ANNALES DE L'INSTITUT FOURIER

THOMAS BLOOM

JEAN-PAUL CALVI

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Annales de l'institut Fourier, tome 48, nº 1 (1998), p. 205-222

<http://www.numdam.org/item?id=AIF_1998__48_1_205_0>

Ann. Inst. Fourier, Grenoble
48, 1 (1998), 205–222

THE DISTRIBUTION OF EXTREMAL POINTS FOR KERGIN INTERPOLATION: REAL CASE

by T. BLOOM and J.-P. CALVI

1. Introduction.

The general purpose of this note is to study, in some cases, the sequences of Kergin interpolation operators that are the best (see below) for approximating holomorphic functions. Let K be a $\mathbb{C}$-convex (for the definition see [1], [2] or [3]) compact set in $\mathbb{C}^n$, $n \geq 1$. We say that an infinite triangular array of points in K

$$A = \{A_j^d;\ j = 0, 1, \ldots, d; d = 1, 2, \ldots\} \tag{1.1}$$

is *extremal* for Kergin interpolation on K if, for every function f holomorphic on a neighborhood of K (i.e. $f \in H(K)$), the Kergin interpolation polynomial $\mathcal{K}_{A^d} f$ of f with respect to the points $A_0^d, \ldots, A_d^d$ converges to f uniformly on K as $d \to \infty$. If such an array exists, we say that K admits an extremal array. The question of knowing whether a given array A is extremal or not is related, as we shall see, to the study of the distribution of the points, that is to the behavior of the sequence of probability measures

$$\mu_d^A = \mu_d := \frac{1}{d} \sum_{j=0}^{d} [A_j^d], \qquad (d = 1, 2, \ldots) \tag{1.2}$$

where $[x]$ stands for the Dirac measure of the point x.

The first author was supported by NSERC of Canada.
Key words: Kergin interpolation - Polynomial approximation of holomorphic functions - Logarithmic potentials.
Math. classification: 32A05 - 41A05 - 41A63.

Recently, examples of extremal arrays have been found in the case of circular convex sets (see [4] and below). Here, we shall study the case of (totally) real sets, the definition of which follows.

One says that a real subspace V of $\mathbb{C}^n$ is *totally real* if $V \cap iV = \{0\}$. A compact set is said to be totally real if it is contained in a translate of a totally real subspace, in particular its interior as a subset of $\mathbb{C}^n$ is empty. A compact set of the form

$$E = \{a + r\cos\theta e_1 + r\sin\theta e_2,\ 0 \le r \le 1,\ \theta \in [0, 2\pi]\} \tag{1.3}$$

is said to be a (totally real) *ellipse* if the space $V := \mathrm{vect}_{\mathbb{R}}(e_1, e_2)$ is a totally real plane. The measure $d\sigma_E$ is then defined, by $\int_E f d\sigma_E = \frac{1}{2\pi}\int_0^{2\pi} f(\cos\theta e_1 + \sin\theta e_2)d\theta$ for all functions f continuous on E. $d\sigma_E$ is supported on the boundary of E as a subset of V. In fact, if A is an affine automorphism from $\mathbb{R}^2$ to V that maps the unit disc of center 0 onto E, then the measure $d\sigma_E$ is only the image by A of the standard $\frac{1}{2\pi}d\theta$ measure on the unit circle. A segment $E = \{a + te_1, t \in [-1, 1]\}$ (not reduced to one point) is said to be a degenerate ellipse, the measure $d\sigma_E$ is defined by $\int f d\sigma_E = \frac{1}{\pi}\int_0^{\pi} f(a + \cos\theta e_1)d\theta$. Thus $d\sigma_E$ is the image of the arcsin distribution on $[-1, 1]$ by the map $t \to a + te_1$.

The main result of this paper is the characterization of those totally real compact convex sets which admit an extremal array.

THEOREM 1. — *Let $n \ge 1$. A totally real convex compact set K in $\mathbb{C}^n$ (not reduced to one point) admits an extremal array if and only if it is a (possibly degenerate) ellipse. Furthermore, in this case, an array A is extremal for K if and only if the sequence μ_d^A converges weakly to $d\sigma_K$.*

Using basic properties of Kergin interpolation, we shall easily reduce the statement to the simpler

THEOREM 2. — *Let K be a convex compact set in $\mathbb{R}^n \subset \mathbb{C}^n$ of non void interior (as a subset of $\mathbb{R}^n$).*

(1) *If $n = 1$, every K (which must be an interval) admits extremal arrays.*

(2) *If $n = 2$, K admits extremal arrays if and only if it is an ellipse.*

(3) *If $n > 2$, there is no extremal array in K.*

In the first two cases, an array A is extremal if and only if μ_d^A converges weakly to $d\sigma_K$ as $d \to \infty$.

We shall assume that the reader is familiar with the definition of Kergin interpolation. We refer to [1], [2], or for a brief elementary survey, to [4]. Let us just recall the fundamental *invariance formula.* Let A be an affine mapping from $\mathbb{C}^m$ to $\mathbb{C}^n$, let $X = \{x^0, \ldots, x^d\}$ be a finite subset of a compact $\mathbb{C}$-convex set $K \subset \mathbb{C}^m$, and let $A(X) = \{A(x^0), \ldots A(x^d)\}$. Then for every $f \in H(K)$ we have

$$\mathcal{K}_X(f \circ A) = \mathcal{K}_{A(X)}(f) \circ A. \tag{1.4}$$

In particular if n is equal to 1 then $\mathcal{K}_{A(X)}$ is the Lagrange-Hermite interpolation polynomial with respect to the points $A(x^0), A(x^1), \ldots, A(x^d)$ in the plane.

