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NON-SUNADA GRAPHS

by Robert BROOKS[(1)]

In memory of Hubert Pesce

In [Su], Sunada presented a method for constructing pairs of non-isometric manifolds $\{M_1, M_2\}$ such that the eigenvalues of the Laplace operator satisfy

$$\lambda_i(M_1) = \lambda_i(M_2) \quad \text{for all } i.$$

His method was based on interpreting the isospectrality condition in terms of finite group theory: if G is a finite group acting freely on a manifold M^G, with M_1 and M_2 quotients of M^G by subgroups H_1 and H_2 respectively, then M_1 and M_2 will be isospectral if the induced representations of the trivial representation $\mathrm{ind}_{H_1}^G(\mathbf{1})$ and $\mathrm{ind}_{H_2}^G(\mathbf{1})$ are equivalent as G-representations, where "$\mathbf{1}$" denotes the trivial representation.

In [Pe2], Hubert Pesce raised the question of whether there might be a converse to the Sunada Theorem. It was known that in general two isospectral manifolds need not arise from the Sunada construction, and today one knows that there exist isospectral manifolds which even have different local geometry, see [GGSWW]. However, he was able to prove a "generic" converse to the Sunada Theorem, in the event that one knows that the manifolds have a common finite covering:

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THEOREM 0.1 (see [Pe2]). — *Let G be a finite group acting freely on a manifold M, and let $\mathcal{M}^G$ denote the metrics on M invariant under G. Let $\mathcal{S}^G$ denote the set of metrics $\boldsymbol{m}$ in $\mathcal{M}^G$ with the following property: if Γ_1 and Γ_2 are two subgroups of G such that the quotients $(\Gamma_1\backslash M,\boldsymbol{m})$ and $(\Gamma_2\backslash M,\boldsymbol{m})$ are isospectral, then $\mathrm{ind}_{\Gamma_1}^G(\mathbf{1})$ and $\mathrm{ind}_{\Gamma_2}^G(\mathbf{1})$ are equivalent G-representations. Then $\mathcal{S}^G$ is open and dense in $\mathcal{M}^G$.*

See [Pe1], [Pe3] for a detailed description, together with extensions to more general situations.

If Γ_1 and Γ_2 are two graphs, then one may define a Laplacian on them, and ask whether they are isospectral. There are many constructions of isospectral graphs which appear to have little to do with the Sunada construction, see [CDGT] for a survey. However, if we impose the condition that Γ_1 and Γ_2 be k-regular, which is a natural condition to impose from the point of view of geometry, then most of these constructions do not apply. An exception to this is the construction of Seidel switching [CDGT], which we will review in §3 below.

It is therefore meaningful to investigate the extent to which there is a converse to Sunada's Theorem in the context of k-regular graphs. We will find that the situation is somewhat delicate.

We first observe that any two k-regular graphs have a common finite covering, by the Leighton Theorem [Le], [AG]. Then the groups G, H_1, and H_2 which were introduced by assumption in Pesce's Theorem 0.1 are available for free in the setting of k-regular graphs.

We then provide a converse to the Sunada Theorem for graphs of the following form:

DEFINITION 0.1. — *Let G be a group of automorphisms of a graph Γ, and H_1 and H_2 two subgroups of G which act freely on Γ. For all paths γ in Γ, let $G_i(\gamma)$ be the set of g in G such that $g(\gamma)$ descends to a closed path on Γ/H_i, $i = 1,2$. The quadruple (Γ,G,H_1,H_2) satisfies the Sunada condition up to length n if, for all paths γ of length $\leq n$,*

$$\#\big(G_1(\gamma)\big) = \#\big(G_2(\gamma)\big).$$

We will then show in §2 below that

THEOREM 0.2. — *Let Γ_1 and Γ_2 be isospectral k-regular graphs. Then, for each n, there exists a graph $\Gamma^{(n)}$, a group $G^{(n)}$ of graph automorphisms* *of $\Gamma^{(n)}$, and two subgroups $H_1^{(n)}$ and $H_2^{(n)}$ of $G^{(n)}$ which act freely on $\Gamma^{(n)}$, such that*

(i) $\Gamma_1 = \Gamma^{(n)}/H_1^{(n)}$ *and* $\Gamma_2 = \Gamma^{(n)}/H_2^{(n)}$;

(ii) $(\Gamma^{(n)}, G^{(n)}, H_1^{(n)}, H_2^{(n)})$ *satisfies the Sunada condition up to length n.*

