ANNALES DE L'INSTITUT FOURIER

YVES LASZLO

About G-bundles over elliptic curves

Annales de l'institut Fourier, tome 48, nº 2 (1998), p. 413-424
<http://www.numdam.org/item?id=AIF_1998__48_2_413_0>

Ann. Inst. Fourier, Grenoble
48, 2 (1998), 413–424

ABOUT G-BUNDLES OVER ELLIPTIC CURVES

by Yves LASZLO(*)

1. Introduction.

In this note, we study principal bundles over a complex elliptic curve X with reductive structure group G. As in the vector bundle case, we first show that a non semistable bundle has a canonical semistable L-structure with L some Levi subgroup of G reducing the study of G-bundles to the study of semistable bundles (Proposition 3.2). We then look at the coarse moduli space M_G of topologically trivial semistable bundles on X (there is not any stable topologically trivial G-bundle) and prove that it is isomorphic to the quotient $[\Gamma(T)\otimes_{\mathbf{Z}} X]/W$ where $\Gamma(T)$ is the group of one parameter subgroups of a maximal torus T and $W=N(G,T)/T$ is the Weyl group (Theorem 4.16). Suppose that G is simple and simply connected and let θ be the longest root (relative to some basis $(\alpha_1,\dots,\alpha_l)$ of the root system $\Phi(G,T)$). The coroot $\theta^\vee$ of θ has a decomposition $\theta^\vee=\sum_i g_i\alpha_i^\vee$ with g_i a positive integer. Using Theorem 4.16 and Looijenga's isomorphism

$$[\Gamma(T)\otimes_{\mathbf{Z}} X]/W \xrightarrow{\sim} \mathbf{P}(1,g_1,\dots,g_l),$$

one gets that M_G is isomorphic to the weighted projective space $\mathbf{P}(1,g_1,\dots,g_l)$ (see 4.17), generalizing the well-known isomorphism $M_{\mathbf{SL}_{l+1}} \xrightarrow{\sim} \mathbf{P}^l$ (see [T] for instance). One recovers for instance the Verlinde formula in this case.

We know that these results are certainly well-known from experts, but we were unable to find any reference in the literature, except of course when G is either **SL** or **GL**. For another point of view, see [BG].

During the referee process of this paper, an independent paper of Friedman, Morgan and Witten has appeared in Duke's eprints (see [FMW]), where the link between Looijenga's result and bundles on elliptic curves is studied from another point of view.

I would like to thank M. Brion and J. Le Potier for their comments on a preliminary version of this note and the referee for valuable suggestions.

(*) Partially supported by the European HCM Project "Algebraic Geometry in Europe" (AGE).
Key words: Semistable – Moduli space – Elliptic curve.
Math. classification : 14-01 – 14H60 – 14D20.

Notations. — By scheme, we implicitly mean a complex scheme. Let G be a reductive group with a Borel subgroup (resp. a maximal torus) B (resp. $T \subset B$). The corresponding Lie algebras will be denoted by $\mathfrak{t}, \mathfrak{b}$ and $\mathfrak{g}$.

We denote by $W = N(G,T)/T$ the Weyl group and by $\Gamma(T)$ the W-module $\mathrm{Hom}(G_m, T)$.

2. Review on the Harder-Narasimhan reduction.

Let X be an algebraic curve which is smooth, projective and connected and E be a G-bundle on X. Recall that E is semistable if and only if the adjoint bundle $\mathcal{E} = \mathrm{Ad}(E)$ is a semistable vector bundle. Following [AB], let me recall how to define the Harder-Narasimhan reduction of E. Pick a non degenerate invariant quadratic form q on the Lie algebra of G. Then q defines a non degenerate quadratic form on $\mathcal{E}$.

LEMMA 2.1. — *The length $r-1$ of the Harder-Narasimhan filtration*

$$0 = \mathcal{E}_0 \subset \ldots \subset \mathcal{E}_i \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

of $\mathcal{E}$ is even.

