

KRZYSZTOF KURDYKA

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## ON GRADIENTS OF FUNCTIONS DEFINABLE IN O-MINIMAL STRUCTURES

by Krzysztof KURDYKA

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### 0. Introduction.

Many results in subanalytic or semialgebraic geometry of  $\mathbb{R}^n$  hold true in a more general setting called “the theory of o-minimal structures on the real field” (see [DM]). This theory has presented a strong interest since 1991 when A. Wilkie [W1] proved that a natural extension of the family of semialgebraic sets containing the exponential function ( $(\mathbb{R}, \exp)$ -definable sets) is an o-minimal structure. A similar extension of subanalytic sets ( $(\mathbb{R}_{\text{an}}, \exp)$ -definable sets) was then treated by L. van den Dries, A. Macintyre, D. Marker in [DMM] (geometric proofs of these facts were found recently by J.-M. Lion and J.-P. Rolin [LR1], [LR2]). Another type of o-minimal structure ( $(\mathbb{R}_{\text{an}}^K)$ -definable sets) was obtained by C. Miller [Mi], by adding to subanalytic sets all functions  $x \rightarrow x^r$ ,  $r \in K$ , where  $K$  is a subfield of  $\mathbb{R}$ . We give a list and examples of o-minimal structures in section 1. An extension of semialgebraic and subanalytic geometry was also undertaken by M. Shiota [S1], [S2].

Theorem 1 (Section 2), the first main result of this paper, is an o-minimal generalization of the famous Łojasiewicz inequality  $\|\mathbf{grad} f\| \geq |f|^\alpha$  with  $\alpha < 1$ , where  $f$  is an analytic function in a neighborhood of  $a \in \mathbb{R}^n$ ,  $f(a) = 0$ . We prove that if  $f$  is a differentiable function in a

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bounded domain, definable in some o-minimal structure, then there exists a  $C^1$  function  $\Psi$  in one variable such that  $\|\mathbf{grad} \Psi \circ f\| \geq c > 0$ . It is rather surprising that the result holds also for infinitely flat functions. Theorem 1 implies that the set of asymptotic critical values of  $f$  is finite (Proposition 2). We recall in the beginning of the section the already known o-minimal version of another Lojasiewicz inequality for continuous definable functions on a compact set.

The main result of Section 3 is Theorem 2 which states: if  $U$  is an open, bounded subset of  $\mathbb{R}^n$ ,  $f : U \rightarrow \mathbb{R}$  is a  $C^1$  function definable in some o-minimal structure, then all trajectories of  $-\mathbf{grad} f$  (i.e. solutions of the equation  $\dot{x} = -\mathbf{grad} f$ ) have their length bounded by a constant independent of the trajectory. The function  $f$  may be unbounded and may not have a continuous extension on  $\bar{U}$ . We prove also, that for a non negative definable  $g$ , the flow of  $-\mathbf{grad} g$  defines a deformation retraction onto  $g^{-1}(0)$ . Some applications of this result in the real analytic case can be found in [Si], [Sj]. We finish the paper by a discussion of Thom's Gradient Conjecture for o-minimal structures.

In Section 1 we gather basic facts on o-minimal structures. To make the paper self-contained and accessible for a wider audience we add a proof of Lemma 2 (on definable functions in one variable). We give also an elementary proof (suggested by C. Miller and J-M. Lion) of the curve selection lemma, the crucial tool in the proof of Theorem 1.

General references of various facts, when not specified, will be as follows: for semialgebraic geometry – [BCR], for subanalytic geometry – [BM] or [L4], for o-minimal structures – [DM].

In this paper we take the gradient with respect to the canonical euclidian metric in  $\mathbb{R}^n$ .

## 1. o-minimal structures on the real field.

DEFINITION 1. — Let  $\mathcal{M} = \bigcup_{n \in \mathbb{N}} \mathcal{M}_n$ , where each  $\mathcal{M}_n$  is a family of subsets of  $\mathbb{R}^n$ . We say that the collection  $\mathcal{M}$  is an o-minimal structure on  $(\mathbb{R}, +, \cdot)$  if:

- (1) each  $\mathcal{M}_n$  is closed under finite set-theoretical operations;
- (2) if  $A \in \mathcal{M}_n$  and  $B \in \mathcal{M}_m$ , then  $A \times B \in \mathcal{M}_{n+m}$ ;

- (3) let  $A \in \mathcal{M}_{n+m}$  and  $\pi : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$  be projection on the first  $n$  coordinates, then  $\pi(A) \in \mathcal{M}_n$ ;
- (4) let  $f, g_1, \dots, g_k \in \mathbb{Q}[X_1, \dots, X_n]$ , then  $\{x \in \mathbb{R}^n : f(x) = 0, g_1(x) > 0, \dots, g_k(x) > 0\} \in \mathcal{M}_n$ ;
- (5)  $\mathcal{M}_1$  consists of all finite unions of open intervals and points.

For a fixed o-minimal structure  $\mathcal{M}$  on  $(\mathbb{R}, +, \cdot)$  we say that  $A$  is an  $\mathcal{M}$ -set if  $A \in \mathcal{M}_n$  for some  $n \in \mathbb{N}$ . We say that a function  $f : A \rightarrow \mathbb{R}^m$ , where  $A \subset \mathbb{R}^n$ , is an  $\mathcal{M}$ -function if its graph is an  $\mathcal{M}$ -set.

