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# $SL_2$, THE CUBIC AND THE QUARTIC

by Yannis Y. PAPAGEORGIOU

## Introduction.

A basic problem in representation theory is that of branching: How do the irreducible representations of a given group decompose upon restriction to a given subgroup? Over the past fifty or so years, much work has been devoted to understanding this question [K], [KT], [LP1], [LP2], [LDE]. If our groups are reductive, and representations rational, then it is known how to solve this problem 'in principle'. This caveat is of course rather prevalent in Lie theory, and is the difference between knowing *how* to calculate something, and *actually* calculating it... As for explicit formulas, few are known, and the rules which are known, such as Kostant's multiplicity formula and the Littlewood-Richardson rule, among others, are unwiedly as regards computation.

The goal of this study is modest: the action of $\mathrm{SL}_2$ on the space of binary cubics yields an embedding of $\mathrm{SL}_2$ into $\mathrm{Sp}_4$, and our goal is to describe how the finite dimensional $\mathrm{Sp}_4$-modules decompose upon restriction to this $\mathrm{SL}_2$. There are two main parts to our description. The first is a numerical multiplicity formula. In contrast to the known examples cited above, one *is* able to calculate with this formula. This is the content of §2. The second approach is more algebro-geometric in nature.

Let $N \subseteq \mathrm{SL}_2$, $U \subseteq \mathrm{Sp}_4$ be maximal unipotent subgroups, and consider $\mathcal{R}(\mathrm{Sp}_4/U)$, the ring of regular functions on the variety $\mathrm{Sp}_4/U$. This ring is a model for the rational representations of $\mathrm{Sp}_4$, and so provides the natural geometric setting in which to understand the branching rule. We shall

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describe a minimal system of generators for the subalgebra of $N$-invariant functions. To completely understand the geometric branching one would have to compute a resolution for this ring, so ours is a first step in this direction...

As $\mathcal{R}(\mathrm{Sp}_4/U)$ may be realized as a quotient of the polynomial ring $\mathcal{P}(\mathbb{C}^4 \oplus \mathbb{C}^5)$, our first task is to provide a system of generators for $\mathcal{P}(\mathbb{C}^4 \oplus \mathbb{C}^5)^N$. The calculation of a generating set for this algebra was first made by Gordan in his important paper [G], (see also [GI], [Gu]), and was subsequently refined by Sylvester [S]. This huge computation should be considered as one of the high points of invariant theory in the 19th century. Our calculation is based on a new algorithm described in §1. Although our result is rougher than the classical one, it does have the benefit of making the situation more transparent than do the classical methods. Using the numerics of §2, and the structure of the ring $\mathcal{R}(\mathrm{Sp}_4/U)$ we are able to refine the generating system of $\mathcal{P}(\mathbb{C}^4 \oplus \mathbb{C}^5)^N$ to one for $\mathcal{R}(\mathrm{Sp}_4/U)^N$. An extremely pleasing aspect of this refinement is that the numerics and structure are strong enough so that we do no have to make any explicit calculations involving these functions.

*Aknowledgements.* — This work is a version of my doctoral dissertation, Yale University, 1996, which was done under the supervision of Roger Howe. I would like to thank Professor Howe for his many suggestions, and his generosity with his time. Anyone who knows his work will feel his influence thoughout the following pages...

I would also like to thank the reader for several suggestions which, I feel, have improved this manuscript; in particular, the remarks following 2.4 and 3.2 are due to questions he posed. Finally, I thank Bram Broer for listening to me as this paper was revised...

## 0.1. Preliminaries.

**0.1.0.** — We have tried to keep this work as a self-contained as possible, and in those cases in which we do not give full explanations, references are provided.

We do assume that the reader knows some basic facts on the representation theory of complex semisimple Lie algebras and groups, the most important and relevant fact being the theorem of the highest weight (see [Hum]).

**0.1.1.** — Let $G$ be a reductive group and $U$ a maximal unipotent subgroup. Let $V$ be a finite-dimensional $G$-module and denote by $\mathcal{P}(V)$ the algebra of polynomial functions on $V$. Then $G$ acts on $\mathcal{P}(V)$ in the usual way:

$$g \cdot p(v) = p(g^{-1} \cdot v). \tag{0.1.1.1}$$

The space $\mathcal{P}(V)^G$ will denote the algebra of $G$-invariant polynomial functions. $\mathcal{P}(V)^U$ will denote the algebra of highest weight vectors which we also call (after Sylvester) the algebra of covariants.

In case $V$ is a variety upon which $G$ acts, $\mathcal{R}(V)$ will denote the ring of regular functions on $V$ and the invariants and covariants are defined in an analogous manner.

**0.1.2.** — Since we shall be studying representations of $SL_2(\mathbb{C})$, we remind the reader that the irreducible finite dimensional representations of $SL_2$ ara parametrized by nonnegative integers $k$; which we denote by $V_k$. Remember that dim $V_k = k+1$.

We shall of course also be discussing the finite dimensional representations of $Sp_4(\mathbb{C})$. The irreducible representations here are parametrized by pairs of integers $(m,n)$ where $m \geq n \geq 0$, and we denote these by $\sigma_4^{(m,n)}$. Note that the fundamental representations are in this case $\sigma_4^{(1,0)} = \mathbb{C}^4$ and $\sigma_4^{(1,1)} = \mathbb{C}^5$.

**0.1.3.** — Let $V$ be a finite dimensional $SL_2$-module, and consider $\mathcal{P}(V)$. This space has a natural grading given by the usual degree. Consider the subalgebra of covariants $\mathcal{P}(V)^N$. In addition to the grading given by degree, this subalgebra admits another grading which we call the $SL_2$-grading. Since $\mathcal{P}(V)^N$ consists of highest weight vectors, the $SL_2$-grading is simply the grading by $SL_2$-highest weight.

## 1.1. A lemma.

**1.1.0.** — Let $\mathcal{A}$ denote an algebra upon which $SL_2$ acts semisimply by algebra automorphisms. We shall call such an algebra a rational $SL_2$-algebra. Let $\mathcal{A}_n$ denote the isotypic component of $\mathcal{A}$ consisting of all $SL_2$-submodules of highest weight $n$. Then we have

$$\mathcal{A} = \sum_{n \in \mathbb{Z}_+} \mathcal{A}_n.$$

Clebsch-Gordan tells us that this is not a grading in general. However, putting

$$\mathcal{A}^{(n)} = \sum_{k \leq n} \mathcal{A}_n,$$

we obtain a filtration on $\mathcal{A}$ from which we may construct the associated graded algebra in the usual way: put $\mathcal{A}^n = \mathcal{A}^{(n)}/\mathcal{A}^{(n-1)}$ and set

$$\mathrm{gr}_{\mathrm{SL}_2} \mathcal{A} = \sum_{n \in \mathbb{Z}_+} \mathcal{A}^n.$$

Then $\mathrm{gr}_{\mathrm{SL}_2} \mathcal{A}$ is a graded algebra upon which $\mathrm{SL}_2$ acts by algebra automorphisms, and each homogeneous component $\mathcal{A}^n$ is naturally isomorphic to the isotypic component $\mathcal{A}_n$ as $\mathrm{SL}_2$-modules.

Now let $\mathcal{B}$ be a graded algebra, *i.e.*, $\mathcal{B} = \sum_{n \in \mathbb{Z}_+} \mathcal{B}^n$. Then there is a unique rational $\mathrm{SL}_2$-algebra $\mathcal{C}$ such that

1) $\mathcal{C}^N = \mathcal{B}$ as graded algebras;

2) $\mathcal{C}$ is graded and the homogeneous component $\mathcal{C}^n$ consists of all $\mathrm{SL}_2$-submodules of highest weight $n$.

We note that if $\mathcal{A}$ is a rational $\mathrm{SL}_2$-algebra such that $\mathcal{A}^N = \mathcal{B}$ as graded algebras, then $\mathrm{gr}_{\mathrm{SL}_2} \mathcal{A} = \mathcal{C}$.

**1.1.1.** — Let $V$ and $W$ denote finite dimensional representations of $\mathrm{SL}_2$. Our goal in the next several sections will be to show how one may obtain a set of generators for the algebra of simultaneous covariants $\mathcal{P}(V \oplus W)^N$ from knowledge of generating sets for $\mathcal{P}(V)^N$ and $\mathcal{P}(W)^N$.

Let $m_1, \ldots, m_k$ be the weights of the generators of $\mathcal{P}(V)^N$ which are not invariants, and let $m_{k+1}, \ldots, m_{k+\ell}$ similarly denote those of $\mathcal{P}(W)^N$. Set

$$\mathcal{Z}_1 = \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_k\mathbb{Z},$$

$$\mathcal{Z}_2 = \mathbb{Z}/m_{k+1}\mathbb{Z} \times \cdots \times \mathbb{Z}/m_{k+\ell}\mathbb{Z},$$

and let $\mathcal{Z}$ be their product. Finally, let $\mathcal{P}_+(V \oplus W)$ denote the quotient of $\mathcal{P}(V \oplus W)$ by the ideal generated by $\mathcal{P}(V)^{\mathrm{SL}_2}$ and $\mathcal{P}(W)^{\mathrm{SL}_2}$. The following is the essential result which will enable us to calculate the simultaneous covariants:

LEMMA. — *There is a surjection of* $\mathrm{SL}_2$*-graded algebras*

$$\mathrm{gr}_{\mathrm{SL}_2} \mathcal{P}\big((\mathbb{C}^2)^{k+\ell}\big)^{\mathcal{Z}} \longrightarrow \mathrm{gr}_{\mathrm{SL}_2} \mathcal{P}_+(V \oplus W). \tag{1.1.1.1}$$

We must explain how $\mathcal{Z}$ acts on $\mathcal{P}((\mathbb{C}^2)^{k+\ell})$. Firstly, each factor acts componentwise, and the action of the factor $\mathbb{Z}/m_i\mathbb{Z}$ on $\mathcal{P}(\mathbb{C}^2)$ is simply by scalar dilation by the $m_i$-th roots of unity. In fact, since we have the $\mathrm{SL}_2$-module decomposition

$$\mathcal{P}(\mathbb{C}^2) \simeq \sum_{j \geq 0} V_j,$$

where $V_j$ denotes the irreducible $\mathrm{SL}_2$-module with highest weight $j$, restricting to the $\mathbb{Z}/m_i\mathbb{Z}$-invariants one gets

$$\mathcal{P}(\mathbb{C}^2)^{\mathbb{Z}/m_i\mathbb{Z}} \simeq \sum_{j \geq 0} V_{jm_i}.$$

Let $\mathcal{P}_+(V)$ (respectively, $\mathcal{P}_+(W)$) denote the quotient of $\mathcal{P}(V)$ (respectively, $\mathcal{P}(W)$) by the ideal generated by $\mathcal{P}(V)^{\mathrm{SL}_2}$ (respectively, $\mathcal{P}(W)^{\mathrm{SL}_2}$). Then it is clear that there are surjective $\mathrm{SL}_2$-module maps

$$\mathcal{P}\big((\mathbb{C}^2)^k\big)^{\mathcal{Z}_1} \longrightarrow \mathcal{P}_+(V) \tag{1.1.1.2}$$

and

$$\mathcal{P}\big((\mathbb{C}^2)^\ell\big)^{\mathcal{Z}_2} \longrightarrow \mathcal{P}_+(W). \tag{1.1.1.3}$$

Combining these, we obtain a surjective $\mathrm{SL}_2 \times SL_2$-module map

$$\mathcal{P}\big((\mathbb{C}^2)^{k+\ell}\big)^{\mathcal{Z}} \longrightarrow \mathcal{P}_+(V \oplus W). \tag{1.1.1.4}$$

This of course is also a surjective $\mathrm{SL}_2$-module map, where $\mathrm{SL}_2$ is diagonally embedded inside $\mathrm{SL}_2 \times SL_2$. The highest weight theory tells us that restriction to the $N$-invariants yields a surjective homomorphism, so the lemma follows from the discussion of 1.1.0.