As a simple consequence, we have

COROLLARY 0.1. — *Two k-regular graphs are isospectral if and only if, for some (equivalently, any) $n > \#(\Gamma_1)$, there is a quadruple $(\Gamma^{(n)}, G^{(n)}, H_1^{(n)}, H_2^{(n)})$ satisfying* (i) *and* (ii) *above.*

Corollary 0.1 thus gives a necessary and sufficient characterization of isospectral graphs in terms of a "Sunada-like" condition. It remains, however, to ask whether we can replace this "Sunada-like" condition with the Sunada condition itself. To that end, we proceed in §3 and §4 below to present examples of pairs of graphs which are isospectral, but do not arise from the Sunada construction in a strong sense defined below. We leave open the question of whether they arise from a weaker Sunada construction. These examples are constructed from fairly straightforward examples of Seidel switching.

Acknowledgements. — Our thinking on these questions was very heavily influenced by the papers of Hubert Pesce [Pe1], [Pe2], [Pe3] on the analogous question in the Riemannian case. Indeed, as a member of the jury at Hubert's habilitation, we raised the issue of whether there were graph-theoretic analogues of his results.

We were shocked to hear of his untimely death so soon afterwards. We will greatly miss his influence on these and many other questions in the future.

We would also like to thank Alex Lubotzky for many helpful conversations, and in particular for pointing us in a number of the directions pursued in this paper, and Shahar Mozes for many helpful conversations. We would also like to thank Greg Quenell for his helpful remarks on an earlier version of this paper.

1. The Sunada condition.

Let G be a finite group, and H_1, H_2 two subgroups of G. Then the triple (G, H_1, H_2) is said to satisfy the *Sunada condition* if, for all $g \in G$,

$$\#([g] \cap H_1) = \#([g] \cap H_2),$$

where $[g]$ denote the conjugacy class of g in G.

There are a number of equivalent formulations of the Sunada condition. It is equivalent to the condition that

$$\mathrm{ind}_{H_1}^{G}(\mathbf{1}) = \mathrm{ind}_{H_2}^{G}(\mathbf{1}).$$

as G-representations.

Now let Γ_1 and Γ_2 be two k-regular graphs. According to the Leighton Theorem (*cf.* [Le], see also [AG] for the k-regular case, and [Br] for a discussion oriented towards present purposes), there is a common finite covering Γ of Γ_1 and Γ_2. We may pick Γ to carry a group of graph automorphisms G, such that

$$\Gamma_1 = \Gamma/H_1 \quad \text{and} \quad \Gamma_2 = \Gamma/H_2.$$

DEFINITION 1.1. — *Γ_1 and Γ_2 are weakly Sunada equivalent if there is a graph Γ, a group of graph automorphisms G of Γ, and subgroups H_1 and H_2 which act freely on Γ such that*

$$\Gamma_1 = \Gamma/H_1, \quad \Gamma_2 = \Gamma/H_2,$$

and the triple (G, H_1, H_2) satisfies the Sunada condition.

An important special case of this the following: if G is a group, with a choice of symmetric set of generators $\{g_1, \ldots, g_k\}$, the Cayley graph $\Gamma(G)$ is the graph whose vertices are the elements of G, with two vertices joined by an edge if they differ by left-multiplication by some g_i. Note that G acts on $\Gamma(G)$ on the right by graph automorphisms. G also acts on the vertices of $\Gamma(G)$ on the left, but this is not in general an action via graph automorphisms. For H a subgroup of G, the Schreier graph $\Gamma(G/H)$ is defined by

$$\Gamma(G/H) = \big(\Gamma(G)\big)/H.$$

DEFINITION 1.2. — Γ_1 and Γ_2 are *strongly Sunada equivalent* if there is a triple of groups (G, H_1, H_2) satisfying the Sunada condition such that

$$\Gamma_1 = \Gamma(G/H_1) \quad \text{and} \quad \Gamma_2 = \Gamma(G/H_2).$$

The Sunada Theorem for graphs then says:

THEOREM 1.1. — *Suppose that Γ_1 and Γ_2 are weakly Sunada equivalent. Then Γ_1 and Γ_2 are isospectral.*

Proof. — We begin by the observation that two k-regular graphs Γ_1 and Γ_2 are isospectral if, for all n, the number of closed paths of length n on Γ_1 is equal to the number of closed paths of length n on Γ_2. It is standard that if this condition is verified for all $n \leq \max(\#(\Gamma_1), \#(\Gamma_2))$, then it is verified for all n.

If γ is a closed path on Γ_1, we may lift it in $\#(H_1)$ ways to a not necessarily closed path on Γ. Conversely, a path on Γ arises as a lift of a closed path on Γ_1 if and only if its endpoints differ by an element of H_1.

