ANNALES DE L'INSTITUT FOURIER

PAUL GAUDUCHON
LIVIU ORNEA

Locally conformally Kähler metrics on Hopf surfaces

Annales de l'institut Fourier, tome 48, n° 4 (1998), p. 1107-1127
<http://www.numdam.org/item?id=AIF_1998__48_4_1107_0>

Ann. Inst. Fourier, Grenoble
48, 4 (1998), 1107–1127

LOCALLY CONFORMALLY KÄHLER METRICS ON HOPF SURFACES

by P. GAUDUCHON & L. ORNEA

To the memory of Franco Tricerri

1. Introduction.

Let M be an even-dimensional, oriented, smooth manifold. A Hermitian structure on M is a pair (J, g) consisting of an *integrable* almost-complex structure J, and a Riemannian metric g such that $g(JX, JY) = g(X, Y)$ for any vector fields X, Y. The Hermitian structure is *Kähler* if J is parallel with respect to the Levi-Civita connection D^g of g; equivalently, as J is integrable, (J, g) is *Kähler* if the *Kähler form* ω, defined by $\omega(X, Y) = g(JX, Y)$, is closed. More generally, (J, g) is called *locally conformally Kähler*, *l.c.K* for short, if for each point x of M there exist an open neighbourhood $\mathcal{U}$ of x and a positive function f on $\mathcal{U}$ so that the pair $(J, f^{-2}g)$ is Kähler, see [7] for a general overview.

When J is understood, we say that g is Kähler, l.c.K. etc. whenever the corresponding Hermitian structure (J, g) is Kähler, l.c.K. etc.

When M is four-dimensional, the only case considered in this paper, the defect for a Hermitian structure (J, g) to be Kähler is measured by the *Lee form*, the real 1-form θ determined by

$$(1) \qquad d\omega = -2\theta \wedge \omega,$$

Key words : Hopf surface – Locally conformal Kähler metric – Sasakian structure – Contact form – Killing vector field – Deformation.
Math. classification : 53C55 – 53C25.

see [18], [23], [8]. (*Warning*: The definition of the Lee form in the literature may differ from the above definition by a factor $\pm\frac{1}{2}$.)

Then, (J, g) is l.c.K if and only if the Lee form θ is closed, i.e. the defect for (J, g) to be l.c.K. is measured by the 2-form $d\theta$.

A special occurence of l.c.K. metrics is when the Lee form θ is *parallel* with respect to the Levi-Civita connection of g. This case reduces to the Kähler case (θ identically zero) if M is compact, with even first Betti number b_1 [25]; if b_1 is odd, θ is (everywhere) non-zero and (M, J, g) is usually called a *generalized Hopf manifold.* The reason for this name is a God-given series of examples on a sub-class of Hopf surfaces (see below); however, in the present context, when the complex surfaces of interest are Hopf surfaces and not all are *generalized Hopf manifolds* (see Remark 1), we may prefer to call them *Vaisman surfaces*, as *e.g.* in [7].

Notice that not all complex surfaces admitting l.c.K. metrics admit l.c.K. metric with parallel Lee form, see [22], [3]. However, if the surface is compact, the Lee form of any l.c.K. structure can always be made *harmonic* by a conformal change of the metric, see [8].

The l.c.K. condition is conformally invariant i.e. concerns the *conformal Hermitian structure* $(J, [g])$, where $[g]$ denotes the conformal class of g. In particular, $d\theta$ is conformally invariant, but θ and ω are not: the Lee form $\tilde{\theta}$ and the Kähler form $\tilde{\omega}$ of the Hermitian structure $(\tilde{g} = f^{-2}g, J)$ are given by $\tilde{\theta} = \theta + \frac{df}{f}$ and $\tilde{\omega} = f^{-2}\omega$. These rules of transformation can be interpreted as follows. On any n-dimensional smooth manifold M, let L be the *bundle of real scalars of weight 1*, the oriented real line determined (via the $G\ell(n, \mathbb{R})$-principal bundle of (all) frames on M) by the representation $A \in G\ell(n, \mathbb{R}) \mapsto |\det A|^{\frac{1}{n}}$. Then, each Riemannian metric g in the conformal class $[g]$ determines a positive section, hence a trivialization, ℓ, of L and the Kähler form ω appears as the expression of a conformally invariant L^2-valued 2-form, say ω_{conf}, with respect to ℓ. In the same way, the Lee form θ can be viewed as the connection 1-form with respect to ℓ of a conformally invariant linear connection ∇^L on L, the so-called *Weyl derivative* determined by the conformal Hermitian structure $(J, [g])$. Finally, the identity (1) means that the *conformal Kähler form* ω_{conf} is *closed* as a L^2-valued 2-form, when L^2 comes equipped with the linear connection ∇^{L^2} induced by ∇^L. Then, the 2-form $d\theta$ is equal (up to the sign) to the curvature 2-form of ∇^L, so that the l.c.K. condition just means that the connection ∇^L is *flat.*

In the case when M is compact and b_1 is even it is well-known that any l.c.K. Hermitian structure is actually *globally* conformal to a Kähler Hermitian structure [25]. Conversely, it is also well-known (but much more difficult to prove, see for example [20], [10], [2]) that any compact complex surface (M, J) with even first Betti number admits a Kähler metric; moreover, many explicit examples of Kähler Hermitian structures are known, many of them provided by the complex algebraic geometry.

The situation when M is compact with *odd* first Betti number is quite different: it is still unknown whether there exist compact complex surfaces with b_1 odd *not* admitting a l.c.K. metric, but, on the other hand, only few examples have been constructed by now in an explicit way: apart from the case of Hopf surfaces considered in the present paper, these are essentially the l.c.K. metrics constructed by F. Tricerri on some classes of *Inoue surfaces* [22] and examples appearing in [5], [6], [26].

A *Hopf surface* is a compact complex surface whose universal covering is $W = \mathbb{C}^2 - \{(0,0)\}$. More precisely, *primary* Hopf surfaces have their fundamental group isomorphic to $\mathbb{Z}$, generated by the transformation γ defined by

$$x = (u, v) \mapsto (\alpha u + \lambda v^m, \beta v), \tag{2}$$

for any $x = (u, v) \in \mathbb{C}^2 - \{(0,0)\}$; here, m is some integer and α, β, λ are complex numbers such that

$$|\alpha| \geq |\beta| > 1, \tag{3}$$

and

$$(\alpha - \beta^m)\lambda = 0. \tag{4}$$

All primary Hopf surfaces are diffeomorphic to the product $S^3 \times S^1$ and any Hopf surface is finitely covered by a primary one, [13], [14].

Following [11], the primary Hopf surfaces fall into two disjoint classes, according to their *Kähler rank*:

– The *Hopf surfaces of class* 1, whose fundamental group is generated by a transformation γ as above for which $\lambda = 0$ and α, β are any two complex numbers satisfying (3); the corresponding Hopf surface will be denoted by $M_{\alpha,\beta}$.

– The *Hopf surfaces of class* 0, corresponding to $\lambda \neq 0$ and $\alpha = \beta^m$ for some positive integer m; the corresponding Hopf surface will be denoted by $\tilde{M}_{\beta,m,\lambda}$; as observed in [11], for any β, m and any two λ, μ in $\mathbb{C}^*$, $\tilde{M}_{\beta,m,\lambda}$ and $\tilde{M}_{\beta,m,\mu}$ are isomorphic as complex surfaces [11, Proposition 60].

For all Hopf surfaces, the complex structure induced by the natural complex structure of $\mathbb{C}^2 - \{(0,0)\}$ is denoted by J; a l.c.K. metric is always understood with respect to J.