Proof. — Let

$$0 = \mathcal{E}_0 \subset \ldots \subset \mathcal{E}_i \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

be the Harder-Narasimhan filtration of $\mathcal{E}$. The Harder-Narasimhan filtration of $\mathcal{E}^*$ is

$$0 \subset \ldots \subset (\mathcal{E}/\mathcal{E}_{r-i})^* \subset \ldots \subset \mathcal{E}^*.$$

Because $\mathcal{E}$ is self-dual, the Harder-Narasimhan filtration is self-dual and one has an isomorphism

$$\mathcal{E}_i \xrightarrow{\sim} (\mathcal{E}/\mathcal{E}_{r-i})^* \tag{2.1}$$

inducing isomorphisms

$$gr_i \xrightarrow{\sim} (gr_{r+1-i})^*, i = 1, \ldots r \tag{2.2}$$

where $gr_i = \mathcal{E}_i/\mathcal{E}_{i-1}$. Assume that the length $r-1$ is odd. Let us consider the morphisms

$$m_k : \mathcal{E}_{r/2} \otimes \mathcal{E}_k \to \mathcal{E}/\mathcal{E}_{k-1},\ 0 \le k \le r-1$$

deduced from the Lie bracket of $\mathcal{E}$. The equality (2.2) gives the inequalities

$$\mu_1 > \ldots > \mu_{r/2} > -\mu_{r/2} > \ldots > -\mu_1 \tag{2.3}$$

where μ_i is the slope of the semistable vector bundle $gr_i = \mathcal{E}_i/\mathcal{E}_{i-1}$. In particular, the slopes $\mu_i + \mu_j$ of the subquotients $gr_i \otimes gr_j, i \leq r/2$ and $j \leq k$ which appear in $\mathcal{E}_{r/2} \otimes \mathcal{E}_k$ are not less than $\mu_{r/2} + \mu_k > \mu_k$ though the slopes of the subquotients $gr_i, i \geq k$ which appear $\mathcal{E}/\mathcal{E}_{k-1}$ are $\leq \mu_k$. This shows that m_k is zero for all k and that all elements of$\mathcal{E}_{r/2}$ are *nilpotent.* By (2.1), this algebra is also lagrangian. Suppose that the center of G is of positive dimension. Then $\mathcal{E}$ contains a trivial sub-bundle (of positive rank) as a direct summand which implies that some of the μ_i's is zero, contradicting (2.3). The Lie algebra bundle $\mathcal{E}$ is therefore semisimple and therefore has no no non trivial lagrangian sub-Lie algebra (with respect of the Killing form form) consisting of nilpotent elements, just by a dimension argument. □

It follows that one can index the Harder-Narasimhan filtration of $\mathcal{E}$ such that

$$0 = \mathcal{E}_{-r} \subset \mathcal{E}_{-r+1} \subset \ldots \subset \mathcal{E}_{-1} \subset \mathcal{E}_0 \subset \ldots \subset \mathcal{E}_{r-1} = \mathcal{E} \tag{2.4}$$

where $\mathcal{E}_{-j}$ is the orthogonal of $\mathcal{E}_{j-1}$. One checks that $\mathcal{E}_0$ is a subalgebra of $\mathcal{E}$. Notice that $\mathcal{E}_0/\mathcal{E}_{-1}$ is self-dual and therefore has slope zero. In particular, the slope of $gr_{-j}, j > 0$ is > 0. As in the proof of the preceding lemma, this immediately implies the sequence of inclusions

$$[\mathcal{E}_{-j}, \mathcal{E}_{-1}] \subset \mathcal{E}_{-j-1} \text{ for all } j. \tag{2.5}$$

In particular, all elements of $\mathcal{E}_{-1}$ are nilpotent. For sake of completeness, let me prove this easy lemma (cf. Th VIII.10.1 of [Bo]).

LEMMA 2.2. — *Let $\mathfrak{g}$ be a reductive algebra endowed with an invariant non degenerate bilinear form. Let $\mathfrak{n}'$ be a subalgebra of $\mathfrak{g}$ whose elements are nilpotent. Then, if the orthogonal of $\mathfrak{n}'$ is a sub-Lie algebra of $\mathfrak{g}$, it is parabolic.*

Proof. — The Lie algebra $\mathfrak{n}'$ is nilpotent. Let $\mathfrak{b}$ be a maximal solvable Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{n}'$ and $\mathfrak{n}$ its nilpotent ideal. By [Bo], Definition VIII.3.3.1, $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$. Because all elements of $\mathfrak{n}'$ are nilpotent, $\mathfrak{n}'$ is contained in $\mathfrak{n}$. By [Bo], proposition VII.1.3.10 (iii), the orthogonal of $\mathfrak{n}$ is $\mathfrak{b}$ and the lemma follows. □