It also follows that the computation of Kergin interpolation is independent of the choice of coordinates for the vector space. We can therefore define the Kergin interpolants of a function defined on an abstract finite-dimensional complex vector space Y. Furthermore if Y is a subspace of $\mathbb{C}^n$ and if K is a $\mathbb{C}$-convex compact set of $\mathbb{C}^n$ included in Y then K is as well, $\mathbb{C}$-convex as a subset of Y. Now if X is a finite subset of such a $\mathbb{C}$-convex set K and if f is holomorphic on a neighborhood of K (as a subset of $\mathbb{C}^n$) then, it follows easily from formulas defining Kergin interpolants that

$$\mathcal{K}_X(f_{|Y}) = \mathcal{K}_X(f)_{|Y}, \tag{1.5}$$

where $g_{|Y}$ denotes the restriction of g to Y. We shall call this formula, the *restriction formula* for Kergin interpolation.

It is known that every totally real $\mathbb{C}$-convex compact set is actually convex in the usual geometric sense. Therefore, in this paper, there will be no loss of generality in considering only such standard convex sets. (There is a survey on $\mathbb{C}$-convexity in [3].)

In Section 2, using a convergence theorem that we proved in [4], we give a new general criterion for deciding whether an array is extremal or not. It is used in Section 3 to prove Theorem 2. Lemma 2 of Section 3 shows that Theorem 1 follows from Theorem 2. The rate of convergence of Kergin interpolation for an extremal array is investigated in the final Section 4.

We will review some concepts from potential theory in the complex plane. An excellent general reference is the book of Ransford [11].

Let μ be a finite positive Borel measure on $\mathbb{C}$ and let

$$p_\mu(z) \;=\; \int_{\mathbb{C}} \log|z-t|d\mu(t) \tag{1.6}$$

be the negative of its logarithmic potential.

Let K be a compact subset of $\mathbb{C}$. We will assume K is polynomially convex, which is equivalent to $\mathbb{C} \setminus K$ being connected. We let $G_K(z)$ denote the Green's function of $\mathbb{C} \setminus K$ with a logarithmic pole at ∞. For K non-polar we let μ_K denote the equilibrium measure of K. It is known that $\mathrm{supp}(\mu_K) \subset \partial K$. We let $\mathrm{cap}\,(K)$ denote the logarithmic capacity of K.

The set K is said to be regular (for the exterior Dirichlet problem) if $G_K(z)$ has a continuous extension (by zero) to ∂K (which we also denote by $G_K(z)$). A regular set K is non-polar, so $\mathrm{cap}\,(K) > 0$ and we have

$$G_K(z) = p_{\mu_K}(z) - \log(\mathrm{cap}\,(K)) \quad \text{for} \quad z \in (\mathbb{C} \setminus K) \cup \partial K \tag{1.7}$$

and

$$p_{\mu_K}(z) - \log|z| \to 0 \quad \text{as} \quad |z| \to \infty. \tag{1.8}$$

For compact sets $K \subset \mathbb{C}$, K is $\mathbb{C}$-convex, by definition, if and only if K and $\mathbb{C}\setminus K$ are connected. Thus K is $\mathbb{C}$-convex if and only if it is connected and polynomially convex. Furthermore, it is known ([11, th. 4.2.1]) that if K is polynomially convex, connected and contains at least two points, then K is regular.

For K a compact subset of $\mathbb{C}$ we let $Ha(K)$ denote the functions, continuous on K and harmonic on $\mathrm{Int}\,(K)$. The proposition below defines a balayage of measures (from K to ∂K) similar to what is done in [9, p. 205-208]. However we do not assume $\mathrm{Int}\,(K)$ is dense in K.

PROPOSITION 1. — *Let $K \subset \mathbb{C}$ be compact, polynomially convex, connected and contain at least two points. Let μ be a finite positive Borel measure on K. Then, there is a unique finite positive Borel measure, denoted $b(\mu)$, on ∂K such that, for all $f \in Ha(K)$*

$$\int_K f d\mu = \int_{\partial K} f db(\mu). \tag{1.9}$$

Proof. — By [11, Cor. 6.3.6] given ϕ continuous on ∂K (denoted $\phi \in \mathcal{C}(\partial K)$) there is a unique function $H(\phi) \in Ha(K)$ such that $H(\phi)|_{\partial K} = \phi$. The linear functional on $\mathcal{C}(\partial K)$ given by

$$\phi \longrightarrow \int_K H(\phi) d\mu \tag{1.10}$$

is positive and hence, by the Riesz representation theorem, given by integration with respect to a measure, denoted $b(\mu)$, on ∂K. □

The next proposition gives equivalent characterizations of measures obtained by the balayage procedure of Proposition 1.

PROPOSITION 2. — *Let μ (resp. ν) be a finite positive Borel measure on K (resp. ∂K). Let K be as in Proposition 1. Then the following are equivalent:*

(i) $\nu = b(\mu)$.

(ii) $p_\mu(z) = p_\nu(z)$ *for all* $z \in \mathbb{C} \setminus K$.

(iii) $\int_K z^m d\mu = \int_K z^m d\nu$ *for* $m = 1, 2, 3, \ldots$

Proof. — (i) $\Rightarrow$ (ii) applying (1.9) since for fixed $z \in \mathbb{C} \setminus K$, the function $t \longrightarrow \log|z - t|$ is in $Ha(K)$. Similarly (i) $\Rightarrow$ (iii) since for any positive integer m, $\mathrm{Re}\,(z^m)$ and $\mathrm{Im}\,(z^m)$ are in $Ha(K)$.

Now (iii) $\Rightarrow$ (i) since by [11, Cor. 6.3.4] every function $f \in Ha(K)$ can be uniformly approximated on K by functions of the form $\mathrm{Re}\,(q)$ where q is a polynomial in z.

Also (ii) $\Rightarrow$ (i) by the reasoning in [11, p. 175]. □

2. A general convergence criterion.

Let K be a compact $\mathbb{C}$-convex set in $\mathbb{C}^n$, $n \geq 1$. K is said to be *regular* $\mathbb{C}$-convex (this is, of course, a property distinct from regularity for the exterior Dirichlet problem for compact sets in the plane) if it admits a basis of neighborhoods that are also $\mathbb{C}$-convex but having smooth (C^2) boundary. Every convex (geometric sense) compact set is regular [5]. Given an infinite triangular array A of points in K (as in (1.1)), we let $\mathcal{M}_A$ denote the set of all the weak limits of the sequence μ_d^A (see (1.2)). This is a closed subset of $\mathcal{M}(K)$, the convex cone of the probability measures supported on K and endowed with the weak-$\star$ topology. If l is a non-zero linear form on $\mathbb{C}^n$ (we shall write $l \in (\mathbb{C}^n)^*$) and ν a probability measure on K, $l \star \nu$ is the probability (measure) on $l(K) \subset \mathbb{C}$ defined by $(l \star \nu)(f) = \nu(f \circ l)$ for f continuous on $l(K)$.