Axiom (5) will be called the o-minimality of  $\mathcal{M}$ .

*Examples.* — We give below a list of o-minimal structures on  $(\mathbb{R}, +, \cdot)$  (see also [DM] for detailed definitions and comparisons between the above examples) with examples of functions definable in those o-minimal structures:

- (1) Semialgebraic sets (by Tarski-Seidenberg);  $f(x, y) = \sqrt{x^4 + y^4}$ .
- (2) Global subanalytic sets (by Gabrielov);  
 $f(x, y) = \frac{y}{\sin x}, x \in (0, \pi)$ .
- (3)  $(\mathbb{R}, \exp)$ -definable sets (by Wilkie);  
 $f(x, y) = x^2 \exp\left(-\frac{y^2}{x^4 + y^2}\right) \ln x$ .
- (4)  $(\mathbb{R}_{an}, \exp)$ -definable sets (by van den Dries, Macintyre, Marker);  
 $f(x, y) = x^{\sqrt{2}} \ln(\sin y), x > 0, y \in (0, \pi)$ .
- (5)  $(\mathbb{R}_{an}^{\mathbb{R}})$ -definable sets (by Miller);  
 $f(x, y) = x^{\sqrt{2}} \exp\left(\frac{x}{y}\right), 0 < x < y < 1$ .

Recently another example of an o-minimal structure was found by van den Dries and Speissegger [DS] which is larger than  $\mathbb{R}_{an}^{\mathbb{R}}$  but polynomially bounded (i.e any definable function in one variable is bounded by a polynomial at infinity). Finally we mention a result of Wilkie [W2] in which he gives a general method for construction of o-minimal structures; this method can be applied to Pfaffian functions.

**In the rest of this paper  $\mathcal{M}$  will denote some fixed, but arbitrary, o-minimal structure on  $(\mathbb{R}, +, \cdot)$ .** We will give now several elementary properties of  $\mathcal{M}$ -sets and  $\mathcal{M}$ -functions.

*Remark 1.* — Let  $E$  be an  $\mathcal{M}$ -set in  $\mathbb{R}^{n+1}$ . Axioms (1)–(4) imply

that the sets

$$\{x \in \mathbb{R}^n : \exists x_{n+1} (x, x_{n+1}) \in E\} \quad \text{and} \quad \{x \in \mathbb{R}^n : \forall x_{n+1} (x, x_{n+1}) \in E\}$$

are  $\mathcal{M}$ -sets. Actually the first set is the image of  $E$  by projection, the second is the complement of the image of the complement of  $E$  by projection.

*Remark 2.* — The sum, product, inverse, composition of  $\mathcal{M}$ -functions is again an  $\mathcal{M}$ -function. Also the image and inverse image of an  $\mathcal{M}$ -set by an  $\mathcal{M}$ -function are again  $\mathcal{M}$ -sets. Proofs of these facts are quite standard applications of Remark 1 and axioms (1)–(4) and actually the same as in the semialgebraic case (see e.g. [BCR]).

LEMMA 1. — Let  $f : A \rightarrow \mathbb{R}$  be an  $\mathcal{M}$ -function such that  $f(x) \geq 0$  for all  $x \in A$ . Let  $G : A \rightarrow \mathbb{R}^m$  be an  $\mathcal{M}$ -mapping and define a function  $\varphi : G(A) \rightarrow \mathbb{R}$  by

$$\varphi(y) = \inf_{x \in G^{-1}(y)} f(x).$$

Then  $\varphi$  is an  $\mathcal{M}$ -function.

*Proof.* — Write a formula for the graph of the function  $\varphi$  and apply Remark 1.

COROLLARY 1. — Let  $A$  be an  $\mathcal{M}$ -set in  $\mathbb{R}^n$ . Then the distance function  $d_A : \mathbb{R}^n \rightarrow \mathbb{R}$  is an  $\mathcal{M}$ -function, where  $d_A(x) = \inf_{y \in A} |x - y|$ .

COROLLARY 2. — Let  $A$  be an  $\mathcal{M}$ -set in  $\mathbb{R}^n$ . Then  $\overline{A}$  and  $\text{Int } A$  are  $\mathcal{M}$ -sets.

*Proof.* — Actually by Corollary 1 we know that  $d_A$  is an  $\mathcal{M}$ -function, hence  $\overline{A} = d_A(0)^{-1}$  is an  $\mathcal{M}$ -set. To prove that the interior of  $A$  is an  $\mathcal{M}$ -set we use the fact that by axiom (1) the complement of an  $\mathcal{M}$ -set is an  $\mathcal{M}$ -set.

LEMMA 2 (Monotonicity Theorem). — Let  $f : (a, b) \rightarrow \mathbb{R}$  be an  $\mathcal{M}$ -function. Then there exist real numbers  $a = a_0 < a_1 < \dots < a_k = b$  such that  $f$  is continuously differentiable on each interval  $(a_i, a_{i+1})$ . Moreover  $f'$  is an  $\mathcal{M}$ -function and the function  $f$  is strictly monotone or constant on every interval  $(a_i, a_{i+1})$ .

*Proof* (Due essentially to van den Dries [vD]). — We may assume that the set  $f((a, b))$  is infinite. First we prove that  $D(f)$ , the set of discontinuity points of  $f$ , is finite.