**1.1.2.** — The importance of the above lemma is that it yields the following

COROLLARY. — *The associated map*

$$\big(\mathcal{P}\big((\mathbb{C}^2)^{k+\ell}\big)^{\mathcal{Z}}\big)^N \longrightarrow \mathcal{P}_+(V \oplus W)^N \tag{1.1.2.1}$$

*takes a set of generators of* $(\mathcal{P}((\mathbb{C}^2)^{k+\ell})^{\mathcal{Z}})^N$ *to a set of generators of the algebra* $\mathcal{P}_+(V \oplus W)^N$.

Thus, we have reduced the calculation of generating set of covariants in $\mathcal{P}(V \oplus W)$ to the calculation of the invariants of the finite abelian group $\mathcal{Z}$ inside the algebra $\mathcal{P}((\mathbb{C}^2)^{k+\ell})^N$.

*Remark.* — This situation generalizes immediately to the case of general reductive algebraic groups, and this will be the focus of a future paper.

**1.1.3.** — The algebra $\mathcal{P}((\mathbb{C}^2)^{k+\ell})^N$ was classically well-understood, and has the following presentation in terms of generators and relations: let $x_{i1}, x_{i2}$ denote the standard coordinates on the $i$-th copy of $\mathbb{C}^2$, and define

$$\delta_{ij} = \begin{vmatrix} x_{i1} & x_{j1} \\ x_{i2} & x_{j2} \end{vmatrix} \qquad \text{for } 1 \le i, j \le k+\ell.$$

The generators are then given by

$$x_i = x_{i1} \qquad \text{for } 1 \le i \le k+\ell$$

and

$$\delta_{ij} \qquad \text{for } 1 \le i, j \le k+\ell.$$

The ideal of relations is generated by

$$x_i\delta_{jr} - x_j\delta_{ir} + x_r\delta_{ij} = 0 \tag{1.1.3.1}$$

and

$$\delta_{ij}\delta_{rs} - \delta_{ir}\delta_{js} + \delta_{is}\delta_{jr} = 0. \tag{1.1.3.2}$$

**1.1.4.** — We proceed to calculate the $\mathcal{Z}$-invariants in $\mathcal{P}((\mathbb{C}^2)^{k+\ell})^N$. At first we shall disregard the relations (1.1.3.1), (1.1.3.2), and work in the polynomial algebra generated by $x_i, \delta_{rs}$ for $1 \le i \le k+\ell, 1 \le r < s \le k+\ell$.

Remember that $\mathbb{Z}/m_i\mathbb{Z}$ only acts on the $i$-th copy of $\mathbb{C}^2$, and therefore affects only $x_i$ and $\delta_{rs}$ with either $r = i$ or $s = i$. Let $\zeta \in \mathbb{Z}/m_i\mathbb{Z}$, and consider the monomial

$$\prod_{j,r<s} {x_j}^{a_j} \delta_{rs}^{b_{rs}}. \tag{1.1.4.1}$$

The effect of $\zeta$ on this monomial is just scalar multiplication by $\zeta^{a_i + \sum\limits_{r<s} b_{rs}}$, where the summation runs over all $(r, s)$ such that either one or the other equals $i$. Therefore, for this monomial to be invariant under $\mathcal{Z}$, we need

$$a_i + \sum_{r<s} b_{rs} \equiv 0 \pmod{m_i} \tag{1.1.4.2}$$

where $r = i$ or $s = i$, for each $i = 1, \ldots, k + \ell$.

We can do a little better as regards the set of $(r, s)$ over which the sums vary for each $i$. First note that there is an obvious restriction on the pairs $(r, s)$: since the first $k$ covariants come from $\mathcal{P}_+(V)$ and the last $\ell$ come from $\mathcal{P}_+(W)$, the map (1.1.2.1) actually factors through the quotient of $(\mathcal{P}((\mathbb{C}^2)^{k+\ell})^N)^{\mathcal{Z}}$ by the ideal $\mathcal{I}(\delta)$ generated by those $\delta_{rs}$ for which $1 \leq r < s \leq k$ or $k+1 \leq r < s \leq k+\ell$. Hence if $1 \leq i \leq k$ we only need to look at the congruence

$$a_i + \sum_{j=k+1}^{k+\ell} b_{ij} \equiv 0 \pmod{m_i}, \tag{1.1.4.3}$$

while if $k+1 \leq i \leq k+\ell$, the relevant congruence is

$$a_i + \sum_{j=1}^{k} b_{ji} \equiv 0 \pmod{m_i}. \tag{1.1.4.4}$$

The set of solutions to the system defined by (1.1.4.3) and (1.1.4.4) is an additive submonoid of $\mathbb{Z}_+^{k+\ell+k\ell}$. Moreover, finding a set of generators for this monoid will clearly yield a set of generators for the $\mathcal{Z}$-invariants. It is not hard to give an initial region where to look for these monoid generators: firstly, it is clear that ${x_i}^{m_i}$ must be a generator for each $i$, and secondly, as the smallest power of $\delta_{rs}$ which is a solution to the system is the least common multiple of $m_r$ and $m_s$, l.c.m.$\{m_r, m_s\}$, we see that we can restrict our attention to the box

$$\begin{gathered}\mathcal{B}(V,W) \text{ defined by } 0 \leq a_i \leq m_i,\ \ 0 \leq b_{rs} \leq \text{l.c.m.}\{m_r, m_s\} \\ \text{where } 1 \leq i \leq k+\ell,\ 1 \leq r \leq k \text{ and } k+1 \leq s \leq k+\ell.\end{gathered}$$

There is another way that we can simplify our search for solutions to (1.1.4.3) and (1.1.4.4). Up to now, we have not taken into account either of the relations (1.1.3.1) and (1.1.3.2). These tell us that

$$x_i\delta_{jr} = x_j\delta_{ir} - x_r\delta_{ij},$$
$$\delta_{ij}\delta_{rs} = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr},$$

which imply that we do not need to consider certain monomials. The effect of these relations is that they further restrict an initial search for solutions to certain faces of the box.

**1.1.5.** — *Algorithm*:

1) Find all solutions to (1.1.4.3), (1.1.4.4) inside $\mathcal{B}(V,W)$.

2) Find all indecomposable solutions from the list obtained in 1.

The output, of course, is a system of generators for the $\mathcal{Z}$-invariants. Combining this with the map (1.1.2.1) yields a set of generators for $\mathcal{P}(V \oplus W)^N$.

*Remark.* — The system of generators for the $\mathcal{Z}$-invariants is minimal. However, the map (1.1.2.1) in general does have a kernel, so the generators obtained for $\mathcal{P}(V \oplus W)^N$ do not necessarily form a minimal system. We shall see in §3 that this method is good enough for our purposes.

## 1.2. A remembrance of things past...

**1.2.0.** — The covariants of the binary cubic and of the binary quartic were intensively studied in the middle of the 19th century by several mathematicians. In the excellent survey article of Meyer [M] we find that the early work was done in a piecemeal fashion.

> "*Etwas früher (1844) hatte schon Eisenstein die einfachsten Invarianten und Covarianten einer kubischen und biquadratischen binären Form erkannt; es hatte Hesse, in eleganter Handhabung des von Jacobi zur Vollendung gebrachten Determinantenapparats, der nach ihm benannten Covariante ein eingehendes Studium gewidmet und die Rolle aufgedeckt, welche dieselbe in der Theorie der ebenen Curven, insonderheit derer von der 3. Ordnung spielt.*"

By 1854, Cayley had already completely understood both these algebras of covariants. From his Memoirs on Quantics in the Philosophical Transactions of the same year, one finds a description not only of the generators in each case, but also of the relations they satisfy. Most probably he had knowledge of these at an earlier date [S], pp. 343–345.

We shall recall the descriptions of these algebras, and discuss the structure of a certain quotient in the quartic case which will be of concern in the subsequent study.

**1.2.1.** — We do not derive a system of generators and relations for either the algebra of covariants of the cubic, $\mathcal{P}(\mathbb{C}^4)^N$, or for $\mathcal{P}(\mathbb{C}^5)^N$, the covariants of the quartic, but simply content ourselves with descriptions of these algebras. These can be found in many places; for classical descriptions, Salmon's treatise [S] is excellent, or for a modern point of view, one can look at Springer's exposition [Sp].

The algebras $\mathcal{P}(\mathbb{C}^4)^N$ and $\mathcal{P}(\mathbb{C}^5)^N$ have bigradings given by $\mathrm{SL}_2$-weight and degree. It turns out that in both these cases, the generators of the algebra of covariants are specified by their bidegrees.

In the cubic case, there are four generators which we denote by $\alpha, \beta, \gamma, \delta$ with bidegrees (3,1), (2,2), (3,3) and (0,4) respectively. Moreover, these satisfy one relation of the form

$$\alpha^2\delta + \beta^3 + \gamma^2 = 0. \tag{1.2.1.1}$$

The quartic case is not much more complicated. Precisely, there are five generators $a, b, c, d, e$ with bidegrees (4,1), (4,2), (0,2), (6,3) and (0,3) respectively and these satisfy the relation

$$a^3e + a^2bc + b^3 + d^2 = 0. \tag{1.2.1.2.}$$

**1.2.2.** — At this point we need to recall some facts about the structure of $\mathcal{P}(\mathbb{C}^5)$ as an $O_5$-module. Firstly, recall that $O_5$ is defined as the isometry group of a nondegenerate symmetric bilinear form on $\mathbb{C}^5$, and hence fixes the associated quadratic form $r^2$ under its action on $\mathcal{P}(\mathbb{C}^5)$. Let $\Delta$ denote the Laplacian dual to $r^2$; then $\Delta$ is an $O_5$-invariant differential operator. The invariance of $\Delta$ tells us that the subspace of harmonics $\mathcal{H}(\mathbb{C}^5) = \ker\Delta$ is preserved by $O_5$. Let $\mathcal{H}^m(\mathbb{C}^5) = \mathcal{H}(\mathbb{C}^5) \cap \mathcal{P}^m(\mathbb{C}^5)$, and let $\mathcal{I}$ denote the ideal generated by $r^2$. Then the classical theory of spherical harmonics tells us that each homogeneous component $\mathcal{H}^m(\mathbb{C}^5)$ is an irreducible $O_5$-module, and that $\mathcal{P}(\mathbb{C}^5) = \mathcal{H}(\mathbb{C}^5) \oplus \mathcal{I}$. Finally, note that multiplication in the algebra $\mathcal{H}(\mathbb{C}^5) \simeq \mathcal{P}(\mathbb{C}^5)/\mathcal{I}$ can be realized by the usual multiplication in $\mathcal{P}(\mathbb{C}^5)$ followed by harmonic projection, the latter a rather unpleasant complication for explicit calculation.