Now let γ be a path of length n in Γ. We would like to count the number $N_{\Gamma_1}(\gamma)$ of closed paths on Γ_1 which lift to a translate $g(\gamma)$ of γ, and similarly for $N_{\Gamma_1}(\gamma)$.

We will begin by counting

$$L_{\Gamma_1}(\gamma) = \#\big(g \in G\colon g(\gamma) \text{ descends to a closed path in } \Gamma_1\big).$$

Denote by x_0 and x_1 the endpoints of γ. If x_1 is not equal to $g(x_0)$ for any $g \in G$, then clearly $L_{\Gamma_1}(\gamma)$ and hence $N_{\Gamma_1}(\gamma)$ are both 0.

If $x_1 = g(x_0)$, then the translate $g(\gamma)$ will descend to a closed path in Γ_1 provided that

$$hg(x_0) = g(x_1) = gg_0(x_0)$$

for some $h \in H_1$. If we denote by $S = S(x_0)$ the stabilizer of x_0 in G, then this will occur provided

$$g(g_0 s)g^{-1} \in H_1.$$

Thus,

$$L_{\Gamma_1}(\gamma) = \sum_{s \in S} \#\big(g\colon g(g_0 s)g^{-1} \in H_1\big).$$

On the other hand,

$$\#\big(g\colon g(g_0 s)g^{-1} \in H_1\big) = C_G(g_0 s)\,\#\,\big([g_0 s] \cap H_1\big),$$

where $C_G(g_0 s)$ denotes the centralizer of $g_0 s$ in G.

To obtain $N_{\Gamma_1}(\gamma)$, we must divide by $\#(H_1)$, and must also take into account that two distinct translates $g_1(\gamma)$ and $g_2(\gamma)$ may descend to the same closed path in Γ_1 which start at different points along the path. To count this last, let $g_1(\gamma)$ be a translate of γ which descends to a closed path on Γ_1, and let $\widetilde{g_1(\gamma)}$ be the full inverse image in Γ of the path to which $g_1(\gamma)$ descends in Γ_1. Let

$$F(g_1(\gamma)) = \#\{g \in G\colon g \text{ sends } \widetilde{g_1(\gamma)} \text{ to itself}\}\left[\frac{\text{length}(\gamma)}{\text{length}(\widetilde{g_1(\gamma)})}\right],$$

where the term in brackets is to correct for those elements of H_1 which fix γ, which we have already taken into account. Clearly, $F(g_1(\gamma))$ is independent of g_1. Then

$$N_{\Gamma_1}(\gamma) = \sum_{s\in S} \frac{C_G(g_0 s)\,\#\,([g_0 s] \cap H_1)}{F(g_1\gamma)\,\#\,(H_1)}.$$

Since $\#([g_0 s] \cap H_1) = \#([g_0 s] \cap H_2)$, we see that for all γ

$$N_{\Gamma_1}(\gamma) = N_{\Gamma_2}(\gamma).$$

To complete the proof, we partition the set of paths of length n into equivalence classes under the action of G, and pick one representative from each equivalence class. Summing over all the representatives gives that the number of closed paths of length n is the same for Γ_1 and Γ_2.

This completes the proof of the theorem. □

We remark that we could have proved this theorem in a somewhat dual manner. We could divide out the graph Γ by the action of G to obtain an "orbifold graph" Γ/G. We could then count orbifold lifts of closed orbifold paths of Γ/G to closed paths on Γ_1 and Γ_2. The Sunada condition would then insure that this count would be the same for the two graphs. Indeed, this would be the same calculation carried out above.

The problem with this approach is in making precise the notion of "orbifold graph." Our approach in this paper will be to avoid entering into the technicalities of "orbifold graphs," and in each instance replace the argument with an argument involving graphs with a non-free group action. Nonetheless, "orbifold graphs" form an important part of our thinking on these questions, and we will make use of this line of thought for motivational purposes.

It is not difficult to give a characterization of those graphs which are strongly Sunada equivalent. To state it, we recall the notion of a *coloring* of a graph, which is the assignment to each edge of either an ordered color and a direction or an unordered color, in such a way that each vertex has precisely one incoming and one outgoing edge of each ordered color, and one edge of each unordered color. See [Br] for a discussion.

Cayley graphs may be characterized as colored graphs admitting a vertex-transitive group of fixed-point-free color-preserving graph automorphisms. In general, a colored graph may be described as a covering graph of the one-point "orbifold" colored graph, where each unordered edge corresponds to an "orbifold edge," and one sees easily that Schreier graphs inherit colorings from the corresponding Cayley graphs. Indeed, given a colored graph, it is not difficult to write it as $\Gamma(G/H)$ for some G and H, showing that Schreier graphs and colored graphs are essentially the same thing.

