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ON NORMAL ABELIAN SUBGROUPS IN PARABOLIC GROUPS

by Gerhard RÖHRLE (*)

1. Introduction.

Throughout, G denotes a (connected) reductive algebraic group defined over an algebraically closed field k of characteristic $p \geq 0$ and P is a parabolic subgroup of G with unipotent radical P_u. The aim of this note is the following result.

THEOREM 1.1. — *Let G be a reductive algebraic group, P a parabolic subgroup of G, and A a closed connected normal subgroup of P in P_u. If A is abelian, then P has finitely many orbits on A.*

The particular case when A is in the center of P_u is well-known. Then the action factors through a Levi subgroup of P. In characteristic 0 the finiteness follows from a result of Vinberg [41, § 2] on gradings of Lie algebras (see also Kac [15]) and in general from work of Richardson [28, § 3]. For a detailed account of the orbit structure in this situation, see [24] and [29, § 2, § 5].

Observe that for abelian P-invariant sub-factors in P_u, the analogous statement of the theorem is false in general. Indeed, this fact is the basis for constructing entire families of parabolic subgroups which admit an infinite number of orbits on the unipotent radical, or its Lie algebra, *e.g.*, see [25], [26], [30], and [31]. Examples in this context also show that a parabolic

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subgroup may have an infinite number of orbits on a normal subgroup of nilpotency class two, *cf.* [9], [16].

The *modality* of the action of P on the normal subgroup A is the maximal number of parameters upon which a family of P-orbits on A depends; likewise for the adjoint action of P on the Lie algebra of A, *cf.* [26]. The basic machinery for investigating the modality of parabolic subgroups of reductive groups was introduced in [26]. Apart from [26] there are several recent articles related to this subject. For instance, all parabolic subgroups P of classical algebraic groups with a finite number of orbits on P_u are determined in [12] and [13]. Similar results for exceptional groups are obtained in [14]. In [8] all such P in $\mathrm{GL}_n(k)$ are classified with a finite number of orbits on a given term of the lower central series of P_u. More generally, in [25] and [32] the modality of the action of P on the Lie algebra of P_u is investigated for any reductive G.

The proof of Theorem 1.1 readily reduces to the case when G is simple, P is a Borel subgroup B of G, and A is a maximal closed connected normal abelian subgoup of B. In Section 3 we classify all such A, up to G-conjugacy (Theorem 3.1), and in Section 5 we show in each instance that B acts on A with a finite number of orbits.

Our proof of Theorem 1.1 uses an extension of the classification of spherical Levi subgroups of reductive groups to arbitrary characteristic due to Brundan [4, § 4] (see Section 4). From that we immediately obtain a proof of those cases of Theorem 1.1 where A is contained in the unipotent radical of a parabolic subgroup whose Levi factor is spherical in G. In the cases where we cannot appeal to spherical Levi subgroups directly, a construction from [1] allows us to apply these results partially.

In the two final sections we discuss the situation for the adjoint action of P on abelian ideals in the Lie algebra of P_u as well as a connection between Theorem 3.1 and Mal'cev's classification of abelian subalgebras of the Lie algebra of G of maximal dimension [21].

Both, our classification of the maximal closed connected normal abelian subgroups A of B, as well as the fact that in each of these instances B operates on A with a finite number of orbits, are obtained in case studies. It would be highly desirable to have a uniform proof of Theorem 1.1 free of case analysis, even for the expense of some characteristic restrictions.

2. Notation and preliminaries.

We denote the Lie algebra of G by $\operatorname{Lie} G$ or $\mathfrak{g}$ and the identity element of G by e; likewise for subgroups of G. Let T be a fixed maximal torus in G and $\Psi = \Psi(G)$ the set of roots of G with respect to T. Fix a Borel subgroup B of G containing T and let $\Sigma = \{\sigma_1, \sigma_2, \ldots\}$ be the set of simple roots of Ψ defined by B such that the positive integral span of Σ in Ψ is $\Psi^+ = \Psi(B)$. The highest (long) root in Ψ is denoted by ϱ. If all roots in Ψ are of the same length, they are all called long. A subset I of Ψ^+ is an *ideal* in Ψ^+ (see [38, p. 24]) provided I is closed under addition by elements from Ψ^+. For a root β of G we denote by U_β the corresponding one-parameter unipotent subgroup of G normalized by T, and the root subspace $\operatorname{Lie} U_\beta$ of $\mathfrak{g}$ by $\mathfrak{u}_\beta$. The members $U_\beta(\xi)$, where $\xi \in k$, of U_β are called *root elements*.

Suppose that G is simple (over its center). A prime is said to be *bad* for G if it divides the coefficient of a simple root in ϱ, else it is called *good* for G [37, § I,4]. Furthermore, we say that a prime is *very bad* for G if it divides a structure constant of the Chevalley commutator relations for G. Thus, if p is very bad for G, there are degeneracies in these relations. This only occurs if $p = 2$ and G is of type B_r, C_r, F_4, or G_2, or $p = 3$ and G is of type G_2. The same notions apply to reductive groups by means of simple components [39, 3.6].

We may assume that each parabolic subgroup P of G considered contains B.

Let N be a closed connected normal subgroup of P in P_u. Since N is normalized by $T \subset P$, *i.e.*, N is T-regular [11], the root spaces of $\mathfrak{n}$ relative to T are also root spaces of $\mathfrak{g}$ relative to T, and the set of roots of N with respect to T, denoted by $\Psi(N)$, is a subset of Ψ. Suppose that $\Psi(N)$ is closed under addition in Ψ. Note that this is automatically satisfied provided p is not very bad for G. Then $\mathfrak{n} = \bigoplus \mathfrak{u}_\beta$ $(\beta \in \Psi(N))$ and consequently, $N = \prod U_\beta$, where the product is taken in some fixed order over $\Psi(N)$. The *support* of an element x in N, denoted by $\operatorname{supp} x$, consists of all roots β for which the projection $N \to U_\beta$ is nontrivial when evaluated at x.

By the *shape of a root* $\beta = \sum n_\sigma(\beta)\sigma$ $(\sigma \in \Sigma)$ *relative to* P, we mean the sub-sum over the elements of $\Sigma(P_u) = \Psi(P_u) \cap \Sigma$, and by the *level of* β *relative to* P, the sub-sum of the coefficients $n_\sigma(\beta)$ over the same set $\Sigma(P_u)$, *cf.* [1].

The descending central series of P_u is defined as usual by $\mathcal{C}^0 P_u := P_u$ and $\mathcal{C}^{i+1} P_u := (\mathcal{C}^i P_u, P_u)$ for $i \geq 0$. Since P_u is nilpotent, the smallest integer m such that $\mathcal{C}^m P_u = \{e\}$ is the class of nilpotency of P_u, *i.e.*, the *length* of this series, and is also denoted by $\ell(P_u)$. If p is not a very bad prime for G, then $\Psi(\mathcal{C}^i P_u)$ consists precisely of all roots whose P-level is at least $i+1$, see [1].

Throughout, we use the labeling of the Dynkin diagram of G (*i.e.* of Σ) as well as the notation for roots in systems of exceptional type as in Bourbaki [3]. Our general reference for algebraic groups is Borel's book [2].

3. The maximal normal abelian subgroups of Borel subgroups.

In this section we determine all maximal closed connected normal abelian subgroups A of our fixed Borel subgroup B of G and record them in the subsequent table. Here we specify the roots α such that A is the normal closure in B of the corresponding root subgroups U_α. The fact that A is abelian follows either from the observation that the sum of two roots in $\Psi(A)$ is not a root, because it exceeds ϱ in some coefficient, and thus, by the commutator relations, A is commutative, or else because p is a very bad prime for G leading to commutation degeneracies. As indicated in the table, some extra cases do occur for very bad primes.

The simple roots σ_i are labeled as in [3]. Moreover, we use the following abbreviations: in type B_r set

$$\beta_i = \sigma_1 + \cdots + \sigma_i, \quad \gamma_i = \sigma_{i-1} + 2\sigma_i + \cdots + 2\sigma_r, \quad \delta_i = \sigma_i + \cdots + \sigma_r,$$

where $2 \leq i \leq r$, and finally $\eta = \beta_r + \sigma_r$. Similarly, for type D_r we define

$$\beta_i = \sigma_1 + \cdots + \sigma_i, \quad \gamma_i = \sigma_{i-1} + 2\sigma_i + \cdots + 2\sigma_{r-2} + \sigma_{r-1} + \sigma_r$$

for $3 \leq i \leq r-2$ and

$$\beta = \beta_{r-2} + \sigma_{r-1}, \quad \gamma = \beta_{r-2} + \sigma_r, \quad \delta = \sigma_{r-2} + \sigma_{r-1} + \sigma_r.$$

The normalizer of A in G is a parabolic subgroup of G, since it contains B. In the third column of Table 1 we indicate the set of simple roots of the standard Levi subgroup of $N_G(A)$. Here the notation $\{\sigma_1', \sigma_2', \ldots\}$ simply means $\Sigma \setminus \{\sigma_1, \sigma_2, \ldots\}$. Finally, we list $\dim A$ in each instance.

THEOREM 3.1. — *Let G be a simple algebraic group. Every maximal closed connected normal abelian subgroup of the Borel subgroup B of G is listed in Table* 1.