**1.2.3.** — Since the harmonics of fixed degrees are irreducible $O_5$-modules, we wish to understand the algebra of harmonic $\mathrm{SL}_2$-covariants, $\mathcal{H}(\mathbb{C}^5)^N$. Of course, this follows immediately from the description of $\mathcal{P}(\mathbb{C}^5)^N$. The first thing to note is that since the action of $\mathrm{SL}_2$ on $\mathbb{C}^5$ yields an embedding $\mathrm{SL}_2 \hookrightarrow O_5$, the quadratic invariant $c$ is none other than the invariant form $r^2$. Moreover, it is not difficult to decompose the first three homogeneous components to see that the covariants $a, b, d, e$ each have nonzero harmonic projection. Finally, since $c = 0$ in $\mathcal{H}(\mathbb{C}^5)$, the relation (1.2.1.2) becomes

$$a^3 e + b^3 + d^2 = 0. \tag{1.2.3.1}$$

LEMMA. — *The algebra $\mathcal{H}(\mathbb{C}^5)^N$ of harmonic covariants is isomorphic to*

$$\mathbb{C}[a, b, d, e]/(a^3 e + b^3 + d^2).$$

## 1.3. An example.

**1.3.0.** — Using our algorithm, we shall make a first approximation to the calculation of a set of generators for the $\mathrm{SL}_2$-covariants in $\mathcal{R}(\mathrm{Sp}_4 / U)$ by computing a set of generators for $\mathcal{P}(\mathbb{C}^4 \oplus \mathbb{C}^5)^N$.

**1.3.1.** — As we saw in 1.2, the covariants of positive weight in $\mathcal{P}(\mathbb{C}^4)^N$ and $\mathcal{P}(\mathbb{C}^5)^N$ have weights 3, 2, 3 and 4 ,4, 6 respectively. Following the notations of 1.1, the system (1.1.4.3), (1.1.4.4) translates to our present situation as

$$\begin{cases} a_1 + b_{14} + b_{15} + b_{16} \equiv 0 & (3), \\ a_2 + b_{24} + b_{25} + b_{26} \equiv 0 & (2), \\ a_3 + b_{34} + b_{35} + b_{36} \equiv 0 & (3), \\ a_4 + b_{14} + b_{24} + b_{34} \equiv 0 & (4), \\ a_5 + b_{15} + b_{25} + b_{25} \equiv 0 & (4), \\ a_6 + b_{16} + b_{26} + b_{36} \equiv 0 & (6). \end{cases} \tag{1.3.1.1}$$

We seek solutions to this system inside the box $\mathcal{B}(\mathbb{C}^4, \mathbb{C}^5) \subseteq \mathbb{Z}_+^{15}$ defined by

$$\begin{cases} 0 \leq a_1 \leq 3, & 0 \leq a_2 \leq 2, & 0 \leq a_3 \leq 3, \\ 0 \leq a_4 \leq 4, & 0 \leq a_5 \leq 4, & 0 \leq a_6 \leq 6, \\ 0 \leq b_{14} \leq 12, & 0 \leq b_{15} \leq 12, & 0 \leq b_{16} \leq 6, \\ 0 \leq b_{24} \leq 4, & 0 \leq b_{25} \leq 4, & 0 \leq b_{26} \leq 6, \\ 0 \leq b_{34} \leq 12, & 0 \leq b_{35} \leq 12, & 0 \leq b_{36} \leq 6. \end{cases} \tag{1.3.1.2}$$

**1.3.2.** — As we noted in 1.1.4, the relations (1.1.3.1) and (1.1.3.2) restrict the search for solutions to (1.3.1.1) to certain faces of the integral cone $\mathbb{Z}_+^{15}$. Let us first consider a relation of the form (1.1.3.2):

$$\delta_{ij}\delta_{rs} - \delta_{ir}\delta_{js} + \delta_{is}\delta_{jr} = 0.$$

Remember that for $\delta_{ab}$ to be nonzero modulo $\mathcal{I}(\delta)$, we need $1 \le a \le 3$, $4 \le b \le 6$. If $\delta_{ir}$ and $\delta_{js}$ are both nonzero, this implies that $\delta_{ij}$ and $\delta_{rs}$ must be zero, so that our relation becomes

$$\delta_{ir}\delta_{js} = \delta_{is}\delta_{jr}. \tag{1.3.2.1}$$

As (1.1.3.1) has the form

$$x_i\delta_{jr} - x_j\delta_{ir} + x_r\delta_{ij} = 0,$$

if $\delta_{ir}$ and $\delta_{jr}$ are both nonzero, then $\delta_{ij}$ is forced to be zero, so

$$x_i\delta_{jr} = x_j\delta_{ir}. \tag{1.3.2.2}$$

In terms of the exponents, (1.3.2.1) can be taken to mean that either $b_{is} = 0$ or $b_{jr} = 0$, while (1.3.2.2) implies that we may take either $a_j = 0$ or $b_{ir} = 0$.

In the case under consideration, there are 9 relations of the form (1.3.2.1) and 18 of type (1.3.2.2), hence, there are many choices of which exponents to take to be zero. The choices we made for our calculations were the following: considering $b_{ir} + b_{js}$, at least one of these must be zero if $(i, r; j, s)$ is one of the following vectors:

$$\begin{cases} (1,4;2,5), & (1,6;2,4), & (2,4;3,5), \\ (1,5;2,6), & (1,6;3,4), & (2,6;3,4), \\ (1,5;3,4), & (1,6;3,5), & (2,6;3,5). \end{cases} \tag{1.3.2.3}$$

Furthermore, if we consider now $a_j + b_{ir}$, either $a_j = 0$ or $b_{ir} = 0$ whenever $(j; i, r)$ is one of

$$\begin{cases} (2;1,4), & (2;1,5), & (2;1,6), & (3;1,4), & (3;1,5), & (3;1,6), \\ (3;2,4), & (3;2,5), & (3;2,6), & (5;1,4), & (5;2,4), & (5;3,4), \\ (6;1,4), & (6;1,5), & (6;2,4), & (6;2,5), & (6;3,4), & (6;3,5). \end{cases} \tag{1.3.2.4}$$

**1.3.3.** — The relations among the generators of $\mathcal{P}(\mathbb{C}^4)^N$ and $\mathcal{H}(\mathbb{C}^5)^N$ also contribute to reducing calculations. Relation (1.2.1.1) says that $\gamma$ can appear with power at most one, while (1.2.3.1) indicates that the same is true for $d$. These relations in turn restrict the congruences

$$a_3 + b_{34} + b_{35} + b_{36} \equiv 0 \quad (3),$$

$$a_6 + b_{16} + b_{26} + b_{36} \equiv 0 \quad (6),$$

to the following equalities:

$$a_3 + b_{34} + b_{35} + b_{36} = 0 \text{ or } 3, \tag{1.3.3.1}$$

$$a_6 + b_{16} + b_{26} + b_{36} = 0 \text{ or } 6. \tag{1.3.3.2}$$

**1.3.4.** — The calculation of the solutions to (1.3.1.1) in the box $\mathcal{B}(\mathbb{C}^4, \mathbb{C}^5)$ subject to the restrictions introduced in 1.3.2 and 1.3.3 is a simple matter. To find the indecomposable solutions one just needs to check which solutions may be written as a sum of two others. This is not so difficult, but it is time-consuming. So as not to have to check all possibilities, we use a norm. If $(a_i, b_{rs})$ is a solution to (1.3.1.1), its norm is defined as the sum of its coordinates, $\sum a_i + \sum b_{rs}$. Since norms are additive, given a solution, to determine its indecomposability, one only has to consider pairs of solutions with suitable norms.

**1.3.5.** — Implementation of the algorithm yields 123 generators for the $\mathcal{Z}$-invariants. To calculate their bidegrees is easy: write

$$w_i = a_i + b_{i4} + b_{i5} + b_{i6} \qquad \text{for } i = 1, 2, 3;$$

$$w_j = a_j + b_{1j} + b_{2j} + b_{3j} \qquad \text{for } j = 4, 5, 6;$$

then the bidegree of the monomial $\prod x_{i1}^{a_i} \delta_{rs}^{b_{rs}}$ is given by

$$\left(\tfrac{1}{3}w_1 + w_2 + w_3,\ \tfrac{1}{4}w_4 + \tfrac{1}{2}w_5 + \tfrac{1}{2}w_6\right).$$

In the tables below we arrange the generators for the $\mathcal{Z}$-invariants. In each table, the $\mathrm{SL}_2$-weight is fixed and specified below the table, and the generators are located in their bidegree homogeneous components. Remember that each $x_i$ has $\mathrm{SL}_2$-weight one and each $\delta_{rs}$ is an $\mathrm{SL}_2$-invariant.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 6 | | | $\delta_{15}^{12}$ | $\delta_{15}^{9}\delta_{35}$ | | | $\delta_{35}^{12}$ |
| 5 | | | $\delta_{14}^{4}\delta_{15}^{8}$ | $\delta_{14}^{8}\delta_{16}^{4}\delta_{26}^{2}$<br>$\delta_{14}^{4}\delta_{15}^{5}\delta_{35}^{3}$ | $\delta_{14}^{8}\delta_{16}\delta_{26}^{2}\delta_{36}^{3}$ | | |
| 4 | | | $\delta_{14}^{6}\delta_{25}^{2}$<br>$\delta_{14}^{8}\delta_{15}^{4}$ | $\delta_{14}^{4}\delta_{16}^{2}\delta_{26}^{4}$<br>$\delta_{15}^{3}\delta_{25}^{2}\delta_{35}^{3}$<br>$\delta_{14}^{8}\delta_{15}\delta_{35}^{3}$ | $\delta_{14}^{3}\delta_{24}\delta_{26}^{3}\delta_{36}^{3}$ | | |
| 3 | | $\delta_{16}^{6}$ | $\delta_{14}^{12}$<br>$\delta_{14}^{2}\delta_{15}^{4}\delta_{24}^{2}$<br>$\delta_{16}^{3}\delta_{36}^{3}$ | $\delta_{14}^{9}\delta_{34}^{3}$<br>$\delta_{26}^{6}$ | | | $\delta_{34}^{12}$ |
| 2 | | | $\delta_{14}^{6}\delta_{24}^{2}$<br>$\delta_{25}^{4}$ | $\delta_{14}^{3}\delta_{24}^{2}\delta_{34}^{3}$ | | | |
| 1 | | | $\delta_{24}^{4}$ | | | | |
| 0 | | | | | | | |
| | 0 | 2 | 4 | 6 | 8 | 10 | 12 |

0

| | | | | | |
|---|---|---|---|---|---|
| 5 | | | $x_4\delta_{14}\delta_{15}^{8}\delta_{24}^{2}$ | $x_5\delta_{16}^{3}\delta_{25}^{3}\delta_{26}^{3}$ | $x_5\delta_{25}^{3}\delta_{26}^{3}\delta_{36}^{3}$ |
| 4 | | $x_1\delta_{15}^{8}$ | $x_1\delta_{14}^{4}\delta_{16}^{4}\delta_{26}^{2}$<br>$x_1\delta_{15}^{5}\delta_{35}^{3}$ | $x_1\delta_{14}^{4}\delta_{16}\delta_{26}^{2}\delta_{36}^{3}$ | $x_4\delta_{24}^{3}\delta_{26}^{3}\delta_{36}^{3}$ |
| 3 | | $x_6\delta_{16}^{3}\delta_{26}^{2}$<br>$x_4\delta_{15}^{3}\delta_{24}\delta_{25}$<br>$x_1\delta_{14}^{4}\delta_{15}^{4}$ | $x_1\delta_{14}^{4}\delta_{15}\delta_{35}^{3}$<br>$x_1\delta_{16}^{2}\delta_{26}^{4}$<br>$x_4\delta_{15}^{3}\delta_{24}^{3}\delta_{25}$<br>$x_6\delta_{26}^{2}\delta_{36}^{3}$ | $x_2\delta_{26}^{3}\delta_{36}^{3}$ | |
| 2 | $x_5\delta_{15}^{3}$ | $x_1\delta_{14}^{8}$<br>$x_1\delta_{15}^{2}\delta_{25}^{2}$<br>$x_5\delta_{35}^{3}$ | $x_1\delta_{14}^{5}\delta_{34}^{3}$<br>$x_2\delta_{25}\delta_{35}^{3}$ | | |
| 1 | $x_4\delta_{14}^{3}$ | $x_1\delta_{14}^{2}\delta_{24}^{2}$<br>$x_4\delta_{34}^{3}$ | $x_2\delta_{24}\delta_{34}^{3}$ | | |
| 0 | | | | | |
| | 1 | 3 | 5 | 7 | 9 |