Writing the definition of continuity of a function at a point and using Remark 1 we deduce that  $D(f)$  is an  $\mathcal{M}$ -set in  $\mathbb{R}$ , hence by o-minimality, it is enough to prove that  $f$  is continuous at some point of  $(a, b)$ . Since the set  $f((a, b))$  is an infinite  $\mathcal{M}$ -set it contains an open interval. Thus by induction we can construct a descending sequence of intervals  $[\alpha_n, \beta_n] \subset (a, b)$  such that  $\alpha_n < \alpha_{n+1}$ ,  $\beta_{n+1} < \beta_n$ ,  $\beta_n - \alpha_n < 1/n$  and  $f([\alpha_n, \beta_n])$  is contained in an interval of length smaller than  $1/n$ . Clearly  $f$  is continuous at the point  $\bigcap_{n \in \mathbb{N}} [\alpha_n, \beta_n]$ . So we have proved that the complement of  $D(f)$  is dense in  $(a, b)$ , hence  $D(f)$  is finite.

We can assume now that  $f$  is continuous on  $(a, b)$ . To prove differentiability observe first that by o-minimality we have:

OBSERVATION. — For each  $x \in (a, b)$  and each  $c \in \mathbb{R}$  there exists an  $\varepsilon > 0$  such that  $f(t) \geq f(x) + c(t - x)$  for all  $t \in (x, x + \varepsilon)$  or  $f(t) \leq f(x) + c(t - x)$  for all  $t \in (x, x + \varepsilon)$ .

Let us write  $f'_-(x) = \lim_{t \searrow 0} \frac{1}{t}(f(x+t) - f(x))$  for  $x \in (a, b]$  and  $f'_+(x) = \lim_{t \searrow 0} \frac{1}{t}(f(x+t) - f(x))$  for  $x \in [a, b)$ . Note that  $f'_+$  and  $f'_-$  are  $\mathcal{M}$ -functions, by Remark 1. From the above observation it is not difficult to obtain the following consequences:

- i) for each  $x \in (a, b)$  the values of  $f'_-(x)$  and  $f'_+(x)$  are well defined (possibly equal to  $+\infty$  or  $-\infty$ ),
- ii) for each  $x \in (a, b)$  there exists  $y$  arbitrary close to  $x$ ,  $y > x$  such that  $f'_+(y) \leq f'_+(x)$ ,  $f'_-(y) \leq f'_-(x)$  or  $f'_+(y) \geq f'_+(x)$ ,  $f'_-(y) \geq f'_-(x)$ .

Clearly the sets

$$\{x \in (a, b); f'_+(x) = +\infty\}, \{x \in (a, b); f'_+(x) = -\infty\}$$

are  $\mathcal{M}$ -sets, hence are finite unions of open intervals and points. By ii) these sets are finite. So we can assume that  $f'_+$  and  $f'_-$  take values in  $\mathbb{R}$ . Since  $f'_+$  and  $f'_-$  are  $\mathcal{M}$ -functions we may also assume that these functions are continuous on  $(a, b)$ . It follows easily now from ii) that  $f'_+ = f'_-$  on  $(a, b)$ , but this means that  $f$  is  $\mathcal{C}^1$  on  $(a, b)$ .

We proved also that  $f'$  is an  $\mathcal{M}$ -function, hence the claim on monotonicity follows from the fact that  $\{f' = 0\}$  is an  $\mathcal{M}$ -set and so is a finite union of points and open intervals.

Writing the definition of partial derivatives and using Remark 1 we obtain:

LEMMA 3. — Let  $f : U \rightarrow \mathbb{R}^k$  be a differentiable  $\mathcal{M}$ -function, where  $U$  is open in  $\mathbb{R}^n$ . Then  $\partial f / \partial x_j$ ,  $j = 1, \dots, n$  are  $\mathcal{M}$ -functions, and hence  $\text{grad } f$  is an  $\mathcal{M}$ -mapping.

PROPOSITION 1 (Curve Selection Lemma). — Let  $A$  be an  $\mathcal{M}$ -set in  $\mathbb{R}^n$  and suppose that  $a \in \overline{A \setminus \{a\}}$ . Then there exists an  $\mathcal{M}$ -function  $\gamma : [0, \varepsilon) \rightarrow \mathbb{R}^n$  which is  $C^1$  on  $(0, \varepsilon)$  and such that

$$a = \gamma(0) \quad \text{and} \quad \gamma((0, \varepsilon)) \subset A \setminus \{a\}.$$

*Proof.* — The key point is to construct a “definable” selection operator  $e$ , which assigns to each nonempty set  $A \in \mathcal{M}_n$  an element  $e(A) \in A$ . Let  $n = 1$ . Then  $e(A)$  is the smallest element of  $A$  if  $A$  has one. Otherwise, let  $a := \inf A$  and let  $b \in \mathbb{R} \cup \{+\infty\}$  be maximal such that  $(a, b) \subseteq A$ . If  $a, b \in \mathbb{R}$ , then  $e(A) := (a + b)/2$ . If  $a \in \mathbb{R}$  and  $b = +\infty$ , then  $e(A) := a + 1$ . If  $a = -\infty$  and  $b \in \mathbb{R}$ , then  $e(A) := b - 1$ . If  $a = -\infty$  and  $b = +\infty$  (i.e.,  $A = \mathbb{R}$ ), then  $e(A) := 0$ . Assume  $e(A)$  has been defined for all nonempty  $A \in \mathcal{M}_n$ . Let  $B \in \mathcal{M}_{n+1}$  be nonempty, and let  $A$  be its image in  $\mathbb{R}^n$  under the projection map  $(x_1, \dots, x_n, x_{n+1}) \mapsto (x_1, \dots, x_n)$ . Put  $a := e(A)$ . Then  $e(B) := (a, e(B_a))$  where  $B_a := \{r \in \mathbb{R} : (a, r) \in B\}$ .