Proof. — First we assume that p is not very bad for G. Then, our aim to determine each maximal closed connected normal abelian subgroup A of B is equivalent to the purely combinatorial task to determine all maximal abelian ideals $\Psi(A)$ in $\Psi(B)$ (*i.e.*, all those ideals I of $\Psi(B)$ which are maximal with respect to the property that no two roots in I sum up to a root in Ψ). If A is the normal closure in B of a single root subgroup U_α, then 2α exceeds ϱ in some coefficient, because of the commutator relations for root subgroups in A. This quickly leads to a complete list of all the maximal $\Psi(A)$'s of this nature. If A is the normal closure of two distinct root subgroups, then the normal closure of each one of them is abelian and no sum of any two roots from the supports of these two subgroups is again a root. One checks that while in type A_r or C_r any such A is already contained in one of the first kind, in type B_r and D_r there are new occurrences of the second type which are not contained in ones of the first kind. The maximal ones then are easily determined which lead to the families in the second entry for B_r and the third one for D_r. As indicated, there are also new cases here for the exceptional types. Abelian normal subgroups which are the normal closure in B of three distinct root subgroups and which are not already contained in one of the first two kinds only arise in type D_r, E_6, E_7, and E_8. There are no maximal incidences for any type when $\Psi(A)$ is generated by four or more distinct roots.

For the exceptional groups these records were obtained with the aid of a computer algorithm which, for any given G, computes all maximal abelian ideals of $\Psi(B)$. Consequently, as the rank of G is bounded in these cases, this yields the desired subgroups A in B in these cases.

Now we consider the situation when p is a very bad prime for G. Here we can only have additional cases when there are two root lengths in Ψ. Since in characteristic 2 the simple groups of type B_r and C_r are isomorphic as abstract groups [38, Thm 28], it suffices to only list the new occurrences for type B_r. In type C_r we only record the single generic case. For G_2 there is only one additional occurrence for $p = 3$ (none for $p = 2$). One checks these remaining events directly.

Note, if I is the ideal in $\Psi(B)$ generated by some positive roots, then U_γ, for some $\gamma \in I$, need not be contained in the normal closure of the corresponding root subgroups in B, because of commutator degeneracies in the presence of very bad primes. This necessitates that in some of the cases in Table 1 we have to include additional generating roots. For instance, this is the case for G_2 and in the third instance for F_4. This is not required in the third case for B_r, as the structure constants of the commutator of the root subgroups relative to the simple roots in a group of type B_2 are all equal to ± 1, *e.g.*, see [38, Lemma 33].

The last entry for B_r is the only event when $\Psi(A)$ fails to be an ideal in Ψ^+. Observe that the roots σ_{i-1} and δ_i form a basis of a subsystem of Ψ of type B_2. Thus by [38, Lemma 33] $\gamma_i = \sigma_{i-1} + 2\delta_i$ is in $\Psi(A)$ in this instance. Whence, if $p = 2$, the subgroup from the second entry for B_r is properly contained in the one from the fourth case (for a fixed i). $\square$

Remark 3.2. — Suppose that p is not a very bad prime for G. Let A be as in Table 1 below. Let $N_G(A) = P = LP_u$ with standard Levi subgroup L. The simple roots $\Sigma(L)$ canonically define a grading of $\mathfrak{g}$ as follows [15]: Define a function $d\colon \Psi \to \mathbb{Z}$ by setting $d(\sigma) := 0$ if σ is in $\Sigma(L)$ and $d(\sigma) := 1$ if σ is in $\Sigma \setminus \Sigma(L)$, and extend d linearly to all of Ψ. Then we define

$$\mathfrak{g}(i) := \begin{cases} \bigoplus\limits_{d(\alpha)=i} \mathfrak{u}_\alpha, & \text{for } i \neq 0; \\ \mathfrak{t} \oplus \bigoplus\limits_{d(\alpha)=0} \mathfrak{u}_\alpha, & \text{for } i = 0. \end{cases}$$

Thus we have $\mathfrak{g} = \bigoplus_i \mathfrak{g}(i)$ and moreover,

$$\mathfrak{p} = \bigoplus_{i\geq 0} \mathfrak{g}(i) \quad \text{and} \quad \mathfrak{p}_u = \bigoplus_{i>0} \mathfrak{g}(i).$$

D. Panyushev observed that for each A from Table 1 below $d(\varrho)$ is odd and for $m = \left[\frac{1}{2}d(\varrho)\right] + 1$ we have $\operatorname{Lie} A = \bigoplus_{i\geq m} \mathfrak{g}(i)$. Using the description of P furnished in the third column in Table 1, the value of $d(\varrho)$ is readily determined.

Remark 3.3. — It is interesting to observe that if p is not very bad for G, then the number of closed connected maximal abelian normal subgroups of B equals the number of long simple roots of G.

Unfortunately, our proof of Theorem 3.1, involving case by case considerations, is less than satisfactory. It would be very desirable to have a uniform proof of this result given that p is not a very bad prime for G.

Type of G	A	$N_G(A)$	$\dim A$
A_r	$\sigma_i \ (1 \leq i \leq r)$	σ_i'	$i(r-i+1)$
B_r	σ_1	σ_1'	$2r-1$
	$\beta_i, \gamma_i \ (3 \leq i \leq r) \ (p \neq 2)$	σ_1', σ_i'	$\frac{1}{2}(4r+i^2-5i+2)$
	$\sigma_r \ (p=2)$	σ_r'	$\frac{1}{2}(r^2+r)$
	$\beta_i, \delta_i, \eta \ (1 < i < r) \ (p=2)$	σ_1', σ_i'	$\frac{1}{2}(4r+i^2-3i)$
C_r	σ_r	σ_r'	$\frac{1}{2}(r^2+r)$
D_r	σ_1	σ_1'	$2r-2$
	σ_{r-1}, or σ_r	σ_{r-1}', or σ_r'	$\frac{1}{2}(r^2-r)$
	$\beta_i, \gamma_i \ (3 \leq i \leq r-2)$	σ_1', σ_i'	$\frac{1}{2}(4r-5i+i^2)$
	β, γ, δ	$\sigma_1', \sigma_{r-1}', \sigma_r'$	$\frac{1}{2}(r^2-3r+6)$
E_6	σ_1, or σ_6	σ_1', or σ_6'	16
	$\begin{smallmatrix}0&1&2&1&0\\&&1&&\end{smallmatrix}$	σ_4'	11
	$\begin{smallmatrix}1&1&1&1&0\\&&0&&\end{smallmatrix}$, $\begin{smallmatrix}0&1&2&2&1\\&&1&&\end{smallmatrix}$	σ_1', σ_5'	13
	$\begin{smallmatrix}0&1&1&1&1\\&&0&&\end{smallmatrix}$, $\begin{smallmatrix}1&2&2&1&0\\&&1&&\end{smallmatrix}$	σ_3', σ_6'	13
	$\begin{smallmatrix}1&1&1&1&1\\&&0&&\end{smallmatrix}$, $\begin{smallmatrix}0&1&2&1&1\\&&1&&\end{smallmatrix}$, $\begin{smallmatrix}1&1&2&1&0\\&&1&&\end{smallmatrix}$	$\sigma_2, \sigma_3, \sigma_5$	12
E_7	σ_7	σ_7'	27
	$\begin{smallmatrix}1&2&2&1&0&0\\&&1&&&\end{smallmatrix}$	σ_3'	17
	$\begin{smallmatrix}0&1&2&2&1&0\\&&1&&&\end{smallmatrix}$	σ_5'	20
	$\begin{smallmatrix}0&1&2&2&2&1\\&&1&&&\end{smallmatrix}$, $\begin{smallmatrix}1&2&2&1&1&0\\&&1&&&\end{smallmatrix}$	σ_3', σ_6'	18
	$\begin{smallmatrix}0&1&2&1&1&1\\&&1&&&\end{smallmatrix}$, $\begin{smallmatrix}1&2&3&2&1&0\\&&1&&&\end{smallmatrix}$	σ_4', σ_7'	20
	$\begin{smallmatrix}0&0&1&1&1&1\\&&1&&&\end{smallmatrix}$, $\begin{smallmatrix}1&2&3&2&1&0\\&&2&&&\end{smallmatrix}$	σ_2', σ_7'	22
	$\begin{smallmatrix}0&1&2&2&1&1\\&&1&&&\end{smallmatrix}$, $\begin{smallmatrix}1&2&2&2&1&0\\&&1&&&\end{smallmatrix}$, $\begin{smallmatrix}1&2&2&1&1&1\\&&1&&&\end{smallmatrix}$	$\sigma_3', \sigma_5', \sigma_7'$	19