Because the orthogonal of $\mathcal{E}_{-1}$ is $\mathcal{E}_0$, the Lie subalgebra $\mathcal{E}_0$ is therefore parabolic with radical $\mathcal{E}_{-1}$ (see [AB], p. 589). Let P be the unique standard parabolic subgroup defined by $\mathcal{E}_0$. If F is the bundle of local trivialization $G_S \xrightarrow{\sim} E_S$ ($S \to X$ étale) whose differential sends $\mathrm{Lie}(P)_S$ to $\mathcal{E}_0$, then F is a P-structure of E. Let us denote by U the unipotent radical of P and by $\bar{P}$ (resp. $\bar{F}$) the quotient P/U (resp. F/U). By construction, $\bar{F}$ is semistable (because $\mathrm{Ad}(\bar{F}) = \mathcal{E}_0/\mathcal{E}_{-1}$).

DEFINITION 2.3. — *With the notation above, the P-bundle F is the Harder-Narasimhan reduction of E.*

Remark 2.4. — It is easy to check that the filtration and therefore the corresponding reduction does not depend on the particular choice of the invariant non degenerate quadratic form on $\mathrm{Lie}(G)$.

Example 2.5. — Suppose that E is the $\mathbf{GL}_n$-bundle of local frames of a vector bundle $\mathcal{E}$ on X with Harder-Narasimhan filtration $0 = \mathcal{E}_0 \ldots \subset \mathcal{E}_i \ldots \subset \mathcal{E}_k = \mathcal{E}$. Let P be the quasi-triangular subgroup of $\mathbf{GL}_n$-defined by the partition $[r_i = \mathrm{rk}(\mathcal{E}_{i+1}) - \mathrm{rk}(\mathcal{E}_i)]_{0 \leq i < k}$ of $\mathrm{rk}(\mathcal{E})$. Then F is the P-bundle of local frames compatible with the filtration and $\bar{F}$ is the $\times_i \mathbf{GL}_{r_i}$-bundle of local frames of $\bigoplus_i \mathcal{E}_{i+1}/\mathcal{E}_i$.

We suppose once for all that X is an elliptic curve.

3. Non semistable G-bundles.

Let E be a G-bundle on X and let F be the Harder-Narasimhan reduction of E. Let us consider a Levi factor of P thought as a section $\sigma : \bar{P} \to P$ of the canonical projection $P \twoheadrightarrow \bar{P} = P/U$.

Remark 3.1. — Following Humphreys (see [Hu], 30.2) a Levi factor is a factor of the unipotent radical and not of the radical itself (as in Bourbaki for instance).

PROPOSITION 3.2. — *With the notations above, the P-bundle $\sigma_*(\bar{F})$ is isomorphic to F.*

Remark 3.3. — This is the generalization of the well-known (and easy) fact that any vector bundle on X is a direct sum of semistable vector bundles.

Proof. — Let us denote by

$$1 \to \mathcal{U} \to \mathcal{P} \to \bar{\mathcal{P}} \to 1$$

be the twist of

$$1 \to U \to P \to \bar{P} \to 1$$

by F (see [S], chap. I & 5). Geometrically, $\mathcal{P}$ (resp. $\bar{\mathcal{P}}$) is the group scheme $\mathcal{A}ut_P(F)$ (resp. $\mathcal{A}ut_{\bar{P}}(\bar{F})$). The twisted group $\mathcal{U}$ is the unipotent radical of $\mathcal{P}$ and is isomorphic $F \times_P U$ (P acts on the normal subgroup U by conjugation). As usual, the map

$$\begin{cases} H^1(X,P) & \to & H^1(X,\mathcal{P}) \\ F'' & \longmapsto & \mathcal{I}som_P(F,F'') \end{cases} \text{resp.} \begin{cases} H^1(X,\bar{P}) & \to & H^1(X,\bar{\mathcal{P}}) \\ \bar{F}'' & \longmapsto & \mathcal{I}som_{\bar{P}}(\bar{F},\bar{F}'') \end{cases}$$