The class 1 contains the subclass of $M_{\alpha,\beta}$'s such that $|\alpha| = |\beta|$. For these special Hopf manifolds, a l.c.K. metric is easily constructed as follows. Let ρ be the *distance function to the origin*, i.e. the (smooth) positive real function on W defined by $\rho(x) = \sqrt{|u|^2 + |v|^2}$ for any $x = (u, v)$ in $W = \mathbb{C}^2 - \{(0,0)\}$. Then, $\frac{1}{4} dd^c \rho^2$ is the Kähler form of the natural flat Hermitian metric (here and henceforth, the operator d^c, acting on functions, is defined by $d^c f(X) = -df(JX)$ so that $dd^c f = 2i\partial\bar{\partial} f$). The 2-form $\frac{1}{4\rho^2} dd^c \rho^2$ clearly descends as the Kähler form of a well-defined, obviously l.c.K., Hermitian metric $g_{\alpha,\beta}$ on the Hopf surface $M_{\alpha,\beta}$. It is then easy to check that the Lee form is parallel.

The main goal of this paper is to prove

THEOREM 1. — *Each primary Hopf surface admits a l.c.K. metric. Moreover, each primary Hopf surface of class* 1 *admits a l.c.K. metric with parallel (non-zero) Lee form.*

An explicit construction of a l.c.K. metric with parallel Lee form on each (primary) Hopf surfaces of class 1, as well as an explicit description of the corresponding Sasakian geometry, are given in Sections 2 and 3 (see, in particular, Proposition 1, Corollary 1, Proposition 3 and Remark 6).

Previous attempts to write (non globally defined) l.c.K. metrics on Hopf surfaces appear in [19].

The *existence* of l.c.K. Hermitian metrics on $M_{\alpha,\beta}$ for $|\alpha|$ and $|\beta|$ different, but close to each other, has been proved by C. LeBrun [17]. The argument goes as follows. First, notice that the line bundle L of any Hopf surface $M_{\alpha,\beta}$ is naturally identified to the real line bundle $(W \times \mathbb{R})/\mathbb{Z}$, where the action of $\mathbb{Z}$ on $W \times \mathbb{R}$ is described by $1.((u,v),a) = ((\alpha u, \beta v), |\alpha|^{\frac{1}{2}} |\beta|^{\frac{1}{2}} a)$ (observe that $|\alpha|^{\frac{1}{2}} |\beta|^{\frac{1}{2}}$ is equal to $|\det \gamma_*|^{\frac{1}{4}}$, where γ_* is the differential of γ). The line bundle so defined admits a natural flat connection which coincides with the Weyl derivative ∇^L; in particular the pullback of (L, ∇^L) on W coincides with the trivial line bundle $W \times \mathbb{R}$ equipped with the trivial connection. Finally, the 2-form $\frac{1}{4} dd^c \rho^2$ on W descends on M as a L^2-valued 2-form on $M_{\alpha,\beta}$, which is obviously closed with respect to ∇^L. Then, a deformation argument using the identification $L = (W \times \mathbb{R})/\mathbb{Z}$

(see Section 4) shows that $M_{\alpha,\beta}$ still admits a l.c.K. Hermitian metric for $|\alpha| \neq |\beta|$, provided that $|\alpha|$ and $|\beta|$ are close enough) to each other [17].

In Section 2, following a suggestion in [17], we give an explicit formulation for these l.c.K. metrics and show that the same formulation actually provides a l.c.K. Hermitian metric on *any* (primary) Hopf surface of class 1, see Proposition 1 and Corollary 1.

It then appears that all the l.c.K. Hermitian metrics obtained in this way have a parallel, non-zero, Lee form, and are related to a very simple class of Sasakian structures on the sphere S^3, of which a precise description is given in Section 3 (Proposition 3 and Remark 6).

Finally, LeBrun's argument still applies to prove the second statement of Theorem 1, see Section 4.

Remark 1. — Theorem 1 gives no information as to the existence or the non-existence of (l.c.K.) Hermitian metric with parallel Lee form on Hopf surfaces of class 0. However, this question has been solved recently by F. Belgun [3], so that Theorem 1 can actually be completed by the following statement: *Hopf surfaces of class 0 admit no (l.c.K.) Hermitian metrics with parallel Lee form.*

2. Construction of l.c.K. metrics on $M_{\alpha,\beta}$.

Fix any two complex numbers α, β satisfying (3) and let $\phi_{\alpha,\beta}$ the (smooth) function determined on W by

$$|u|^2|\alpha|^{-2\phi_{\alpha,\beta}(x)} + |v|^2|\beta|^{-2\phi_{\alpha,\beta}(x)} = 1, \tag{5}$$

for any $x = (u, v)$ in W. Notice that $\phi_{\alpha,\beta}$ is well-defined since, for x fixed, the function $t \mapsto |u|^2|\alpha|^t + |v|^2|\beta|^t$ is strictly increasing from 0 to $+\infty$.

Then, $\phi_{\alpha,\beta}$ satisfies the following equivariance property:

$$\phi_{\alpha,\beta}(\gamma \cdot x) = \phi_{\alpha,\beta}(x) + 1 \tag{6}$$

for any x in $\tilde{M}$.

Indeed, we have

$$\begin{aligned} 1 &= |\alpha u|^2|\alpha|^{-2\phi_{\alpha,\beta}(\gamma\cdot x)} + |\beta v|^2|\beta|^{-2\phi_{\alpha,\beta}(\gamma\cdot x)} \\ &= |u|^2|\alpha|^{2-2\phi_{\alpha,\beta}(\gamma\cdot x)} + |v|^2|\beta|^{2-2\phi_{\alpha,\beta}(\gamma\cdot x)} \\ &= |u|^2|\alpha|^{-2\phi_{\alpha,\beta}(x)} + |v|^2|\beta|^{-2\phi_{\alpha,\beta}(x)} \end{aligned}$$

for any x in W.

We then get a diffeomorphism from the corresponding Hopf surface (of class 1) $M_{\alpha,\beta}$ onto the product $S^3 \times S^1$ as follows (here S^3 denotes the 3-dimensional sphere, viewed as the unit sphere in $\mathbb{C}^2$, and S^1 denotes the circle, identified with the quotient $\mathbb{R}/\mathbb{Z}$). Let $\tilde{\psi}$ be the map from W to $S^3 \times S^1$ defined by

$$x = (u, v) \mapsto ((u\,\alpha^{-\phi_{\alpha,\beta}(x)}, v\,\beta^{-\phi_{\alpha,\beta}(x)}), \phi_{\alpha,\beta}(x) \bmod \mathbb{Z}). \tag{7}$$

Then, due to (6), $\tilde{\psi}$ is γ -invariant hence determines a map, ψ, from $M_{\alpha,\beta}$ to $S^3 \times S^1$, which is clearly a diffeomorphism; the inverse ψ^{-1} is given by

$$(z, t \bmod \mathbb{Z}) \mapsto [u = \alpha^t z_1, v = \beta^t z_2], \tag{8}$$

where $z = (z_1, z_2)$, $|z_1|^2 + |z_2|^2 = 1$, is a point of $S^3 \subset \mathbb{C}^2$ and $t \bmod \mathbb{Z}$ is an element of $S^1 = \mathbb{R}/\mathbb{Z}$; here, $[u, v]$ denotes the class of (u, v) mod. $\Gamma_{\alpha,\beta}$. Observe that the diffeomorphism ψ depends on the choice of an argument for α and for β, say $\mathfrak{Arg}\,\alpha$ and $\mathfrak{Arg}\,\beta$.