1

| | | | | |
|---|---|---|---|---|
| 5 | | | | $x_1x_5\delta_{16}^5\delta_{25}^3\delta_{26}$ |
| 4 | | | $x_4^2\delta_{14}^2\delta_{16}^4\delta_{26}^2$ | $x_4^2\delta_{14}^2\delta_{16}\delta_{26}^2\delta_{36}^3$ |
| 3 | $x_4^2\delta_{14}^2\delta_{15}^4$ | $x_1x_6\delta_{16}^5$<br>$x_1^2\delta_{16}^4\delta_{26}^2$ | $x_1x_4\delta_{14}\delta_{15}^4\delta_{24}^2$<br>$x_2x_6\delta_{26}^5$<br>$x_1x_6\delta_{16}^2\delta_{36}^3$<br>$x_4^2\delta_{14}^2\delta_{15}\delta_{35}^3$<br>$x_6^2\delta_{26}^4$ | $x_1^2\delta_{16}\delta_{26}^2\delta_{36}^3$ |
| 2 | | $x_1^2\delta_{15}^4$<br>$x_5^2\delta_{25}^2$ | $x_1^2\delta_{15}\delta_{35}^3$<br>$x_2x_5\delta_{25}^3$ | |
| 1 | | $x_1^2\delta_{14}^4$<br>$x_4^2\delta_{24}^2$ | $x_1^2\delta_{14}\delta_{34}^3$<br>$x_2x_4\delta_{24}^3$ | |
| 0 | | $x_2^2$ | | |
| | 0 | 2 | 4 | 6 |

| | | |
|---|---|---|
| 5 | $x_1x_5\delta_{16}^2\delta_{25}^3\delta_{26}\delta_{36}^3$ | |
| 4 | | $x_1^2\delta_{14}\delta_{24}^3\delta_{26}^3\delta_{36}^3$ |
| 3 | | |
| 2 | | |
| 1 | | |
| 0 | | |
| | 8 | 10 |

2

| | | | | | |
|---|---|---|---|---|---|
| 5 | | $x_4^3\delta_{14}\delta_{15}^8$ | $x_4^3\delta_{14}\delta_{15}^5\delta_{35}^3$<br>$x_5^3\delta_{16}^3\delta_{25}\delta_{26}^3$ | $x_5^3\delta_{25}\delta_{26}^3\delta_{36}^3$ | $x_1^2x_5\delta_{16}\delta_{25}^3\delta_{26}^5$ |
| 4 | | | $x_4^3\delta_{14}\delta_{16}^2\delta_{26}^4$ | $x_4^3\delta_{24}\delta_{26}^3\delta_{36}^3$ | |
| 3 | $x_6^3\delta_{16}^3$ | $x_1x_6^2\delta_{16}^2\delta_{26}^2$<br>$x_6^3\delta_{36}^3$ | $x_4^3\delta_{25}^2\delta_{34}\delta_{35}^2$<br>$x_2x_6^2\delta_{26}\delta_{36}^3$ | $x_1^2x_4\delta_{15}\delta_{24}^3\delta_{25}^3$<br>$x_2x_4^2\delta_{25}^3\delta_{34}^2\delta_{35}$ | |
| 2 | $x_1x_5^2\delta_{15}^2$ | $x_1^2x_5\delta_{15}\delta_{25}^2$<br>$x_3x_5^2\delta_{35}^2$ | | | |
| 1 | $x_1x_4^2\delta_{14}^2$ | $x_1^2x_4\delta_{14}\delta_{24}^2$<br>$x_3x_4^2\delta_{34}^2$ | | | |
| 0 | $x_1^3$ | $x_3^3$ | | | |
| | 1 | 3 | 5 | 7 | 9 |

3

| | | | | | |
|---|---|---|---|---|---|
| 5 | | | $x_1x_5^3\delta_{16}^5\delta_{25}\delta_{26}$ | $x_1x_5^3\delta_{16}^2\delta_{25}\delta_{26}\delta_{36}^3$ | |
| 4 | | | $x_1x_4^3\delta_{14}\delta_{16}^4\delta_{26}^2$ | $x_1x_4^3\delta_{14}\delta_{16}\delta_{26}^2\delta_{36}^3$ | $x_1^2x_4^2\delta_{14}\delta_{24}\delta_{26}^3\delta_{36}^3$ |
| 3 | | $x_6^4\delta_{26}^2$<br>$x_1^2x_6^2\delta_{16}^4$<br>$x_1x_4^3\delta_{14}\delta_{15}^4$ | $x_1x_4^3\delta_{14}\delta_{15}\delta_{35}^3$<br>$x_1^2x_6^2\delta_{16}\delta_{36}^3$<br>$x_2x_6^3\delta_{26}^3$ | | |
| 2 | $x_5^4$ | $x_2x_5^3\delta_{25}$ | | | |
| 1 | $x_4^4$ | $x_2x_4^3\delta_{24}$ | | | |
| 0 | | | | | |
| | 0 | 2 | 4 | 6 | 8 |

4

| | | | | |
|---|---|---|---|---|
| 5 | | | | $x_1^2x_5^3\delta_{16}\delta_{25}\delta_{26}^5$ |
| 4 | | | | |
| 3 | $x_1x_6^4\delta_{16}^2$ | $x_1^2x_6^3\delta_{16}\delta_{26}^2$ <br> $x_3x_6^4\delta_{36}^2$ | $x_1^2x_4^3\delta_{15}\delta_{24}\delta_{25}^3$ | |
| 2 | $x_1^2x_5^3\delta_{15}$ | $x_3^2x_5^3\delta_{35}$ | | |
| 1 | $x_1^2x_4^3\delta_{14}$ | $x_3^2x_4^3\delta_{34}$ | | |
| 0 | | | | |
| | 1 | 3 | 5 | 7 |

5

| | | |
|---|---|---|
| 3 | $x_6^6$ | $x_2x_6^5\delta_{26}$ |
| 2 | | |
| 1 | | |
| 0 | | |
| | 0 | 2 |

6

| | | |
|---|---|---|
| 3 | $x_1^2x_6^5\delta_{16}$ | $x_3^2x_6^5\delta_{36}$ |
| 2 | | |
| 1 | | |
| 0 | | |
| | 1 | 3 |

7

## 2.1. Numerics.

**2.1.1.** — The main result of this section is a numerical description of the decomposition of an irreducible finite dimensional representation of $\mathrm{Sp}_4$ upon restriction to $\mathrm{SL}_2$. The formula we obtain is strikingly simple and compact.

In order to state the theorem, we need to introduce some notation; we shall be brief and not discuss the provenance of these objects until the subsequent sections.

**2.1.2.** — Let $p_{23}(x)$ denote the number of partitions of the nonnegative integer $x$ into 2's and 3's. Note in particular that $p_{23}(x)$ is zero if $x$ is negative. We can obtain a simple formula for $p_{23}(\ell)$ rather easily. Since the least common multiple of 2 and 3 is 6, one can deduce the fact that $p_{23}(\ell+6) = p_{23}(\ell)+1$. From this we obtain

$$p_{23}(\ell) = \left[\frac{\ell}{6}\right] + 1 - \delta_1(\ell)$$

where $\left[\frac{1}{6}\ell\right]$ is the greatest integer part of $\frac{1}{6}\ell$ and where $\delta_1(\ell) = 1$ if $\ell \equiv 1 \bmod 6$, 0 otherwise. Although the right-hand side makes sense for any $\ell$, we emphasize that $p_{23}$ can only be nonzero on nonnegative integers $\ell$. Set

$$P(x) = \sum_{\ell \geq 0} p_{23}\big(\tfrac{1}{2}x - \ell\big);$$

since $p_{23}$ may only be nonzero on nonnegative integers, the right-hand side is actually a finite sum, so $P(x)$ makes sense.

Finally, define

$$\begin{aligned}(2.1.2.1)\qquad H(k,m,n) = P\big(3(m-n)+4n-k\big)&\\ - P\big(3(m-n)+2n-k-2\big)&\\ - P\big((m-n)+4n-k-2\big)&\\ + P\big((m-n)-2n-k-8\big)&\end{aligned}$$

and

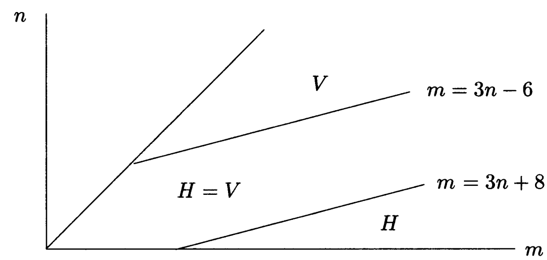
$$\begin{aligned}(2.1.2.2)\qquad V(k,m,n) = P(4n+3(m-n)-k)&\\ - P(4n+(m-n)-k-2)&\\ - P(2n+3(m-n)-k-2)&\\ + P(2n-(m-n)-k-6).&\end{aligned}$$

**2.1.3.** — Set $\mu(k,m,n) = \dim \mathrm{Hom}_{\mathrm{SL}_2}\big(V_k, \sigma_4^{(m,n)}\big)$; this is just the multiplicity of $V_k$ in $\sigma_4^{(m,n)}$.

THEOREM (the Branching Rule). — *One has*

$$\mu(k,m,n) = \begin{cases} H(k,m,n) & \text{if } m > 3n-6,\\ V(k,m,n) & \text{if } m < 3n+8.\end{cases}$$

*Remark.* — Note that for $m$ in the strip $3n-6<m<3n+8$ the last terms in the formulas (2.1.2.1), (2.1.2.2) both vanish, so that for such $m$, $H(k,m,n)=V(k,m,n)$.




## 2.2. A tensor product decomposition.

**2.2.1.** — If $(m,n)$ sits in the interior of the fundamental Weyl chamber of $\mathrm{Sp}_4$, it is easy to see that

$$\sigma_4^{(1,0)}\otimes\sigma_4^{(m,n)}=\sigma_4^{(m+1,n)}\oplus\sigma_4^{(m,n+1)}\oplus\sigma_4^{(m-1,n)}\oplus\sigma_4^{(m,n-1)},\tag{2.2.1.1}$$

and if we agree that $\sigma_4^{(a,b)}=0$ if $(a,b)$ is not dominant, the Weyl dimension formula tells us that this is the decomposition of this tensor product for any dominant $(m,n)$.

The importance of this relation is that it will yield a homogeneous difference equation to which our branching rule will be the unique solution. More precisely, if $k\geq 3$ and $(m,n)$ is in the interior of the Weyl chamber, Clebsch-Gordan tells us that

$$\begin{aligned}\mu(k-3,m,n)&+\mu(k-1,m,n)\\&+\mu(k+1,m,n)+\mu(k+3,m,n)\\&\qquad=\mu(k,m+1,n)+\mu(k,m,n+1)\\&\qquad\qquad+\mu(k,m-1,n)+\mu(k,m,n-1).\end{aligned}\tag{2.2.1.2}$$

Of course, if $k\leq 3$, or if $(m,n)$ lies on the wall of the Weyl chamber, then the analogue of equation (2.2.1.2) will be different, and in fact one easily sees that including the aforementioned, there are twelve such equations. These variations may be accounted for by specifying initial conditions to the difference equation given by (2.2.1.2), as we shall see in our discussion.