are bijective. The image of $\mathcal{I}som_P(F,\sigma_*\bar{F})$ in $H^1(X,\bar{\mathcal{P}})$ is the trivial torsor $\mathcal{I}som_{\bar{P}}(\bar{F},\bar{F})$ and it is enough to show the equality $H^1(X,\mathcal{U}) = \{[\mathcal{U}]\}$ to prove the isomorphism $F \xrightarrow{\sim} \sigma_*(\bar{F})$. With the notations of (2.4), the Lie algebra of $\mathcal{U}$ is $\mathcal{E}_{-1}$. By (2.5), the Lie algebra $\mathcal{E}_{-j}/\mathcal{E}_{-j-1}$ is abelian for any $j \geq 1$. This induces a filtration

$$1 = \mathcal{U}_{-r} \subset \mathcal{U}_{-r+1} \subset \ldots \subset \mathcal{U}_{-2} \subset \mathcal{U}_{-1} = \mathcal{U}$$

by unipotent group schemes where the exponential defines isomorphisms

$$\mathcal{U}_{-j}/\mathcal{U}_{-j-1} \xrightarrow{\sim} \mathcal{E}_{-j}/\mathcal{E}_{-j-1} \; j \geq 1$$

of abelian group schemes. By construction, $\mathcal{E}_{-j}/\mathcal{E}_{-j-1}, j \geq 1$ is semistable of positive slope and therefore

$$H^1(X, \mathcal{E}_{-j}/\mathcal{E}_{-j-1}) = 0, j \geq 1$$

because $g(X) = 1$. This implies the equality

$$H^1(X,\mathcal{U}) = \{[\mathcal{U}]\}.$$

□

4. The coarse moduli space M_G.

Let M_G be the coarse moduli space of semistable G-bundle of trivial topological type (what is the same, the component containing the trivial torsor G_X). Recall that the (closed) points of M_G are S-equivalence classes of semistable G-bundles. The only thing which will be needed about this equivalence relation is the following (cf. [Ra1], Corollary 3.12.1):

4.1. Every class ξ defines a Levi subgroup L such that there exists a stable L-bundle F with $F(G) \in \xi$. Moreover, $F(G)$ is well defined up to isomorphism.

Remark 4.2. — Ramanathan's construction of M_G is written for a curve of genus ≥ 2, but the construction can be made in general (see for instance [LeP] in the case of $G = GL_n$ from which the general case follows).

4.3. We denote by $a \otimes b$ the product of two T-bundles a and b (for the natural structure of abelian group of $H^1(X,T)$). Let $\underline{\psi} = (\psi)_{i \in I}$ be a finite family of one parameter subgroups and $\underline{L} = (L_i \in I)$ a family of line bundles of degree 0 on X (thought as G_m-torsors). Then, $\underset{i \in I}{\otimes} L_i(\psi_i)$ is a T-structure of a G-bundle $\underline{L_{\psi}}$ on X which is semistable. This defines a morphism of abelian groups

$$p : \Gamma(T) \otimes_{\mathbf{Z}} X \to H^1(X,T).$$

Chose a (closed) point x of X which defines an isomorphism $\mathrm{Pic}^0(X) \xrightarrow{\sim} X$ and a Poincaré line bundle $\mathcal{P}$ on $X \times \mathrm{Pic}^0(X)$. This allows to construct a universal semistable T-bundle $\mathbf{L}$ on $X \times \Gamma(T) \underset{\mathbf{Z}}{\otimes} X$.

Remark 4.4. — The theta line bundle Θ on $X = \mathrm{Pic}^1(X)$ becomes through the isomorphism $X \xrightarrow{\sim} \mathrm{Pic}^0(X)$ the determinant bundle $\det(R\Gamma\mathcal{P})^*$.

The family of semistable bundles $\mathbf{L}(G)$ defines a morphism of (reduced) schemes

$$\Gamma(T) \otimes_{\mathbf{Z}} X \to M_G.$$

The action of the Weyl group W on $\Gamma(T)$ defines an action $\Gamma(T) \underset{\mathbf{Z}}{\otimes} X$ such that $w.L_{\psi} \xrightarrow{\sim} L_{\psi}$ for all $w \in W$. Let

$$\pi : \ [\Gamma(T) \otimes_{\mathbf{Z}} X]/W \to M_G$$

be the induced morphism. We want to prove that π is an isomorphism.

