viewed as the crossed Leibniz \mathfrak{g} -algebra $(\mathfrak{g}, \text{id}_{\mathfrak{g}})$, and let (\mathfrak{N}, ν) be a crossed Leibniz \mathfrak{g} -algebra.

DEFINITION-PROPOSITION 6.3. — *The map $\Psi_{\mathfrak{N}} : \mathfrak{N} \star \mathfrak{g} \rightarrow \mathfrak{N}$ given on generators by*

$$\Psi_{\mathfrak{N}}(n \star g) := n^g \quad \text{and} \quad \Psi_{\mathfrak{N}}(g \star n) := {}^g n, \quad g \in \mathfrak{g}, \quad n \in \mathfrak{N},$$

is a morphism of crossed Leibniz \mathfrak{g} -algebras. We define the low-degrees non-abelian homology of \mathfrak{g} with coefficients in \mathfrak{N} to be

$$\mathfrak{H}\mathcal{L}_0(\mathfrak{g}, \mathfrak{N}) := \text{coker } \Psi_{\mathfrak{N}} \quad \text{and} \quad \mathfrak{H}\mathcal{L}_1(\mathfrak{g}, \mathfrak{N}) := \ker \Psi_{\mathfrak{N}}.$$

Proof. — To see that the map $\Psi_{\mathfrak{N}}$ is a Leibniz algebra morphism is equivalent to the fact that the Leibniz action of \mathfrak{N} on \mathfrak{g} is well-defined. The definition of the crossing homomorphism $\eta_{\mathfrak{N}} : \mathfrak{N} \star \mathfrak{g} \rightarrow \mathfrak{g}$ implies that $\Psi_{\mathfrak{N}}$ is a morphism of crossed Leibniz \mathfrak{g} -algebras. □

PROPOSITION 6.4. — *For any exact sequence of crossed Leibniz \mathfrak{g} -algebras*

$$0 \rightarrow (\mathfrak{A}, \circ) \xrightarrow{\alpha} (\mathfrak{B}, \lambda) \xrightarrow{\beta} (\mathfrak{C}, \mu) \rightarrow \circ,$$

there exists an exact sequence of \mathbb{K} -modules

$$\begin{aligned} \mathfrak{H}\mathcal{L}_1(\mathfrak{g}, \mathfrak{A}) \rightarrow \mathfrak{H}\mathcal{L}_1(\mathfrak{g}, \mathfrak{B}) \rightarrow \mathfrak{H}\mathcal{L}_1(\mathfrak{g}, \mathfrak{C}) \xrightarrow{\partial} \mathfrak{H}\mathcal{L}_0(\mathfrak{g}, \mathfrak{A}) \rightarrow \mathfrak{H}\mathcal{L}_0(\mathfrak{g}, \mathfrak{B}) \\ \rightarrow \mathfrak{H}\mathcal{L}_0(\mathfrak{g}, \mathfrak{C}) \rightarrow \circ. \end{aligned}$$

Proof. — We know that the functor $-\star \mathfrak{g}$ is right exact (Proposition 4.5). Therefore Proposition 6.4 is nothing but the “snake-lemma” applied to diagram

$$\begin{array}{ccccccc} \mathfrak{A} \star \mathfrak{g} & \longrightarrow & \mathfrak{B} \star \mathfrak{g} & \longrightarrow & \mathfrak{C} \star \mathfrak{g} & \longrightarrow & 0 \\ & & \downarrow \Psi_{\mathfrak{A}} & & \downarrow \Psi_{\mathfrak{B}} & & \downarrow \Psi_{\mathfrak{C}} \\ 0 & \longrightarrow & \mathfrak{A} & \longrightarrow & \mathfrak{B} & \longrightarrow & \mathfrak{C} & \longrightarrow & 0 \end{array}$$

which is obviously commutative. □

6.3. Universal central extension.

Let \mathfrak{g} be a Leibniz algebra and let $\Psi := \Psi_{\mathfrak{g}}$ be the morphism defining the homology $\mathfrak{H}\mathcal{L}_*(\mathfrak{g}, \mathfrak{g})$. From the relations $v)$ of Definition-Theorem 4.1, it is clear that $\Psi : \mathfrak{g} \star \mathfrak{g} \rightarrow [\mathfrak{g}, \mathfrak{g}]$ is a central extension of Leibniz algebras (see [4]).