We are now able to state a criterion characterizing the extremal arrays for Kergin interpolation.

THEOREM 3. — *Let K be a regular $\mathbb{C}$-convex set in $\mathbb{C}^n$ which is not included in a complex hyperplane and A, a triangular array of points in K. Then A is an extremal array for Kergin interpolation (on K) if and only if for every non zero linear form l and every weak limit $\mu \in \mathcal{M}_A$ one has $b(l \star \mu) = \mu_{l(K)}$.*

Let us make a few comments on this statement.

First, for every non zero l, $l(K)$ is compact and $\mathbb{C}$-convex [3, th. 2.3.4]. It follows from the restriction formula (see also the proof of Lemma 3) that there is no loss of generality in assuming K not to be contained in an hyperplane, so we may assume $l(K)$ contains at least two points. Thus $l(K)$ is regular, non-polar and satisfies the hypothesis of Proposition 1.

It is not necessary to verify the hypothesis for *every* l, but just for l in a subset S of $(\mathbb{C}^n)^*$ satisfying the following property:

$$l \in (\mathbb{C}^n)^*, l \neq 0 \Longrightarrow \exists(\lambda, q) \in \mathbb{C} \times S \text{ such that } l = \lambda q. \tag{2.1}$$

This comes from the relations $b(h \star \nu) = h \star b(\nu)$ and $\mu_{h(E)} = h \star \mu_E$ where E is any non-polar plane compact set and $h = h_\lambda : z \in \mathbb{C} \to \lambda z \in \mathbb{C}$. In particular, in the one dimensional case $(n = 1)$, we may take $S = \{\mathrm{Id}\}$.

In the case $n = 1$, Theorem 3 follows from the classical Kalmar-Walsh theorem [7, p. 65] or [15] on convergence of Lagrange-Hermite interpolants. We give a version of that result below:

THEOREM 4 (Kalmar-Walsh). — *Let $K \subset \mathbb{C}$ be polynomially convex, compact and regular. A triangular array $A \subset K$ is extremal (for Lagrange-Hermite interpolation) if and only if $\lim_{d\to\infty} \frac{1}{d+1} \log |w_d(z)| = p_{\mu_K}(z)$ uniformly on compact subsets of $\mathbb{C} \setminus K$. Here $w_d(z) = \prod_{i=0}^{d} (z - A_i^d)$.*

COROLLARY 1. — *Let K be as in Proposition 1. Then A is extremal for K if and only if for any $\mu \in \mathcal{M}_A$ we have $b(\mu) = \mu_K$.*

Proof. — Let $\mu \in \mathcal{M}_A$. By definition of $\mathcal{M}_A$, there exists a sequence $\{\mu_{n_j}\}$ which converges weakly to μ. (We will use the notation $\mu_{n_j} \xrightarrow{*} \mu$.) It follows from (1.6) that $\lim_{j\to\infty} p_{\mu_{n_j}}(z) = p_\mu(z)$ for $z \in \mathbb{C} \setminus K$. Thus, since A is extremal, by the Kalmar-Walsh theorem $p_\mu(z) = p_{\mu_K}(z)$ for all $z \in \mathbb{C} \setminus K$ and by Proposition 2(ii), $b(\mu) = \mu_K$.

Conversely, suppose $b(\mu) = \mu_K$ for all $\mu \in \mathcal{M}_A$. We will show, by contradiction, that $\lim_{d\to\infty} p_{\mu_d}(z) = p_{\mu_K}(z)$ uniformly on compact subsets of $\mathbb{C} \setminus K$, so, by the Kalmar-Walsh theorem, we may conclude that A is extremal.

Thus, suppose for some sequence n_j and compact set $L \subset \mathbb{C} \setminus K$ we have $\overline{\lim_{j\to\infty}} \|p_{\mu_{n_j}} - p_{\mu_K}\|_L = \delta > 0$. Then, by Helly's Theorem, there is a subsequence n_k of n_j such that the sequence of measures $\{\mu_{n_k}\}_{k\in\mathbb{N}}$ converges weakly to a measure $\mu \in \mathcal{M}_A$. Using Proposition 2(ii) we conclude that $\{p_{\mu_{n_k}}\}_{k\in\mathbb{N}}$ converges uniformly to p_{μ_K} on L thus arriving at a contradiction. $\square$

It follows in particular that when $\mathrm{Int}(K) = \emptyset$ then A is extremal if and only if $\{\mu_d\}$ converges to μ_K. More generally, if $A \subset \partial K$ then A is extremal if and only if $\{\mu_d\}$ converges to μ_K. (In these cases, one must have $\mathcal{M}_A = \{\mu_K\}$.) In general however $\mathcal{M}_A$ can be much larger.

The proof of the sufficient part of Theorem 3 makes use of the following general convergence theorem proved in [4, Th. 3.1 and Cor. 3.11]. We recall it now as well as its (rather technical) "machinery".

THEOREM 5. — *Let $\Omega = \{\rho < 0\}$ be a $\mathbb{C}$-convex open neighborhood of K with C^2 boundary i.e ρ is a C^2 defining function.*

We assume that the (sub)sequence

$$\mu_{d_k} := \frac{1}{d_k+1}\sum_{j=0}^{d_k}[A_j^{d_k}] \quad (k = 0, 1, 2, \ldots) \tag{2.2}$$

converges weakly to some probability μ (in $\mathcal{M}(K)$).