4.5. Let us prove that π is finite. Let $G \to \mathbf{SL}_N$ be a faithful representation of G inducing a morphism $M_G \to M_{\mathbf{SL}_N}$. Let L be the inverse of the determinant bundle on $M_{\mathbf{SL}_N}$.

Remark 4.6. — Notice that in this case, $M_{\mathbf{SL}_N} = \mathbf{P}^{N-1}$ and that the determinant bundle is just $\mathcal{O}(1)$ (see [Tu], Theorem 7 for instance).

LEMMA 4.7. — *The line bundle $\pi^*(L)$ is ample.*

Proof. — One can assume that G is semisimple. Let q be the natural morphism

$$q : \ \Gamma(T) \otimes_{\mathbf{Z}} X \to M_G.$$

It is enough to prove that $q^*(L)$ is ample. Let us choose a basis of $\Gamma(L)$ identifying $\Gamma(T) \underset{\mathbf{Z}}{\otimes} X$ with X^l (l is the rank of G). Let $\gamma : G_m \to T$ be a non trivial element in $\Gamma(T)$. Let $q_\gamma : X \to M_G$ be the morphism defined by γ. One can assume that $\gamma(z) = \operatorname{diag}(z^{\gamma_1}, \ldots, z^{\gamma_l})$ for $z \in \mathbf{C}^*$ (with $\sum \gamma_i = 0$). Then (see Remark 4.4),

$$q_\gamma^*(L) = \Theta^{\sum_i \gamma_i^2}$$

which is ample because $\sum\limits_i \gamma_i^2 > 0$ (recall that γ is non trivial). The rank-N vector bundle bundle parameterized by $(x_1, \ldots, x_l)$ is

$$\bigoplus_i \mathcal{O}\Big(\sum_{\gamma \in \gamma} \gamma_i (x_\gamma - x)\Big).$$

By additivity of the determinant bundle, $q^*(L)$ is of the form

$$\underset{1 \leq i \leq l}{\boxtimes} \Theta^{b_i} \text{ with } b_i > 0$$

and therefore is ample. □

The fibers of π are therefore finite, and the proper morphism π is finite.

4.8. Let $\pi^{-1}(0)$ be the fiber of π at the trivial bundle G_X. Let us first prove that $\pi^{-1}(0)$ is set-theoretically reduced to $[0]$, the class $W.0$. Let us first prove the following general result.

4.9. Let us consider the following situation: let $p : \mathcal{X} \to S$ be a proper morphism such that $\mathcal{O}_S \to p_*\mathcal{O}_\mathcal{X}$ is an isomorphism. Assume that p has a section $\sigma :\ S \to \mathcal{X}$. Let $A \subset B$ be a reductive subgroup of a linear group B.

LEMMA 4.10. — *Let α be an A-bundle trivial along σ. Then, if the associated B-bundle $\beta = \alpha(B)$ is trivial, the A-bundle α is so.*

Proof. — Let s be the section of β/A defined by α. Because β/A is affine over S, the section s factors through p in a section $\tilde{s}$. Because α is trivial along σ, the section $\tilde{s}$ comes from a section of the restriction to σ of the trivial bundle β and can be lifted to a section s' of β. The section s' mod A ofβ/A is equal to s and defines a trivialization of $\alpha = s^*\beta$. □

4.11. Choose an embedding G in a product $G' = \prod\limits_i \mathbf{GL}_{n_i}$ of linear groups such that $Z_0(G) \subset Z_0(G')$ (Z_0 denotes the neutral component). Let T' be a maximal torus of G' containing T. Let $f : M_G \to M_{G'}$ be a natural morphism (see [Ra2], Corollary of Theorem 7.1). Let E be a T-bundle such that $E \in \pi^{-1}(0)$ and let E' be the corresponding T'-bundle.

Because $f(E(G)) = [E'(G')]$, the semistable bundle is equivalent to the trivial bundle and is therefore trivial (a direct sum of line bundles of degree 0 is equivalent to the trivial bundle if and only if all summands are trivial). Applying the preceding lemma with $\alpha = E, A = T, B = G'$ and $\mathcal{X} = X$ for instance, one gets that E is trivial.