The graphs Γ_1 and Γ_2 are then strongly Sunada equivalent if they admit colorings such that, for any pattern of colors, the number of closed paths of a given pattern on Γ_1 is the same as the number of closed paths of a given coloring on Γ_2. This can be seen by counting lifts of paths (all such paths are closed) in the corresponding one-point "orbifold graph" to closed paths in Γ_1 and Γ_2 respectively.

Alternatively, given colorings of Γ_1 and Γ_2, the corresponding coverings exhibit the fundamental groups of Γ_1 and Γ_2 as finite-index subgroups of the "orbifold fundamental group" of the one-point "orbifold graph." The graph Γ then is the covering of this "orbifold graph" corresponding to the largest normal subgroup contained in the intersection of the image of the fundamental group of Γ_1 and the fundamental group of Γ_2.

We remark that there is a similar characterization of weakly Sunada equivalent graphs. Namely, we may think of a non-free action of G as providing us with equivalence classes of colorings on the graph. Then two graphs are weakly Sunada equivalent if, for some equivalence class of colorings, the set of closed paths with a given equivalence class of patterns is the same for the two graphs. We leave the details of this to the interested reader.

2. A converse to the Sunada theorem.

In this section, we will prove

THEOREM 2.1. — *Let Γ_1 and Γ_2 be isospectral graphs. Then, for any n, there exist a graph $\Gamma^{(n)}$, a group $G^{(n)}$ of graph automorphisms of $\Gamma^{(n)}$, and two subgroups $H_1^{(n)}$ and $H_2^{(n)}$ of $G^{(n)}$ which act freely on $\Gamma^{(n)}$, such that*

(i) $\Gamma_1 = \Gamma^{(n)}/H_1$ *and* $\Gamma_2 = \Gamma^{(n)}/H_2$.

(ii) $\left(\Gamma^{(n)}, G^{(n)}, H_1^{(n)}, H_2^{(n)}\right)$ *satisfies the Sunada condition up to length n.*

The strategy of the proof may be explained simply, as follows: suppose that we could find a common covering $\Gamma^{(n)}$ of Γ_1 and Γ_2 and a group of graph automorphisms $G^{(n)}$ of $\Gamma^{(n)}$ with the property that, for any $k \leq n$, all the lifts of all closed paths of length k of Γ_1 and Γ_2 are orbit equivalent under $G^{(n)}$. Then the isospectrality condition forces the Sunada condition up to length n, since for each k there is only one $G^{(n)}$-orbit to consider.

We could consider by way of example the following pair of graphs, see figures 1 and 2 below, which are known to be strongly Sunada equivalent (see [BPP]). With the coloring shown, they are not in fact pattern equivalent, for the simple reason that we made a bad choice of what coloring to give the closed loops. We may think of this as saying that the covering of the one-point graph corresponding to the coloring sent the closed loops to the "wrong" closed loops. However, if we divide out by the automorphism of the one-point graph which interchanges the loops, then this difficulty disappears, and the graphs become Sunada equivalent up to length 1.

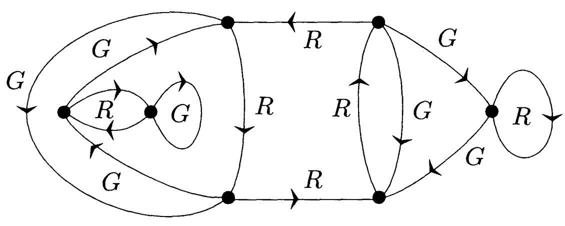

Figure 1. The first graph

We will show essentially that one may continue this argument inductively, by considering automorphisms which identify paths of longer length.

We now proceed with the proof.

We will need some results from [LMZ] concerning k-regular graphs and automorphisms of trees. To state them, recall that if Γ is a colored graph, we may identify its universal covering with a colored k-tree, with the fundamental group $\pi = \pi_1(\Gamma)$ acting on the tree as color-preserving transformations.

If π is any cocompact group acting as color-preserving transformations on the k-tree, we may associate to π its commensurator group $C(\pi)$, given by

$$C(\pi) = \{g \in \mathrm{Aut}(T_k)\colon g\pi g^{-1} \cap \pi \text{ is of finite index in } \pi \text{ and } g\pi g^{-1}\}.$$

It is easily seen that $C(\pi)$ is independent of π up to conjugacy, and will be denoted simply by C.