**2.2.2.** — Let $f(k,m,n)$ be a $\mathbb{Z}_+$-valued function defined on the integral cone in $\mathbb{Z}^3$ given by $k \geq -3$, $m \geq n-1$ and $n \geq -1$. Define the difference operator $D$ on such functions by

$$(2.2.2.1)\quad \begin{aligned} Df(k,m,n) =& f(k-3,m,n) + f(k-1,m,n) \\ &+ f(k+1,m,n) + f(k+3,m,n) \\ &- \big(f(k,m+1,n) + f(k,m,n+1) \\ &+ f(k,m-1,n) + f(k,m,n-1)\big). \end{aligned}$$

Consider the homogeneous difference equation

$$(2.2.2.2)\qquad Df(k,m,n) = 0$$

subject to the initial conditions

$$(2.2.2.3)\qquad \begin{cases} f(k,m,0) = H(k,m,0) & \text{for all } k,m; \\ f(k,m,m) = V(k,m,m) & \text{for all } k,m; \\ f(k,m,-1) = 0 & \text{for all } k,m; \\ f(k,m,m+1) = 0 & \text{for all } k,m; \\ f(-1,m,n) = 0 & \text{for all } m,n; \\ f(-2,m,n) = \begin{cases} -H(0,m,n) \\ -V(0,m,n) \end{cases} & \begin{array}{l} \text{for } m \geq 3n-2, \\ \text{for } m \leq 3n+4; \end{array} \\ f(-3,m,n) = \begin{cases} -H(1,m,n) \\ -V(1,m,n) \end{cases} & \begin{array}{l} \text{for } m \geq 3n-1, \\ \text{for } m \leq 3n+3. \end{array} \end{cases}$$

These boundary conditions at first look rather unnatural, but closer inspection should convince the reader the last seven should take care of the degeneracies incurred by the special tensor product decompositions, whereas the first two will describe the branching rules of the cubic and quartic respectively, as we shall show in the next sections...

### 2.3. Rules on the faces.

**2.3.1.** — Recall from 1.2.1 that the generators of the covariants of the cubic are specified by SL$_2$-weight and degree: $\alpha = (3,1)$, $\beta = (2,2)$, $\gamma = (3,3)$ and $\delta = (0,4)$, and these satisfy the relation $\alpha^2\delta + \beta^3 + \gamma^2 = 0$. Using this relation it is possible to describe a linear basis for the algebra of covariants. There are several possible choices for such a basis. Our choice enables us to easily count how many basis elements of fixed weight and degree there are, and hence obtain a branching rule for the cubic case. We take as basis for $\mathcal{P}(\mathbb{C}^4)$ all monomials of the form $\alpha^*\beta^*\gamma^*$, $\beta^*\gamma^*\delta^*$ and $\alpha\beta^*\gamma^*\delta^*$.

**2.3.2.** — To understand how many copies of the $SL_2$-module $V_k$ there are inside the $Sp_4$-module $\sigma_4^{(\ell,0)}$, we must count the number of monomials of the above type of weight $k$ and degree $\ell$.

First note that the monomials $\beta^*\gamma^*$ are the only ones lying on the ray $k=\ell$, where $\ell \geq 0$. Multiplication of these by $\alpha^m$ has the effect of translating this ray to $k=\ell+2m$, with $\ell \geq m$; multiplication by $\delta^n$ translates $k=\ell$, $\ell \geq 0$ to $k=\ell-4n$, $\ell \geq 4n$, and the monomials $\alpha\beta^*\gamma^*\delta^*$ all lie on the ray $k=\ell-4n+2$ where $\ell \geq 4n+1$.

**2.3.3.** — Counting the number of monomials of a fixed degree $\ell$ along the ray $k=\ell$ is very simple: since only monomials of the form $\beta^*\gamma^*$ lie here, the number of such is clearly the number $p_{23}(\ell)$ of partitions of $\ell$ into 2's and 3's.

**2.3.4.** — From 2.3.2 and 2.3.3 it is clear that $\mu(k,\ell,0)$ will vary as a suitable translate of $p_{23}$ as $k$ varies along a ray. We wish to understand how $\mu(k,\ell,0)$ varies for fixed $\ell$, so fix $\ell$ and first consider $k \geq \ell$, *i.e.* $k=\ell+2m$. Since $(\ell+2m,\ell)=(\ell-m,\ell-m)+(3m,m)$, $\mu(\ell+2m,\ell,0)=\mu(\ell-m,\ell-m,0)$ yielding

$$\mu(\ell+2m,\ell,0)=p_{23}\Big(\frac{3\ell-k}{2}\Big).$$

For $k<\ell$, there are two cases to consider:

- First, if $k=\ell-4n+2$, we have

$$(\ell-4n+2,\ell)=(\ell-4n-1,\ell-4n-1)+(3,1)+(0,4n),$$

so for the multiplicities we get

$$\mu(\ell-4n+2,\ell,0)=\mu(\ell-4n-1,\ell-4n-1,0),$$

which we may also write as $\mu(\ell+2n-1,\ell+2n-1,0)-n$ since $(\ell+2n-1,\ell+2n-1)=(\ell-4n-1,\ell-4n-1)+(6n,6n)$.

- The second case is that in which $k=\ell-4n$. Here we have $(\ell-4n,\ell)=(\ell-4n,\ell-4n)+(0,4n)$, which on the level of multiplicities becomes

$$\mu(\ell-4n,\ell,0)=\mu(\ell-4n,\ell-4n,0)=\mu(\ell+2n,\ell+2n,0)-n.$$

Writing these multiplicities in terms of $\ell$ and $k$ we obtain the

LEMMA (Branching Rule for the Cubic). — *One has*

$$\mu(k,\ell,0) = H(k,\ell,0).$$

**2.3.5.** — Recall from 1.2.3, that $\mathcal{H}^\ell(\mathbb{C}^5)$ is isomorphic to the $\mathrm{Sp}_4$-module $\sigma_4^{(\ell,\ell)}$, that the algebra of harmonic $\mathrm{SL}_2$-covariants is again generated by elements specified by $\mathrm{SL}_2$-weight and degree: $a = (4,1)$, $b = (4,2)$, $d = (6,3)$, $e = (0,3)$, and that these satisfy the relation $a^3e + b^3 + d^2 = 0$. Our basis of $\mathcal{H}(\mathbb{C}^5)^N$ will consist of monomials of the following types:

$$a^*b^*d^*, \quad b^*d^*e^*, \quad ab^*d^*e^*, \quad a^2b^*d^*e^*.$$

**2.3.6.** — Using the above basis as in the analysis of the cubic case, the analogue for the quartic easily follows.

LEMMA (Branching Rule for the Quartic). — *One has*

$$\mu(k,\ell,\ell) = V(k,\ell,\ell).$$

## 2.4. Solving the initial value problem.

**2.4.1.** — Suppose that $H$ and $V$ satisfy the appropriate initial conditions from (2.2.2.3). To show that they will satisfy (2.2.2.2) in the appropriate regions is then a purely formal calculation: considering $H$, the left-hand side of (2.2.1.2) is equal to

$$\begin{aligned}
&P\big(3(m-n)+4n-(k-3)\big) && -P\big(3(m-n)+2n-(k-3)-2\big)\\
&-P\big((m-n)+4n-(k-3)-2\big) && +P\big((m-n)-2n-(k-3)-8\big)\\
&+P\big(3(m-n)+4n-(k-1)\big) && -P\big(3(m-n)+2n-(k-1)-2\big)\\
&-P\big((m-n)+4n-(k-1)-2\big) && +P\big((m-n)-2n-(k-1)-8\big)\\
&+P\big(3(m-n)+4n-(k+1)\big) && -P\big(3(m-n)+2n-(k+1)-2\big)\\
&-P\big((m-n)+4n-(k+1)-2\big) && +P\big((m-n)-2n-(k+1)-8\big)\\
&+P\big(3(m-n)+4n-(k+3)\big) && -P\big(3(m-n)+2n-(k+3)-2\big)\\
&-P\big((m-n)+4n-(k+3)-2\big) && +P\big((m-n)-2n-(k+3)-8\big),
\end{aligned}$$

while the right-hand side is

$$\begin{aligned}
&P\big(3(m+1-n)+4n-k\big) - P\big(3(m+1-n)+2n-k-2\big)\\
&- P\big((m+1-n)+4n-k-2\big) + P\big((m+1-n)-2n-k-8\big)\\
&+ P\big(3(m-(n+1))+4(n+1)-k\big)\\
&- P\big(3(m-(n+1))+2(n+1)-k-2\big)\\
&- P\big((m-(n+1))+4(n+1)-k-2\big)\\
&+ P\big((m-(n+1))-2(m+1)-k-8\big)\\
&+ P\big(3(m-1-n)+4n-k\big) - P\big(3(m-1-n)+2n-k-2\big)\\
&- P\big((m-1-n)+4n-k-2\big) + P\big((m-1-n)-2n-k-8\big)\\
&+ P\big(3(m-(n-1))+4(n-1)-k\big)\\
&- P\big(3(m-(n-1))+2(n-1)-k-2\big)\\
&- P\big((m-(n-1))+4(n-1)-k-2\big)\\
&+ P\big((m-(n-1)-2(n-1)-k-8\big),
\end{aligned}$$

and comparing the two tells us precisely that $H$ satisfies (2.2.2.2). Similarly one obtains the analogous fact for $V$. Thus, the theorem will follow if we can show that the $H$ and $V$ satisfy the remaining initial conditions of (2.2.2.3).

**2.4.2.** — Immediately from their definitions one obtains, for all $k, m$,

$$H(k,m,-1) = 0 \quad \text{and} \quad V(k,m,m+1) = 0.$$

**2.4.3.** — As the verification of the remaining conditions is a simple task, which unfortunately involves many cases and calculations. We shall show what happens for $H$, and then state the analogues for $V$. Beginning with the fifth condition of (2.2.2.3),

$$\begin{aligned}
&H(-1,m,n)\\
&\quad = P(3(m-n-1)+4n+4) - P(3(m-n-1)+2n+2)\\
&\qquad\quad - P((m-n-1)+4n) + P((m-n-1)-2n-6)\\
&\quad = p_{23}\big(\tfrac{3}{2}(m-n-1)+2n+2\big) + \cdots + p_{23}\big(\tfrac{3}{2}(m-n-1)+n+2\big)\\
&\qquad\quad - \big(p_{23}\big(\tfrac{1}{2}(m-n-1)+2n\big) + \cdots + p_{23}\big(\tfrac{1}{2}(m-n-1)-n-2\big)\big),
\end{aligned}$$

and putting $w = \frac{1}{2}(m-n-1)$, $v = n$,

$$H(-1,m,n) = \sum_{\ell=0}^{v} p_{23}(3w+v+2+\ell) - \sum_{\ell=0}^{v} p_{23}(3w-3v+9\ell). \tag{2.4.3.1}$$

To explain the appearance of the last summand, we hope the reader will believe the fact that

$$p_{23}(x) + p_{23}(x-1) + p_{23}(x-2) = p_{23}(3x).$$

We claim that for each $\ell$,

$$\begin{aligned} p_{23}(3w+2v+2-\ell) &+ p_{23}(3w+v+2+\ell) \\ &= p_{23}(3w+6v-9\ell) + p_{23}(3w-3v+9\ell). \end{aligned} \tag{2.4.3.2}$$

Clearly the identity will follow from this claim.

Let $q$ denote the function on $\mathbb{Z}$ defined by

$$q(d) = \left[\frac{d}{6}\right] + 1 - \delta_1(d)$$

where $\delta_1(d)$ is as in 2.1.2. Since $q(d)$ agrees with $p_{23}(d)$ for $d \geq -4$, $d = -6$ and since $q$ is periodic on all of $\mathbb{Z}$ (in contrast to $p_{23}$ which is 0 for $d < 0$), we find it much more convenient to verify the claim for $q$ replacing $p_{23}$, and then worry about the implications for $p_{23}$.