4.12. It remains to show that π is étale at the origin: this will follow from the fact that the completion of π at the origin can be identified to the completion at the origin of the Chevalley isomorphism $\mathfrak{t}/W \xrightarrow{\sim} \mathfrak{g}/G$.

LEMMA 4.13. — *The morphism* $\pi : \ (\Gamma(T) \otimes \mathrm{Pic}^0(X))/W \to M_G$ *is étale at the origin.*

Proof. — Let's briefly recall how to construct the moduli space M_G (see [Ra1], [BLS]), or better of an affine neighborhood M of the trivial bundle $X \times G$ as a GIT quotient Y/H of a smooth affine scheme Y by some reductive group H (with Lie algebra $\mathfrak{h}$). One choose first a faithful representation $G \hookrightarrow \mathbf{GL}_n$ inducing an embedding $\Gamma(T) \otimes \mathrm{Pic}^0(X) \hookrightarrow (\mathrm{Pic}^0(X))^n$. For m big enough, one knows that the canonical morphism

$$\iota_P : \ H^0(P(\mathbf{C}^n) \otimes \mathcal{O}(mx)) \otimes \mathcal{O} \to P(\mathbf{C}^n) \otimes \mathcal{O}(mx)$$

is surjective for all semistable bundles P and that $H^0(\iota_P)$ is bijective. Let χ be the Euler characteristic of some $P(\mathbf{C}^n) \otimes \mathcal{O}(mx)$. By the theory of Hilbert schemes, the pairs (P, ι) where P is a semistable G-bundle and ι an isomorphism

$$H^0(P(\mathbf{C}^n) \otimes \mathcal{O}(mx)) \xrightarrow{\sim} \mathbf{C}^\chi$$

are parameterized by a smooth scheme Y and M_G is a GIT quotient of this scheme by $H = \mathbf{GL}_\chi$ (see [BLS]). Notice that the stabilizer of the "trivial pair" is G itself.

Let $\mathcal{U}$ be the universal T-bundle on $X \times (\Gamma(T) \otimes \mathrm{Pic}^0(X))$. Let us chose a trivialization of the vector bundle $R\Gamma(\mathcal{L} \boxtimes \mathcal{O}(mx))$ on some symmetric affine neighborhood S^0 of 0 in $\mathrm{Pic}^0(X)$. Therefore, the direct image of $\mathcal{U}(\mathbf{C}^n) \boxtimes \mathcal{O}(m)$ is trivial on $S = (S^0)^n \cap \Gamma(T) \otimes \mathrm{Pic}^0(X)$ and the trivialization is W-equivariant. The induced morphism $\pi \colon S \to M_G = Y/H$ is therefore induced by a W-equivariant morphism $S \to Y$ mapping 0 to y. Notice that the orbit $H.y$ is closed. By considering some H-invariant affine open neighborhood of y, one can assume that Y is affine (the quotient Y/H is now a neighborhood of $H.y$ in M_G).

Let's consider the following commutative diagram:

$$\begin{array}{ccc} \mathbf{C}[Y]_+ & \longrightarrow & \mathbf{C}[S]_+ \\ \Big\downarrow & & \Big\downarrow \\ V = (T_y^*Y)/\mathfrak{h} & \xrightarrow{k} & T_0^*S \end{array} \tag{4.1}$$

where $\mathbf{C}[Y]_+$ (resp. $\mathbf{C}[S]_+$) denotes the maximal ideal of y (resp. 0). The transpose of k is the tangent map of $S \to \mathcal{M}_G$ from S to the stack of G-bundles on X, namely the Kodaira-Spencer map

$$k: \ \mathfrak{t} = \mathfrak{t} \otimes H^1(X, \mathcal{O}_X) = T_0S \to (T_yY)/\mathfrak{h} = \mathfrak{g} \otimes H^1(X, \mathcal{O}) = \mathfrak{g}.$$

LEMMA 4.14. — *The Kodaira-Spencer map k is the canonical inclusion $\mathfrak{t} \hookrightarrow \mathfrak{g}$.*