Table 1 (first part): the maximal normal abelian subgroups of Borel subgroups

Type of G	A	$N_G(A)$	$\dim A$
E_8	$\begin{smallmatrix}0&1&2&2&2&2&1\\&&1\end{smallmatrix}$	σ'_7	29
	$\begin{smallmatrix}1&2&3&3&2&1&0\\&&1\end{smallmatrix}$	σ'_5	34
	$\begin{smallmatrix}1&2&3&2&1&0&0\\&&2\end{smallmatrix}$	σ'_2	36
	$\begin{smallmatrix}1&1&2&2&2&2&1\\&&1\end{smallmatrix}$, $\begin{smallmatrix}2&3&4&3&2&1&0\\&&2\end{smallmatrix}$	σ'_1, σ'_7	30
	$\begin{smallmatrix}1&2&2&2&2&2&1\\&&1\end{smallmatrix}$, $\begin{smallmatrix}1&3&4&3&2&1&0\\&&2\end{smallmatrix}$	σ'_3, σ'_7	31
	$\begin{smallmatrix}1&2&3&2&2&2&1\\&&1\end{smallmatrix}$, $\begin{smallmatrix}1&2&4&3&2&1&0\\&&2\end{smallmatrix}$	σ'_4, σ'_7	32
	$\begin{smallmatrix}1&2&3&3&3&2&1\\&&1\end{smallmatrix}$, $\begin{smallmatrix}1&2&3&2&2&1&0\\&&2\end{smallmatrix}$	σ'_2, σ'_6	34
	$\begin{smallmatrix}1&2&3&3&2&2&1\\&&1\end{smallmatrix}$, $\begin{smallmatrix}1&2&3&2&2&2&1\\&&2\end{smallmatrix}$, $\begin{smallmatrix}1&2&3&3&2&1&0\\&&2\end{smallmatrix}$	$\sigma'_2, \sigma'_5, \sigma'_7$	33
F_4	1220	σ'_2	8
	$1221, 0122$ $(p \neq 2)$	σ_1, σ_3	9
	$0121, 0122$ $(p = 2)$	σ_1, σ_2	11
	$1111, 0122$ $(p = 2)$	σ_2, σ_3	11
G_2	21 $(p \neq 3)$	σ_2	3
	$11, 21$ $(p = 3)$	$\oslash$	4

Table 1 (second part): the maximal normal abelian subgroups of Borel subgroups

4. Spherical Levi subgroups.

The following basic result is due to Brion [4] and Vinberg [42] in characteristic 0 and in arbitrary characteristic to Knop [17, 2.6].

THEOREM 4.1. — *Let G be a reductive algebraic group and B a Borel subgroup of G. Let X be an irreducible G-variety admitting a dense B-orbit. Then B has finitely many orbits on X.*

A closed subgroup H of G is called *spherical* if H has a dense orbit on G/B, or equivalently, if there is a dense B-orbit on G/H, or equivalently, by Theorem 4.1, if there is a finite number of B-orbits on G/H. Concerning recent results on spherical subgroups in positive characteristic, consult [7]. The associated *spherical varieties* G/H have been studied extensively in the literature; see [6] for a survey.

In characteristic 0 all reductive spherical subgroups have been classified in [20], [5], and [23]. When $p \neq 2$ many instances are known as centralizers of involutions by a result of Springer [36]. Brundan's method of "integral embeddings" [7, § 2] and the representation theoretic fact that Levi subgroups of G are good filtration subgroups (see [10], [22]) enable him to extend the classification of spherical Levi subgroups from characteristic zero [20] to arbitrary characteristic [7, Thm 4.1]:

THEOREM 4.2. — *Let L be a Levi subgroup of G. Let $G = \prod_{i=1}^{r} G_i$ as a commuting product of simple factors and $L_i := L \cap G_i$. Then L is spherical if and only if, for each i, either $L_i = G_i$, or (G_i, L_i') is one of $(A_r, A_m A_{r-m-1})$, (B_r, B_{r-1}), (B_r, A_{r-1}), (C_r, C_{r-1}), (C_r, A_{r-1}), (D_r, D_{r-1}), (D_r, A_{r-1}), (E_6, D_5), or (E_7, E_6).*

Theorem 4.2 and the next result [5, Prop. I.1] (or [7, Lemma 4.2]) yield a classification of all parabolic subgroups $P = LP_u$ of G for which a Borel subgroup B_L of L has a dense orbit on P_u, whence finitely many orbits thanks to Theorem 4.1.

LEMMA 4.3. — *Let $P = LP_u$ be a parabolic subgroup of G and B_L a Borel subgroup of L. Then L is spherical in G if and only if there is a dense B_L-orbit on P_u.*

Remark 4.4. — A Levi subgroup L of P is spherical in G if and only if either P_u is abelian, or $p \neq 2$ and one of the pairs (G_i, L_i') as in Theorem 4.2 equals (B_r, A_{r-1}) or (C_r, C_{r-1}). This is immediate from Theorem 4.2 and the well-known instances when P_u is abelian, *e.g.*, see [29, Rem. 2.3] when p is not very bad for G. In this latter case, Vavilov gave a direct proof of the finiteness of the number of B_L-orbits on P_u in [40, § 4].

5. Proof of Theorem 1.1.

Clearly, we may assume that G is simple (over its center) and that A is a maximal closed connected abelian subgroup of B_u normalized by B. We have compiled all possibilities for such A in Table 1 in Section 3 above. Since A is B-invariant, the normalizer of A in G is parabolic in G. Throughout this section, we write $P = LP_u$ for $N_G(A)$ with standard Levi subgroup L.

5.1. — By inspection of the list in Theorem 4.2, one checks that in each of the cases in Table 1 where A is generated by a single root subgroup relative to a *simple root* A is contained in the unipotent radical of a parabolic subgroup $Q = MQ_u$ of G whose Levi subgroup M is spherical in G. Hence, by Lemma 4.3 and Theorem 4.1, B_M and thus B act on Q_u and thus on A with a finite number of orbits. In fact, in each of these events $Q = P$ and $A = Q_u$. In particular, this covers all instances for A_r, C_r, and the first entries for B_r, D_r, E_6, and E_7. Also, if G is of type B_r and A is generated by the root subgroups relative to β_r and γ_r, then A is (properly) contained in the unipotent radical of the maximal parabolic subgroup Q of G corresponding to $\Sigma \setminus \{\sigma_r\}$. Here $N_G(A)$ is of semisimple corank 1 in Q. This corresponds to the third case listed in Theorem 4.2 which equally applies when $p = 2$ and then, Q_u itself is abelian. This leads to the third B_r entry in this table.

5.2. — Now we turn to the remaining cases in Table 1. Here we cannot appeal directly to the results from Section 4. However, a construction from [1] allows us to apply these results in part. Throughout this paragraph, suppose that p is not a very bad prime for G. In each of the cases we are concerned with $A = \mathcal{C}^s P_u$ for some $s \geq 1$. For each $i \in \mathbb{N}$ define

$$V_i : = \mathcal{C}^{s+i-1} P_u / \mathcal{C}^{s+i} P_u.$$

Because A is abelian, we may regard each V_i as a subgroup of A. Let

$$t : = \ell(P_u) - s.$$

Then $V_t = Z(P_u)$ is the last term in the descending central series of P_u. By the commutator relations for G, each root in $\Psi(V_i)$ is of P-level $s+i$. Let $S_i^1, S_i^2, \ldots$ be the different shapes among roots of P-level $s+i$. For each shape S_i^n there is a unique root α_i^n in $\Psi(V_i)$ of minimal height of that shape. Let V_i^n be the product of all the root subgroups of shape S_i^n. Then each V_i^n naturally is an L-module of lowest weight α_i^n and as L-modules $V_i \cong V_i^1 \oplus V_i^2 \ldots$, *cf.* [1]. In particular, V_i^n is the product of all root subgroups of shape S_i^n and V_i is the product of all root subgroups of level $s+i$. Denote by Ψ_i the set of all roots whose shapes are integral multiples of all the occurring shapes S_i^n

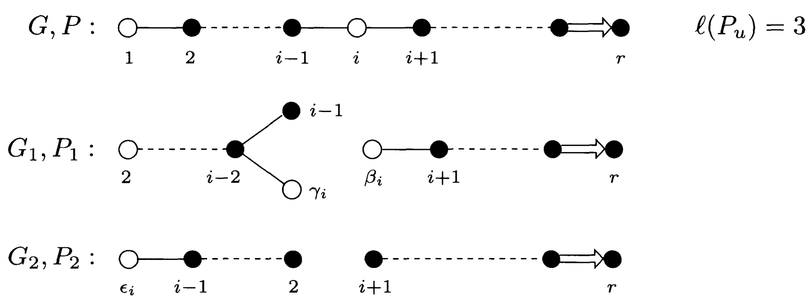
of level $s+i$. This is a closed semisimple subsystem of Ψ, and it contains $\Psi(L)$ (the set of roots of shape 0) as a subsystem, since its elements are all multiples of the various S_i^n's modulo the integral span of $\Psi(L)$. Thus the positive simple system Σ_i for Ψ_i consists of $\Sigma(L)$ and the α_i^n's for $n \geq 1$. Since $2S_i^n$ is not the shape of any root in Ψ, the union of $\pm\Psi(V_i)$ and $\Psi(L)$ equals Ψ_i. Therefore, if G_i is the connected reductive subgroup of G corresponding to Ψ_i (*i.e.* $\Psi(G_i) = \Psi_i$) and containing the maximal torus T, then $P_i := LV_i$ is the standard parabolic subgroup of G_i corresponding to $\Sigma(L)$ with unipotent radical V_i. The number of simple components of G_i' equals the number of different shapes S_i^n of level $s+i$. Since V_i is abelian, L is a spherical Levi subgroup of G_i by Remark 4.4. Thus, the previous results applied to $P_i = LV_i$ in G_i yield that B_L has a finite number of orbits on V_i for each $i = 1, \ldots, t$. Therefore, since $A = V_1 \cdots V_t$, we are able to conclude our desired finiteness statement, once we have proved that

$$(\dagger) \quad \begin{cases} \text{there are only finitely many } B\text{-orbits passing through each} \\ \text{coset of the form } vV_i \cdots V_t, \text{ where } v \text{ is in } V_{i-1} \setminus \{e\}, \text{ for} \\ 1 < i \leq t. \end{cases}$$

As $B = B_L P_u$, one method to establish this is to (possibly) first replace v by a suitable B_L-conjugate v' of v and then to show that each element in $v'V_i \cdots V_t$ is already conjugate to v' under P_u. Since P_u is connected and unipotent, it has a finite number of orbits on $v'V_i \cdots V_t$ precisely if this coset is a single P_u-orbit [33]. Sometimes another way to establish $(\dagger)$ is more convenient. Since $B = TB_u$, we first aim to show that an element of $vV_i \cdots V_t$ is B_u-conjugate to an element x which is supported by at most $\operatorname{rank} G$ linearly independent roots. Then $(\dagger)$ follows, as each of the coefficients in the root elements of x can be scaled to 1 using the action of T in this event. We combine these techniques with inductive arguments below.