This selection operator  $e$  has several applications, and Curve Selection is only one of them: let  $A \in \mathcal{M}_n$  and  $a \in \overline{A \setminus \{a\}}$ . By  $\mathcal{O}$ -minimality the set  $\{|a - x| : x \in A\} \in \mathcal{M}_1$  contains an interval  $(0, \varepsilon)$ ,  $\varepsilon > 0$ . For  $0 < t < \varepsilon$ , let  $\gamma(t) := e(\{x \in A : |a - x| = t\})$ . It is routine to check that  $\gamma : (0, \varepsilon) \rightarrow A$  belongs to  $\mathcal{M}$ . By the monotonicity theorem  $\gamma$  is  $C^1$  after suitable shrinking of  $\varepsilon$ . After composition on the right with a sufficiently flat (at 0) function in  $\mathcal{M}$  (e.g. the inverse of the biggest component of  $\gamma$ ) we can further arrange that  $\gamma$  extends to a  $C^1$ -function on  $[0, \varepsilon)$ .

## 2. Lojasiewicz inequalities for $\mathcal{O}$ -minimal structures.

We begin this section recalling an already well-known generalization of the Lojasiewicz inequality for continuous  $\mathcal{M}$ -functions on a compact set. This result was observed by T. Loi [Lo] for  $(\mathbb{R}, \text{exp})$ -definable sets (actually his version is more precise than the theorem stated below); M. Shiota [S1], [S2] and L. van den Dries and C. Miller [DM] also noticed this fact.

THEOREM 0. — Let  $K$  be a compact subset of  $\mathbb{R}^n$  and let  $f, g : K \rightarrow \mathbb{R}$  be two continuous  $\mathcal{M}$ -functions. If  $f^{-1}(0) \subset g^{-1}(0)$ , then there

exists a strictly increasing positive  $\mathcal{M}$ -function  $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}$  of class  $C^1$ , such that for any  $x \in K$  we have

$$|f(x)| \geq \sigma(g(x)).$$

The idea of the proof goes back to the original argument of Łojasiewicz (see [L2] , [KLZ]). Let  $\Sigma \subset \mathbb{R}^2$  be the image of  $K$  by the mapping  $K \ni u \rightarrow (g(u), f(u)) = (x, y)$ . Clearly  $\Sigma$  is an  $\mathcal{M}$ -set; moreover it is compact and  $\Sigma \cap \{y = 0\} = \{(0, 0)\}$ . It is not difficult to find (by Lemma 2) a strictly increasing positive  $\mathcal{M}$ -function  $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}$  of class  $C^1$ , such that  $\Sigma \subset \{y \geq \sigma(x), x \geq 0\}$ . It is proved in [DM] that for each  $k \in \mathbb{N}$  one can find  $\sigma$  of class  $C^k$ .

We state now the main result of this section. Recall that  $\mathcal{M}$  is any fixed o-minimal structure on  $(\mathbb{R}, +, \cdot)$ .

**THEOREM 1.** — *Let  $f : U \rightarrow \mathbb{R}$  be a differentiable  $\mathcal{M}$ -function, where  $U$  is an open and bounded subset of  $\mathbb{R}^n$ . Suppose that  $f(x) > 0$  for all  $x \in U$ . Then there exists  $c > 0, \rho > 0$  and a strictly increasing positive  $\mathcal{M}$ -function  $\Psi : \mathbb{R}_+ \rightarrow \mathbb{R}$  of class  $C^1$ , such that*

$$\|\mathbf{grad} (\Psi \circ f)(x)\| \geq c,$$

for each  $x \in U, f(x) \in (0, \rho)$ .

The proof is given in the end of the section. We shall see now that in the subanalytic case our Theorem 1 is equivalent to the classical Łojasiewicz inequality for gradients of analytic functions (see [L1], [L2], [BM]). We state this result in the form generalized in [KP]:

**THEOREM (LI).** — *Let  $f : \Omega \rightarrow \mathbb{R}$  be a subanalytic function which is differentiable in  $\Omega \setminus f^{-1}(0)$ , where  $\Omega$  is an open bounded subset of  $\mathbb{R}^n$ . Then there exist  $C > 0, \rho > 0$  and  $0 \leq \alpha < 1$  such that:*

$$\|\mathbf{grad} f(x)\| \geq C|f(x)|^\alpha,$$

for each  $x \in \Omega$  such that  $|f(x)| \in (0, \rho)$ . If in addition  $\lim_{x \rightarrow a} f(x) = 0$  for some  $a \in \overline{\Omega}$  (which holds in the classical case, where  $f$  is analytic and  $a \in \Omega, f(a) = 0$ ), then the above inequality holds for each  $x \in \Omega \setminus f^{-1}(0)$  close to  $a$ .