We then have

PROPOSITION (see [LMZ], Prop. 2.9). — *Let x and y be non-trivial elements of π. Then the following are equivalent*:

(a) *x and y are conjugate in $Aut(T_k)$.*

(b) *$\ell_T(x) = \ell_T(y)$, where $\ell_T(z)$ denotes the translation length along the axis of z.*

(c) *x and y are conjugate in C.*

The proof in [LMZ] shows more: if we let ϕ denote the element in C given in (c), then ϕ and π generate a group which contains π as a (non-normal) subgroup of finite index.

We now apply these considerations to the graph Γ which covers Γ_1 and Γ_2. After passing to a double covering if necessary, we may assume that Γ carries a coloring.

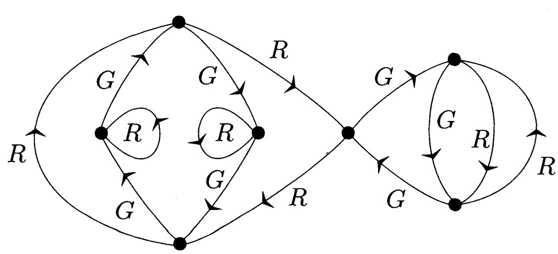

Figure 2. The second graph

If γ_1 and γ_2 are closed paths of the same length in Γ_1 and Γ_2 respectively, we may lift them to closed paths in Γ, which may multiply their lengths. They will then correspond to elements x_1 and x_2 of π, such that, for some m and n, x_1^n and x_2^m have the same translation length.

It follows from Proposition 2.1 that we may adjoin to π an element ϕ_{γ_1,γ_2} which conjugates x_1^n to x_2^m.

We continue in this way with all pairs of closed paths of the same length $\leq n$ in Γ_1 and Γ_2, to obtain a subgroup $\mathcal{G}^{(n)}$ of $\mathrm{Aut}(T_k)$ which contains π as a subgroup of finite index.

We would like to obtain from this a subgroup π' of finite index in π, such that π' is a normal subgroup of $\mathcal{G}^{(n)}$. But this is easily done: if π has index k in $\mathcal{G}^{(n)}$, then we may set π' to be the intersection of all subgroups of $\mathcal{G}^{(n)}$ of index k. This subgroup is clearly normal in $\mathcal{G}^{(n)}$, and is contained in π as a subgroup of finite index.

We may now choose $\Gamma^{(n)}$ to be T_k/π', $G^{(n)}$ the finite group $\mathcal{G}^{(n)}/\pi'$, and $H_i^{(n)}$ the subgroups of $G^{(n)}$ such that $\Gamma_i = \Gamma^{(n)}/H_i$.

This concludes the proof of the theorem. □

3. Seidel switching.

In this section, we describe a method, called *Seidel switching* [CDGT], which produces pairs of isospectral k-regular graphs. In the next section, we will give evidence that the pairs of graphs constructed this way do not arise from the Sunada construction.

For evidence of a different type, see [BGG], where Seidel switching is used to construct quite large families of mutually isospectral sets of graphs. See also [Qu] for interesting explicit constructions with Seidel switching.

The main construction is the following: let Γ_1 and Γ_2 be two graphs with the following properties:

(a) They have the same even number V of vertices.

(b) They are both k-regular, for the same value of k.

Let P be a collection of order pairs of vertices (v_1, v_2) with v_1 a vertex in Γ_1 and v_2 a vertex in Γ_2, with the following properties:

(i) For all $v_1 \in \Gamma_1$, the collection of v_2 such that $(v_1, v_2) \in P$ has cardinality $\frac{1}{2}V$.

(ii) For all $v_2 \in \Gamma_2$, the collection of $v_1 \in \Gamma_1$ such that $(v_1, v_2) \in P$ has cardinality $\frac{1}{2}V$.

We now construct graphs Δ_1 and Δ_2 as follows:

(i) The set of vertices of both Δ_1 and Δ_2 is the disjoint union of the set of vertices in Γ_1 and the set of vertices of Γ_2.

(ii) If v_1 and v_2 are two vertices which either both lie in Γ_1 or both lie in Γ_2, then the set of edges joining them in Δ_1 (resp. Δ_2) is the set of edges joining them in the union of Γ_1 and Γ_2.

(iii) Suppose $v_1 \in \Gamma_1$ and $v_2 \in \Gamma_2$. Then there is an edge joining v_1 to v_2 in Δ_1 if and only if $(v_1, v_2) \in P$.

(iv) Suppose $v_1 \in \Gamma_1$ and $v_2 \in \Gamma_2$. Then there is an edge joining v_1 to v_2 in Δ_2 if and only if $(v_1, v_2) \notin P$.