5.3. — We first attend to the remaining classical occurrences. According to 5.1 the only ones left here are the second and fourth entries for B_r as well as the last two for D_r. Here we use the notation from Section 3.

5.3.1. — Let G be of type B_r and let A be the normal closure in B of U_{β_i} and U_{γ_i}, where $3 \leq i \leq r$ and suppose that $p \neq 2$. Then $A = \mathcal{C}^1 P_u$, $t = 2$, and $\Sigma(L)$ consists of all simple roots but σ_1 and σ_i, *i.e.*, L' is of type $A_{i-2}B_{r-i}$. If $i = r$, then L is of type A_{r-2}. But this case was already discussed in 5.1. There are precisely two different P-shapes of roots of P-level 2 and the unique roots of minimal height of these two shapes are β_i and γ_i respectively. Note that G_1 is of type $D_{i-1}B_{r-i+1}$ and $P_1 = LV_1$ is the parabolic subgroup of G_1 of semisimple corank 2 corresponding to β_i and γ_i. Written additively, V_1 is the sum of the alternating square of the natural module of the A_{i-2} component of L' (denoted here by V_1^2) and of the natural module for the B_{r-i} factor (V_1^1). The unique root of minimal height of P-level 3 is $\epsilon_i := \beta_r + \delta_i$, and G_2 is of type $A_{i-1}B_{r-i}$ with simple positive system $\Sigma(L) \cup \{\epsilon_i\}$. Here V_2 is the natural module for the A_{i-2} component of L' with the second component of L acting passively. We summarize this information in the following figure. Here we indicate $\Sigma(L)$ by coloring the corresponding nodes in the diagram of G, likewise for G_1 and G_2.

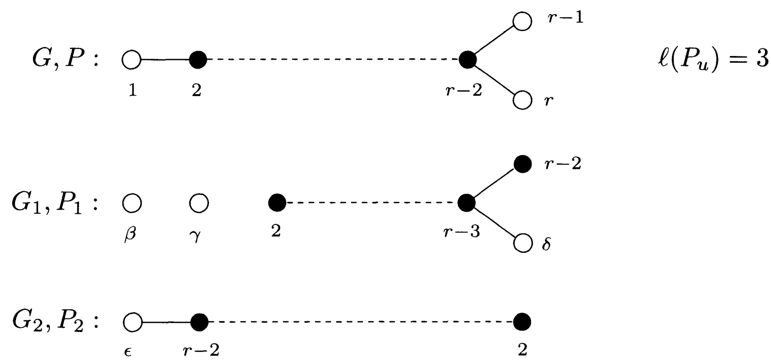


Thus $A = V_1V_2$ and B_L has a finite number of orbits on each V_i by 5.2. Let $v = v^1v^2$ be in $V_1 \setminus \{e\}$, where $v^n \in V_1^n$ for $n = 1, 2$. If $v^1 = e$, then $vV_2 = v^2V_2$ is contained in the unipotent radical of the parabolic subgroup of G corresponding to $\Sigma \setminus \{\sigma_r\}$ (as every root in $\operatorname{supp} v^2V_2$ has coefficient 2 at σ_r). If $v^2 = e$, then $vV_2 = v^1V_2$ is contained in the unipotent radical of the parabolic subgroup of G corresponding to $\Sigma \setminus \{\sigma_1\}$ (as every root in $\operatorname{supp} v^1V_2$ has coefficient 1 at σ_1). In each one of these cases, the desired finiteness statement follows from 5.1. Now we may suppose that $v^1 \neq e \neq v^2$. But then every element in vV_2 is P_u-conjugate to v: Let $x = v^1v^2x'$ be in vV_2 with $x' \in V_2$ and let τ be a root of minimal height in $\operatorname{supp} v^2$. Then for each root ν in $\Psi(V_2)$ there is a unique root μ in $\Psi(P_u)$ such that $\tau + \mu = \nu$. By induction on height we can thus remove x' completely using the action of root elements in U_μ. Since σ_1 is a summand of μ, any such operation fixes v^1, and thus x is indeed P_u-conjugate to v in this event. Thus (†) is fulfilled. This completes the argument for the second B_r entry.

Next we address the fourth entry for B_r. So here $p = 2$ and A is the normal closure in B of U_{β_i}, U_{δ_i}, and U_η, where $2 \leq i \leq r-1$. Recall that here $\Psi(A)$ is no longer an ideal in $\Psi(B)$. Let x be in A. If there is no β_j in $\operatorname{supp} x$ for $i \leq j < r$, then x is contained in the unipotent radical of the maximal parabolic subgroup corresponding to $\Sigma \setminus \{\sigma_r\}$ (as then every root in $\operatorname{supp} x$ has a non-zero coefficient at σ_r). By 5.1 there is a finite number of B-orbits in A of this nature. So we may suppose that there is a β_j in $\operatorname{supp} x$ for some $i \leq j < r$. Without loss, we may suppose that β_j is the unique such root of minimal height in $\operatorname{supp} x$. Any of the remaining roots in $\operatorname{supp} x$ with coefficient 1 at σ_1, which is either β_r or long, can then be removed from $\operatorname{supp} x$ by applying suitable root elements from B_u to x. Thus we may suppose that we have an orbit representative of the form $x = U_{\beta_j}(\xi)x'$ ($\xi \in k^*$), where all roots in $\operatorname{supp} x'$ have coefficient 0 at σ_1 and a non-zero coefficient at σ_r. Let H be the simple subgroup of G of type B_{r-1} defined by the simple system $\Sigma' = \Sigma \setminus \{\sigma_1\}$. Then x' belongs to the unipotent radical of the maximal parabolic subgroup Q of H corresponding to $\Sigma' \setminus \{\sigma_r\}$. By Theorem 4.2 the Levi subgroup M of Q is spherical in H and thus x' belongs to a finite set of orbits in A of the Borel subgroup B_M of M. By the nature of the root system $\Psi(A)$ and since $p = 2$, any two elements in $Q_u \cap A$ which are $(B_M)_u$-conjugate are conjugate by an element of $(B_M)_u$ which fixes the factor $U_{\beta_j}(\xi)$ of x. Since the maximal torus in B_M normalizes U_{β_j}, its action on the factor $U_{\beta_j}(\xi)$ of x merely may result in a different coefficient $\zeta \in k^*$. Hence we may assume that x' belongs to a finite set of B_M-orbit representatives in $Q_u \cap A$. Note that each root in $\operatorname{supp} x'$ is orthogonal to β_j. Thus we may use the 1-dimensional torus S associated to β_j to scale the coefficient ζ to equal 1. Ultimately, there is only a finite number of B-orbits on A in this instance as well.

5.3.2. — Next we treat the cases left for type D_r ($r \geq 4$). The finiteness result for the first two entries follows from 5.1. The argument for the third entry in D_r is completely analogous to the second case for B_r. We leave the details to the reader.

In the last entry for D_r in our Table A is the normal closure in B of U_β, U_γ, and U_δ. Here we have $A = \mathcal{C}^1 P_u$ and $\Sigma(L) = \Sigma \setminus \{\sigma_1, \sigma_{r-1}, \sigma_r\}$, *i.e.*, L' is of type A_{r-3}. There are three different P-shapes of roots of P-level 2 and the unique roots of minimal height of these shapes are precisely β, γ, and δ, respectively. Note that G_1 is of type $A_1 A_1 D_{r-2}$ and $P_1 = LV_1$ is the parabolic subgroup of G_1 of semisimple corank 3 corresponding to β, γ, and δ. Written additively, V_1 is the sum of two copies of the trivial module k, one for each A_1 component (these correspond to β and γ, respectively, with L acting by scalars), and the alternating square of the natural module of $L' = A_{r-3}$. The unique root of minimal height of P-level 3 is $\epsilon : = \beta_{r-3} + \delta$ and G_2 is of type A_{r-2} with simple positive system $\Sigma(L) \cup \{\epsilon\}$. Here V_2 is the natural module for L'. We collect this data in our next figure.