This verification, as all the subsequent ones, is rather tedious, but we include one case here to convince our reader.

First note that it is clear that we only need to check the claimed identity (2.4.3.2) for $0 \leq v,\ \ell \leq 5$ by periodicity modulo 6. If $\ell = 0$, then (2.4.3.2) becomes

$$\begin{aligned} q(3w+2v+2) &+ q(3w+v+2) \\ &= q(3w+6v) + q(3w-3v) = q(3w) + q(3w+3v). \end{aligned}$$

Fixing $v$ in the above range, it is trivial to check the resulting six identities:

$$\begin{aligned} q(3w+2) + q(3w+2) &= q(3w) + q(3w), \\ q(3w+4) + q(3w+3) &= q(3w) + q(3w+3), \\ q(3w+6) + q(3w+4) &= q(3w) + q(3w+6), \\ q(3w+8) + q(3w+5) &= q(3w) + q(3w+9), \\ q(3w+10) + q(3w+6) &= q(3w) + q(3w+12), \\ q(3w+12) + q(3w+7) &= q(3w) + q(3w+15). \end{aligned}$$

To minimize effort we note that one only needs to check these for two consecutive $w$'s since $q(3x+y)$ has period 2 as a function of $x$ for fixed $y$. Similarly for the remaining cases for $\ell$.

Thus, the analogue of (2.4.3.1) with $q$, instead of $p_{23}$, is identically zero for any $w$ and $v$. To see what this tells us about (2.4.3.1) itself, note that the summand in (2.4.3.1) with smallest argument (assuming $w, v \geq 0$) is $p_{23}(3w-3v) = p_{23}(3(w-v))$. Furthermore, note that $p_{23}(3x) = q(3x)$ only for $x \geq -2$; hence (2.4.3.1) is valid whenever $w - v \geq -2$, which gives us $m \geq 3n-3$. To summarize:

$$H(-1,m,n) = 0 \quad \text{in the region} \quad m \geq 3n-3.$$

**2.4.4.** — We continue our assault on the desired identities...

$$\begin{aligned} H(0,m,n) + H(-2,m,n) \\ &= P\big(3(m-n)+4m\big) - P\big(3(m-n)+2n-2\big) \\ &\quad - P\big((m-n)+4n-2\big) + P\big((m-n)-2n-8\big) \\ &\quad + P\big(3(m-n)+4n+2\big) - P\big(3(m-n)+2n\big) \\ &\quad - P\big((m-n)+4n\big) + P\big((m-n)-2n-6\big), \end{aligned}$$

and putting $\frac{1}{2}(m-n) = w$, $n = v$, we have

$$\begin{aligned} H(0,m,n) &+ H(-2,m,n) \\ &= p_{23}(3w+2v) + \cdots + p(3w+v) \\ &\quad - \big(p_{23}(w+2v-1) + \cdots + p_{23}(w-v-3)\big) \\ &\quad + p_{23}(3w+2v+1) + \cdots + p_{23}(3w+v+1) \\ &\quad - \big(p_{23}(w+2v) + \cdots + p_{23}(w-v-2)\big) \\ &= \sum_{\ell=0}^{v} p_{23}(3w+v+\ell) - \sum_{\ell=0}^{v} p_{23}(3w-3v-3+9\ell) \\ &\quad + \sum_{\ell=0}^{v} p_{23}(3w+v+1+\ell) - \sum_{\ell=0}^{v} p_{23}(3w-3v+9\ell). \end{aligned} \tag{2.4.4.1}$$

Again, putting everything in terms of $q$ we get

$$\begin{aligned} &\sum_{\ell=0}^{v} q(3w+v+\ell) - \sum_{\ell=0}^{v} q(3w-3v-3+9\ell) \\ &\quad + \sum_{\ell=0}^{v} q(3w+v+1+\ell) - \sum_{\ell=0}^{v} q(3w-3v+9\ell). \end{aligned} \tag{2.4.4.2}$$

But note that from 2.4.3 we may conclude that

$$\sum_{\ell=0}^{v} q(3w - 3v + 9\ell) = \sum_{\ell=0}^{v} q(3w + v + 2 + \ell)$$

and

$$\sum_{\ell=0}^{v} q(3w - 3v - 3 + 9\ell) = \sum_{\ell=0}^{v} q(3w + v - 1 + \ell)$$

so (2.4.4.2) becomes

$$q(3w + 2v) - q(3w + v - 1) + q(3w + v + 1) - q(3w + 2v + 2)$$

and to check that this is identically zero is a trivial matter.

As before, to translate this back into (2.4.4.1), we see that the smallest argument is $3(w - v - 1)$, and just as before, we may replace $q$ by $p_{23}$ whenever $w - v - 1 \geq -2$, yielding $m \geq 3n - 2$.

**2.4.5.** — The calculation showing $H(1, m, n) + H(-3, m, n) = 0$ is very similar to that of 2.4.4

$$\begin{aligned} H(1,m,n) &+ H(-3,m,n) \\ &= P(3(m-n) + 4n - 1) - P(3(m-n) + 2n - 3) \\ &\quad - P((m-n) + 4n - 3) + P((m-n) - 2n - 9) \\ &\quad + P(3(m-n) + 4n + 3) + P(3(m-n) + 2n + 1) \\ &\quad - P((m-n) + 4n + 1) + P((m-n) - 2n - 5), \end{aligned}$$

and putting $w = \frac{1}{2}(m - n - 1)$, $v = n$ we obtain

$$\begin{aligned} &\sum_{\ell=0}^{v} p_{23}(3w + v + 1 + \ell) - \sum_{\ell=0}^{v} p_{23}(3w - 3v + 3 + 9\ell) \\ &\quad + \sum_{\ell=0}^{v} p_{23}(3w + v + 3 + \ell) - \sum_{\ell=0}^{v} p_{23}(3w - 3v + 3 + 9\ell). \end{aligned}$$

Using the calculation of 2.4.3 this becomes

$$\begin{aligned} \text{(2.4.5.1)} \quad &\sum_{\ell=0}^{v} p_{23}(3w + v + 1 + \ell) - \sum_{\ell=0}^{v} p_{23}(3w + v - 1 + \ell) \\ &\quad + \sum_{\ell=0}^{v} p_{23}(3w + v + 3 + \ell) - \sum_{\ell=0}^{v} p_{23}(3w + v + 5 + \ell). \end{aligned}$$

Replacing $p_{23}$ by $q$, the identity to be verified becomes

$$\begin{aligned} & q(3w+2v+5)+q(3w+2v+4)+q(3w+v)+q(3w+v-1) \\ & \quad = q(3w+2v+1)+q(3w+2v)+q(3w+v+4)+q(3w+v+3) \end{aligned}$$

which is easily checked. The only point which remains is the determination of the region in which (2.4.5.1) is identically zero — just as before the condition is $w-v-1 \geq -2$ which translates to $m \geq 3n-1$.

**2.4.6.** — The results of 2.4.3 through 2.4.5 show that $H$ satisfies the initial conditions in the region $m \geq 3n-2$. The observant reader might wonder why we have $m \geq 3n-2$ instead of $m \geq 3n-1$. The answer is simple: the parity of $\mathrm{SL}_2$-types occurring on lines of the form $m = 3n+j$ is determined by the parity of $j$, so, while the identities of 2.4.3 and 2.4.5 hold simultaneously only for $m \geq 3n-1$, since no odd $\mathrm{SL}_2$-type occurs on $m = 3n-2$, the three identities of 2.4.3, 2.4.4, 2.4.5 are valid in the region $m \geq 3n-2$.

**2.4.7.** — Since the arguments for $V$ are exactly as those in the previous sections for $H$, we shall leave them to the interested reader, and simply state the results here:

$$\begin{aligned} V(-1,m,n) &= 0 \quad \text{when} \quad m \leq 3n+5, \\ V(0,m,n)+V(-2,m,n) &= 0 \quad \text{when} \quad m \leq 3n+4, \\ V(1,m,n)+V(-3,m,n) &= 0 \quad \text{when} \quad m \leq 3n+3. \end{aligned}$$

As in the discussion of 2.4.6, it follows that these identities for $V$ are simultaneously valid in the region $m \leq 3n+4$.

**2.4.8.** — It is easy to see that since $H$ satisfies the initial conditions in the region $m \geq 3n-2$, for $k \geq 0$, we have

$$DH(k,m,n) = 0 \text{ in the region } m \geq 3n-5,$$

and similarly from 2.4.7,

$$DV(k,m,n) = 0 \quad \text{in the region} \quad m \leq 3n+7.$$

In other words, the theorem follows...

*Remark.* — The reader wizened in the lore of Lie theory may ask: Why not simply use the Weyl character formula? for there is *even* a very simple form that the formula takes...

The character formula tells us that at the element of the maximal torus in $Sp_4$ with eigenvalues $x$, $y$, $y^{-1}$, $x^{-1}$, the character of $\sigma_4^{(m,n)}$ is given by

$$\frac{\begin{vmatrix} x^{m+2} - x^{-(m+2)} & x^{n+1} - x^{-(n+1)} \\ y^{m+2} - y^{-(m+2)} & y^{n+1} - y^{-(n+1)} \end{vmatrix}}{\begin{vmatrix} x^2 - x^{-2} & x - x^{-1} \\ y^2 - y^{-2} & y^- y^{-1} \end{vmatrix}}.$$

Restricting to the torus of $SL_2$ yields $x = t^3$, $y = t$, and the determinants split as

$$\frac{(t^{m+n+3} - t^{-m-n-3})(t^{m-n+1} - t^{-m+n-1})(t^{m+2} - t^{-m-2})(t^{n+1} - t^{-n-1})}{(t^3 - t^{-3})(t^2 - t^{-2})(t - t^{-1})^2}$$

which implies that we may calculate $\mu(k, m, n)$ as the coefficient of $t^{k+1}$ in the polynomial

$$\frac{(t^{m+n+3} - t^{-m-n-3})(t^{m-n+1} - t^{-m+n-1})(t^{m+2} - t^{-m-2})(t^{n+1} - t^{-n-1})}{(t^3 - t^{-3})(t^2 - t^{-2})(t - t^{-1})}.$$

To calculate this coefficient, one must understand the form of this polynomial in the various cases depending upon how the denominator divides the numerator. This is elementary, and one may obtain a collection of formulas describing the branching rule. But there are several difficulties with this approach. Firstly, in the rule given in Theorem 2.1, the multiplicities are determined by the two functions $H(k, m, n)$ and $V(k, m, n)$, whereas using the character method, one has more than ten different formulas... Moreover, the regions on which these formulas are valid, depending on certain congruences, are more complicated than the two nice domains one obtains using our approach, implying that the organization of these formulas is also much more complicated... Furthermore, there is a lot of overlap in these regions of definition in the character approach, hence much redundancy; our regions also overlap, but the common domain of definition is very simple, and on this domain we see from the formulas easily that the restrictions agree. Of course, these two approaches will yield the same answer, but to go from the answer of the character formula approach to the one derived above is a bit of work, and not illuminating in any way; at least from our discussion above, one can see that our formula is really a result of our method of proof — the two simple recursion formulas implied by 2.2.1.2...