Proof. — By functoriality, one is reduced to the case where $G = \mathbf{GL}_n$ and T is the torus of invertible diagonal matrices. Consider the one parameter subgroup of differential $aE_{i,i}$ for some integer a ($E_{i,i}$ is the standard diagonal rank 1 matrix). If $(\lambda_{\alpha,\beta})$ is a Cech-cocycle representing $\lambda \in H^1(\mathcal{O})$, the derivative

$$\frac{\partial \pi}{\partial(\gamma \otimes \lambda)}(0)$$

is defined by the vector bundle on $X[\epsilon]/(\epsilon^2 = 0)$ with cocycle $1 + a\epsilon\lambda_{\alpha,\beta}E_{i,i}$. In other words,

$$\frac{\partial \pi}{\partial(\gamma \otimes \lambda)}(0) = \lambda \mathrm{d}\gamma,$$

which proves the lemma. □

Notice that k is $N(G,T)$-equivariant. By Luna's results ([Lu]), one obtains an étale slice of $Y \to Y/H$ as follows.

One choose an $H_y = \mathrm{Aut}_G(X \times G) = G$-invariant section σ of $\mathbf{C}[Y]_+ \to (T_y^*Y)/\mathfrak{h}$ and the induced morphism $Y \to V$ identifies étale locally Y/H and V/H_y. The group $N(G,T)$ being reductive and ${}^tk$ being surjective, one pick an invariant section τ of ${}^tk : \mathfrak{g}^* \twoheadrightarrow \mathfrak{t}^*$ which defines a morphism (still denoted by τ)

$$\mathfrak{t}^* \to \mathbf{C}[Y]_+ \to \mathbf{C}[S]_+$$

which is W-equivariant. This is a W-equivariant section of $\mathbf{C}[S]_+ \to T_0^*S$ and therefore defines an étale slice of $S \to S/W$. Shrinking S and Y if necessary, one obtains from the diagram (4.1) the commutative diagram

$$\begin{tikzcd} \mathbf{C}[Y]_+^H \arrow[r, "\pi"] & \mathbf{C}[S]_+^W \\ (\mathbf{S}\,\mathfrak{g}^*)^G \arrow[u, "\mathbf{S}(\sigma)"] \arrow[r, "\mathbf{S}({}^t k)"] & (\mathbf{S}\,\mathfrak{t}^*)^W \arrow[u, "\mathbf{S}(\tau)"'] \end{tikzcd}$$

where $\mathbf{S}(\sigma)$ and $\mathbf{S}(\tau)$ are étale. By Chevalley's theorem, π is therefore étale at the origin. □

4.15. The morphism π is therefore a finite morphism between normal varieties and is of degree 1. We have proved the

THEOREM 4.16. — *The morphism*

$$\pi : \ [\Gamma(T) \otimes_{\mathbf{Z}} X]/W \to M_G$$

is an isomorphism.

4.17. Assume that G is simple and simply connected. Let θ be the longest root and $\alpha_i, i = 1, \dots l$ the basis of the root system $\Phi(B, G)$. The coroot $\theta^\vee$ is a sum

$$\theta^\vee = \sum_i g_i \alpha_i^\vee$$

where $\alpha_i^\vee$ is the coroot of α. Then Looijenga [Lo] has proved that $[\Gamma(T) \otimes_{\mathbf{Z}} X]/W$ is the weighted projective space $\mathbf{P}(1, g_1, \dots g_l)$.

Remark 4.18. — The proof in [Lo] is not correct. See [BS] for a more general result and hints for a complete proof.

4.19. It is interesting to remark ([D], remarques 1.8) that $\mathcal{O}(l)$ is locally free if and only if l is a multiple of $\mathrm{lcm}(g_i)$ although it is reflexive (Lemme 4.1 of *loc. cit.*). In particular, M_G is locally factorial if and only if $\mathrm{lcm}(g_i) = 1$, condition which is equivalent to G special in the sense of Serre (look at the table of [Bo]). If one notice that $\mathrm{lcm}(g_i)$ is also the minimal Dynkin index of the representations of G (see [LS]), this funny characterization of special groups in terms of M_G is the version in the genus one case of Proposition 13.2 of [BLS] (which deals with the genus > 1). In all the cases, one has the formula

$$\dim H^0(M_G, \mathcal{O}(l)) = \mathrm{card}(P_l) \tag{4.2}$$

where P_l is the number of dominant weights w such that $< \theta^\vee, w > \leq l$, as predicted by the Verlinde formula (see [Be]).