So, $A = V_1V_2$ and by 5.2 there is a finite number of B_L-orbits on each V_i. Let v be in $V_1 \setminus \{e\}$. If β and γ are not in $\operatorname{supp} v$, then vV_2 is contained in the unipotent radical of the parabolic subgroup of G corresponding to $\Sigma \setminus \{\sigma_r\}$ (as then every root in $\operatorname{supp} vV_2$ has coefficient 1 at σ_r), and it follows from 5.1 that there is a finite number of B-orbits passing through vV_2 in this instance. On the other hand if either β or γ is in $\operatorname{supp} v$, then each element in vV_2 is P_u-conjugate to v: Let vx' be in vV_2 with $x' \in V_2$. Without loss, suppose that $\beta \in \operatorname{supp} v$. Then for each root ν in $\Psi(V_2)$ there is a unique root μ in $\Psi(P_u)$ such that $\beta + \mu = \nu$. By induction on height we can thus remove x' completely using the action of root elements in U_μ. Observe that, since β has coefficient 0 at σ_r, each root μ involved has coefficient 1 at σ_r. Thus each operation by a root element from U_μ on vx' fixes the other factor of v (as it consists of root elements whose support involve σ_r as a summand), and so x is P_u-conjugate to v in this case. Thus (†) is satisfied.

We have now established Theorem 1.1 in all classical instances.

5.4. — We now address the bulk of the exceptional cases, the ones not covered by 5.1. For that purpose we consider the following condition: suppose that $t = 2$ (*i.e.*, $A = V_1V_2$):

$$(\diamondsuit)\qquad\left\{\begin{array}{l}\text{Suppose that for every pair of roots } (\beta,\gamma) \text{ in } \Psi(V_1)\times\Psi(V_2)\\ \text{the difference } \gamma-\beta \text{ is again a root in } \Psi \ (i.e., \text{ in } \Psi(P_u)) \text{ and}\\ \text{moreover, if this is the case, then we further assume that}\\ \text{for any two roots } \beta_1 \text{ and } \beta_2 \text{ in } \Psi(V_1) \text{ of the same height,}\\ \gamma-\beta_1+\beta_2 \text{ is not a root for any } \gamma\in\Psi(V_2).\end{array}\right.$$

5.4.1. — If $(\diamondsuit)$ is satisfied and $\operatorname{char} k$ is not very bad for G, then every B-orbit passing through an element of $A = V_1V_2$ with nontrivial support in V_1 already has a representative in V_1. The second condition in $(\diamondsuit)$ ensures that every V_2-factor can be removed using a succession of conjugations by suitable root elements in P_u arguing by induction on the height of the roots in $\Psi(V_2)$. Then, since B_L has a finite number of orbits on V_1 and on V_2, there is a finite number of B-orbits on all of $A = V_1V_2$. The advantage of $(\diamondsuit)$ is that it is a purely combinatorial condition.

One checks that in the exceptional cases with $t = 2$ the conditions in $(\diamondsuit)$ are satisfied precisely in the second E_6 and E_7 instances, as well as in the first entries for E_8, F_4, and G_2. Thus, our finiteness result follows in these instances provided $p \neq 2$ if G is of type F_4 or $p \neq 3$ for G_2. Note that for G_2 only the prime 3 leads to an obstruction here. But in the first G_2 entry in Table 1 we require that $p \neq 3$.

5.4.2. — We proceed with the remaining entries for E_6. By 5.1 and 5.4.1 the finiteness result holds for the cases from the first two entries. Recall that we set $P = LP_u = N_G(A)$. In the third case $\Sigma(L) = \Sigma\backslash\{\sigma_1, \sigma_5\}$; in particular, L is of type A_1A_3, $A = \mathcal{C}^1P_u$, and $t = 2$. Let $\alpha_1^1 = {}^{11110}_{\ \ \ 0}$ and $\alpha_1^2 = {}^{01221}_{\ \ \ 1}$. These are the unique roots of different P-shapes and minimal height of P-level 2. Note that V_1^1 is the tensor product of the natural modules for the simple factors of L' and $V_1^2 = k$, while V_2 is the dual of the natural module for the A_3-component of L'. Thus $\dim V_1 = 9$ and $\dim V_2 = 4$. Let v be in $V_1 \setminus \{e\}$. We consider the set of B-orbits passing through vV_2. Let N be the intersection of A with the unipotent radical of the parabolic subgroup corresponding to $\Sigma \setminus \{\sigma_1\}$. Then $A = U_{\alpha_1^2}N$ and B acts on N with a finite number of orbits, by 5.1. Thus we may assume that α_1^2 is in $\operatorname{supp} v$. But if $\alpha_1^2 \in \operatorname{supp} v$, then every element in vV_2 is P_u-conjugate to v: Let vx' be in vV_2 with $x' \in V_2$. One checks that for each root ν in $\Psi(V_2)$ there is a unique root μ in $\Psi(P_u)$ such that $\alpha_1^2 + \mu = \nu$. By induction on height we can thus remove x' completely using the action of root elements in U_μ. Observe that, since α_1^2 has coefficient 0 at σ_1, each root μ involved has coefficient 1 at σ_1. Thus each operation by a root element from U_μ on vx' fixes the other factor of v (as it consists of root elements whose support involve σ_1), and so x is P_u-conjugate to v in this case, as claimed. Whence (†) is fulfilled.

Since the subgroup A from the fourth case for E_6 is conjugate to the one from the third entry by the graph automorphism of G, the result follows readily from the previous discussion by duality.

For the final entry for E_6 we have $\Sigma(L) = \{\sigma_2, \sigma_3, \sigma_5\}$; in particular, L is of type A_1^3, $A = \mathcal{C}^2 P_u$, and $t = 3$. The three "generating roots" for A are precisely the unique ones of minimal height and distinct P-shapes of P-level 3. Observe that V_1 is the direct sum of three copies of natural modules for the three A_1-factors of L'. Thus $\dim V_1 = 6$, $\dim V_2 = 4$, and $\dim V_3 = 2$. Let v be in V_1 and write $v = v^1 v^2 v^3$, where $v^n \in V_1^n$ for $n = 1, 2, 3$. Let N be the intersection of A and the abelian subgroup studied in the second case. Then $A = U_{\alpha_1^1} U_{\alpha_1^1 + \sigma_2} N$ and B acts on N with a finite number of orbits by 5.4.1. Thus we may assume that $\operatorname{supp} v^1$ consists of just one root, *i.e.*, either α_1^1, or $\alpha_1^1 + \sigma_2$. Furthermore, we may also suppose that both $v^2 \neq e$ and $v^3 \neq e$, as otherwise vV_2V_3 is contained in the unipotent radical of the maximal parabolic subgroup corresponding to $\Sigma \setminus \{\sigma_1\}$, respectively $\Sigma \setminus \{\sigma_6\}$, and thus again, there is a finite number of B-orbits passing through vV_2V_3 by 5.1 in this event. One checks that under these assumptions every element in vV_2V_3 is conjugate to v under P_u: Let $x = v^1 v^2 v^3 x'$ be in vV_2V_3 with $x' \in V_2V_3$. Since each of the modules V_1^n is the natural representation for one of the A_1-components in L', the support of each v^n consists of at most two roots for each $n = 1, 2, 3$. While fixing the factor $v^2 v^3$ all roots in $\operatorname{supp} x'$ except ϱ or $\varrho - \sigma_2$ can be removed acting suitably on v^1. The factor $v^2 v^3$ is fixed by this procedure, as each of the roots involved in the conjugation has coefficient 0 at σ_1 and at σ_6. Furthermore, the single remaining root element in x' can be removed acting on v^2 while fixing $v^1 v^3$ (as the conjugating root has σ_1 as a summand). Hence we have established (†).

5.4.3. — The desired finiteness result for the first two cases for E_7 was established in 5.1 and 5.4.1. We now treat the 5 remaining ones in a similar inductive manner as for E_6. For A in the third entry $\Sigma(L) = \Sigma \setminus \{\sigma_5\}$, so L is of type $A_2 A_4$, where $A = \mathcal{C}^1 P_u$, and $t = 2$. Here V_1 is the tensor product of the natural modules of the simple components of L' and V_2 is the dual of the natural module of the A_4-factor of L'. Thus $\dim V_1 = 15$ and $\dim V_2 = 5$. Suppose that v is in $V_1 \setminus \{e\}$. We may assume that there is at least one root in $\operatorname{supp} v$ with coefficient 0 at σ_7 (there are 5 such roots), as else vV_2 is in the abelian subgroup treated in the first case. In fact, we may suppose that there is precisely one such root in $\operatorname{supp} v$, the one of minimal height. The other ones can be removed using suitable root group elements from $(B_L)_u$. Moreover, we may suppose that there is a root in $\operatorname{supp} v$ whose coefficient at σ_3 is 1, (there are six such roots), as else vV_2 is contained in the subgroup considered in the second case. With these constraints one checks that either every element in vV_2 is conjugate to v under B_u, or else the support of an orbit representative consists either of ${}^{01\,2210}_{1}$ and ϱ, or of ${}^{11\,2210}_{1}$ and $\varrho - \sigma_1$. Then, using the action of T, we can scale the coefficients of the associated root elements to equal 1 in each event. Whence the condition in (†) is satisfied.