To see that in the subanalytic case (LI)  $\Rightarrow$  Theorem 1 it is enough to put  $\Psi(t) = t^{1-\alpha}$ . To prove the converse in the subanalytic case, recall first that every subanalytic function in one variable is actually



semianalytic (see [L2], [KLZ]). Hence  $\Psi$  has the Puiseux expansion of the form  $\Psi(t) = \sum_{\nu=0}^{\infty} a_{\nu} t^{\frac{\nu}{k}}$ . Thus, for  $t$  small enough we have  $|\Psi'(t)| \leq D t^{\frac{1}{k}-1}$  for some  $D > 0$ . The last inequality and Theorem 1 yield

$$\|\mathbf{grad} f(x)\| = \frac{\|\mathbf{grad}(\Psi \circ f)(x)\|}{|\Psi' f(x)|} \geq \frac{c}{D} |f(x)|^{1-\frac{1}{k}}.$$

*Remark.* — The above argument and Theorem 1 imply that (LI) holds in any polynomially bounded o-minimal structure on  $(\mathbb{R}, +, \cdot)$ .

We discuss now a consequence of Theorem 1. Let  $f : U \rightarrow \mathbb{R}$  be a differentiable function, where  $U$  is an open subset of  $\mathbb{R}^n$ . We shall say that  $\lambda \in \mathbb{R} \cup \{-\infty, +\infty\}$  is an *asymptotic critical value* of  $f$  if there exists a sequence  $x_n \in U$  such that

$$f(x_n) \rightarrow \lambda \quad \text{and} \quad \mathbf{grad} f(x_n) \rightarrow 0.$$

Clearly any “true” critical value of  $f$  (i.e.  $\lambda = f(x)$  and  $\mathbf{grad} f(x) = 0$ , for some  $x \in U$ ) is also an asymptotic critical value. Notice that this notion depends heavily on the domain  $U$ , in particular on whether  $U$  is bounded or not.

Suppose now that  $U$  is bounded and that our  $f$  is an  $\mathcal{M}$ -function, where  $\mathcal{M}$  is an o-minimal structure on  $(\mathbb{R}, +, \cdot)$ . Let  $\lambda$  be an asymptotic critical value of  $f$ . It follows immediately from Theorem 1 that  $f$  has no asymptotic critical values in  $(\lambda - \rho, \lambda) \cup (\lambda, \lambda + \rho)$  for some  $\rho > 0$ . But on the other hand the set of all asymptotic critical values of  $f$  is an  $\mathcal{M}$ -subset of  $\mathbb{R}$ , so it must be finite. Thus we have proved:

**PROPOSITION 2.** — *If  $U$  is bounded and  $f$  is an  $\mathcal{M}$ -function, then the set of all asymptotic critical values of  $f$  is finite.*

It is easily seen that  $-\infty$  and  $+\infty$  cannot be an asymptotic critical value of an  $\mathcal{M}$ -function defined in a bounded set. As the following example shows the assumption of boundedness on  $U$  is necessary.

*Example.* — The function  $f(x, y) = \frac{x}{y}$  on  $U = \{y > 0\} \subset \mathbb{R}^2$ , being semialgebraic, belongs to any o-minimal structure on  $(\mathbb{R}, +, \cdot)$ . But clearly any  $\lambda \in \mathbb{R}$  is an asymptotic critical value of  $f$ .

*Proof of Theorem 1.* — It follows from Lemma 3 that  $U \ni x \mapsto \|\mathbf{grad} f(x)\|$  is an  $\mathcal{M}$ -function. We may suppose that  $f^{-1}(t) \neq \emptyset$  for any

small enough  $t > 0$ , since otherwise, by o-minimality, the theorem is trivial. Hence the function

$$\varphi(t) = \inf\{\|\mathbf{grad} f(x)\| : x \in f^{-1}(t)\}$$

is well-defined in some interval  $(0, \varepsilon)$ . By Lemma 1,  $\varphi$  is an  $\mathcal{M}$ -function.

CLAIM. — *There exists  $\varepsilon' > 0$  such that  $\varphi(t) > 0$  for any  $t \in (0, \varepsilon')$ .*

Assume that this is not the case and put

$$\Sigma = \{x \in U : \|\mathbf{grad} f(x)\| < (f(x))^2\}.$$

Clearly  $\Sigma$  is an  $\mathcal{M}$ -set. Let  $f|_{\Sigma}$  denote the graph of  $f$  restricted to  $\Sigma$ . If the claim doesn't hold, then there exists a sequence of positive numbers  $t_n \rightarrow 0$  such that  $\varphi(t_n) = 0$  for all  $n \in \mathbb{N}$ . Let  $x_n \in \Sigma$  be a sequence such that  $f(x_n) = t_n$ , in other words  $(x_n, t_n) \in f|_{\Sigma}$ . Let  $b$  be an accumulation point of  $\{x_n\}$ , then  $(b, 0)$  belongs to the closure of the set  $(f|_{\Sigma} \setminus \{(b, 0)\})$ . By the curve selection lemma (Proposition 1) we have an  $\mathcal{M}$ -function (arc)  $\tilde{\gamma} : (-\delta, \delta) \rightarrow \mathbb{R}^n \times \mathbb{R}$  of class  $C^1$ , such that  $\tilde{\gamma}(0) = (b, 0)$ , and  $\tilde{\gamma}(0, \delta) \subset f|_{\Sigma}$ . Write  $\tilde{\gamma}(s) = (\gamma(s), f \circ \gamma(s))$ , where  $\gamma(s) \in \Sigma \subset \mathbb{R}^n$ . Let  $h(s) = f \circ \gamma(s)$  for  $s \in (0, \delta)$ , then clearly  $\lim_{s \rightarrow 0} h(s) = 0 = \lim_{s \rightarrow 0} h'(s)$ , since  $\gamma(s) \in \Sigma$ . Of course  $h$  and  $h'$  are  $\mathcal{M}$ -functions, so by Lemma 2 we may suppose that  $h$  and  $h'$  are monotone; actually they must be strictly increasing. Thus we have

$$0 < h'(s) \leq A(h(s))^2, \text{ for } s \in (0, \delta),$$

where  $A$  is a bound for  $\|\gamma'(s)\|$ . But by the Mean Value Theorem we have  $h(s) \leq s h'(s)$ , because  $h'$  is increasing. Finally, we get  $0 < h'(s) \leq A s^2 (h'(s))^2$  for any  $s \in (0, \delta)$ , which is impossible since  $\lim_{s \rightarrow 0} h'(s) = 0$ .