In other words, Δ_2 is constructed from Δ_1 by switching connections between Γ_1 and Γ_2 into non-connections, and *vice versa*.

The somewhat confusing conditions of the construction are illustrated in Figure 3 below, where Γ_1 is the 2-regular graph consisting of a circle of length 3 and a circle of length 1, while Γ_2 is a circle of length 2 and two circles of length 1. The set P is shown in the drawing as well.

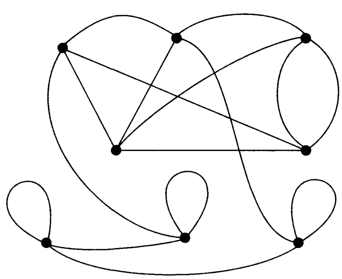

Figure 3. Seidel switching: the graphs Γ_1 and Γ_2

The resulting graphs Δ_1 and Δ_2 are shown in Figures 4 and 5 below.

It is not difficult to see that the graphs Δ_1 and Δ_2 are both k'-regular, where $k' = k + \frac{1}{2}V$. We now claim:

THEOREM 3.1 (see [CDGT]). — *The graphs Δ_1 and Δ_2 so constructed are isospectral.*

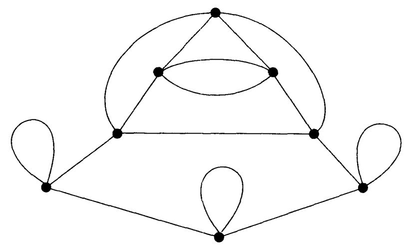

Figure 4. Seidel switching: the graph Δ_1

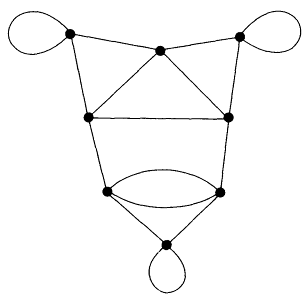

Figure 5. Seidel switching: the graph Δ_2

Proof. — The proof of this theorem is well-known, but is simple and elegant enough to present here.

Let γ be a closed path in Δ_1 (resp. Δ_2). We pick some vertex v_0 of γ as the starting vertex, and number all the edges e_i of γ in the order in which they occur as the path γ is traversed.

Let $\text{type}_i(\gamma)$ denote the set of closed paths γ' with edges e_i' on Δ_i with the following properties:

(i) If $e_i \in \Gamma_1$, then $e_i' = e_i$.

(ii) If $e_i \in \Gamma_2$, then $e_i' \in \Gamma_2$

(iii) If $e_i \notin \Gamma_1$ or Γ_2, then $e' \notin \Gamma_1$ or Γ_2.

The theorem will follow if we can show that $\text{type}_1\,\gamma = \text{type}_2(\gamma)$, since it will then follow that, for any n, the number of paths of length n in Δ_1 is equal to the number of paths of length n in Δ_2.

The assertion that $\text{type}_1(\gamma) = \text{type}_2(\gamma)$ will follow from the following assertion: given two vertices (possibly the same) v_1 and v_2 in Γ_1, and given a number ℓ (possibly 0), let R denote the set of paths in Γ_2 of length ℓ which begin at a vertex w such that $(v_1, w) \in P$, and let B denote the set of paths of length ℓ beginning at a point w such that $(v_1, w) \notin P$. Similarly, let G (resp. Y) denote the set of paths of length ℓ in Γ_2 which end at a point w such that $(v_2, w) \in P$ (resp. $(v_2, w) \notin P$). Then

CLAIM. — *One has* $\#(R \cap G) = \#(B \cap Y)$.

Proof. — We have

$$\#(R) = \#(R \cap G) + \#(R \cap Y),$$

while

$$\#(Y) = \#(R \cap Y) + \#(B \cap Y).$$

On the other hand,

$$\#(R) = \#(Y) = \left(\tfrac{1}{2} \# (\Gamma_2)\right) \cdot k \cdot (k-1)^{\ell-1},$$

from the k-regularity of Γ_2 and the condition on P.

This completes the proof of Theorem 3.1. □

4. The graphs Δ_1 and Δ_2.

In this section, we will show:

THEOREM 4.1. — *The graphs Δ_1 and Δ_2 are not strongly Sunada equivalent.*

The strategy of the proof is as follows: we will show that there is no coloring of the graphs Δ_1 and Δ_2 with the property that the number of closed loops of a given pattern is the same for both graphs. We will establish this by an enumeration of the various possibilities.

To aid in the argument, we present a numbering of the vertices in Figures 6 and 7 below. We will label an edge joining vertices v_1 and v_2 by $e(v_1, v_2)$.