If for every non zero linear form l, we have

$$l(\Omega) \supset F_\mu(l) \tag{2.3}$$

where

$$F_\mu(l) = \{u \in \mathbb{C} \text{ such that } p_{l\star\mu}(u) \le \sup_{w\in l(K)} p_{l\star\mu}(w)\} \tag{2.4}$$

then, for every function f holomorphic in a neighborhood of Ω,

$$\lim_{k\to\infty} \|\mathcal{K}_{A^{d_k}}(f) - f\|_K = 0. \tag{2.5}$$

Here again, it suffices to check the property (2.3), for l in a set S as above.

Proof of Theorem 3. — Let us first make a preliminary remark. For $l \in (\mathbb{C}^n)^*$, $l(A)$ is a one dimensional array for which we may consider $\mathcal{M}_{l(A)}$. Then setting $l\star\mathcal{M}_A := \{l\star\mu, \mu \in \mathcal{M}_A\}$, we claim that $l\star\mathcal{M}_A = \mathcal{M}_{l(A)}$. That $l\star M_A \subset \mathcal{M}_{l(A)}$ is obvious. Let us prove the reverse inclusion. Let $\nu \in \mathcal{M}_{l(A)}$. There exists a subsequence μ_{d_k} such that $l\star\mu_{d_k} \xrightarrow{\star} \nu$. But the subsequence μ_{d_k} has itself, by Helly's theorem, a convergent subsequence say $\mu_{d_{k'}}$ to μ. Hence $\nu = \lim_{k'\to\infty} l\star\mu_{d_{k'}} = l\star\mu$ and $\nu \in l\star\mathcal{M}_A$.

Now if A is extremal for K then, for $l \in (\mathbb{C}^n)^*$ and $g \in H(l(K))$ we have $g \circ l \in H(K)$ whence $\mathcal{K}_{A^d}(g \circ l)$ converges uniformly on K to $g \circ l$. However by the invariance property (1.4), we have $\mathcal{K}_{A^d}(g \circ l) = \mathcal{K}_{l(A^d)}(g) \circ l$. Therefore $\mathcal{K}_{l(A^d)}(g)$ converges to g uniformly on $l(K)$. Thus $l(A)$ is extremal for Lagrange-Hermite interpolation in the plane and consequently, by Corollary 1, $b(\mathcal{M}_{l(A)}) = \{\mu_{l(K)}\}$. This is what was to be proved since $\mathcal{M}_{l(A)} = l \star \mathcal{M}_A$.

Let us now establish the sufficiency. Using the notation of Theorem 4, we first prove that

$$F_\mu(l) = l(K) \tag{2.6}$$

for $l \in (\mathbb{C}^n)^*$. Clearly we have $l(K) \subset F_\mu(l)$. Since $b(l \star \mu) = \mu_K$ we have (by Proposition 2(ii)) $p_{l\star\mu} = p_{\mu_{l(K)}}$. Since $l(K)$ is regular, $\sup_{w \in l(K)} p_{\mu_{l(K)}}(w) = 0$ and $p_{\mu_{l(K)}}(z) > 0$ for $z \in \mathbb{C} \setminus K$. Hence (2.6) follows. Now, if there exists a function $f \in H(K)$ for which $\|\mathcal{K}_{A^d} f - f\|_K$ does not converge to zero then, using Helly's theorem, one can find a subsequence μ_{d_k} converging to, say, μ such that

$$\lim_{k \to \infty} \|\mathcal{K}_{A^{d_k}} f - f\|_K = \delta > 0.$$

This together with (2.6) leads to a contradiction. Indeed, all the hypothesis of Theorem 5 are satisfied in taking for Ω any smooth $\mathbb{C}$-convex neighborhood of K such that $f \in H(\overline{\Omega})$ and the conclusion of Theorem 5 yields a contradiction. □

It is of interest to state the hypothesis on A in Theorem 3 in another form. The hypothesis (on A) holds true if and only if for every $\mu \in \mathcal{M}_A$ and every $l \in (\mathbb{C}^n)^*$, we have

$$\sum_{|\alpha|=k} \binom{n}{\alpha} l^\alpha \mu(z^\alpha) = \mu_{l(K)}(w^k), \qquad (k = 1, 2, 3, \ldots) \tag{2.7}$$

where if $l(z) := \sum_{i=1}^{n} l_i z_i$ then $l^\alpha = l_1^{\alpha_1} \ldots l_n^{\alpha_n}$ and $\binom{n}{\alpha} = n!/\alpha_1! \ldots \alpha_n!$. Indeed, by Proposition 2(iii) $b(l \star \mu) = \mu_{l(K)}$ if and only if both measures agree on the (holomorphic) polynomials. Thus the existence of extremal arrays for Kergin interpolation on K requires that the equilibrium measures $\mu_{l(K)}$ behaves very regularly as a function of l and clearly, this cannot be expected of a general $\mathbb{C}$-convex compact set in $\mathbb{C}^n$.

We showed in [4] that extremal arrays exist on compact sets that are circular and convex. Indeed, supposing that K is circular of centre 0 then for every l, $l(K)$ is a disc centre 0 whose equilibrium measure is the standard invariant measure on the boundary of the disc so that $\mu_{l(K)}(w^k) = 0$ for $k = 1, 2, \ldots$. Therefore A is extremal if and only every $\mu \in \mathcal{M}_A$ represents 0 on the polynomials (i.e., $\int_K p(z)d\mu = p(0)$ for all polynomials p). This is the case when μ is invariant, that is for every $\theta \in \mathbb{R}$ and every continuous function f on K we have $\int f(e^{i\theta}z)d\mu(z) = \int f(z)d\mu(z)$. Several natural examples are given in [4]. However there are many measures that represent 0 without being invariant.

Example. — Let us consider the euclidean unit ball in $\mathbb{C}^2$ i.e

$$B_2 := \{z = (z_1, z_2) | |z_1|^2 + |z_2|^2 \leq 1\},$$

and let Ψ be an automorphism of D, the closed unit disc in the plane. Next we define the function $\varphi : D \to B_2$ by $\varphi(u) = \left(\frac{\sqrt{2}}{2}u, \frac{\sqrt{2}}{2}u\Psi(u)\right)$. Let ν be the measure on ∂B_2 defined by

$$\int f d\nu := \int_0^{2\pi} (f \circ \varphi)(e^{i\theta}) \frac{d\theta}{2\pi}.$$

Now we can easily verify that ν represents zero while ν is not invariant (its support is not invariant).