### 3.1. Refinement of an old result.

**3.1.1.** — In this section we describe the geometric branching rule: the determination of a minimal set of generators for $\mathcal{R}(N \setminus \mathrm{Sp}_4 /U)$, where, as usual, $N$ is a maximal unipotent subgroup of $\mathrm{SL}_2$. Since $\mathcal{R}(N \setminus \mathrm{Sp}_4 /U)$ is isomorphic to the algebra of $\mathrm{SL}_2$-covariants inside $\mathcal{R}(\mathrm{Sp}_4 /U)$, we may view this result as a refinement of the classical computation of Gordan [G], or, of course, of our calculation in 1.3.

This ring is triply-graded by $\mathrm{SL}_2$-weight and $\mathrm{Sp}_4$-weight — if the former is $k$ and the latter $(m,n)$, as in §2, we write the triple-grading as $(k,m,n)$. The fact that it is graded by $\mathrm{SL}_2$-weight is clear, as it is an algebra of $\mathrm{SL}_2$ covariants; the fact that it is graded by $\mathrm{Sp}_4$-weight is also well-known but shall be explained in 3.2.1 for completeness.

Our result is summarized in the

THEOREM. — *$\mathcal{R}(N \setminus \mathrm{Sp}_4 /U)$ is minimally generated by the following elements which are specified by their triple gradings :*

$$\begin{array}{llll}
(0,4,0), & (0,3,3), & (0,6,2), & (0,9,3), \\
(1,2,1), & (1,4,1), & (1,5,2), & (1,7,2), \\
(2,2,0), & (2,3,1), & (2,5,1), & \\
(3,1,0), & (3,3,0), & (3,3,2), & \\
(4,1,1), & (4,2,2), & (4,4,2), & \\
(5,2,1), & & & \\
(6,3,3). & & &
\end{array}$$

Note that the numerical branching rule tells us that this algebra has dimension one in each of the above triply homogeneous components.

### 3.2. Some properties of $\mathcal{R}(G/U)$.

**3.2.1.** — Let $G$ be a connected reductive group over $\mathbb{C}$, $A \subseteq G$ a maximal torus and $U$ a maximal unipotent subgroup normalized by $A$. The group $G$ acts by left translation on the quotient space $G/U$, hence also on the ring of regular functions $\mathcal{R}(G/U)$. Let $\widehat{A}^+$ denote the lattice of dominant characters of $A$, and if $\lambda \in \widehat{A}^+$, let $V_\lambda$ denote the irreducible $G$-module with highest weight $\lambda$. The theorem of the highest weight tells us that $\dim V_\lambda^U = 1$, so by Frobenius reciprocity, $\mathcal{R}(G/U)$ has the following decomposition as a $G$-module:

$$\mathcal{R}(G/U) \simeq \sum_{\lambda \in \widehat{A}^+} V_\lambda. \tag{3.2.1.1}$$

The decomposition (3.2.1.1) also describes the decomposition of $\mathcal{R}(G/U)$ into eigenspaces for the action of the torus $A$ by *right* translation, and because of this, $\mathcal{R}(G/U)$ has an $\widehat{A}^+$-grading

$$V_\lambda \cdot V_\mu \subseteq V_{\lambda+\mu}. \tag{3.2.1.2}$$

Since $G/U$ is an irreducible variety, $\mathcal{R}(G/U)$ is an integral domain.

**3.2.2.** — In our study we shall need more information about the nature of multiplication inside the ring $\mathcal{R}(\mathrm{Sp}_4/U)$.

PROPOSITION. — *If $G$ is a connected, simply connected group over $\mathbb{C}$, then $\mathcal{R}(G/U)$ is a unique factorization domain.*

The proof of this propostion is not difficult: the fact that $\mathcal{R}(G)$ is a unique factorization domain when $G$ is simply connected is basically a consequence of the Bruhat decomposition; then as $\mathcal{R}(G/U)$ consists of those functions on $G$ which are right-invariant under translation by $U$, the facts that $U$ has no nontrivial multiplicative characters and is connected ensure that this property passes to $\mathcal{R}(G/U)$. For details, see [H-Per], [KKLV].

**3.2.3.** — The observant reader may have looked at the title of the previous section and wondered: why a *refinement* of an old result? This is so because $\mathcal{R}(\mathrm{Sp}_4/U)$ may be realized as a quotient of $\mathcal{P}(\mathbb{C}^4 \oplus \mathbb{C}^5)$... The fact is that each bidegree homogeneous component $\mathcal{P}^{(m,n)}(\mathbb{C}^4 \oplus \mathbb{C}^5)$ decomposes as an $\mathrm{Sp}_4$-module as

$$\mathcal{P}^{(m,n)}(\mathbb{C}^4 \oplus \mathbb{C}^5) \simeq \sigma_4^{(m+n,n)} \oplus \text{ lower order terms}, \tag{3.2.3.1}$$

where the order is the usual one on the set of dominant weights. Projection onto these highest $\mathrm{Sp}_4$-modules, the so-called Cartan component, yields a surjective $\mathrm{Sp}_4$-equivariant map

$$\mathcal{P}(\mathbb{C}^4 \oplus \mathbb{C}^5) \longrightarrow \sum_{a \geq b \geq 0} \sigma_4^{(a,b)}. \tag{3.2.3.2}$$

This projection is of course an algebra map, and as its image is $\mathbb{Z}_+^2$-graded, we get an equivariant graded algebra isomorphism of the image with $\mathcal{R}(\mathrm{Sp}_4/U)$. In order to avoid possible confusion, it is important to note that the gradings on $\mathcal{P}(\mathbb{C}^4 \oplus \mathbb{C}^5)$ by bidegree and on $\mathcal{R}(\mathrm{Sp}_4/U)$ by $\widehat{A}^+$ are precisely related by (3.2.3.1).

We may even be more precise about the map (3.2.3.2). Let $\mathcal{I}$ denote its kernel. It is a consequence of a theorem of M. Brion [Br] that as an ($\mathrm{Sp}_4$-invariant) ideal, $\mathcal{I}$ is generated by quadratic elements. Explicitly, these are the invariant form sitting inside $\mathcal{P}^{(0,2)}(\mathbb{C}^4 \oplus \mathbb{C}^5)$ and by the copy of $\mathbb{C}^4$ lying in $\mathcal{P}^{(1,1)}(\mathbb{C}^4 \oplus \mathbb{C}^5)$. From these remarks we see that we have a rather good description of $\mathcal{R}(\mathrm{Sp}_4/U)$.

*Remark.* — It would be both interesting and satisfying to give a description of a generating set of $\mathcal{I}^N$, as we have explicit descriptions of $\mathcal{P}(\mathbb{C}^4 \oplus \mathbb{C}^5)^N$ and $\mathcal{R}(\mathrm{Sp}_4/U)^N$. In order to do so, we would need to use our description of the generating system of the $\mathcal{Z}$-invariants, but there is a difficulty: understanding how the images of the $\mathcal{Z}$-invariants lie with respect to the $\mathrm{Sp}_4$-structure in $\mathcal{P}(\mathbb{C}^4 \oplus \mathbb{C}^5)$. In fact, it is not at all evident how to determine this, since it is not even clear how to give a formula for the map (1.1.2.1), and moreover, a given $\mathcal{Z}$-invariant may have projections both on and off the Cartan component in $\mathcal{P}(\mathbb{C}^4 \oplus \mathbb{C}^5)$... To get around this problem, we would need to give a more structured description of the $\mathcal{Z}$-invariants, a task which is beyond the scope of the present study...

### 3.3. The invariants.

**3.3.1.** — Our first step in obtaining a system of generators for $\mathcal{R}(N \backslash \mathrm{Sp}_4/U)$ will be to describe the subalgebra $\mathcal{R}(SL_2 \backslash \mathrm{Sp}_4/U)$ of $\mathrm{SL}_2$-invariants.

**3.3.2.** — Let $r^2 \in \mathcal{P}^2((\mathbb{C}^4)^2)$ denote the symplectic form defining $\mathrm{Sp}_4$ and denote by $\Delta$ its associated Laplacian. Classical invariant theory tells us that $r^2$ is the only $\mathrm{Sp}_4$-invariant polynomial in $\mathcal{P}((\mathbb{C}^4)^2)$, and if $\mathcal{H}((\mathbb{C}^4)^2)$ denotes the subspace annihilated by $\Delta$, the harmonics, then we may write

$$\mathcal{P}\big((\mathbb{C}^4)^2\big) \simeq \mathcal{H}\big((\mathbb{C}^4)^2\big) \otimes \mathcal{P}(r^2), \tag{3.3.2.1}$$

where $\mathcal{P}(r^2)$ consists of all polynomials in $r^2$. Moreover, the natural action of $\mathrm{GL}_2$ on $\mathcal{P}((\mathbb{C}^4)^2)$ commutes with that of $\mathrm{Sp}_4$, and this action preserves the space of harmonics. Hence there is a description of $\mathcal{H}((\mathbb{C}^4)^2)$ as an $\mathrm{Sp}_4 \times \mathrm{GL}_2$ module:

$$\mathcal{H}((\mathbb{C}^4)^2) \simeq \sum_{a \geq b \geq 0} \sigma_4^{(a,b)} \otimes \rho_2^{(a,b)} \tag{3.3.2.2}$$

where $\rho_2^{(a,b)}$ denotes the irreducible representation of $\mathrm{GL}_2$ with highest weight $(a, b)$, $a \geq b \geq 0$. Note that in this parametrization, the standard representation of $\mathrm{GL}_2$ has weight $(1, 0)$, while the character determinant has weight $(1, 1)$.

The decomposition 3.3.2.2 implies that we may regard $\sigma_4^{(a,b)}$ as consisting of $\mathrm{GL}_2$ highest weight vectors, and in fact, the subspace of harmonic $\mathrm{GL}_2$ highest weight vectors is a model for the representations of $\mathrm{Sp}_4$:

$$\mathcal{H}((\mathbb{C}^4)^2)^{U_2} \simeq \sum_{a \geq b \geq 0} \sigma_4^{(a,b)} \tag{3.3.2.3}$$

as $\mathrm{Sp}_4$ modules, where $U_2$ denotes a maximal unipotent subgroup of $\mathrm{GL}_2$.

The space $\mathcal{H}((\mathbb{C}^4)^2)$ has a natural algebra structure. This product may be explicitly described as the usual multiplication of polynomials followed by projection to the subspace of harmonics. This may also be seen by a perhaps more familiar description of the harmonics as

$$\mathcal{H}((\mathbb{C}^4)^2) \simeq \mathcal{P}((\mathbb{C}^4)^2)/(r^2), \tag{3.3.2.4}$$

where $(r^2)$ denotes the ideal generated by $r^2$. In terms of this identification it is easily seen that the multiplication just described is the same as the usual multiplication on the quotient. Moreover, as this ideal is prime, the multiplication in the space of harmonics is non-degenerate. For further details, the reader is referred to [H-Per], [H-Rem].

The subspace $\mathcal{H}((\mathbb{C}^4)^2)^{U_2}$ of harmonic $\mathrm{GL}_2$ highest weight vectors then clearly becomes a subalgebra, and this subalgebra is of course isomorphic to $\mathcal{R}(\mathrm{Sp}_4 / U)$. Therefore to calculate the $\mathrm{SL}_2$-invariants in $\mathcal{R}(\mathrm{Sp}_4 / U)$, it is the same to calculate the $\mathrm{GL}_2$-covariants inside $\mathcal{H}((\mathbb{C}^4)^2)^{SL_2}$.

**3.3.3.** — To determine the structure of $\mathcal{R}(SL_2 \backslash \mathrm{Sp}_4 / U)$ we first recall an old result of Salmon's [S] on the invariant theory of a system of two cubics. For a modern derivation, see [Sch].