THEOREM 6.5. — *If the Leibniz algebra \mathfrak{g} is perfect and free as a \mathbb{K} -module, then the morphism $\Psi : \mathfrak{g} \star \mathfrak{g} \rightarrow [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ is the universal central extension of \mathfrak{g} . Moreover, we have an isomorphism of \mathbb{K} -modules*

$$\mathfrak{H}\mathcal{L}_1(\mathfrak{g}, \mathfrak{g}) \cong \text{HL}_2(\mathfrak{g}).$$

Proof. — It is enough to prove the universality of the central extension $\Psi : \mathfrak{g} \star \mathfrak{g} \rightarrow [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Let $\alpha : \mathfrak{C} \rightarrow \mathfrak{g}$ be a central extension of \mathfrak{g} . Since $\ker(\alpha)$ is central in \mathfrak{C} , the quantity $[\alpha^{-1}(x), \alpha^{-1}(y)]$ does not depend on the choice of the pre-images $\alpha^{-1}(x)$ and $\alpha^{-1}(y)$ where $x, y \in \mathfrak{g}$. One easily checks that the map $\phi : \mathfrak{g} \star \mathfrak{g} \rightarrow \mathfrak{C}$ given on generators by

$$\phi(x \star y) := [\alpha^{-1}(x), \alpha^{-1}(y)]$$

is a well-defined Leibniz algebra morphism such that $\alpha\phi = \Psi$. The uniqueness of the map ϕ follows from Lemma 2.4 of [4] since the perfectness of \mathfrak{g} implies that of $\mathfrak{g} \star \mathfrak{g}$:

$$x \star y = \left(\sum_i [x_i, x'_i] \right) \star \left(\sum_j [y_j, y'_j] \right) = \sum_{i,j} [x_i \star x'_i, y_j \star y'_j].$$

By definition we have $\mathfrak{H}\mathcal{L}_1(\mathfrak{g}, \mathfrak{g}) = \ker(\cdot)$. After [4] the kernel of the universal central extension of a Leibniz algebra \mathfrak{g} is canonically isomorphic to $\text{HL}_2(\mathfrak{g})$. Therefore we have

$$\mathfrak{H}\mathcal{L}_1(\mathfrak{g}, \mathfrak{g}) \cong \text{HL}_2(\mathfrak{g}). \quad \square$$

7. The Milnor-type Hochschild homology.

Let A be an associative algebra viewed as a Leibniz (in fact Lie) algebra for the bracket given by $[a, b] := ab - ba, a, b \in A$. Recall that the \mathbb{K} -module $L(A) := A^{\otimes 2} / \text{im}(b_3)$ is a Leibniz (non-Lie) algebra for the bracket defined by

$$[x \otimes y, x' \otimes y'] := (xy - yx) \otimes (x'y' - y'x'), \forall x, y, x', y' \in A.$$

PROPOSITION 7.1. — *The operations given by*

$$A \times L(A) \rightarrow L(A), \quad {}^a(x \otimes y) := [a, x] \otimes y - [a, y] \otimes x,$$

$$L(A) \times A \rightarrow L(A), \quad (x \otimes y)^a := [x, a] \otimes y + x \otimes [y, a]$$

confer to $L(A)$ a structure of Leibniz A -algebra. Moreover the map

$$\mu_A : L(A) \rightarrow A, \quad x \otimes y \mapsto [x, y] = xy - yx$$

equips $L(A)$ with a structure of crossed Leibniz A -algebra.