Remark 2. — For any choice of $\mathfrak{Arg}\alpha$ and $\mathfrak{Arg}\beta$, the above action of $\mathbb{Z}$ on $\tilde{M}$ is the restriction of an action of the (additive) group $\mathbb{R}$ defined by

$$t \cdot (u, v) = (\alpha^t u, \beta^t v), \tag{9}$$

for any t in $\mathbb{R}$. Then, $\phi_{\alpha,\beta}$ can be described as follows: for any $x = (u, v)$ in $\tilde{M}$, $\phi_{\alpha,\beta}(x)$ is the unique element of $\mathbb{R}$ such that $(-\phi_{\alpha,\beta}(x)) \cdot x$ belongs to the unit sphere of $\mathbb{C}^2$.

We denote by $\Phi_{\alpha,\beta}$ the real positive function on W defined by $\Phi_{\alpha,\beta} = e^{(k_1+k_2)\phi_{\alpha,\beta}}$; alternatively, $\Phi_{\alpha,\beta}$ is determined by

$$\rho_1^2\,\Phi_{\alpha,\beta}^{-\frac{2k_1}{k_1+k_2}} + \rho_2^2\,\Phi_{\alpha,\beta}^{-\frac{2k_2}{k_1+k_2}} = 1, \tag{10}$$

where k_1, k_2 are the (positive) real numbers given by

$$k_1 = \ln|\alpha|,\ k_2 = \ln|\beta|, \tag{11}$$

and ρ_1, ρ_2 are the functions on $\tilde{M}$ defined by

$$\rho_1(x) = |u|,\ \rho_2(x) = |v|. \tag{12}$$

In this notations (3) translates to

$$k_1 \geq k_2 > 0. \tag{13}$$

Then, by (6), $\Phi_{\alpha,\beta}$ satisfies the following equivariance property with respect to the action of γ:

$$\Phi_{\alpha,\beta}(\gamma \cdot x) = |\alpha||\beta| \cdot \Phi_{\alpha,\beta}(x). \tag{14}$$

In other words, $\Phi_{\alpha,\beta}$ descends on $M_{\alpha,\beta}$ as a (positive) section of L^2.

PROPOSITION 1. — *For any pair of complex numbers α, β satisfying (3), the real 2-form $\frac{1}{4}dd^c\Phi_{\alpha,\beta}$ is the Kähler form of a Hermitian metric on $\tilde{M}$.*

Proof. — For simplicity, $\Phi_{\alpha,\beta}$ will be denoted by Φ. Then, it follows readily from (10) that the differential of Φ is given by

$$d\Phi = \frac{1}{\Delta}\big(\Phi^{\frac{k_2-k_1}{k_1+k_2}}(ud\bar{u} + \bar{u}du) + \Phi^{\frac{k_1-k_2}{k_1+k_2}}(vd\bar{v} + \bar{v}dv)\big), \tag{15}$$

where Δ is the positive function defined by

$$\Delta = \frac{2k_1\,\rho_1^2\,\Phi^{\frac{-2k_1}{k_1+k_2}} + 2k_2\,\rho_2^2\,\Phi^{\frac{-2k_2}{k_1+k_2}}}{k_1+k_2}. \tag{16}$$

From (15), we infer:

$$\partial^2_{u,\bar{u}}\Phi = \frac{2\Phi^{\frac{k_2-k_1}{k_1+k_2}}}{\Delta^3(k_1+k_2)^2}\big(k_1(k_1+k_2)\rho_1^4\Phi^{\frac{-4k_1}{k_1+k_2}} + 2k_2^2\rho_2^4\Phi^{\frac{-4k_2}{k_1+k_2}} + k_2(k_1+3k_2)\rho_1^2\rho_2^2\Phi^{-2}\big); \tag{17}$$

$$\partial^2_{v,\bar{v}}\Phi = \frac{2\Phi^{\frac{k_1-k_2}{k_1+k_2}}}{\Delta^3(k_1+k_2)^2}\big(k_2(k_1+k_2)\rho_2^4\Phi^{\frac{-4k_2}{k_1+k_2}} + 2k_1^2\rho_1^4\Phi^{\frac{-4k_1}{k_1+k_2}} + k_1(k_2+3k_1)\rho_1^2\rho_2^2\Phi^{-2}\big); \tag{18}$$

$$\partial^2_{u,\bar{v}}\Phi = \frac{2\bar{u}v\Phi^{-1}}{\Delta^3(k_1+k_2)^2}(k_1-k_2)\big(k_1\rho_1^2\Phi^{\frac{-2k_1}{k_1+k_2}} - k_2\rho_2^2\Phi^{\frac{-2k_2}{k_1+k_2}}\big). \tag{19}$$

The claim is that the Hermitian matrix

$$A = \begin{pmatrix} \partial^2_{u,\bar{u}}\Phi & \partial^2_{u,\bar{v}}\Phi \\ \partial^2_{v,\bar{u}}\Phi & \partial^2_{v,\bar{v}}\Phi \end{pmatrix}$$

is *positive*; this in turn is equivalent to the fact that the trace and the determinant are positive.

By (17) and (18), $\partial^2_{u,\bar{u}}\Phi$ and $\partial^2_{v,\bar{v}}\Phi$ are both positive; it then remains to check that the determinant of A is positive. By a straightforward calculation, this determinant is equal to

$$\begin{aligned}\det A = \frac{8}{\Delta^6(k_1+k_2)^3}\big(&k_1^3\rho_1^8\Phi^{\frac{-8k_1}{k_1+k_2}} + k_2^3\rho_2^8\Phi^{\frac{-8k_2}{k_1+k_2}} \\ &+ 3k_1k_2(k_1+k_2)\rho_1^4\rho_2^4\Phi^{-4} \\ &+ k_1^2(k_1+3k_2)\rho_1^6\rho_2^2\Phi^{\frac{-6k_1-2k_2}{k_1+k_2}} \\ &+ k_2^2(k_2+3k_1)\rho_1^2\rho_2^6\Phi^{\frac{-2k_1-6k_2}{k_1+k_2}}\big) \\ = \frac{1}{\Delta^3},&\end{aligned} \tag{21}$$

which is obviously positive for any k_1, k_2 positive.

COROLLARY 1. — *For any pair of complex numbers α, β satisfying (3), the 2-form $\omega_{\alpha,\beta} = \dfrac{1}{4\Phi_{\alpha,\beta}} dd^c\Phi_{\alpha,\beta}$ is well-defined on $M_{\alpha,\beta}$ and is the Kähler form of a locally conformally Kähler structure, $(g_{\alpha,\beta}, J)$.*

Proof. — The fact that $\omega_{\alpha,\beta}$ is well-defined on $M_{\alpha,\beta}$ follows readily from (14). By the above proposition, $\omega_{\alpha,\beta}$ is the Kähler form of a, clearly l.c.K., Hermitian structure. □

Remark 3. — The 2-form $\frac{1}{4} dd^c\Phi_{\alpha,\beta}$ descends on $M_{\alpha,\beta}$ as a L^2-valued 2-form, equal to the conformal Kähler form of the l.c.K. Hermitian structure $(g_{\alpha,\beta}, J)$. In the special case that $|\alpha| = |\beta|$ or, equivalently, $k_1 = k_2$, we recover $\Phi_{\alpha,\beta} = \rho^2$.

In the general situation, we actually get a 1-parameter family of l.c.K. Hermitian structures obtained by choosing any positive real number ℓ and by considering, instead of $\dfrac{1}{4\Phi_{\alpha,\beta}} dd^c\Phi_{\alpha,\beta}$, the new Kähler form $\dfrac{1}{4\Phi^\ell_{\alpha,\beta}} dd^c\Phi^\ell_{\alpha,\beta}$. This amounts to replacing k_1, k_2 by ℓk_1, ℓk_2 in the above formulae.