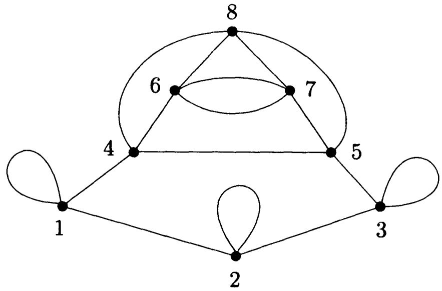

Figure 6. The graph Δ_1 with vertices labelled

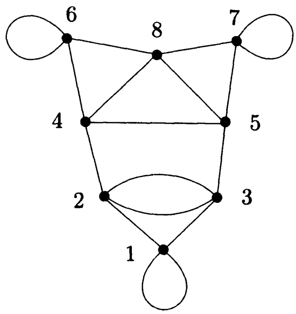

Figure 7. The graph Δ_2 with vertices labelled

Since the graphs are 4-regular, there are three possible types of colorings to consider:

Case (a): There are four unoriented colors.

Case (b): There is one oriented color (Red) and two unoriented colors (Green and Yellow).

Case (c): There are two oriented colors, Red and Green.

We may rule out case (a) immediately, since there are closed loops of length one on both graphs. If case (a) applied, then the two ends of such a loop would have to be given different colors, a contradiction.

We now consider case (b).

By the remarks above, all closed loops must be given the oriented color R.

We may now complete the coloring of the graph Δ_2, up to a small amount of ambiguity, as follows: $e(6,7)$ must be either G or Y. By symmetry, the choice doesn't matter, so we can choose G. Then $e(7,8)$ and $e(4,6)$ must be Y, and $e(5,8)$ must be G. Hence $e(4,7)$ and $e(5,7)$ must be R, with undetermined orientation.

It follows that $e(4,5)$ must be R, since 4 already has a Y and 5 already has a G.

We then have that $e(2,4)$ is G and $e(3,5)$ is Y, from which it follows that both edges $e(2,3)$ are R, with opposite orientation. Hence $e(1,2)$ is Y, and $e(1,3)$ is G.

Note that the directions of the R's at the vertices $4,5$, and 7 must be chosen so that going from 4 to 5 to 7 to 4 will always either agree or disagree with the direction or disagree, and the two choices correspond too changing G to Y and Y to G, and flipping the graph.

We show in Figure 8 below the resulting coloring of the graph Δ_2.

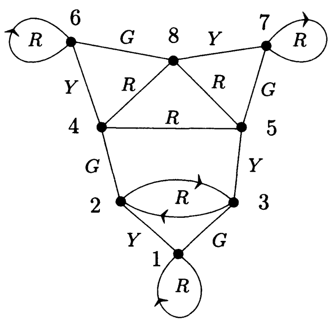

Figure 8. Case (b): The graph Δ_2

The coloring of the graph Δ_1 is not so completely determined, but we may choose $e(2,3)$ to be G, from which it follows that $e(1,2)$ and $e(3,5)$ are Y, and so $e(1,4)$ is G and $e(4,5)$ is R (with some direction). These colorings are given in Figure 9 below.

We now note that at this stage we already get a contradiction, because Δ_1 has at least one closed path of the form R^+GR+G, where "+" denotes crossing in the positive direction, which starts at vertex 2, whereas Δ_2 has no closed paths of this form for either choices of orientation for R in the triangle at points $4,5$, and 7.

This concludes case (b).

We now consider case (c).

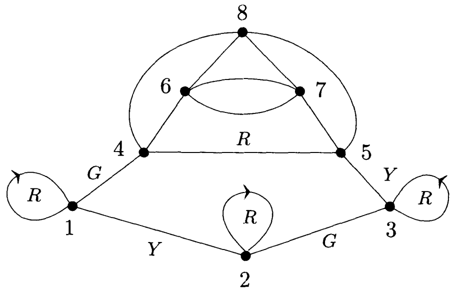

Figure 9. Case (b): The graph Δ_1

By considering the vertices 1, 2, and 3 of graph Δ_1, we see that all the closed loops must be given the same color R. Hence the same must be true for the three closed loops of Δ_2.

We now have two possiblities for completing the coloring of Δ_2, neglecting directions:

Case (c2a): $e(4,5)$ is G. In this case, one of the two $e(2,3)$'s is G, and the other is R. The remaining edges may be described as: $e(1,2)$, $e(1,3)$, $e(4,6)$, $e(6,7)$, $e(5,8)$, and $e(7,8)$ are G. The remaining edges are R.