For A from the fourth entry for E_7 we have $\Sigma(L) = \Sigma \setminus \{\sigma_3, \sigma_6\}$, so L is of type $A_1^2 A_3$, $A = \mathcal{C}^2 P_u$, and $t = 3$. Observe that the two roots defining A are the unique ones of minimal height of P-level 3 and different P-shapes, so we denote them by α_1^1 and α_1^2 respectively. Let v be in V_1 with $v = v^1 v^2$, where $v^n \in V_i^n$ for $n = 1, 2$. Let N be the intersection of A and the abelian subgroup studied in the second E_7 entry. Then $A = U_{\alpha_1^1} U_{\alpha_1^1 + \sigma_1} N$ and B acts on N with a finite number of orbits by 5.4.1. So we may suppose that $v^1 \neq e$. More specifically, we may assume that $\operatorname{supp} v^1$ consists of either α_1^1, or $\alpha_1^1 + \sigma_1$. Moreover, we may also assume that there is one root in $\operatorname{supp} v^2$ whose coefficient at σ_5 is 1 (there are two such roots), as else vV_2V_3 is contained in the abelian subgroup from the third entry. This leads to a small list of possibilities and one checks that in each one of them any element in vV_2V_3 is conjugate to v under P_u, and so (†) is satisfied: Let $x = vx'$ be in vV_2V_3, with $x' \in V_2V_3$. Then one checks that all roots in $\operatorname{supp} x'$ can be removed except possibly one (it is either ϱ, or $\varrho - \sigma_1$, depending on the single root in $\operatorname{supp} v^1$). However, since $\operatorname{supp} v$ also contains roots with coefficient 1 at σ_5, the single root possibly remaining in $\operatorname{supp} x'$ can be removed without reintroducing any new ones. Thus $x = vx'$ is B_u-conjugate to v in this event. We have established (†) also in this case.

Next we consider the sixth case for E_7. Here $\Sigma(L) = \Sigma \setminus \{\sigma_2, \sigma_7\}$, so L is of type A_5, $A = \mathcal{C}^1 P_u$, and $t = 2$. Let $v = v^1 v^2$ be in $V_1 \setminus \{e\}$. Note that V_1^2 is the trivial module k, and thus $\operatorname{supp} v^2$ consists of at most one element. If $\alpha_1^2 = {}^{12\,3210}_{2}$ is not in $\operatorname{supp} v$, then vV_2 is contained in the unipotent radical of the maximal parabolic associated to $\Sigma \setminus \{\sigma_7\}$ and we are done. Else, (*i.e.*, when $v^2 \neq e$) every element in vV_2 is P_u-conjugate to v itself: Let x be in vV_2. For every root ν in $\Psi(V_2)$ there is a unique root μ in $\Psi(P_u)$ such that $\alpha_1^2 + \mu = \nu$. By induction on height we can remove the factor of x in V_2 completely using the action of suitable root elements in U_μ. Since σ_7 must be a summand of μ, any such operation fixes the factor v^1, and thus x is indeed P_u-conjugate to v in this instance.

Now let A be as in the fifth entry for E_7. Here $\Sigma(L) = \Sigma \setminus \{\sigma_4, \sigma_7\}$, so L is of type $A_1 A_2^2$, $A = \mathcal{C}^2 P_u$, and $t = 3$. The two roots defining A are the unique ones of minimal height of P-level 3 and of different P-shapes. We denote them by α_1^1 and α_1^2 respectively. Let v be in V_1 with $v = v^1 v^2$, where $v^n \in V_1^n$ for $n = 1, 2$ and let $x = vx'$ be in vV_2V_3, with $x' \in V_2V_3$. Let N be the intersection of A and the abelian subgroup from the sixth E_7 entry just treated. Then $A = U_{\alpha_1^2} N$ and B acts on N with a finite number of orbits by the result of the previous paragraph. So we may suppose that $\alpha_1^2 = \operatorname{supp} v^2$. Observe that all but possibly three roots can then already be removed from $\operatorname{supp} x'$. Furthermore, we may assume that there is precisely one root in $\operatorname{supp} v^1$ whose coefficient at σ_5 is 1 (there are three such roots, and those of larger height can be removed using root elements from $(B_L)_u$), as otherwise vV_2V_3 is contained in the abelian subgroup treated the third case. One checks that then the two roots of smaller height of the three possible remaining ones in $\operatorname{supp} x'$ can be removed without introducing any new ones. If there are any other roots left in $\operatorname{supp} v$ then the final possible root remaining in $\operatorname{supp} x'$ can also be removed and consequently x is B_u-conjugate to v. If there are no further roots in $\operatorname{supp} v$, then x is supported by at most three linearly independent roots. Once again this establishes (†) in this instance.

Finally, we address the last E_7 entry. In this case $\Sigma(L) = \{\sigma_1, \sigma_2, \sigma_4, \sigma_6\}$, so L is of type $A_1^2 A_2$, $A = \mathcal{C}^3 P_u$, and $t = 4$. Let v be in V_1. Let $V = V_2V_3V_4$. We may suppose that ${}^{122111}_{1}$ is in $\operatorname{supp} v$ (note $V_1^3 = k$), as else vV is contained in the subgroup we studied in the third case (this also applies for $v = e$). Furthermore, we may suppose that there is a root in $\operatorname{supp} v$ whose coefficient at σ_3 equals 1, else vV is contained in the abelian subgroup from the second entry (there are four such roots). These two conditions together already lead to a small list of possible configurations in this event, and it turns out that then every element in vV is already P_u-conjugate to v: Let $x = vx'$ be in vV with $x' \in V$. Since ${}^{122111}_{1}$ is in $\operatorname{supp} v$, all but possibly one of the roots in $\operatorname{supp} x'$ can be removed using root subgroups from P_u. The remaining root in $\operatorname{supp} x'$ can then be removed by acting on one of the root elements in v relative to a root with coefficient 1 at σ_3 without introducing any new roots from $\Psi(V)$. Thus we have (†) also in this case. This completes the discussion for E_7.

5.4.4. — For E_8 the finiteness result for the first case follows from 5.4.1. We treat the remaining 7 cases again in the same inductive manner as done for E_6 and E_7. In the second case $\Sigma(L) = \Sigma \setminus \{\sigma_5\}$, so L is of type A_3A_4, $A = \mathcal{C}^2 P_u$, and $t = 3$. Here $\dim V_1 = 20$, $\dim V_2 = 10$, and $\dim V_3 = 4$. Let v_1 be in V_1. We may suppose that there is a root in $\operatorname{supp} v_1$ whose coefficient at σ_7 equals 1, as else $v_1 V_2 V_3$ is contained in the subgroup from the first case (also for $v_1 = e$). There are ten such roots. Using the action of $(B_L)_u$ we may suppose that there are at most two such (orthogonal) roots in $\operatorname{supp} v_1$. Let $x = v_1 v_2 v_3$ be in $V_1 V_2 V_3$, where $v_i \in V_i$ for $i = 2, 3$ and $v_2 v_3 \neq e$. One checks that for a fixed $\beta \in \Psi(V_1)$ there are precisely four roots $\gamma \in \Psi(V_2)$ such that $\gamma - \beta$ is not a root and there is precisely one root $\delta \in \Psi(V_3)$ such that $\delta - \beta$ is not a root in Ψ. Let τ be a root in the support of v_1 of minimal height and with coefficient 1 at σ_7. Thus, we may assume that all roots of the form $\tau + \eta$ with $\eta \in \Psi(B_L)$ have been removed from the support of v_1 and that v_2 consists of at most four root elements and that either $v_3 = e$, or v_3 is a single root element (using the action of P_u). If there are no further roots in the support of v_1, then x is supported by at most six linearly independent roots and we can use the maximal torus T in B to scale the coefficients in the remaining root elements of x to equal 1. Now suppose there are two (orthogonal) roots in $\operatorname{supp} v_1$ each with coefficient 1 at σ_7, say τ and γ, and that all roots of the form $\tau + \eta$ with $\eta \in \Psi(B_L)$ have been removed from the support of v_1, likewise for γ. Then all of the 5 remaining roots but possibly one in $\operatorname{supp} v_2$ can be removed from $\operatorname{supp} v_2 v_3$ without introducing any new roots. If there are no further roots in $\operatorname{supp} v_1$, then x is supported by three linearly independent roots. Furthermore, if there are still any additional roots in $\operatorname{supp} v_1$ (necessarily orthogonal to τ), then (using the action of P_u) we can remove the final root element factor from v_2 without reintroducing any new root elements or ones which have already been removed. Thus, in this latter instance we are left with an orbit representative in V_1. Consequently, we have a finite number of B-orbits on all of $A = V_1 V_2 V_3$, as desired.