This can be done in particular in the case $|\alpha| = |\beta|$; then, the Kähler form on W is equal to $\frac{1}{4} dd^c\rho^{2\ell}$ and the corresponding Riemannian metric g_ℓ can be described as follows:

$$g_\ell(X, X) = \ell\, \rho^{(2\ell-2)}(\ell |X_{\mathrm{rad}}|^2 + |X^\perp_{\mathrm{rad}}|^2), \tag{22}$$

with the following notation: For any vector X at the point x of W, X_{rad} denotes the *radial component* of X, i.e. the orthogonal projection of X on the complex line $\mathbb{C} \cdot x$ (viewed as a real 2-plane), and $X^\perp_{\mathrm{rad}}$ denotes the *transversal component* of X, i.e. the orthogonal projection of X on the orthogonal complex line $(\mathbb{C} \cdot x)^\perp$ (here, *orthogonal* means orthogonal with respect to the natural flat metric of $\mathbb{C}^2$).

The Lee form $\theta_{\alpha,\beta}$ of the Hermitian structure $(g_{\alpha,\beta}, J)$ is clearly equal to $\frac{1}{2}\Phi^{-1}_{\alpha,\beta}\, d\Phi_{\alpha,\beta}$.

Let $V_{\alpha,\beta}$ denote the dual vector field of $\theta_{\alpha,\beta}$ with respect to $g_{\alpha,\beta}$, the so-called *Lee vector field*.

A direct computation using (17), (18), (19) shows that the pull-back vector field of $V_{\alpha,\beta}$ on W, still denoted by $V_{\alpha,\beta}$, is expressed by

$$V_{\alpha,\beta}(x) = \left(\frac{2k_1}{k_1 + k_2} u, \frac{2k_2}{k_1 + k_2} v \right). \tag{23}$$

Again, a direct, but lengthy, computation shows that $V_{\alpha,\beta}$ is of norm 1 with respect to $g_{\alpha,\beta}$ and is parallel with respect to the Levi-Civita connection of $g_{\alpha,\beta}$. These facts will however become more easily apparent in the framework of the next section.

3. Associated Sasakian structures.

3.1. Three-dimensional Sasakian structures.

We begin this section with some general considerations concerning three-dimensional Sasakian structures (see e.g. [4]) for more information).

A *Sasakian structure* on some oriented, three-dimensional smooth manifold N is a pair (g, Z), where g is a Riemannian metric and Z a unit Killing vector field with respect to g, such that

$$D^g Z = *Z \; ; \tag{24}$$

here, D^g is the Levi-Civita connection of g, $*$ is the Hodge operator determined by the metric and the chosen orientation and $*Z$ is viewed as a skew-symmetric operator, also denoted by I; we thus have $I(Z) = 0$ and the restriction of I to $Q := Z^\perp$ coincides with the uniquely defined complex structure compatible with the metric and the induced orientation.

The distribution Q constitutes a *contact structure* and the Riemannian dual 1-form of Z with respect to g, η, is a *contact 1-form* for Q.

Notice that (3) implies

$$g(X,Y) = \frac{1}{2} d\eta(X, IY), \tag{25}$$

for any sections X, Y of Q.

In general, for any contact structure Q and any choice of a contact 1-form η, the corresponding *Reeb vector field* is the vector field V determined by the two conditions: $\eta(V) = 1$, $i_V d\eta = 0$. In the case of a Sasakian structure as above, the Reeb vector field of the contact structure Q with respect to the contact 1-form η is clearly equal to Z.

We denote by R^g the curvature tensor of D^g. Since Z is a Killing vector field, it satisfies the *Kostant identity*: $D^g_X(D^g Z) = R^g_{X,Z}$ [16]. We thus get

$$R^g_{Z,Y} = Z \wedge Y, \tag{26}$$

for any vector field Y. In particular, the sectional curvature of g is equal to 1 for any 2-plane containing Z.

Since N is three-dimensional, R^g is entirely determined by the Ricci tensor Ric^g and it is easy to deduce from (26) that Z is an eigenvector field for Ric^g (viewed as a symmetric operator) with respect to the constant eigenvalue 2, whereas Q is an eigen-subbundle of Ric^g with respect to the (in general non-constant) eigenvalue $\dfrac{\mathrm{Scal}^g}{2} - 1$, where Scal^g denotes the scalar curvature of g; Ric^g can thus be written as follows:

$$\mathrm{Ric}^g = \Big(\frac{\mathrm{Scal}^g}{2} - 1\Big)\, g + \Big(3 - \frac{\mathrm{Scal}^g}{2}\Big)\, \eta \otimes \eta. \tag{27}$$

The Levi-Civita connection D^g can be computed by using the well-known 6-terms formula, see e.g. [12]; it is given by the following table, where X denotes any *unit* section of Q, and Y any vector field on N:

$$D^g_Y Z = IY, \tag{28}$$

$$\begin{aligned} D^g_Y X = &\big((g([Z,X], IX) - 1)\, \eta(Y) - g([X,IX],Y)\big)\, IX \\ &+ g(Y, IX)\, Z. \end{aligned} \tag{29}$$

It follows that the sectional curvature restricted to Q, $K^g(Q)$, is given by

$$\begin{aligned} K^g(Q) = &1 - 2g([Z,X], IX) - g([X,IX],[X,IX]) \\ &+ X \cdot g([X,IX], IX) - IX \cdot g([X,IX], X), \end{aligned} \tag{30}$$

for any *unit* section, X, of Q. Then, $\mathrm{Scal}^g = 2(2 + K^g(Q))$ is immediately deduced from (30).

Remark 4. — The Levi-Civita connection D^g does not preserve the sub-bundle Q, but induces a linear connection ∇ on Q by orthogonal projection

$$\nabla_Y X = \big((g([Z,X], IX) - 1)\, \eta(Y) - g([X,IX],Y)\big)\, IX, \tag{31}$$

for any unit section X of Q and for any vector field Y on N. This connection is clearly I-linear and preserves the metric g, i.e. is a Hermitian connection when Q is viewed as a Hermitian complex line bundle over N. The (real) connection 1-form of ∇ with respect to X, viewed as a (unit) *gauge* of the Hermitian line bundle Q, is then the real 1-form ζ defined by

$$\zeta = (g([Z,X],IX) - 1)\,\eta - [X,IX]^\flat, \tag{32}$$

where $[X,IX]^\flat$ denotes the dual 1-form of the vector field $[X,IX]$. Then, (30) can be written as follows:

$$\begin{aligned} K(Q) &= -d\zeta(X,IX) - 1 \\ &= \Omega^\nabla(X,IX) - 1, \end{aligned} \tag{33}$$

where $\Omega^\nabla = -d\zeta$ is the (real) curvature form of ∇.

3.2. Sasakian versus Hermitian geometry.

We here describe the well-known correspondence between three-dimensional Sasakian manifolds and l.c.K. Hermitian complex surfaces with parallel unit Lee form, [25].

First, start from a three-dimensional Sasakian manifold (N,g,Z) as above and consider the product manifold $M = N \times \mathbb{R}$; let M be equipped with the product Riemannian metric, still denoted by g, of g and the standard metric of the factor $\mathbb{R}$, and with the almost complex structure J defined as follows:

$$J_{|Q} = I_{|Q},\ JZ = T, \tag{34}$$

where $T := \partial/\partial t$ is the vector field determined by the natural parameter, t, of the factor $\mathbb{R}$. Let again D^g denote the Levi-Civita connection of g on M. Then, it follows from (24) that

$$D^g_U J = dt \wedge JU - \eta \wedge U, \tag{35}$$

for any vector field U on M. This implies that J is integrable and that the Lee form θ of the Hermitian structure (J,g) is the 1-form dt, see e.g. [23] or [1]. In particular, θ is D^g-parallel, of norm 1.

This construction can be compactified in the following manner. Let σ be any Sasakian transformation of (N,g,Z), i.e. any (direct) diffeomorphism of N preserving g and Z, and choose a positive real number ℓ; then the (Riemannian) *suspension* $M_{\sigma,\ell}$ of σ over the circle of length ℓ is obtained by identifying $N \times \{0\}$ and $N \times \{1\}$ via σ in the product $N \times [0,\ell]$ of N by the closed segment $[0,\ell]$.