It is not difficult to find a convex compact set of non-void interior in $\mathbb{C}^n$ which does not admit extremal arrays for Kergin interpolation. But, as a by-product of our main theorem, we shall exhibit in Section 4 the first examples of *non circular* convex compact set of non-void interior in $\mathbb{C}^2$ which admit extremal arrays. Whether or not there is a non-circular convex compact set of non-void interior admitting extremal arrays for $n > 2$ has still not been settled.

Let us finally note that Theorem 3 can be made more precise when restricted to sequences (a sequence A is an array for which $j \leq d \leq d' \Rightarrow A_j^d = A_j^{d'}$). In this case, $\mathcal{M}_A$ is always a closed connected subset of $\mathcal{M}$. (This follows as in [13, p. 35] since the sequence $\{\mu_d\}$ satisfies $\mu_{d+1} - \mu_d \xrightarrow{\star} 0$.) Conversely if $\mathcal{X}$ is a closed connected subset of $\mathcal{M}(K)$ such that $(\nu \in \mathcal{X}, l \in (\mathbb{C}^n)^*) \Rightarrow b(l \star \nu) = \mu_{l(K)}$, then $\mathcal{X}$ is the set of weak limits of the sequence $\{\mu_d\}$ for an extremal sequence in K. Indeed, by a theorem of Totik, (see [13] or [14, p. 35-36]), there exists a sequence A such that $\mathcal{X} = \mathcal{M}_A$.

3. Proof of Theorem 1.

The following basic property has already been used in the case $n = 1$ in the proof of Theorem 3.

LEMMA 1. — *Let K be a $\mathbb{C}$-convex compact set in $\mathbb{C}^m$ and ϕ an affine endomorphism from $\mathbb{C}^m$ to $\mathbb{C}^n$. If A is an extremal array for K, then $\phi(A)$ is an extremal array for $\phi(K)$.*

Proof. — The lemma follows immediately from the invariance property (1.4) together with the observation that $f \in H(\phi(K)) \Rightarrow f\circ\phi \in H(K)$. Note that the $\mathbb{C}$-convexity of K implies the $\mathbb{C}$-convexity of $\phi(K)$.

□

In particular, K admits an extremal array if and only if $a+K$ admits one. This means that one may assume that the origin belongs to K.

LEMMA 2. — *Theorem 2 implies Theorem 1.*

Proof. — Let K be a totally real convex compact set in $\mathbb{C}^m$ that admits an extremal array. Assuming the conclusions of Theorem 2 we are going to prove that (i) K must be an ellipse, (ii) if A is an extremal array in K then $\mu_d \xrightarrow{\star} \sigma_K$, and conversely (iii) if $\mu_d \xrightarrow{\star} \sigma_K$ then A is extremal.

We suppose without loss of generality that the origin belongs to K. Let V be the totally real subspace of minimal (real) dimension, say m, among those containing K.

(i) For some $M \in GL_n(\mathbb{C})$, $V \subset M(\mathbb{R}^n)$. (A totally real subspace is contained in a maximal totally real subspace W of $\mathbb{C}^n$ which is only a copy of $\mathbb{R}^n$ in $\mathbb{C}^n$ that is $V \subset W = M(\mathbb{R}^n)$, $M \in GL_n(\mathbb{C})$.) We take a linear map N from $\mathbb{C}^n$ to $\mathbb{C}^m$ such that $N_{|M^{-1}V}$ is one to one and $N(M^{-1}V) = \mathbb{R}^m$ and define $\phi = N\circ M^{-1}$. According to Lemma 1, $\phi(K)$ admits an extremal array, since $\phi(K)$ is a compact convex set of non void interior in $\mathbb{R}^m$, it follows from Theorem 2, that $m = 1$ or $m = 2$ and K is a (possibly degenerate) ellipse.

(ii) Let A be an extremal array in K and let $\psi = \phi_{|V}$ then ψ is a real isomorphism from V to $\mathbb{R}^m$. By Theorem 2, $\mu_d^{\psi(A)} \xrightarrow{\star} \sigma_{\psi(K)}$, but $\mu_d^{\psi(A)} = \psi\star\mu_d^A$ thus $\mu_d \xrightarrow{\star} \psi^{-1}\star\sigma_{\psi(K)} = \sigma_K$.

(iii) Conversely, if $\mu_d \xrightarrow{\star} \sigma_K$ then $\mu_d^{\psi(A)} \xrightarrow{\star} \sigma_{\psi(K)}$ so that $\psi(A)$ is an extremal array for $\psi(K)$. Let $V^{\mathbb{C}} = \phi^{-1}(\mathbb{C}^m)$, ψ extends to a $\mathbb{C}$-linear isomorphism from $V^{\mathbb{C}}$ onto $\mathbb{C}^m$. If f is holomorphic on a neighborhood of K as a subset of $\mathbb{C}^n$ then it is holomorphic on a neighborhood of K as a subset of $V^{\mathbb{C}}$, consequently, for $f \in H(K)$, we have

$$\begin{aligned}\|f - \mathcal{K}_{A^d}(f)\|_K &= \|f_{|V^{\mathbb{C}}} - \mathcal{K}_{A^d}(f_{|V^{\mathbb{C}}})\|_K \\ &= \|f_{|V^{\mathbb{C}}}\circ\psi^{-1} - \mathcal{K}_{\psi(A^d)}(f_{|V^{\mathbb{C}}}\circ\psi^{-1})\|_{\psi(K)} \to 0 \quad \text{as } d\to\infty.\end{aligned}$$

The first equality follows from the restriction formula (1.5), the second uses the invariance property and the limit holds true because $\psi(A)$ is an extremal array in $\psi(K)$ and $f_{|V^c} \circ \psi^{-1} \in H(\psi(K))$. This shows that A is an extremal array for K. □

Proof of Theorem 2. — Let K be a compact convex set of non void interior in $\mathbb{R}^n$ and let A be an extremal array in K. We shall work in several steps, each of which provides information on A or K, ultimately leading to the conclusion of the theorem. The case (1) is well known, we shall not discuss it.