So we have proved that  $\varphi(t) > 0$  for all  $t \in (0, \varepsilon)$ , provided that  $\varepsilon > 0$  is small enough. We define now:

$$\Delta = \{x \in U \setminus f^{-1}(0) : f(x) < \varepsilon, \|\mathbf{grad} f(x)\| \leq 2\varphi(f(x))\}.$$

Observe that  $\Delta$  is also an  $\mathcal{M}$ -set and moreover  $\Delta \cap f^{-1}(t) \neq \emptyset$  for every  $t \in (0, \varepsilon)$ . Hence as before there exists  $d \in \overline{U}$  such that  $(d, 0) \in \overline{f|_{\Delta} \setminus \{(d, 0)\}}$ . Applying again the curve selection lemma to  $f|_{\Delta}$  at the point  $(d, 0)$  we obtain an  $\mathcal{M}$ -function (arc)  $\tilde{\eta} : (-\delta, \delta) \rightarrow \mathbb{R}^n$  of class  $C^1$ , such that  $\tilde{\eta}(0) = (d, 0)$ , and  $\tilde{\eta}(0, \delta) \subset f|_{\Delta}$ . Write as before  $\tilde{\eta}(s) = (\eta(s), f \circ \eta(s))$ , where  $\eta(s) \in \Delta \subset \mathbb{R}^n$ . Let  $g(s) = f \circ \eta(s)$  for  $s \in (0, \delta)$ , then clearly  $\lim_{s \rightarrow 0} g(s) = 0$  and  $g(s) > 0$  for each  $s \in (0, \delta)$ . It follows from Lemma 2 that for  $\delta' > 0$  small enough the function  $g : (0, \delta') \rightarrow \mathbb{R}$  is a diffeomorphism onto  $(0, \rho)$ , for some  $\rho > 0$ . We put

$$\Psi(t) = g^{-1}(t) \text{ for } t \in (0, \rho).$$

We shall check now the inequality claimed in Theorem 1. Let  $B$  be some bound for  $\|\eta'(s)\|$  in  $(0, \delta')$ . Take any  $x \in U$  such that  $t = f(x) \in (0, \rho)$ , and write  $s = \Psi(t) = g^{-1}(t)$ . Then we have

$$\begin{aligned} \|\mathbf{grad} \Psi \circ f(x)\| &= \Psi'(f(x)) \|\mathbf{grad} f(x)\| \\ &\geq \Psi'(t) \frac{1}{2} \|\mathbf{grad} f(\eta(s))\| \geq \frac{\Psi'(t)}{2B} (f \circ \eta)'(s) = \frac{1}{2B} = c, \end{aligned}$$

since  $\|\mathbf{grad} f(\eta(s))\| \|\eta'(s)\| \geq \langle \mathbf{grad} f(\eta(s)), \eta'(s) \rangle = (f \circ \eta)'(s)$  and  $B \geq \|\eta'(s)\|$ . Theorem 1 follows.

### 3. Trajectories of gradients of $\mathcal{M}$ -functions.

Let  $f : U \rightarrow \mathbb{R}$  be a  $C^1$  function, where  $U$  is an open subset of  $\mathbb{R}^n$ . We shall consider a vector field,

$$U \ni x \mapsto -\mathbf{grad} f(x) \in \mathbb{R}^n.$$

Let  $\alpha, \beta \in \mathbb{R} \cup \{-\infty, +\infty\}$ . We shall say that  $\gamma : (\alpha, \beta) \rightarrow U$  is a *trajectory of the vector field*  $-\mathbf{grad} f$  if it is a maximal differentiable curve verifying  $\gamma'(t) = -\mathbf{grad} f(\gamma(s))$ . Actually we shall consider  $\gamma$  as an equivalence class of all curves obtained from  $\gamma$  by a strictly increasing  $C^1$  reparametrization. Observe that if  $\psi$  is an increasing  $C^1$  diffeomorphism between two intervals in  $\mathbb{R}$ , then the trajectories of  $-\mathbf{grad} \psi \circ f$  and those of  $-\mathbf{grad} f$  are the same.

Let  $a, b \in \gamma$ . We denote by  $|\gamma(a, b)|$  the length of  $\gamma$  between  $a$  and  $b$ .