The natural projection π from $M_{\sigma,\ell}$ to the circle $S^1_\ell = \mathbb{R}/\mathbb{Z}\cdot\ell$ is thus a Riemannian submersion and the natural vector field d/dt on S^1_ℓ admits a natural unit lift, $\tilde{T}$ on $M_{\sigma,\ell}$, orthogonal to the fibers of π, whose flow at time 1 coincides with σ. Applying this construction to the Sasakian three-manifold (N, g, Z) for any σ and any ℓ we eventually get a Hermitian structure with parallel (unit) Lee form on $M_{\sigma,\ell}$ by putting $JZ = \tilde{T}$.

Conversely, let (M, J, g) be a Hermitian complex surface. Let θ and V denote the Lee form and the Lee vector field. Since J is integrable, we have

$$D^g_U J = \theta \wedge JU + J\theta \wedge U, \tag{36}$$

for any vector field U on M (as usual, the RHS of (36) has to be considered as a skew-symmetric operator). If, moreover, θ, hence also V, are D^g-parallel, then the metric g splits locally as a Riemannian product $N \times \mathbb{R}$, where (N, g_N) is a three-dimensional Riemannian manifold and $V = d/dt$. By rescaling g if necessary, we can assume that V is of norm 1, as well as the vector field $Z := JV$; now, Z can be viewed as a vector field on N and (36) directly implies (24), showing that (g_N, Z) is a Sasakian structure on N.

3.3. Deformation of Sasakian structures.

Start from any Sasakian structure (g, Z) on N, fix any real positive function f on N and consider the new contact 1-form $\eta_f = \frac{1}{f}\eta$. Denote by Z_f the Reeb vector field corresponding to the same contact structure Q and the contact 1-form η_f; then,

$$Z_f = f\, Z + Z_f^Q, \tag{37}$$

where Z_f^Q is a section of Q, uniquely determined by the identity

$$df + i_{Z_f^Q} d\eta = 0. \tag{38}$$

It follows that

$$Z_f^Q = \frac{1}{2} I(df_{|Q})^\sharp, \tag{39}$$

where $df_{|Q}$ denotes the restriction of df to Q and $df^\sharp_{|Q}$ the section of Q dual to $df_{|Q}$ with respect to the the restriction of g to Q.

Let g_f be the Riemannian metric on N defined as follows:

1. Z_f is of norm 1 w.r.t g_f;

2. Q and Z_f are orthogonal with respect to g_f;

3. $g_f(X,Y) = \frac{1}{2}d\eta_f(X,IY) = \frac{1}{f}\, g(X,Y)$, for any sections X,Y of Q.

We shall refer to the metric g_f as *the metric obtained by deforming the Sasakian structure* (g,Z) *by the function* f, see [21].

Notice that η_f is the dual 1-form of Z_f with respect to g_f.

Now we have

PROPOSITION 2. — *The pair* (g_f, Z_f) *is a Sasakian structure on* N *if and only if the following condition is satisfied:*

$$\mathrm{Hess}^{\mathrm{g}} f(X,Y) = \mathrm{Hess}^{\mathrm{g}} f(IX,IY), \tag{40}$$

for any sections X,Y *of* Q, *where* $\mathrm{Hess}^{\mathrm{g}} f = D^g df$ *denotes the Hessian of* f *with respect to* g; *equivalently, the restriction of* $\mathrm{Hess}^{\mathrm{g}} f$ *to* Q *is a multiple of the restriction of* g.

Proof. — Let D^{g_f} be the Levi-Civita connection of g_f. We first show that $D^{g_f}_{Z_f} Z_f = 0$. Indeed, $g_f(D^{g_f}_{Z_f} Z_f, Z_f) = 0$, since Z_f is of norm 1 with respect to g_f, and, for any section X of Q, we have $g_f(D^{g_f}_{Z_f} Z_f, X) = g_f([X,Z_f],Z_f) = \eta_f([X,Z_f]) = -d\eta_f(X,Z_f) = 0$. Then, for any sections X,Y of Q, we have

$$\begin{aligned} g_f(D^{g_f}_X Z_f, Y) = &-\frac{1}{2}\eta_f([X,Y]) \\ &+\frac{1}{2}\big(Z_f\cdot g_f(X,Y) - g_f([Z_f,X],Y) - g_f([Z_f,Y],X)\big). \end{aligned}$$

This shows that the pair (g_f,Z_f) is Sasakian if and only if Z_f is Killing with respect to the induced metric g_f; moreover, in the present case that Z is already a Killing vector field with respect to g, Z_f is a Killing vector field with respect to g_f if and only if

$$g(D^g_X Z^Q_f, Y) + g(X, D^g_Y Z^Q_f) = df(Z)\, g(X,Y), \tag{41}$$

holds for any sections X,Y of Q. We then have

$$\begin{aligned} g(D^g_X Z^Q_f, Y) &= \frac{1}{2}X\cdot g(I(df_{|Q})^\sharp, Y) - \frac{1}{2}g(I(df_{|Q})^\sharp, D^g_X Y) \\ &= -\frac{1}{2}X\cdot df(IY) + \frac{1}{2}df(ID^g_X Y) \\ &= -\frac{1}{2}\mathrm{Hess}^{\mathrm{g}} f(X,IY) - \frac{1}{2}df((D^g_X I)Y) \\ &= -\frac{1}{2}\mathrm{Hess}^{\mathrm{g}} f(X,IY) + \frac{1}{2}df(Z)\, g(X,Y), \end{aligned}$$

which shows that (41) is true if and only (40) is satisfied. The last statement comes from Q being of rank 2 (notice that, except for this last statement, the argument holds in any dimension). □

By using (30), we get the following formulation for the scalar curvature of g_f:

$$\widetilde{\mathrm{Scal}^{g_f}} = 2\Big(3 + 4(f-1) + 4\mathrm{Hess}^g f(X,X) - 3\,\frac{((df(X))^2 + (df(JX))^2)}{f}\Big), \tag{42}$$

for any unit section X of Q.

3.4. Sasakian structures attached to $M_{\alpha,\beta}$.

Recall that the Hopf surface $M_{\alpha,\beta}$ as a manifold has been identified to the product $S^3 \times S^1$ by $\psi : M_{\alpha,\beta} \mapsto S^3 \times S^1$, defined by (7) and its inverse $\psi^{-1} : S^3 \times S^1 \mapsto M_{\alpha,\beta}$ described by (8).

We adopt the following notations. The sphere S^3 is realized as the set of elements of $\mathbb{C}^2$ of norm 1: a generic element of S^3 is denoted by $z = (z_1, z_2)$, where z_1, z_2 are complex numbers such that $|z_1|^2 + |z_2|^2 = 1$. Accordingly, a generic vector X of S^3 at z is identified to a pair of complex numbers $(X_1 = \dot{z_1}, X_2 = \dot{z_2})$ satisfying $\mathfrak{Re}(X_1\overline{z_1} + X_2\overline{z_2}) = 0$ ($\mathfrak{Re}$ and $\mathfrak{Im}$ denote respectively the real and imaginary part of a complex number). We denote by Z the vector field on S^3 generated by the natural action of S^1, so that

$$Z = (iz_1, iz_2).$$

We denote by $Q := Z^{\perp}$ the rank 2 vector sub-bundle of TS^3 orthogonal to Z with respect to the standard metric, g, of S^3 (of constant sectional curvature $+1$). The natural complex structure of $Q = Z^{\perp}$ is denoted by i. We denote by E, iE the generators of Q defined by

$$E = (\overline{z_2}, -\overline{z_1}),\ iE = (i\overline{z_2}, -i\overline{z_1}).$$

For any complex number μ, μE stands for the *real* vector field $\mathfrak{Re}\mu E + \mathfrak{Im}\mu\, iE$. The three (real) vector fields Z, E, iE are (unit) Killing vector fields with respect to g and generate the (real) Lie algebra of left-invariant vector fields of S^3, when S^3 is identified to the Lie group $Sp(1)$ of unit quaternions, via the usual identification $\mathbb{H} = \mathbb{C} \oplus j\,\mathbb{C}$. Their brackets are given by

$$[Z, E] = -2iE,\ [E, iE] = -2Z,\ [iE, Z] = -2E. \tag{43}$$

Also recall that, if D^g denotes the Levi-Civita connection of g, we have

$$D^g_Z Z = 0,\ D^g_X Z = iX, \tag{44}$$

for any X in Q, i.e. the pair (g, Z) is a *Sasakian structure*, called the *canonical* Sasakian structure of S^3. The corresponding contact 1-form is denoted by η: $\eta(X) = g(Z, X)$, for any vector field X on S^3.