Step 1. *K must be symmetric* (possibly after translation). Let μ be a probability measure supported on K, satisfying (2.7). We define $a \in \mathbb{R}^n$ by $a_i = \int_K x_i d\mu(x)$, $i = 1, \ldots, n$. Since K is convex, $a \in K$. We may suppose that $a = 0$ (otherwise we work with $K - a$). Therefore, applying (2.7) with $k = 1$, we have

$$0 = \sum_{j=1}^{n} l_i \mu(z_i) = \mu_{l(K)}(w), \quad l \in (\mathbb{C}^n)^*.$$

Let us restrict our attention to those l with real coefficients. Then $l(K)$ is an interval, say, $[a(l), b(l)]$. But

$$\mu_{l(K)}(w) = \int_{[-1,+1]} \left(\frac{a(l) + b(l)}{2} + \frac{b(l) - a(l)}{2} x \right) d\mu_{[-1,+1]} = \frac{a(l) + b(l)}{2}.$$

(Recall that $\mu_{[-1,1]}$ is the arcsin distribution, so $\int x d\mu_{[-1,1]} = 0$.) Consequently $a(l) = -b(l)$ and $l(K)$ is centered at 0 for every real l. In view of the formula

$$K = \bigcap_{l \neq 0} l^{-1}(l(K)),$$

this implies that K is symmetric about 0 as well.

Step 2. *K must be an ellipsoid.* By step 1, we may assume that K is symmetric about 0. By ellipsoid, we mean a set of the form $\{x \in \mathbb{R}^n, ||A(x)|| \leq 1\}$ with $A \in GL_n(\mathbb{R})$. Let us again take a probability measure μ on K satisfying (2.7). For every l with real coefficients, we have $l(K) = [-b(l), b(l)]$ and

$$\int_{l(K)} w^2 d\mu_{l(K)}(w) = \frac{1}{\pi} \int_{-b(l)}^{b(l)} \frac{x^2 dx}{\sqrt{b^2(l) - x^2}} = \frac{b^2(l)}{\pi} \int_0^{\pi} \cos^2(\theta) d\theta = \frac{1}{2} b^2(l).$$

In view of (2.7) and identifying $\mathbb{R}^n$ with its dual we deduce that the function $l \to b^2(l)$ is a quadratic form on $\mathbb{R}^n$. But, we have also

$$b(l) = \max_{x \in K} \langle l, x \rangle$$

which means that K^0, the polar set of K, is given by $b^2(l) \leq 1$ or, b being quadratic, for some matrix A, $K^0 = \{||Ax||^2 \leq 1\}$ and consequently $K = (K^0)^0 = \{||^tA(x)||^2 \leq 1\}$. Since K is bounded and of non empty interior, A must be invertible which proves that K is an ellipsoid.

Step 3. *The case* $n = 2$. It suffices to study the case of the closed unit disc D for every ellipse is the image of D under some affine automorphism of $\mathbb{C}^2$. Next, we observe that, if there exists a measure μ satisfying (2.7), it is unique since the polynomials $p(z_1, z_2)$ are dense in the continuous functions on D. Thus $\mathcal{M}_A = \mu$ and μ_d converges to μ. In view of Theorem 3, it remains to prove that for every $l \in (\mathbb{C}^2)^*$,

$$b(l \star \sigma) = \mu_{l(D)}. \tag{3.1}$$

(Recall that $d\sigma = \dfrac{1}{2\pi} d\theta$.)

It suffices to show that

$$p_{l\star\sigma}(w) = p_{\mu_{l(D)}}(w) \quad \text{for } |w| \text{ large,} \tag{3.2}$$

since both sides of (3.2) are harmonic on the connected open set $\mathbb{C} \setminus l(D)$ we must have equality in (3.2) for all $w \in \mathbb{C} \setminus l(D)$ and so by Proposition 2(ii), (3.1) holds.

Let $l(z) = l_1 z_1 + l_2 z_2 \in \mathbb{C}$. Let $r = \dfrac{l_1 - il_2}{2}$ and $s = \dfrac{l_1 + il_2}{2}$. Now $l(\cos\theta, \sin\theta) = l_1 \cos\theta + l_2 \sin\theta = re^{i\theta} + se^{-i\theta}$. Thus if r or s is zero, $l(D)$ is a disc center the origin and l being linear, (3.1) holds.

Now assume $rs \neq 0$. Then $l(D)$ is an ellipse. Let $z_1(w)$ and $z_2(w)$ be the two branches of $rz^2 - zw + s = 0$ satisfying $|z_1(w)| \longrightarrow \infty$ as $|w| \longrightarrow \infty$ and $|z_2(w)| \longrightarrow 0$ as $|w| \longrightarrow \infty$ i.e., (for an appropriate branch of $\sqrt{}$)

$$z_1(w) = \frac{w + \sqrt{w^2 - 4rs}}{2r} \quad \text{and} \quad z_2(w) = \frac{w - \sqrt{w^2 - 4rs}}{2r}. \tag{3.3}$$

We will first compute $p_{l\star\sigma}(w)$

(3.4)

$$\begin{aligned} p_{l\star\sigma} &= \frac{1}{2\pi}\int_0^{2\pi} \log|l_1\cos\theta + l_2\sin\theta - w|d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log|re^{2i\theta} - we^{i\theta} + s|d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\theta} - z_1(w)|d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\theta} - z_2(w)|d\theta + \log|r| \\ &= \log|w + \sqrt{w^2 - 4rs}| - \log 2 \quad \text{for} \quad |w| \quad \text{large} \end{aligned}$$

since $\frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\theta} - u|d\theta = \mathrm{Max}\{\log|u|, 0\}$.