Case (c2b): $e(4,5)$ is R. Then both edges $e(2,3)$ are R, and the G edges are: $e(1,2)$, $e(1,3)$, $e(2,4)$, $e(3,5)$, $e(4,6)$, $e(6,7)$, $e(5,8)$, and $e(7,8)$.

We now turn to the various possibilities for Δ_1. We first observe that in none of the colorings of Δ_2 is there no closed path of length 2 consisting just of G paths. Hence one of the edges $e(6,7)$ must be R. We now have the following possibilities:

Case (c1a): $e(4,5)$ is G. In this case, the remaining $e(6,7)$ must be G, and the remaining G edges are $e(1,2)$, $e(2,3)$, $e(1,4)$, $e(3,5)$, $e(6,8)$, and $e(7,8)$.

Case (c1b): $e(4,5)$ is R and both $e(6,7)$'s are R. Then the G edges are: $e(1,2)$, $e(2,3)$, $e(1,4)$, $e(3,5)$, $e(4,6)$, $e(6,7)$, $e(6,8)$, and $e(7,8)$.

Case (c1c): $e(4,5)$ is R, one of the two $e(6,7)$'s is G, and $e(4,6)$ is R. Then the remaining G edges are: $e(1,2)$, $e(2,3)$, $e(1,4)$, $e(3,5)$, $e(4,8)$, $e(5,7)$, and $e(6,8)$.

Case (c1d): The mirror image of Case (c1c).

Let us show that possibilities (c1c) and (c1d) cannot obtain. In each of these possibilities, there is a closed path of length 8 in the G's, and no smaller closed path in the G's, and three loops of length 1 in the R's, and one loop of length 5 in the R's. On the other hand, in Case (c2a) we have three loops of length 1 and one loop of length 5 in the R's, and one loop of length 3 and one loop of length 5 in G, while in Case (c2b) we have three loops of length 1, one loop of length 2, and one of length 3 in the R's, and one loop of length 8 in the G's.

So neither (c1c) or (c1d) match either of the possibilities (c2a) or (c2b).

We now consider case (c1a). Since it has no loops of length 8 in the G's, it cannot match case (c2b). It remains to show that it does not match case (c2a).

Counting closed loops of the form $G^+G^+R^\pm$ or $G^-G^-R^\pm$, we see that on (c1a) there are exactly two such paths, beginning at vertices 6 and 7 respectively. On the other hand, in case (c2a) there are six such loops, starting at vertices $2, 3, 4, 5$, and two paths starting at 7. Hence these two colorings cannot correspond.

It remains to consider case (c1b), which can only correspond to case (c2b), by consideration of closed loops all of one color. We recognize the colorings arising from the Seidel construction, if we assign all the edges lying in Γ_1 and Γ_2 the color R, and all the edges going from Γ_1 to Γ_2 the color G. It follows that the assignment of directions of the R edges is essentially unique, but that there are two choices of directions for the G edges. It also follows from the Seidel construction that if one neglects the direction of the G's in counting patterns, the counts of the various patterns will always work out the same for the two graphs. In other words, the only obstacle to these graphs being strongly Sunada equivalent is that, in the process of taking out and replacing sections as in the proof of Theorem 3.1, occasionally a G^+ will be replaced by a G^- in a way that is hard to control.

Let us show that this indeed does happen. Consider the pictures 10 and 11 below.

We consider the word $R^+G^+R^-G^-$. Notice that the number of times this closes up does not depend on an orientation of the G's, since changing G^+ to G^- just rotates the word.

We notice that in the graph (c2b), this never closes, while in the graph (c1b), it closes four times, at the vertices $2, 3, 4$, and 6.

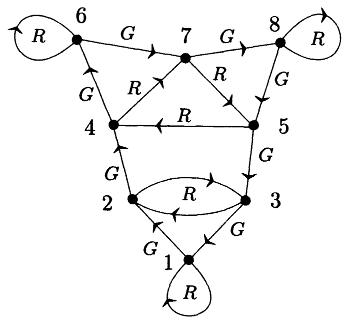

Figure 10. Case (c2b): The graph Δ_2

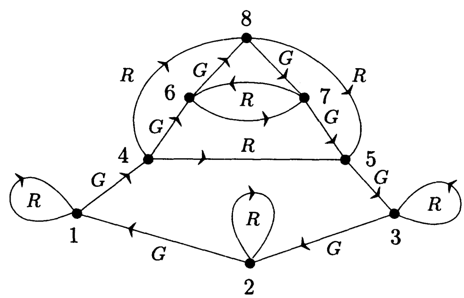

Figure 11. Case (c1b): The graph Δ_1

This contradiction completes the proof of the theorem. □

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