The vector fields Z, E, iE will be also considered as vector fields on $S^3 \times S^1$ (with the same notation). As for the factor $S^1 = \mathbb{R}/\mathbb{Z}$, we denote by t the natural parameter of $\mathbb{R}$ and by T the vector field $\partial/\partial t$, also considered as a vector field on $S^3 \times S^1$.

For later convenience, we consider the complex function, F, on $S^3 \times S^1$ defined by

$$\begin{aligned} F(z) &= \ln\alpha|z_1|^2 + \ln\beta|z_2|^2 \\ &= k_1|z_1|^2 + k_2|z_2|^2 + i\left(\mathfrak{Arg}\,\alpha\,|z_1|^2 + \mathfrak{Arg}\,\beta\,|z_2|^2\right). \end{aligned} \tag{45}$$

Viewed as functions on $M_{\alpha,\beta}$, $|z_1|^2$, $|z_2|^2$ and $\mathfrak{Re}F$ are respectively equal to $\rho_1^2\Phi^{\frac{-2k_1}{k_1+k_2}}$, $\rho_2^2\Phi^{\frac{-2k_2}{k_1+k_2}}$ and $\dfrac{(k_1+k_2)}{2}\Delta$.

The image of any vector field $U = (U_1, U_2)$ of $M_{\alpha,\beta}$ by the differential ψ_* of ψ is of the form (X, aT), where $X = (X_1, X_2)$ is tangent to S^3 and a is a real function on $S^3 \times S^1$. It is easily checked that

$$a = \frac{\mathfrak{Re}(U_1(x)\,\overline{u}\,|\alpha|^{-2t} + U_2(x)\,\overline{v}\,|\beta|^{-2t})}{\mathfrak{Re}F}, \tag{46}$$

$$X_1 = \alpha^{-t}U_1 - a\,\ln\alpha\, z_1,\ X_2 = \beta^{-t}U_2 - a\,\ln\beta\, z_2. \tag{47}$$

Finally, the vector field X can be written uniquely as $bZ + \mu E$, where b is a real function and μ a complex function on $S^3 \times S^1$, given by

$$b = -i\,(X_1\overline{z_1} + X_2\overline{z_2}),\ \mu = X_1\, z_2 - X_2\, z_1. \tag{48}$$

We conclude that the complex structure J of $M_{\alpha,\beta}$, transported on $S^3 \times S^1$ by ψ, is described by the following table:

$$\begin{aligned} JT &= \frac{1}{\mathfrak{Re}F}\left(-\mathfrak{Im}F\,T + |F|^2\,Z + i\overline{F}(\ln\alpha - \ln\beta)\,z_1z_2\,E\right), \\ JZ &= \frac{1}{\mathfrak{Re}F}\left(-T + \mathfrak{Im}F\,Z + (\ln\alpha - \ln\beta)\,z_1z_2\,E\right), \\ JE &= iE. \end{aligned} \tag{49}$$

The Kähler form $\omega_{\alpha,\beta}$, transported on $S^3 \times S^1$ by ψ, is given by

$$\begin{aligned} \omega_{\alpha,\beta} &= \frac{1}{4}e^{-(k_1+k_2)t}\,dd^c e^{(k_1+k_2)t} \\ &= \frac{(k_1+k_2)}{4}dd^c t + \frac{(k_1+k_2)^2}{4}dt \wedge d^c t, \end{aligned} \tag{50}$$

where d^c refers to the operator J defined by (49).

From (49), we obtain

$$dt(T) = 1,\ dt(Z) = dt(E) = dt(iE) = 0, \tag{51}$$

$$d^ct(T) = \frac{\mathfrak{Im}F}{\mathfrak{Re}F},\ d^ct(Z) = \frac{1}{\mathfrak{Re}F},\ d^ct(E) = d^ct(iE) = 0, \tag{52}$$

hence also, the following table for $\omega_{\alpha,\beta}$

$$\begin{aligned}
\omega_{\alpha,\beta}(T,Z) &= \frac{(k_1+k_2)^2}{4\mathfrak{Re}F},\\
\omega_{\alpha,\beta}(T,\lambda E) &= \frac{(k_1+k_2)}{2(\mathfrak{Re}F)^2}\,\mathfrak{Re}(\overline{\lambda}z_1z_2)\,(k_1\mathfrak{Arg}\beta - k_2\mathfrak{Arg}\alpha),\\
\omega_{\alpha,\beta}(Z,\lambda E) &= \frac{(k_1+k_2)}{2(\mathfrak{Re}F)^2}\,(k_1-k_2)\,\mathfrak{Re}(\overline{\lambda}z_1z_2),\\
\omega_{\alpha,\beta}(\lambda E,\mu E) &= -\frac{(k_1+k_2)}{2\mathfrak{Re}F}\,\mathfrak{Im}(\lambda\overline{\mu}),
\end{aligned} \tag{53}$$

for any complex numbers λ, μ. We finally derive the following table for the metric $g_{\alpha,\beta} = \omega_{\alpha,\beta}(\cdot, J\cdot)$:
(54)

$$\begin{aligned}
g_{\alpha,\beta}(T,T) &= \frac{(k_1+k_2)^2|F|^2}{4(\mathfrak{Re}F)^2} + \frac{(k_1+k_2)(k_1\mathfrak{Arg}\beta - k_2\mathfrak{Arg}\alpha)^2}{2(\mathfrak{Re}F)^3}|z_1|^2|z_2|^2,\\
g_{\alpha,\beta}(T,Z) &= \frac{(k_1+k_2)^2\mathfrak{Im}H}{4(\mathfrak{Re}F)^2} + \frac{(k_1^2-k_2^2)(k_1\mathfrak{Arg}\beta - k_2\mathfrak{Arg}\alpha)}{2(\mathfrak{Re}F)^2}|z_1|^2|z_2|^2,\\
g_{\alpha,\beta}(T,\lambda E) &= \frac{(k_1+k_2)(k_1\mathfrak{Arg}\beta - k_2\mathfrak{Arg}\alpha)}{2(\mathfrak{Re}F)^2}\mathfrak{Im}(\overline{\lambda}z_1z_2),
\end{aligned}$$

$$\begin{aligned}
g_{\alpha,\beta}(Z,Z) &= \frac{(k_1+k_2)^2}{4(\mathfrak{Re}F)^2} + \frac{(k_1+k_2)(k_1-k_2)^2}{2(\mathfrak{Re}F)^3}\,|z_1|^2|z_2|^2,\\
g_{\alpha,\beta}(E,E) &= g_{\alpha,\beta}(iE,iE) = \frac{(k_1+k_2)}{2\mathfrak{Re}F}\\
g_{\alpha,\beta}(Z,\lambda E) &= \frac{(k_1^2-k_2^2)}{2(\mathfrak{Re}F)^2}\,\mathfrak{Im}(\overline{\lambda}z_1z_2),\\
g_{\alpha,\beta}(E,iE) &= 0.
\end{aligned} \tag{55}$$

From (23), we infer that the Lee vector field $V_{\alpha,\beta}$, viewed as a vector field on $S^3\times S^1$ via ψ, is written as:

$$V_{\alpha,\beta} = \frac{2}{(k_1+k_2)}\big(T - \mathfrak{Im}F\,Z - (\mathfrak{Arg}\,\alpha - \mathfrak{Arg}\,\beta)\,i\,z_1z_2\,E\big). \tag{56}$$

The vector field $JV_{\alpha,\beta}$ is thus equal to

$$JV_{\alpha,\beta} = \frac{2}{(k_1+k_2)}\mathfrak{Re}F\, Z + (k_1-k_2)iz_1z_2\, E). \tag{57}$$

In particular, $JV_{\alpha,\beta}$ is independent of t and is tangent to the factor S^3, hence can ve viewed as a vector field on S^3; as such, it will be denoted by Z_Δ.