Assume $\left|\frac{s}{r}\right| \leq 1$ and consider the 2:1 map of $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$ given by

$$w = F(z) = rz + \frac{s}{z}. \tag{3.5}$$

Now, $F(\partial D) = l(\partial D)$, in fact $F(\cos\theta, \sin\theta) = l(\cos\theta, \sin\theta)$. The inverse image of $w \in \mathbb{C}$ consists of the two points $\{z_1(w), z_2(w)\}$. Since $|z_1(w)||z_2(w)| = \left|\frac{s}{r}\right| \leq 1$ then for $w \in \mathbb{C} \setminus l(D)$, $|z_1(w)| > 1$ and $|z_2(w)| < 1$. Thus the map $w \longrightarrow z_1(w)$ is analytic, 1-1 from $\mathbb{C} \setminus l(D)$ to $\mathbb{C} \setminus D$ and it extends continuously to the boundary. Thus $\log|z_1(w)|$ is the Green's function of $\mathbb{C} \setminus l(D)$ and, adding a constant so that (1.8) holds we have

$$p_{\mu_{l(D)}}(w) = \log|z_1(w)| + \log|r| = \log|w + \sqrt{w^2 - 4rs}| - \log 2 \tag{3.6}$$

and combining this with (3.4) we see that (3.2) holds.

If $\left|\frac{r}{s}\right| \leq 1$ we replace F by the map $w = G(z) = \frac{r}{z} + sz$. Then $G(\partial D) = l(D)$, in fact $G(\cos\theta, -\sin\theta) = l(\cos\theta, \sin\theta)$. Proceeding similarly as above, and using the two branches of $r - zw + sz^2 = 0$ we obtain (3.6) for $p_{\mu_{l(D)}}(w)$. This completes the proof.

Step 4. *The case $n > 2$.* By step 2 and the invariance property, we may assume that K is the euclidean unit ball. We are going to prove that the existence of an extremal array A leads to a contradiction. Let us take $\mu \in \mathcal{M}_A$, x a point in the support of μ and M a linear map from $\mathbb{R}^n$ to $\mathbb{R}^2$ such that $M(x) = 0$ and $M(K)$ is the unit disc in $\mathbb{R}^2$. Such a map exists since $n > 2$. Note that M extends to a $\mathbb{C}$-linear map from $\mathbb{C}^n$ to $\mathbb{C}^2$. By Lemma 1, $M(A)$ is an extremal array in $M(K) = D$ consequently, since $M \star \mu \in \mathcal{M}_{M(A)} = \{\sigma\}$, we have $M \star \mu = \sigma$. But since x is in the support of μ, $0 = M(x)$ is in the support of $M \star \mu$ and this is impossible. This finishes the proof of the theorem. □

4. Rate of convergence.

Let us first recall some basic facts from complex pluripotential theory. Let K be a polynomially convex, non-pluripolar, compact set in $\mathbb{C}^n$. The Siciak extremal function is defined by

$$\Phi_K(z) = \sup_{p \in \mathcal{P}}\{|p(z)|^{1/\deg p}\}$$

where $\mathcal{P}$ is the set of all the polynomials p of degree at least 1 such that $||p||_K = 1$. When this function is continuous on $\mathbb{C}^n$, K is said to be *regular* and then the family $\{K_r\}_{r>1}$ where $K_r := \{\Phi(z) < r\}$ forms a basis of open neighborhoods of K. It is known that when K is convex then $\overline{K}_r$ is *convex* as well, see [10, p. 160]. When K is regular, $\overline{K}_r$ is regular [12, cor. 5.12] ($\overline{K}_r$ denotes the closure of the open set K_r) and

$$(\overline{K}_{r_1})_{r_2} = K_{r_1 r_2} \qquad (r_1, r_2 \geq 1). \tag{4.1}$$

In the case where K is a non degenerate ellipse in $\mathbb{C}^2$ (which is of most interest here) an explicit formula is available for Φ_K. For example, when $K = D$ we have, see [8, Th. 5.4.6],

$$\Phi_D(z) = \sqrt{h(||z||^2 + |\langle z, z\rangle - 1|)} \tag{4.2}$$

where $\langle z, w\rangle = \sum_{j=1}^{n} z_j w_j$, $(||z||^2 = \langle z, \bar{z}\rangle)$ and h is the inverse of Zhukovski's function, that is,

$$h(\xi) = \xi + \sqrt{\xi^2 - 1}.$$

THEOREM 6. — *Let K be a non degenerate ellipse in $\mathbb{C}^2$ and A an extremal array for K. For every $f \in H(K)$, we have*

$$\overline{\lim_{d\to\infty}} \, ||f - \mathcal{K}_{A^d} f||_K^{1/d} = \frac{1}{R(f)}$$

where $R(f) > 1$ is the supremum of the all the $r > 1$ such that f has an holomorphic continuation to K_r.

Proof. — This result is an immediate consequence of the more general Theorem 7 below. (That the hypotheses are satisfied follows from the usual formulas for Kergin interpolants.) □

Let K be regular as above. We endow $H(K)$ with its usual topology as the inductive limit of the family of Banach spaces $A(K_r) = H(K_r) \cap C(\overline{K}_r)$. Π_d denotes the space of polynomials of degree at most d.

THEOREM 7. — *Let K be a regular polynomially convex compact set in $\mathbb{C}^n$ and T_d, $d = 0, 1, 2, \ldots$ be a sequence of continuous linear projectors from $H(K)$ to $\Pi_d \subset C(K)$. If $T_d(f)$ converges to f (as $d \to \infty$) uniformly on K for every $f \in H(K)$ then the following formula holds true:*

$$\overline{\lim_{d\to\infty}} \, ||f - T_d f||_K^{1/d} = \frac{1}{R(f)} \qquad (f \in H(K))$$

where $R(f) > 1$ has the same meaning as in the previous theorem.

Here the word "projector" means that $T_d(p) = p$ for every $p \in \Pi_d$.