Proof. — The operations are well-defined since we have

$$\begin{aligned} \alpha(b_3(x \otimes y \otimes z)) &= b_3(ax \otimes y \otimes z - a \otimes z \otimes xy - za \otimes x \otimes y \\ &\quad + a \otimes yz \otimes x + a \otimes zx \otimes y - a \otimes y \otimes zx) \end{aligned}$$

and

$$\begin{aligned} (b_3(x \otimes y \otimes z))^a &= b_3(-ax \otimes y \otimes z + xy \otimes a \otimes z + x \otimes y \otimes za \\ &\quad - x \otimes a \otimes yz - zx \otimes a \otimes y - zx \otimes y \otimes a). \end{aligned}$$

One easily checks that the couple $(L(A), \mu_A)$ is a pre-crossed Leibniz A -algebra. Moreover we have

$$\begin{aligned} \mu_A(x \otimes y)(x' \otimes y') - [x \otimes y, x' \otimes y'] &= b_3([x, y] \otimes x' \otimes y' - [x, y] \otimes y' \otimes x') \\ (x \otimes y)^{\mu_A(x \otimes y)} - [x \otimes y, x' \otimes y'] &= b_3(x \otimes [x', y'] \otimes y - x \otimes y \otimes [x', y']). \end{aligned}$$

Thus the Leibniz A -algebra $(L(A), \mu_A)$ is crossed. □

It is clear that the inclusion map $[A, A] \hookrightarrow A$ induces a structure of crossed Leibniz A -algebra on the two-sided ideal $[A, A]$, and that the map $\mu_A : L(A) \rightarrow [A, A]$ is a morphism of crossed Leibniz A -algebras. Moreover we have an exact sequence of \mathbb{K} -modules

$$0 \rightarrow \text{HH}_1(A) \rightarrow L(A) \xrightarrow{\mu_A} [A, A] \rightarrow 0.$$

LEMMA 7.2. — *The Leibniz algebra A acts trivially on $\text{HH}_1(A)$.*

Proof. — One easily checks that

$$\alpha(x \otimes y) = a \otimes [x, y] + b_3(a \otimes x \otimes y - a \otimes y \otimes x) \equiv a \otimes [x, y] \text{ in } L(A)$$

and

$$(x \otimes y)^a = [x, y] \otimes a + b_3(x \otimes a \otimes y - x \otimes y \otimes a) \equiv [x, y] \otimes a \text{ in } L(A).$$

Therefore, if $\omega = \sum \lambda_i(x_i \otimes y_i) \in \text{HH}_1(A)$, that is $\sum \lambda_i[x_i, y_i] = 0$, then we have

$$\alpha\omega = \sum \lambda_i \alpha(x_i \otimes y_i) \equiv \sum \lambda_i(a \otimes [x_i, y_i]) \equiv a \otimes \sum \lambda_i[x_i, y_i] = 0$$

and

$$\omega^a = \sum \lambda_i(x_i \otimes y_i)^a \equiv \sum \lambda_i([x_i, y_i] \otimes a) \equiv (\sum \lambda_i[x_i, y_i]) \otimes a = 0$$

for any $a \in A$. □

As an immediate consequence, we get the following

COROLLARY 7.3. — *The sequence*

$$0 \rightarrow \text{HH}_1(A) \rightarrow L(A) \xrightarrow{\mu_A} [A, A] \rightarrow 0$$

is an exact sequence of crossed Leibniz A -algebras. □

We deduce from Proposition 6.4 an exact sequence of \mathbb{K} -modules

$$\begin{aligned} \mathfrak{H}\mathcal{L}_1(\mathfrak{A}, \text{HH}_1(\mathfrak{A})) &\rightarrow \mathfrak{H}\mathcal{L}_1(\mathfrak{A}, L(\mathfrak{A})) \rightarrow \mathfrak{H}\mathcal{L}_1(\mathfrak{A}, [\mathfrak{A}, \mathfrak{A}]) \rightarrow \\ &\rightarrow \mathfrak{H}\mathcal{L}_0(\mathfrak{A}, \text{HH}_1(\mathfrak{A})) \rightarrow \mathfrak{H}\mathcal{L}_0(\mathfrak{A}, L(\mathfrak{A})) \rightarrow \mathfrak{H}\mathcal{L}_0(\mathfrak{A}, [\mathfrak{A}, \mathfrak{A}]) \rightarrow 0. \end{aligned}$$