We then have

$$\begin{aligned} Z_\Delta &= (\frac{2k_1}{k_1+k_2}iz_1, \frac{2k_2}{k_1+k_2}iz_2) \\ &= \frac{2}{(k_1+k_2)}\big(\mathfrak{Re}F\, Z + (k_1-k_2)\, iz_1z_2\, E\big) \\ &= Z + \frac{2(k_1-k_2)}{(k_1+k_2)}Z_R, \end{aligned} \tag{58}$$

where Z_R is the vector field on S^3 defined by $Z_R = (iz_1, -iz_2)$. It can be seen that Z_R is a *right-invariant* (unit) Killing vector field for the standard metric g of S^3. In particular, Z_Δ is itself a Killing vector field with respect to g.

We observe that the restriction of $g_{\alpha,\beta}$ on each fiber of the natural fibration $\pi : S^3 \times S^1 \to S^1$ is independent of t, hence can be considered as a Riemannian metric on the sphere S^3; this metric is denoted by g_Δ.

PROPOSITION 3. — *The pair (g_Δ, Z_Δ) is a Sasakian structure on S^3, actually coincides with the Sasakian structure obtained by deforming the canonical Sasakian structure (g, Z) of S^3 by the function Δ defined by*

$$\Delta(z) = \frac{2}{(k_1+k_2)}\mathfrak{Re}F = \frac{2k_1|z_1|^2 + 2k_2|z_2|^2}{k_1+k_2}. \tag{59}$$

Proof. — By using (55) we check that Z_Δ is of norm 1 and is orthogonal to Q with respect to the metric g_Δ. Then, with respect to the triple Z_Δ, E, iE, g_Δ is described by the following table:

$$\begin{aligned} &g_\Delta(Z_\Delta, Z_\Delta) = 1, \\ &g_\Delta(Z_\Delta, E) = g_\Delta(Z_\Delta, F) = g_\Delta(E, F) = 0, \\ &g_\Delta(E, E) = g_\Delta(iE, iE) = \frac{1}{\Delta}. \end{aligned} \tag{60}$$

Now, the vector field Z_Δ can be written

$$Z_\Delta = \Delta\, Z - \frac{2(k_1-k_2)}{(k_1+k_2)}\mathfrak{Re}(z_1z_2)\, E + \frac{2(k_1-k_2)}{(k_1+k_2)}\mathfrak{Im}(z_1z_2)\, F. \tag{61}$$

On the other hand, we clearly have

$$d\Delta(E) = \frac{4(k_1-k_2)}{(k_1+k_2)}\mathfrak{Re}(z_1 z_2),\; d\Delta(F) = \frac{4(k_1-k_2)}{(k_1+k_2)}\mathfrak{Im}(z_1 z_2). \tag{62}$$

These prove that Z_Δ is the Reeb vector field of the contact structure Q with respect to the contact 1-form $\frac{\eta}{\Delta}$. By (60), the metric g_Δ coincides with the Riemannian metric determined by the Reeb vector field Z_Δ. It remains to check that Δ satisfies the condition of Proposition 2, which is clear. □

COROLLARY 2. — *For any complex numbers α, β satisfying (3), the Lee form of the l.c.K. Hermitian structure $(g_{\alpha,\beta}, J)$ is parallel with respect to $D^{g_{\alpha,\beta}}$.*

Proof. — As already observed in Section 3.2, (24) together with (36) imply that the Lee vector field $V_{\alpha,\beta}$, hence also the Lee form $\theta_{\alpha,\beta}$, is parallel with respect to $g_{\alpha,\beta}$. □

Remark 5. — By (42) and the above proposition, we infer that the scalar curvature $\mathrm{Scal}^{g_{\alpha,\beta}}$, which is also equal to the scalar curvature Scal^{g_Δ} of g_Δ, is given by

$$\mathrm{Scal}^{g_{\alpha,\beta}} = 6\left(1 - 4\frac{(k_1-k_2)}{(k_1+k_2)}\,\frac{(k_1|z_1|^2 - k_2|z_2|^2)}{(k_1|z_1|^2 + k_2|z_2|^2)}\right). \tag{63}$$

In particular, $\mathrm{Scal}^{g_{\alpha,\beta}}$ is not constant, except in the case that $k_1 = k_2$, *i.e.* $|\alpha| = |\beta|$.

Remark 6. — It follows readily from (23) that the flow $\Psi^{V_{\alpha,\beta}}$ of $V_{\alpha,\beta}$ on $S^3 \times S^1$ is given by

$$\begin{aligned}\Psi_s^{V_{\alpha,\beta}}((z,t)) = \Big(\big(&e^{-i\frac{2s}{(k_1+k_2)}\mathfrak{Arg}\,\alpha}\cdot z_1, e^{-i\frac{2s}{(k_1+k_2)}\mathfrak{Arg}\,\beta}\cdot z_2\big),\\ &t + \frac{2s}{(k_1+k_2)} \operatorname{mod} \mathbb{Z}\Big).\end{aligned} \tag{64}$$

This flow preserves the fibration π and induces an isometry with respect to g_Δ from each fiber to the corresponding target fiber. In particular, after one rotation over S^1, this isometry is the isometry $\sigma_{\alpha,\beta}$ from S^3 to itself defined by

$$\sigma_{\alpha,\beta}((z,t)) = \Big((e^{-i\mathfrak{Arg}\,\alpha}\cdot z_1, e^{-i\mathfrak{Arg}\,\beta}\cdot z_2), t\Big). \tag{65}$$

Finally, the l.c.K. metric $g_{\alpha,\beta}$ on the Hopf surface $M_{\alpha,\beta}$ is obtained by the following procedure (see [9] for the case $k_1 = k_2$):

1. Equip the sphere S^3 with the Riemannian metric g_Δ obtained by deforming the canonical Sasakian structure (g, Z) by the function Δ defined by (59) (see Section 3.3).

2. Realize $(M_{\alpha,\beta}, g_{\alpha,\beta})$ as the suspension of the isometry $\sigma_{\alpha,\beta}$ defined by (65) over the circle of length $\frac{(k_1 + k_2)}{2}$ (see Section 3.2).

Proof of Theorem 1.

The first statement has been proved in the preceding sections, see in particular Proposition 1 and Proposition 3.