For A as in the third entry $\Sigma(L) = \Sigma \setminus \{\sigma_2\}$, so L is of type A_7, $A = \mathcal{C}^1 P_u$, and $t = 2$. Here V_1 is the alternating square of the natural module of A_7, thus $\dim V_1 = 28$ and V_2 is the 8-dimensional natural A_7-module. Let v be in V_1. We may suppose that there is at least one root in $\operatorname{supp} v$ whose coefficient at σ_5 is 2, as otherwise vV_2 is contained in the subgroup studied in the second case. There are just six such roots in $\Psi(V_1)$. We may further suppose that there are at most two such orthogonal roots in $\operatorname{supp} v$. If needed, the other four can be removed using suitable root elements from B_L. Now let $x = vx'$ be a B-orbit representative in $A = V_1V_2$, where $x' \in V_2$. One checks that for a fixed $\beta \in \Psi(V_1)$ there are precisely two roots $\gamma \in \Psi(V_2)$ such that $\gamma - \beta$ is not a root. Let δ be a root in the support of v with coefficient 2 at σ_5. Thus, we may assume that all roots of the form $\delta + \eta$ with $\eta \in \Psi(B_L)$ have been removed from the support of v and that x' consists of at most two root elements. If there is a second root in $\operatorname{supp} v$ with coefficient 2 at σ_5 (orthogonal to δ), then we can remove both remaining root element factors of x' and thus x is B_u-conjugate to v. So we may suppose that δ is the single root in $\operatorname{supp} v$ with coefficient 2 at σ_5. If there are no further roots in the support of v, then x is supported by at most three linearly independent roots and we use the maximal torus T in B to scale the coefficients in the various root elements to 1 and obtain a single orbit representative. Furthermore, one checks that if there are any extra roots in $\operatorname{supp} v$, apart form the ones already removed (*i.e.*, necessarily orthogonal ones to δ), then we can remove the factor x' by conjugating with suitable root elements from P_u, without reintroducing any root factors which have already been removed and we are left with an orbit representative in V_1. Consequently, we have a finite number of B-orbits on $A = V_1V_2$ in each event.

Now consider the fourth case. Here $\Sigma(L) = \Sigma \setminus \{\sigma_1, \sigma_7\}$, so L is of type A_1D_5, $A = \mathcal{C}^2P_u$, and $t = 3$. Let $v = v^1v^2$ be in V_1. Note that V_1^1 is a 16-dimensional spin module for the D_5-component of L' and V_1^2 is the natural module for the A_1 factor. We may suppose that there is precisely one of the two roots in $\operatorname{supp} v^2$ with coefficient 1 at σ_7, as otherwise vV_2V_3 is contained in the subgroup studied in the first case (also for $v = e$). In addition, we may suppose that there are roots in $\operatorname{supp} v$ whose coefficient at σ_5 is 2, otherwise vV_2V_3 is contained in the subgroup from the second case. Let $x = vx'$ be in vV_2V_3 with $x' \in V_2V_3$. Acting on v^2 with suitable root elements from P_u we can remove all but possibly one root from $\operatorname{supp} x'$. This fixes the v^1-factor, as each of the roots involved has coefficient 0 at σ_1. Acting on v^1 we can remove this remaining root from $\operatorname{supp} x'$ while fixing v^2, as any root element needed here has coefficient 1 at σ_1 and thus commutes with v^2. Therefore, these two conditions combined imply that every element in vV_2V_3 is already conjugate to v under P_u and thus (†) is satisfied.

For the fifth E_8 entry $\Sigma(L) = \Sigma \setminus \{\sigma_3, \sigma_7\}$. So L is of type $A_1^2A_4$, $A = \mathcal{C}^3P_u$, and $t = 4$. Let $v = v^1v^2$ be in V_1. We write V for $V_2V_3V_4$. The two defining roots for A are the unique ones of minimal height and different P-shapes of level 4. Note that $\dim V_1 = 14$ and $\dim V = 17$. We may suppose in this instance that there is precisely one of the two roots in the support of $v^2 \in V_1^2$ with coefficient 1 at σ_1, as else vV is contained in the subgroup treated in the fourth case (this also applies for the case $v = e$). Let $x = vx'$ be in vV with $x' \in V$. One checks that all put possibly six roots in $\operatorname{supp} x'$ can be removed acting on v^2. In addition, we may suppose that there is precisely one of the three roots in $\operatorname{supp} v^1$ with coefficient 2 at σ_5, as else vV is contained in the subgroup from the second case. Then we may remove up to four more roots from $\operatorname{supp} x'$. If there are no other roots in $\operatorname{supp} x$, then x is supported by at most four linearly independent roots, and we are done. Else, x is B_u-conjugate to v, and (†) is satisfied.

If A is as in the sixth entry for E_8, then $\Sigma(L) = \Sigma \setminus \{\sigma_4, \sigma_7\}$. So L is of type $A_1^2 A_2^2$, $A = \mathcal{C}^4 P_u$, and $t = 5$. Write V for $V_2 \cdots V_5$. Let $v = v^1 v^2$ be in V_1. The two defining roots for A are the unique ones of minimal height and different P-shapes of level 5. Note that $\dim V_1 = 12$ and $\dim V = 20$. Let $x = vx'$ be in vV with $x' \in V$. We may suppose that there is precisely one (of the two) roots in $\operatorname{supp} v^2$ with coefficient 2 at σ_3, as else vV is contained in the subgroup studied in case 5 (this also applies for the case $v = e$). Then we can remove all but nine roots from $\operatorname{supp} x'$. Furthermore, we may suppose that one of the two roots with coefficient 2 at σ_5 is in $\operatorname{supp} v^1$. Otherwise vV is contained in the subgroup from in case 2. Acting on root elements in v of the second kind we can eliminate up to another six roots in $\operatorname{supp} x'$. If there are no additional roots in $\operatorname{supp} v$, then x is supported by at most five linearly independent roots and we may use the action of T. Otherwise, all of the remaining roots in $\operatorname{supp} x'$ can be removed, so that x is conjugate to v under B_u, as desired.

For A from the seventh E_8 entry $\Sigma(L) = \Sigma \setminus \{\sigma_2, \sigma_6\}$. So L is of type $A_2 A_4$, $A = \mathcal{C}^3 P_u$, and $t = 4$. Write again V for $V_2 V_3 V_4$. Let $v = v^1 v^2$ be in V_1. We may suppose that there is precisely one of the three roots in $\operatorname{supp} v^2$ with coefficient 2 at σ_5, as else vV is contained in the subgroup studied in case 2 (this also applies when $v = e$). Note that $V_1^1 = k$. Moreover, we may assume that $\alpha_1^1 = {}^{1233321}_{1}$ is in $\operatorname{supp} v$, as otherwise vV is contained in the maximal abelian subgroup from case 3. These conditions imply that either an element in vV is P_u-conjugate to v itself, or it belongs to a B_u-orbit with a representative x in vV whose support consists of three linearly independent roots, α_1^1, $\varrho - (\sigma_6 + \sigma_7 + \sigma_8)$, and one of α_1^2, $\alpha_1^2 + \sigma_8$, or $\alpha_1^2 + \sigma_7 + \sigma_8$. In each event we use the action of T to scale the coefficients of the corresponding root elements to 1 and so (†) is satisfied.

Finally, we address the last E_8 case. Here $\Sigma(L) = \Sigma \setminus \{\sigma_2, \sigma_5, \sigma_7\}$ and L is of type $A_1^2 A_3$, $A = \mathcal{C}^5 P_u$, and $t = 6$. Write again V for $V_2 \cdots V_6$. Here

$\dim V_1 = 11$ and $\dim V = 22$. Let $v = v^1v^2v^3$ be in V_1 and as before let $x = vx'$ be in vV with $x' \in V$. Note that $V_1^2 = k$ and so $|\operatorname{supp} v^2| \leq 1$. We may suppose that $\alpha_1^2 = {}^{1232221}_{2}$ is in $\operatorname{supp} v$, as otherwise vV is contained in the maximal abelian subgroup from case 2. This is the single root in $\Psi(V_1)$ with coefficient 2 at σ_5. Acting on v^2 with root elements from P_u we can remove all but possibly up to eight roots from $\operatorname{supp} x'$. Also, we may suppose that there is precisely one of the two roots in $\operatorname{supp} v^3$ with coefficient 3 at σ_4, as else vV is contained in the maximal abelian subgroup from case 6. Then we can act on v^3 and remove an additional four roots from x' without reintroducing any new ones. If there are no other roots in $\operatorname{supp} v$, then x is supported by at most eight linearly independent roots and we are done using the action of T. Otherwise, we can use other root elements to completely eliminate x', and thus x is conjugate under B_u to v. This completes the discussion for E_8.

5.4.5. — We proceed with the remaining events for F_4. The desired result for the first entry in Table 1 for F_4 was established for $p \neq 2$ in 5.4.1. First we consider the same instance when $p = 2$. As before, $A = \mathcal{C}^1 P_u$, where $P = N_G(A)$. Although $(\diamondsuit)$ is satisfied, the commutator relations in F_4 are degenerate when $p = 2$ for short root subgroups spanning a subsystem of Ψ of type B_2. Let v_1 be in $V_1 \setminus \{e\}$. We consider the set of B-orbits passing through v_1V_2. Since the two roots in $\Psi(V_2)$ are long and there are no degeneracies for the commutators of long root subgroups in $\Psi(B)$, we can argue as above, provided there is a long root in $\operatorname{supp} v_1$. Consequently, we only need to consider the case when v_1 is supported entirely on short roots. Since any two short roots in $\Psi(V_1)$ span a subsystem of Dynkin type A_2 (and the structure constants in the corresponding group are ± 1), we may assume (possibly after using the action of $(B_L)_u$) that v_1 is a single short root element. Thus, any element in $v_1V_2 = v_1U_{1342}U_{2342}$ is supported by at most three linearly independent roots, and we can use the action of the torus T in B to obtain a finite set of orbit representatives in this case as well.