Proof. — Roughly, this is a consequence of the uniform boundedness theorem together with the following theorem of Siciak: for every $f \in H(K)$, we have $\overline{\lim}_{d\to\infty} \Delta_d(f)^{1/d} = 1/R(f)$ where $\Delta_d(f)$ is the distance, in the uniform norm on K, between f and Π_d, see [12, Th. 8.5].

To be precise, fix $r > 1$ (eventually, we will let r approach 1), the operators $T_d : f \in A(K_r) \to T_d(f) \in C(K)$ and $\mathcal{I} : f \in A(K_r) \to f_{|K} \in C(K)$ are continuous and, for every $f \in A(K_r)$, the sequence $(\mathcal{I} - T_d)(f)$ is bounded in $C(K)$ for it converges to 0. Hence, by the uniform boundedness theorem, there exists a positive constant $M = M(r)$ such that

$$||(\mathcal{I} - T_d)(f)||_K \le M(r)||f||_{\overline{K}_r} \qquad (f \in A(K_r)). \tag{4.3}$$

Let $f \in H(K)$ and r sufficiently small so that $f \in H(\overline{K}_r)$. If t_d is a best approximation of f in Π_d on K_r, it follows from Siciak's Theorem (applied to $\overline{K}_r$ and taking into account (4.1)) that

$$\overline{\lim_{d\to\infty}} ||f - t_d||_{\overline{K}_r}^{1/d} = \frac{r}{R(f)}. \tag{4.4}$$

Now, since $T_d(t_d) = t_d$ and using (4.3), we obtain

$$||f - T_d(f)||_K = ||(f - t_d) - T_d(f - t_d)||_K \le M(r).||f - t_d||_{\overline{K}_r}$$

whence, by (4.4), it follows

$$\overline{\lim_{d\to\infty}} ||f - T_d||_K^{1/d} \le \frac{r}{R(f)}.$$

Since r can be taken as close as we like to 1, we have as well

$$\overline{\lim_{d\to\infty}} ||f - T_d(f)||_K^{1/d} \le \frac{1}{R(f)}.$$

That the inequality cannot be strict follows from Siciak's Theorem since $T_d(f)$ is a "competitor" for the best approximant from Π_d to f on K. The theorem is proved. □

We can derive another interesting consequence of this result.

COROLLARY. — *Let A be an extremal array for K a regular compact convex set in $\mathbb{C}^n$, then it is also an extremal array for $\overline{K}_r$, $r > 1$.*

Proof. — Fix $r > 1$ and let $f \in H(\overline{K}_r)$, we must prove that $\mathcal{K}_d := \mathcal{K}_{A^d}(f)$ converges uniformly to f on $\overline{K}_r$ as $d \to \infty$. Since A is extremal for K, in view of Theorem 7, there exists $r_1 > r$ such that for d large,

$$||f - \mathcal{K}_d||^{1/d} \le \frac{1}{r_1^d}.$$

Now consider the series

$$\mathcal{K}_0 + \sum_{d=0}^{\infty} (\mathcal{K}_{d+1} - \mathcal{K}_d)$$

whose partial sum of order $d-1$ is exactly $\mathcal{K}_d$. This series is uniformly convergent on $\overline{K}_r$ for we have, using the Bernstein-Walsh inequality, (here, this essentially follows from the definition of the Siciak extremal function),

$$\begin{aligned} \|\mathcal{K}_{d+1} - \mathcal{K}_d\|_{\overline{K}_r} &\leq r^{d+1}.\|\mathcal{K}_{d+1} - \mathcal{K}_d\|_K \\ &\leq r^{d+1}(\|f - \mathcal{K}_{d+1}\|_K + \|f - \mathcal{K}_d\|_K) \\ &\leq 2.\frac{r^{d+1}}{r_1^d}. \end{aligned}$$

Its sum, say g, is therefore a continuous function on $\overline{K}_r$ and holomorphic on K_r. Since the series converges to f on K, which is not pluripolar, g must be equal to f on $\overline{K}_r$ and the corollary is proved. □

Example. — Let $\rho > 1$. Any extremal array for D provides an extremal array of the *non circular* compact convex set E of non-void interior in $\mathbb{C}^2$ defined by

$$z \in E \iff \|z\|^2 - |\langle z, z\rangle - 1| \leq \rho. \tag{4.5}$$

Indeed, since $h^{-1}(x) = x + 1/x$, it follows from (4.1) that for such ρ, there exists $r > 1$ such that $E = D_r$ and the claim follows then from the previous corollary.

This corollary does not enable us to construct similar examples in $\mathbb{C}^n$ for $n > 2$, for a totally real ellipse in $\mathbb{C}^n$ is pluripolar.

Final Remark. — A typical extremal array in the unit disc in $\mathbb{R}^2$ is formed by the $d-th$ roots of unity ($d = 1, 2, ...$). This array (and its images on ellipses) has been recently proved to be "better than extremal". Specifically, the convergence holds true for functions that are only twice continuously differentiable on a neighborhood of D, see [6]. A consequence of our Theorem 1 is that a generalization of this latter result cannot be expected for sets of real dimension greater than two.

However, in this connection we raise the following problem: Given a compact convex set $K \subset \mathbb{R}^n$, does there exist a unique minimal compact $\mathbb{C}$-convex set $\widehat{K} \subset \mathbb{C}^n$ with the property that there exists a triangular array $A \subset K$ and for every function f holomorphic on a neighborhood of $\widehat{K}$ then $K_{A^d}(f)$ converges to f uniformly on K.

Acknowledgments. The research for this paper has been done, in part, when J.-P. Calvi was visiting T. Bloom in the department of Mathematics of Toronto University in March 96. There, the authors benefited from several discussions with Len Bos.

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Manuscrit reçu le 9 avril 1997,
accepté le 21 octobre 1997.

T. BLOOM,
University of Toronto
Department of Mathematics
Toronto M5S 3G3 (Canada).
bloom@math.toronto.edu
&
J.-P. CALVI,
Université Paul Sabatier
Laboratoire Émile Picard
118 route de Narbonne
31062 Toulouse Cedex (France).
calvi@picard.ups-tlse.fr