Lojasiewicz derived (see [L1], [L3]) from (LI) that all trajectories of  $-\mathbf{grad} f$  are of finite length, when  $f$  is analytic in a neighborhood of a compact  $\bar{U}$ . We have:

**THEOREM 2.** — *Let  $f : U \rightarrow \mathbb{R}$  be a function of class  $C^1$ , where  $U$  is an open and bounded subset of  $\mathbb{R}^n$ . Suppose that  $f$  is an  $\mathcal{M}$ -function, for some  $\mathcal{o}$ -minimal structure  $\mathcal{M}$ .*

a) *Then there exists  $A > 0$  such that all trajectories of  $-\mathbf{grad} f$  have length bounded by  $A$ .*

b) *More precisely, there exists  $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  a continuous strictly increasing  $\mathcal{M}$ -function, with  $\lim_{t \rightarrow 0} \sigma(t) = 0$ , such that if  $\gamma$  is a trajectory of  $-\mathbf{grad} f$  and  $a, b \in \gamma$ , then*

$$|\gamma(a, b)| \leq \sigma(|f(b) - f(a)|).$$

*Proof of theorem 2.* — Taking, if necessary the composition  $\psi \circ f$ , where  $\psi(t) = \frac{t}{\sqrt{1+t^2}}$ , we may suppose that  $f$  is bounded; more exactly that the image of  $f$  lies in  $(-1, 1)$ . We consider again the  $\mathcal{M}$ -function  $\varphi : (-1, 1) \rightarrow \mathbb{R}$  defined by

$$\varphi(t) = \inf\{\|\mathbf{grad} f(x)\| : x \in f^{-1}(t)\},$$

when  $f^{-1}(t) \neq \emptyset$ , and  $\varphi(t) = 1$  when  $f^{-1}(t) = \emptyset$ . Let  $\Sigma$  be the set of all asymptotic critical values of  $f$ . Observe that  $\lambda \in \Sigma$  if  $\varphi(\lambda) = 0$ , or  $\lim_{t \nearrow \lambda} \varphi(t) = 0$ , or  $\lim_{t \searrow \lambda} \varphi(t) = 0$ .

Let  $I \subset (-1, 1)$  be an open interval. Assume that  $\varphi$  is bounded from below in  $I$  by some  $c > 0$ . Let  $\gamma$  be a trajectory of  $-\mathbf{grad} f$  and  $a, b \in \gamma$ . Suppose that the part of  $\gamma$  lying between  $a$  and  $b$  is contained in  $f^{-1}(I)$ . We parametrise  $\gamma$  by arc-length (i.e.  $\|\gamma'(s)\| = 1$ ), so by the Mean Value Theorem we have that  $|f \circ \gamma(\beta) - f \circ \gamma(\alpha)| \geq c|\beta - \alpha|$ , in other words

$$|\gamma(a, b)| \leq \frac{1}{c}|f(b) - f(a)|.$$

This observation explains the idea of the proof. By a partition  $-1 = t_0 < t_1 < \dots < t_k = 1$  we shall decompose  $(-1, 1)$  in such a way that  $\varphi$  is strictly monotone on  $(t_i, t_{i+1})$ . Moreover we shall distinguish between two disjoint types of intervals, namely

(1) there exists  $c_i > 0$  such that  $\varphi(t) \geq c_i$  on  $(t_i, t_{i+1})$  (we write  $i \in I_1$  in this case), or

(2) one of  $t_i, t_{i+1}$  is an asymptotic critical value of  $f$ , hence by Theorem 1, there exist  $c_i > 0$  and  $\Psi_i : (t_i, t_{i+1}) \rightarrow \mathbb{R}$  a strictly increasing, bounded  $C^1$  function such that,

$$\|\mathbf{grad}(\Psi_i \circ f)(x)\| \geq c_i$$

for all  $x \in f^{-1}(t_i, t_{i+1})$  (we write  $i \in I_2$  in this case).

Take now any trajectory  $\gamma$  of  $-\mathbf{grad} f$ , and let  $\gamma_i = \gamma \cap f^{-1}(t_i, t_{i+1})$ . We denote by  $|\gamma|$  (resp.  $|\gamma_i|$ ) the length of  $\gamma$  (resp.  $\gamma_i$ ). Clearly  $|\gamma_i| \leq \frac{1}{c_i}|t_i - t_{i+1}|$  if  $i \in I_1$ . Extending by continuity, we may suppose that each  $\Psi_i$  is defined also at  $t_i$  and  $t_{i+1}$ . Hence for  $i \in I_2$  we have  $|\gamma_i| \leq \frac{1}{c_i}|\Psi_i(t_i) - \Psi_i(t_{i+1})|$ , since the trajectories of  $-\mathbf{grad}(\Psi_i \circ f)$  and  $-\mathbf{grad} f$  are the same in  $f^{-1}(t_i, t_{i+1})$ . Finally, we can write

$$|\gamma| = \sum_{i=0}^{k-1} |\gamma_i| \leq \sum_{i \in I_1} \frac{1}{c_i}|t_i - t_{i+1}| + \sum_{i \in I_2} \frac{1}{c_i}|\Psi_i(t_i) - \Psi_i(t_{i+1})| = A,$$

which proves part a) of Theorem 2.

We are now going to construct the function  $\sigma$  of part b). For  $i \in I_2$  we put

$$\sigma_i(r) = \frac{1}{c_i} \sup\{|\Psi_i(p) - \Psi_i(q)| : p, q \in (t_i, t_{i+1}), r = p - q\},$$

and  $\sigma_i(r) = \frac{r}{c_i}$  for  $i \in I_1$ . Extend each  $\sigma_i$  to a continuous strictly increasing  $\mathcal{M}$ -function on  $\mathbb{R}$ . It is easily seen that  $\sigma = \sup \sigma_i$  satisfies b) of Theorem 2.