In order to prove the second statement, *i.e.* the *existence* of l.c.K. metrics on all Hopf surfaces of class 0, we use a specific deformation argument due to C. LeBrun [17]. Here are details. Fix any complex number β such that $|\beta| > 1$ and any positive integer m. Consider the three-dimensional complex manifold $\mathcal{M}$ defined as the quotient of $\mathbb{C} \times (\mathbb{C}^2 - \{(0,0)\})$ by the group $\tilde{\Gamma}_{\beta,m} \equiv \mathbb{Z}$ generated by the transformation $\tilde{\gamma}_{\beta,m} : (\lambda, (u,v)) \mapsto (\lambda, (\beta^m u + \lambda v^m, \beta v))$. Let p be the natural projection from $\mathcal{M}$ onto $\mathbb{C}$ which assigns λ to the class of $(\lambda, (u,v))$. Then, p is a holomorphic fibration whose fiber at $\lambda = 0$ is the Hopf surface of class 1 $M_{\beta^m,\beta}$ whereas fibers at $\lambda \neq 0$ are Hopf surfaces of class 0, all isomorphic to each other as recalled in the first section.

The bundle of scalars of weight 1 on $\mathcal{M}$ (see Section 1) is naturally identified to the quotient of the product bundle $\mathbb{C} \times (\mathbb{C}^2 - \{(0,0)\}) \times \mathbb{R}$ by $\tilde{\gamma}_{\beta,m} \equiv \mathbb{Z}$ acting by $1.(\lambda, (u,v), a) = (\lambda, (\beta^m u + \lambda v^m, \beta v), |\beta|^{\frac{m+1}{2}} a)$. (Notice that $|\beta|^{\frac{m+1}{2}}$ is equal to $|\det(\tilde{\gamma}_{\beta,m}) * |^{\frac{1}{4}}$.) Let $\mathcal{L}$ denote this bundle.

As already observed, the function $\Phi_{\beta^m,\beta}$ introduced in Section 2 can be considered as a section of $\mathcal{L}^2$ over $p^{-1}(0) = M_{\beta^m,\beta}$. It extends to a smooth section, $\tilde{\Phi}$ of $\mathcal{L}^2$ on $\mathcal{U}$ for some neighbourhood $\mathcal{U}$ of 0 in $\mathbb{C}$. For any λ in $\mathcal{U}$, let $\tilde{\Phi}_\lambda$ be the restriction of $\tilde{\Phi}$ to the fiber $p^{-1}(\lambda)$, also viewed as a function on $W = \mathbb{C}^2 - \{(0,0)\}$. Then, $\tilde{\Phi}_0$ is equal to $\Phi_{\beta^m,\beta}$. By Proposition 1, $\frac{1}{4} dd^c \Phi_{\beta^m,\beta}$ is a Kähler form on $\tilde{M}$; by continuity, the same is true for $\frac{1}{4} dd^c \tilde{\Phi}_\lambda$, so that $\frac{1}{4\tilde{\Phi}_\lambda} dd^c \tilde{\Phi}_\lambda$ is the Kähler form of a l.c.K. metric on $p^{-1}(\lambda)$. We thus get a l.c.K. metric on $\tilde{M}_{\beta,m,\lambda}$ for any λ in $\mathcal{U}$, hence for for any λ in $\mathbb{C}$. By varying β and m, we eventually get a l.c.K. metric for each primary Hopf surface of class 0.

Remark 7. — Note that the above deformation argument is specific to the Hopf surface. A general stability theorem for l.c.K. structures, as in the Kähler case, [15], is still lacking in the literature.

Aknowledgement. This paper was written while the second named author was visiting the university of Paris 6 with a research fellowship offered by the French Ministery of Education, Research and Technology. He is especially thankful to A. Boutet de Monvel and C.M. Marle for having made this visit possible and to the Mathematical Institute of Jussieu for hospitality.

BIBLIOGRAPHY

[1] V. APOSTOLOV, P. GAUDUCHON, The Riemannian Goldberg-Sachs Theorem, Int. J. Math., 8 (1997), 421–439.

[2] W. BARTH, C. PETERS, A. VAN DE VEN, Compact complex surfaces, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band, 4, Springer-Verlag, 1984.

[3] F. BELGUN, Complex surfaces admitting no metric with parallel Lee form, preprint.

[4] D.E. BLAIR, Contact manifolds in Riemannian geometry, Lecture Notes in Math., 509, Springer-Verlag, 1976.

[5] L.C. De ANDRES, L.A. CORDERO, M. FERNANDEZ, J.J. MENCIA, Examples of four dimensional locally conformal Kähler manifolds, Geometriae Dedicata, 29 (1989), 227–233.

[6] L.A. CORDERO, M. FERNANDEZ, M. De LEON, Compact locally conformal Kähler nilmanifolds, Geometriae Dedicata, 21 (1986), 187–192.

[7] S. DRAGOMIR, L. ORNEA, Locally conformal Kähler geometry, Progress in Math., 155, Birkhäuser (1998).

[8] P. GAUDUCHON, La 1-forme de torsion d'une variété hermitienne compacte, Math. Ann., 267 (1984), 495–518.

[9] P. GAUDUCHON, Structures de Weyl-Einstein, espaces de twisteurs et variétés de type $S^1 \times S^3$, J. Reine Angew. Math., 469 (1995), 1–50.

[10] Géométrie des surfaces K3: modules et périodes. Séminaire Palaiseau, octobre 1981-janvier 1982, Astérisque, 126 (1985).

[11] R. HARVEY, H. BLAINE LAWSON, Jr, An intrinsic characterisation of Kähler manifolds, Inv. Math., 74 (1983), 139–150.

[12] S. KOBAYASHI, K. NOMIZU, Foundations of differential geometry, Interscience Publishers, New York, vol. I, 1963.

[13] K. KODAIRA, On the structure of compact complex analytic surfaces, II, American J. Math., 88 (1966), 682–722.

[14] K. KODAIRA, Complex structures on $S^1 \times S^3$, Proc. Nat. Acad. Sci. USA, 55 (1966), 240–243.

[15] K. KODAIRA, D.C. SPENCER, On deformations of complex analytic structures, III, stability theorems for complex structures, Ann. of Math., 71 (1960), 43–77.

[16] B. KOSTANT, Holonomy and the Lie algebra of infinitesimal motions of a Riemannian manifold, Trans. Amer. Math. Soc., 80 (1955), 528–542.

[17] C. LEBRUN, Private letter to the first named author, September 22, 1992.

[18] H.C. LEE, A kind of even dimensional differential geometry and its application to exterior calculus, American J. Math., 65 (1943), 433–438.

[19] P. PICCINNI, Attempts of writing metrics on primary Hopf surfaces, private communication, October 1991.

[20] Y.-T. SIU, Every $K3$ surface is Kähler, Inv. Math., 73 (1983), 139–150.

[21] S. TANNO, The standard CR structure on the unit tangent bundle, Tohoku Math. J., 44 (1992), 535–543.

[22] F. TRICERRI, Some examples of locally conformal Kähler manifolds, Rend. Sem. Mat. Univ. Politecn. Torino, 40 (1982), 81–92.

[23] I. VAISMAN, Some curvature properties of Locally Conformal Kähler Manifolds, Trans. Amer. Math. Soc., 259 (1980), 439–447.

[24] I. VAISMAN, On locally and Globally Conformal Kähler Manifolds, Trans. Amer. Math. Soc., 262 (1980), 533–542.

[25] I. VAISMAN, Generalized Hopf manifolds, Geometriae Dedicata, 13 (1982), 231–255.

[26] I. VAISMAN, Non-Kähler metrics on geometric complex surfaces, Rend. Sem. Mat. Univ. Politecn. Torino, Vol. 45, 3 (1987), 117–123.

Manuscrit reçu le 11 février 1998,
accepté le 2 avril 1998.

P. GAUDUCHON,
École Polytechnique
Centre de Mathématiques
CNRS URA 169
91128 Palaiseau (France).
pg@math.polytechnique.fr

L. ORNEA,
University of Bucharest
Faculty of Mathematics
14 Academiei str.
70109 Bucharest (România).
lornea@geo.math.unibuc.ro