Next we turn to the remaining F_4 cases. It is advantageous to first consider a particular normal abelian subgroup of B which, although not maximal, still fits the setting of 5.2. Namely, let V be the normal closure in B of U_δ, where $\delta = 0122$. Let $P = N_G(V)$. Then $P = LP_u$ is the maximal parabolic subgroup of G of type B_3, $V = Z(P_u)$, and $\dim V = 7$. We can apply the same construction as in 5.2. Let Ψ_1 be the closed semisimple subsystem of Ψ formed by $\Psi(L)$ together with $\pm\Psi(V)$ and positive simple system consisting of $\Sigma(L)$ and δ. Let G_1 be the semisimple subgroup of G corresponding to Ψ_1. Then G_1 is of type B_4 and $P_1 = LV$ is the maximal parabolic subgroup of G_1 corresponding to the simple root δ. Since V is abelian, L is a spherical Levi subgroup of G_1 by Remark 4.4. Hence, B_L and thus B has a finite number of orbits on V. Note this is valid also for $p = 2$.

Now in the second F_4 incident on our list $A = U_{1221}U_{1231}V$. Thus, by the last paragraph, it suffices to consider orbit representatives whose support contains at least one of 1221 or 1231. One easily checks that there is only a finite number of such B-orbits passing through A.

In the third case for F_4 we have $A = U_{0121}U_{1121}U_{1221}U_{1231}V$. As above, we only need to consider elements whose support involves at least one of the first four roots shown. Keeping in mind that $p = 2$ in this instance, one checks directly that there is only a finite number of such B-orbits in A.

The commutative groups from the last two F_4 entries (both for $p = 2$ only) share the 10-dimensional normal subgroup of B which is generated by U_{1121} and U_{0122}. Thus, by the previous result, in the fourth case we only need to show that there are finitely many B-orbits passing through elements in A whose support involves 1111. This is easily verified.

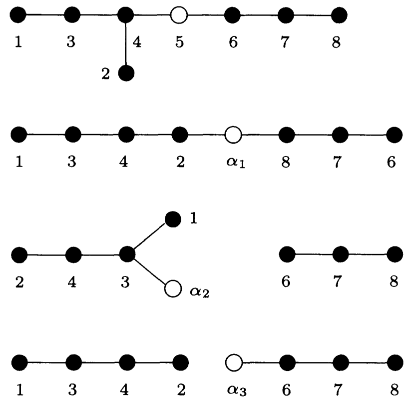
5.4.6. — Finally, the result for the second G_2 case is readily checked directly.

This completes the proof of Theorem 1.1.

We close this section by illustrating the construction from 5.2 once again with two examples from Table 1, this time for exceptional groups.

Examples 5.5. — Our first example is the second entry for E_8 from Table 1. The Levi subgroup of $N_G(A) = P = LP_u$ is of type A_3A_4, $\ell(P_u) = 5$, $A = \mathcal{C}^2P_u$, and $t = 3$. As shown, G_i is of type A_8, A_3D_5, or A_4A_4 for $i = 1, 2, 3$, respectively. As before we indicate $\Sigma(L)$ by coloring the corresponding nodes in the diagram of G and G_i. Moreover, α_i, $\dim V_i$, and the various passive components of G_i are specified as well.

We also present the first case for F_4. Here the Levi subgroup of $N_G(A)$ is of type $A_1\widetilde{A}_2$, where $\widetilde{A}_2$ is the subsystem of Ψ spanned by the short simple roots, and G_1 and G_2 are of type A_1C_3 and $A_2\widetilde{A}_2$, respectively.

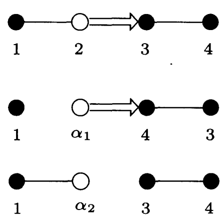


G, P: $\ell(P_u) = 5$

G_1, P_1: $\alpha_1 = \substack{1233210\\1}$, $\dim V_1 = 20$

G_2, P_2: $\alpha_2 = \substack{1244321\\2}$, $\dim V_2 = 10$

G_3, P_3: $\alpha_3 = \substack{2465321\\3}$, $\dim V_3 = 4$



G, P: $\ell(P_u) = 3$

G_1, P_1: $\alpha_1 = 1220$, $\dim V_1 = 6$

G_2, P_2: $\alpha_2 = 1342$, $\dim V_2 = 2$

6. Abelian ideals of Lie P.

Let G, P, and A be as in the Introduction. If $\operatorname{char} k$ is zero, then the exponential mapping is a P-equivariant morphism between $\mathfrak{a} = \operatorname{Lie} A$ and A. If $\operatorname{char} k$ is a good prime for G, we can make use of Springer's map $\varphi : \mathcal{U} \longrightarrow \mathcal{N}$ between the unipotent variety $\mathcal{U}$ of G and the nilpotent variety $\mathcal{N}$ of $\mathfrak{g}$ which is a G-equivariant bijective morphism (see [35]), and, upon "restriction" of φ to A, we obtain a P-equivariant bijective morphism from A onto $\mathfrak{a}$, *e.g.*, see [31, Thm 4.1]. Note that $\mathfrak{a}$ is an abelian ideal in $\mathfrak{p}$. Consequently, we get a statement analogous to Theorem 1.1 for the adjoint action of P on P-invariant linear subspaces of $\operatorname{Lie} P_u$ which are abelian as subalgebras of $\mathfrak{p}$. Thanks to a result of Pyasetskii [27], which is also valid in positive characteristic, we also obtain a statement for the coadjoint action of P on $\mathfrak{a}^*$ similar to Theorem 1.1.

The characteristic restrictions for the adjoint action can be removed completely in the exceptional cases in Table 1 by employing the computer algorithm outlined in [14]. This algorithm is valid provided p is not very bad for G. The remaining cases when $p = 2$ for F_4 and G_2 can be verified directly.

7. Abelian subalgebras of $\mathfrak{g}$.

In [21] A.I. Mal'cev determined all abelian subalgebras of maximal dimension in each simple complex Lie algebra $\mathfrak{g}$, up to G-conjugacy, extending work of I. Schur [34], *i.e.*, the special case for $\mathfrak{sl}_n(k)$.

We give an approach to Mal'cev's result in arbitrary characteristic utilizing the information in Table 1 above. Let $\mathfrak{s}$ be a solvable subalgebra of $\mathfrak{g}$. Without loss, we may assume that $\mathfrak{s} \subset \mathfrak{b}$ (Lie-Kolchin). Let X be the Grassmann variety consisting of flags of subspaces $\mathfrak{s}' \subset \mathfrak{b}'$ in $\mathfrak{g}$, where $\dim \mathfrak{s}' = \dim \mathfrak{s}$ and $\dim \mathfrak{b}' = \dim \mathfrak{b}$. Then X is projective and G operates on it via the adjoint action. Let Y be the closure of the G-orbit in X of the flag $\mathfrak{s} \subset \mathfrak{b}$. Whence Y is complete. Observe that Y consists of flags $\mathfrak{s}' \subset \mathfrak{b}'$ in $\mathfrak{g}$, where $\mathfrak{s}'$ is a solvable subalgebra of $\mathfrak{g}$ and $\mathfrak{b}'$ is a Borel subalgebra of $\mathfrak{g}$. By Borel's fixed point theorem [2, 10.4] there is a flag $\mathfrak{s}_0 \subset \mathfrak{b}_0$ in Y which is fixed by B. Thus $\mathfrak{b} = \mathfrak{b}_0$ and $\mathfrak{s}_0$ is an ideal in $\mathfrak{b}$. In other words, there is always an ideal of $\mathfrak{b}$ in the closure of the G-orbit in $\mathfrak{g}$ of any solvable subalgebra $\mathfrak{s}$ of $\mathfrak{b}$ (possibly of different isomorphism type, but of the same dimension as $\mathfrak{s}$).

Considering the particular case when $\mathfrak{s}$ is an abelian subalgebra of $\mathfrak{g}$, the maximal possible dimensions of these can be read off from Table 1 above by the aforementioned construction. Comparing the information in this table with Mal'cev's list, we observe that in all instances, with the exception of G_2, every abelian subalgebra of $\mathfrak{g}$ of maximal dimension is itself conjugate to an abelian ideal of $\mathfrak{b}$ under G (provided p is not very bad for G). In G_2 there are three classes of abelian subalgebras of $\mathfrak{g}$ of maximal dimension 3 ($p \neq 2$); but there is only one such class containing the abelian ideal $\operatorname{Lie} A$ of $\mathfrak{b}$, where A is as in Table 1.

Although, it is not directly related to the questions addressed in this paper, we should like to mention recent work of B. Kostant [19], extending earlier results from [18], where the family of *all* abelian ideals $\mathfrak{a}$ of the Borel subalgebra $\mathfrak{b}$ of a simple complex Lie algebra $\mathfrak{g}$ plays an important rôle. Motivated by Mal'cev's work [21], Kostant constructs inequivalent irreducible G-submodules in the exterior algebra $\wedge\mathfrak{g}$ of $\mathfrak{g}$, one for each abelian ideal $\mathfrak{a}$ of $\mathfrak{b}$ in [18], where $\operatorname{Lie} G = \mathfrak{g}$. In his recent summary [19], Kostant gives an account of D. Peterson's remarkable theorem that the number of abelian ideals in $\mathfrak{b}$ equals 2^r, where $r = \operatorname{rank} \mathfrak{g}$. He determines the structure of each such ideal $\mathfrak{a}$ in terms of a certain Cartan decomposition of $\mathfrak{g}$, and moreover, exhibits a close connection between the discrete series representations and the abelian ideals $\mathfrak{a}$.

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