We finish this section by a short discussion of some consequences of Theorem 2, which extend and generalize those known in the real analytic (compact) setting.

Observe that if  $\gamma : (\alpha, \beta) \rightarrow U$  is a trajectory then  $x_0 = \lim_{s \rightarrow \beta} \gamma(s)$  exists, and in general  $x_0$  belongs to  $\bar{U}$ . Notice that if  $x_0 \in U$ , then  $x_0$  is a critical point of  $f$ . Let us take  $E$  a closed  $\mathcal{M}$ -subset in an open set  $U$ ; by 4.22 of [DM],  $E$  is the zero set of an  $\mathcal{M}$ -function  $f : U \rightarrow \mathbb{R}$  of class  $C^2$ . Let  $g = f^2$ . We want to show that the flow of  $-\mathbf{grad} g$  defines a strong deformation retraction of a neighborhood of  $E$  onto  $E$ . This is actually a new result even in the subanalytic case since the retraction is global and  $E$  is not necessarily compact. By Proposition 2, taking a neighborhood of  $E$ , we may suppose that 0 is the only asymptotic critical value of  $g$  in  $U$ . Clearly the set

$$V = \{x \in U : \text{dist}(x, \partial U) < \sigma(g(x))\}$$

is an  $\mathcal{M}$ -set, it is an open neighborhood of  $E$ . For each  $x \in V$  we denote by  $\gamma_x : (\alpha_x, \beta_x) \rightarrow U$  the trajectory passing through  $x$ . It is clearly unique if  $g(x) \neq 0$  and constant (hence unique) if  $g(x) = 0$ . Put  $R(x) = \lim_{s \rightarrow \beta_x} \gamma_x(s)$ , and observe that  $R(x) \in E$ . We have:

**PROPOSITION 3.** — *There exists an open neighborhood  $V_1$  of  $E$  such that  $R : V_1 \rightarrow E$  is a strong deformation retraction.*

*Proof.* — First we shall prove that  $R$  is continuous. Take  $x_0 \in V$  and  $\Omega_0$  a neighborhood of  $R(x_0)$ . Let  $x_1 \notin E$  be close to  $R(x_0)$  so that there is (by Theorem 2 b)) a neighborhood  $\Omega_1$  of  $x_1$  with the following property: any trajectory passing through  $\Omega_1$  has its limit in  $\Omega_0$ . By continuity of the flow of  $-\mathbf{grad} g$  there exists a neighborhood  $G$  of  $x_0$  such that any trajectory passing by  $G$  must cross  $\Omega_1$ . So we have  $R(G) \subset \Omega_0$ , which proves the continuity of  $R$ .

Let  $\gamma$  be the trajectory passing through  $x$ . Let  $\gamma_x$  be the part of  $\gamma$  between  $x$  and the limit  $R(x)$ . Assume that  $\gamma_x : [0, \beta_x] \rightarrow U$  is parametrized by arc-length; moreover that  $\gamma_x(0) = x$ , and  $\gamma_x(\beta_x) = R(x)$ . Clearly  $\beta_x$  is the length of  $\gamma_x$ . Notice that the argument in the proof of continuity of  $R$  yields that the function  $V \ni x \rightarrow \beta_x$  is continuous. Let  $V_1$  be the set of all  $x \in V$  such that  $\gamma_x$  lies in  $V$ . We define a homotopy  $F : [0, 1] \times V_1 \rightarrow V_1$  as follows:  $F_t(x) = \gamma_x(t\beta_x)$ .

In general the retraction  $R$  is not an  $\mathcal{M}$ -mapping. Take  $g(x, y) = (x^2 - y^3)^2$ ; it was observed by Hu [Hu] that the retraction  $R$  is not hoelderian (at  $(0, 0)$ ) in this case, hence it cannot be subanalytic. Observe also that, in general, the set  $V_1$  is not an  $\mathcal{M}$ -set. It would be interesting to prove that actually  $R$  belongs to some larger o-minimal structure. Even a weaker problem is open (also in the subanalytic case):

CONJECTURE (F). — *Let  $\gamma$  be a trajectory of  $-\mathbf{grad} f$ , where  $f$  is an  $\mathcal{M}$ -function of class  $C^1$ , and let  $H$  be any  $\mathcal{M}$ -subset. Then  $\gamma \cap H$  has a finite number of connected components.*

This is connected with the Gradient Conjecture of R. Thom, proved recently in [KM]. R. Thom asked whether for an analytic function  $f$  every trajectory  $\gamma$  of  $-\mathbf{grad} f$  has a tangent at the limit point (i.e. whether  $\lim_{s \rightarrow \beta_x} \frac{\gamma(s) - R(x)}{|\gamma(s) - R(x)|}$  exists). We can of course ask the same question for a trajectory of the gradient of any  $\mathcal{M}$ -function of class  $C^1$ .

It is easily seen that (F) implies that  $\lim_{s \rightarrow \beta_x} \frac{\gamma'(s)}{|\gamma'(s)|}$  exists, thus that the tangent to  $\gamma$  at the limit point exists.

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Krzysztof KURDYKA,  
Université de Savoie  
Laboratoire de Mathématiques  
Campus Scientifique  
73376 Le Bourget-du-Lac Cedex (France).  
kurdyka@univ-savoie.fr  
and  
Uniwersytet Jagielloński  
Instytut Matematyki  
ul. Reymonta 4  
30-059 Kraków (Poland).  
kurdyka@im.uj.edu.pl