Since A and $\text{HH}_1(A)$ act trivially on each other, we have

$$\mathfrak{H}\mathcal{L}_0(\mathfrak{A}, \text{HH}_1(\mathfrak{A})) = \text{HH}_1(\mathfrak{A})$$

and

$$\mathfrak{H}\mathcal{L}_1(\mathfrak{A}, \text{HH}_1(\mathfrak{A})) = \mathfrak{A} \star \text{HH}_1(\mathfrak{A}) \cong \mathfrak{A}/[\mathfrak{A}, \mathfrak{A}] \otimes \text{HH}_1(\mathfrak{A}) \oplus \text{HH}_1(\mathfrak{A}) \otimes \mathfrak{A}/[\mathfrak{A}, \mathfrak{A}].$$

On the other hand, it is clear that

$$\mathfrak{H}\mathcal{L}_1(\mathfrak{A}, [\mathfrak{A}, \mathfrak{A}]) \cong [\mathfrak{A}, \mathfrak{A}]/[\mathfrak{A}, [\mathfrak{A}, \mathfrak{A}]].$$

Therefore we can state

THEOREM 7.4. — *For any associative algebra A with unit, there exists an exact sequence of \mathbb{K} -modules*

$$\begin{aligned} A/[A, A] \otimes \text{HH}_1(A) \oplus \text{HH}_1(A) \otimes A/[A, A] &\rightarrow \mathfrak{H}\mathcal{L}_1(\mathfrak{A}, L(\mathfrak{A})) \rightarrow \mathfrak{H}\mathcal{L}_1(\mathfrak{A}, [\mathfrak{A}, \mathfrak{A}]) \\ &\rightarrow \text{HH}_1(A) \rightarrow \text{HH}_1^M(A) \rightarrow [A, A]/[A, [A, A]] \rightarrow 0 \end{aligned}$$

where $\text{HH}_1^M(A)$ denotes the Milnor-type Hochschild homology of A .

Proof. — Recall that $\text{HH}_1^M(A)$ is defined to be the quotient of $A \otimes A$ by the relations

$$a \otimes [b, c] = 0, [a, b] \otimes c = 0, b_3(a \otimes b \otimes c) = 0$$

for any $a, b, c \in A$ (see [6, 10.6.19]). By definition $L(A) = A \otimes A / \text{im}(b_3)$ and from the proof of Lemma 7.2, we get

$$\Psi_{L(A)}(a * (x \otimes y)) = {}^a(x \otimes y) \equiv a \otimes [x, y]$$

and

$$\Psi_{L(A)}((x \otimes y) * a) = (x \otimes y)^a \equiv [x, y] \otimes a.$$

Therefore it is clear that $\mathfrak{H}\mathcal{L}_0(\mathfrak{A}, L(\mathfrak{A})) = \text{coker}(\mathcal{L}(\mathfrak{A}))$ is isomorphic to $\text{HH}_1^M(A)$. □

Remark. — The \mathbb{K} -modules $\mathrm{HH}_1(A)$ and $\mathrm{HH}_1^M(A)$ coincide when the associative algebra A is *superperfect* as a Leibniz algebra that is, $A = [A, A]$ and $\mathrm{HL}_2(A) = 0$. Also, if the associative algebra A is commutative, then we have

$$\mathrm{HH}_1(A) \cong \mathrm{HH}_1^M(A) \cong \Omega_{A|\mathbb{K}}^1.$$

Let us also mention that the Milnor-type Hochschild homology appears in the description of the obstruction to the stability

$$\mathrm{HL}_n(\mathfrak{gl}_{n-1}(A)) \rightarrow \mathrm{HL}_n(\mathfrak{gl}_n(A)) \rightarrow \mathrm{HH}_{n-1}^M(A) \rightarrow 0$$

where $\mathfrak{gl}_n(A)$ is the Lie algebra of matrices with entries in the associative algebra A (see [2], [6, 10.6.20]).

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