4.20. Let us explain briefly the link between the theorem of Narasimhan and Seshadri and our description of M_G. Suppose that G is semisimple with maximal compact subgroup K. The theorem of Narasimhan and Seshadri says that the complex points of M_G are parameterized by equivalence classes of pairs of elements of K which commutes (K acting on these pairs diagonally through the adjoint action). Suppose further that G is simply connected. Then such a class has a representative in $T_{\mathbf{R}} \times T_{\mathbf{R}}$ (where $T_{\mathbf{R}}$ is the maximal torus of K). Suppose that $X(\mathbf{C})$ is a complex torus $\mathbf{C}/\mathbf{Z} \oplus \mathbf{Z}\tau$ of period τ in the Poincaré upper half plane. The complex structure $(a, b) \to a - \tau b$ on $\mathbf{R} \times \mathbf{R}$ induces a complex structure on $T_{\mathbf{R}} \times T_{\mathbf{R}}$ which is naturally the maximal torus T of G. We get a diagram

$$\begin{array}{ccc} \Gamma(T) \otimes \mathbf{C} & \xrightarrow{\sim} & T_{\mathbf{R}} \times T_{\mathbf{R}} \\ \downarrow & & \downarrow \\ \Gamma(T) \otimes X(\mathbf{C}) & \xrightarrow{\pi} & M_G(\mathbf{C}) \end{array}$$

One checks easily that this diagram commutes.

BIBLIOGRAPHY

[AB] M.F. ATIYAH, R. BOTT, The Yang-Mills equations over Riemann surfaces, Phil. Trans. R. Soc. Lond., A 308 (1982), 523-615.

[Be] A. BEAUVILLE, Conformal blocks, fusion rules and the Verlinde formula, Israel Math. Conf. Proc., 9 (1996), 75-96.

[BS] I. N. BERNSHTEIN, O. V. SHVARTSMAN, Chevalley's theorem for complex crystallographic Coxeter groups, Funkt. Anal. i Ego Prilozheniya, 12 (1978), 79-80.

[BLS] A. BEAUVILLE, Y. LASZLO, C. SORGER, The Picard group of the moduli stack of G-bundles on a curve, preprint alg-geom/9608002, to appear in Compos. Math.

[Bo] N. BOURBAKI, Groupes et algèbres de Lie, chap. 7, 8 (1990), Masson.

[BG] V. BARANOVSKY, V. GINZBURG Conjugacy classes in loop groups and G-bundles on elliptic curves, Int. Math. Res. Not., 15 (1966), 733-752.

[D] C. DELORME Espaces projectifs anisotropes, Bull. Soc. Math. France, 103 (1975), 203-223.

[FMW] R. FRIEDMAN, J. MORGAN, E. WITTEN Vector bundles and F Theory, eprint hep-th 9701162.

[Hu] J. E. HUMPHREYS, Linear algebraic groups, GTM 21, Berlin, Heidelberg, New-York, Springer (1975).

[LS] Y. LASZLO, C. SORGER Picard group of the moduli stack of G-bundles, Ann. Scient. Éc. Norm. Sup., 4e série, 30 (1997), 499-525.

[LeP] J. LE POTIER, Fibrés vectoriels sur les courbes algébriques, Publ. Math. Univ. Paris 7, 35 (1995).

[Lo] E. LOOIJENGA, Root systems and elliptic curves, Invent. Math., 38 (1976), 17-32.

[Ra1] A. RAMANATHAN, Moduli for principal bundles over algebraic curves, I and II, Proc. Indian Acad. Sci. Math. Sci., 106 (1996), 301-328 and 421-449.

[Ra2] A. RAMANATHAN, Stable principal bundles on a compact Rieman surface, Math. Ann., 213 (1975), 129-152.

[S] J.-P. SERRE, Cohomologie galoisienne, LNM 5(1964).

[T] L. TU, Semistable bundles over an elliptic curve, Adv. Math., 98 (1993), 1–26.

Manuscrit reçu le 27 juin 1997,
révisé le 15 octobre 1997,
accepté le 13 janvier 1998.

Yves LASZLO,
École Normale Supérieure
DMI (URA 762 du CNRS)
45, rue d'Ulm
F-75230 Paris Cedex 05.