PROPOSITION. — *The algebra* $\mathcal{P}((\mathbb{C}^4)^2)^{\mathrm{SL}_2}$ *is generated by the* $\mathrm{GL}_2$-*modules* $\rho_2^{(1,1)}$, $\rho_2^{(4,0)}$ *and* $\rho_2^{(3,3)}$, *degrees* 2, 4 *and* 6 *respectively.*

It is clear that $\rho_2^{(1,1)}$ is just $r^2$, and that the $\mathrm{GL}_2$-highest weight vector in $\rho_2^{(4,0)}$ is just $\delta \in \mathcal{P}^4(\mathbb{C}^4)^{\mathrm{SL}_2}$ (as in 1.2.1). Let $R$ denote the $\mathrm{GL}_2$ highest weight vector with weight (3,3); since $\mu(0,3,3) = 1$, we see that $R$ has nonzero projection onto $\mathcal{H}((\mathbb{C}^4)^2)$ and we denote its harmonic projection also by $R$.

This reduces our problem to finding the $\mathrm{GL}_2$-covariants in the polynomial algebra generated by the subspaces $\rho_2^{(4,0)} \simeq \mathbb{C}^5$ and $\rho_2^{(3,3)} \simeq \mathbb{C}$. Since $\rho_2^{(3,3)}$ is one-dimensional, we really only need to look for the covariants of $\mathrm{GL}_2$ in $\mathcal{P}(\mathbb{C}^5)$ — the covariants of the quartic — which is something we understand well. Let $\delta = \delta_1, \dots, \delta_5$ be an eigenbasis of $\mathbb{C}^5$ with respect to the standard torus of $\mathrm{GL}_2$. The weights of these vectors are, respectively, (4,0), (3,1), (2,2), (1,3) and (0,4). Let $a(\delta) = \delta$, $b(\delta)$, $c(\delta)$, $d(\delta)$, $e(\delta)$ be the generators of the covariants of the quartic as in (1.2.2). These have $\mathrm{GL}_2$-weight (4,0), (6,2), (4,4), (9,3) and (6,6) respectively. Since $\mu(0,4,4) = 0$, we must have $c(\delta) \equiv 0 \pmod{r^2}$ and as $\mu(0,6,6) = 1$ it follows that $e(\delta) \equiv R^2 \pmod{r^2}$. Hence the relation of the harmonic quartic (1.2.3.1) becomes

$$\delta^3 R^2 + b^3(\delta) + d^2(\delta) = 0. \tag{3.3.3.1}$$

To summarize:

LEMMA. — *One has*

$$\mathcal{R}(SL_2 \setminus \mathrm{Sp}_4 / U) \simeq \mathbb{C}\big[\delta, R, b(\delta), d(\delta)\big] / \big(\delta^3 R^2 + b^3(\delta) + d^2(\delta)\big)$$

*and these generators are specified by their bidegrees* $(4,0)$, $(3,3)$, $(6,2)$ *and* $(9,3)$.

*Remarks.*

1) It is interesting to note that the presentation of this algebra is very similar to both those of the covariants of the cubic 1.2.1 and, of course, of the harmonic quartic 1.2.3.

2) The calculation of the algebra $\mathcal{R}(SL_2 \setminus \mathrm{Sp}_4 / U)$ is not new. It may be found in the thesis of E. Ioannidis (University of Grenoble, 1986, unpublished), a student of D. Luna, and has also been computed by F. Knop. As far as we know, however, our derivation is new.

## 3.4. The structure of certain modules.

**3.4.1.** — Let $\mathbb{V}_k \subseteq \mathcal{R}(\mathrm{Sp}_4 /U)$ denote the subspace of all $\mathrm{SL}_2$ highest weight vectors of weight $k$; then $\mathbb{V}_0 = \mathcal{R}(\mathrm{SL}_2 \setminus \mathrm{Sp}_4 /U)$ is the subalgebra of invariants. Multiplication of a covariant of weight $k$ by an invariant is again a covariant of weight $k$:

$$\mathbb{V}_0 \cdot \mathbb{V}_k = \mathbb{V}_k. \tag{3.4.1.1}$$

Hence $\mathbb{V}_k$ is a module over $\mathbb{V}_0$, for each $k$.

The calculation of 1.3 tells us that the highest weight possible for a generator of $\mathcal{R}(N \setminus \mathrm{Sp}_4 /U)$ is 7. It is then clear that a system of $\mathbb{V}_0$-module generators for the modules $\mathbb{V}_k$, $k = 0, \ldots, 7$, will contain a set of generators for $\mathcal{R}(N \setminus \mathrm{Sp}_4 /U)$.

**3.4.2.** — It is easier to work with polynomial rings, so let $\mathcal{A} \subseteq \mathbb{V}_0$ be the polynomial subalgebra generated by (0,4,0), (0,3,3) and (0,6,2). Since $(0,9,3) = d(\delta)$ is quadratic over $\mathcal{A}$ by (3.3.3.1), we have

$$\mathbb{V}_0 = \mathcal{A} \oplus \mathcal{A}\, d(\delta). \tag{3.4.2.1}$$

Instead of studying the $\mathbb{V}_k$ as modules over $\mathbb{V}_0$ we shall study them as modules over $\mathcal{A}$. Note that the polynomials (0,4,0), (0,3,3) and (0,6,2) are all primes by 3.2.3.

**3.4.3.** — Multiplication by (0,4,0) realizes $\sigma_4^{(m-4,n)}$ as an $\mathrm{SL}_2$-submodule of $\sigma_4^{(m,n)}$. We also have

$$(0,3,3)\sigma_4^{(m-3,n-3)} \subseteq \sigma_4^{(m,n)}$$

and it is clear that

$$(0,4,0)\sigma_4^{(m-4,n)} \cap (0,3,3)\sigma_4^{(m-3,n-3)} = (0,4,0)(0,3,3)\sigma_4^{(m-7,m-3)}.$$

Therefore, we may think of the function

$$\begin{aligned}\kappa(k,m,n) = \mu(k,m,n) &- \mu(k,m-4,n) \\ &- \mu(k,m-3,n-3) + \mu(k,m-7,n-3)\end{aligned} \tag{3.4.3.1}$$

as measuring the dimension of the complement of

$$(0,4,0)\sigma_4^{(m-4,n)} + (0,3,3)\sigma_4^{(m-3,n-3)} \subseteq \sigma_4^{(m,n)}$$

inside $\mathbb{V}_k$.

**3.4.4.** — We study the functions $\kappa(k,m,n)$ for $k=1,\ldots,7$ to propose sets of generators for the $\mathcal{A}$-modules $\mathbb{V}_k$. Although these numerics are not quite enough to guarantee that the systems we obtain are generating sets, using a little more structure we will be able to show that in certain cases they indeed are.

The tables below indicate the values of $\kappa(k,m,n)$ for the various $k$. We include only the calculation for $k=1$; the computations for the other cases are extremely similar.

First consider the region $m \geq 3n+1$, and put $m = 3n+2j+1$. Then

$$\begin{aligned}\mu(1,m,n) &= H(1,m,n)\\ &= p_{23}(5n+3j+1)+\cdots+p_{23}(4n+3j+1)\\ &\quad-(p_{23}(3n+j-1)+\cdots+p_{23}(j-3))\end{aligned} \tag{3.4.4.1}$$

and

$$\begin{aligned}H(1,m-4,n) &= p_{23}(5n+3j-5)+\cdots+p_{23}(4n+3j-5)\\ &\quad-\big(p_{23}(3n+j-3)+\cdots+p_{23}(j-5)\big).\end{aligned} \tag{3.4.4.2}$$

Subtracting 3.4.4.2 from 3.4.4.1, we get, using the identity $p_{23}(2x) = \left[\frac{1}{3}x\right]$,

$$\begin{aligned}&H(1,m,n)-H(1,m-4,n)\\ &\qquad= (n+1)-\big(p_{23}(6n+2j-2)-p_{23}(2j-8)\big)\\ &\qquad= \begin{cases}1 & \text{if } j=0,\\ 0 & \text{if } j\geq 1.\end{cases}\end{aligned} \tag{3.4.4.3}$$

This tells us that

$$\kappa(1,3n+2j+1,n) = \begin{cases}1 & \text{if } j=0,\\ 0 & \text{if } j\geq 1.\end{cases} \tag{3.4.4.5}$$

Similarly, if we look now at the region $m \leq 3n-1$, and set $m = 3n-2j-1$,

$$\begin{aligned}V(1,m,n) &= p_{23}(5n-3j-2)+\cdots+p_{23}(3n-j-1)\\ &\quad-\big(p_{23}(4n-3j-3)+\cdots+p_{23}(j-2)\big)\end{aligned} \tag{3.4.4.6}$$

and

$$\begin{aligned}V(1,m-3,n-3) &= p_{23}(5n-3j-8)+\cdots+p_{23}(3n-j-7)\\ &\quad-\big(p_{23}(4n-3j-6)+\cdots+p_{23}(j-5)\big).\end{aligned} \tag{3.4.4.7}$$

Now using the identity $p_{23}(3x) = \left[\frac{1}{2}x\right]$, we obtain that

$$V(1,m,n) - V(1,m-3,n-3) = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j \geq 1 \end{cases} \tag{3.4.4.8}$$

and so

$$\kappa(1,3n-2j-1,n) = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j \geq 1. \end{cases} \tag{3.4.4.9}$$

The calculations for $k = 2, \ldots, 7$ are completely analogous and so are omitted. We tabulate the results below. For each fixed $k$ from 0 to 7, these tables indicate the value $\kappa(k,m,n)$. We have $\kappa(k,m,n) = 0$ if the position $(m,n)$ is blank.

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | | | | | | | | | | | | | | | | |
| 8 | | | | | | | | | | | | | | | | |
| 7 | | | | | | | | | | | | | | | | |
| 6 | | | | | | | | | | | | | | | | |
| 5 | | | | | | | | | | | | | | | | |
| 4 | | | | | | | | | | | | | 1 | | | |
| 3 | | | | | | | | | | 1 | | | | | | |
| 2 | | | | | | | 1 | | | | | | | | | |
| 1 | | | | 0 | | | | | | | | | | | | |
| 0 | 1 | | | | | | | | | | | | | | | |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | $m$ |

$k = 0$

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | | | | | | | | | | | | | | | | |
| 8 | | | | | | | | | | | | | | | | |
| 7 | | | | | | | | | | | | | | | | |
| 6 | | | | | | | | | | | | | | | | |
| 5 | | | | | | | | | | | | | | | 1 | |
| 4 | | | | | | | | | | | | 1 | | 1 | | |
| 3 | | | | | | | | | 1 | | 1 | | | | | |
| 2 | | | | | | 1 | | 1 | | | | | | | | |
| 1 | | | 1 | | 1 | | | | | | | | | | | |
| 0 | | 0 | | | | | | | | | | | | | | |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | $m$ |

$k = 1$

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | | | | | | | | | | | | | | | | |
| 8 | | | | | | | | | | | | | | | | |
| 7 | | | | | | | | | | | | | | | | |
| 6 | | | | | | | | | | | | | | | | |
| 5 | | | | | | | | | | | | | | 1 | | |
| 4 | | | | | | | | | | | 1 | | 1 | | 1 | |
| 3 | | | | | | | | 1 | | 1 | | 1 | | | | |
| 2 | | | | | 1 | | 1 | | 1 | | | | | | | |
| 1 | | 0 | | 1 | | 1 | | | | | | | | | | |
| 0 | 0 | | 1 | | | | | | | | | | | | | |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | $m$ |

$k = 2$

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | | | | | | | | | | | | | | | | |
| 8 | | | | | | | | | | | | | | | | |
| 7 | | | | | | | | | | | | | | | | |
| 6 | | | | | | | | | | | | | | | | |
| 5 | | | | | | | | | | | | | 1 | | 1 | |
| 4 | | | | | | | | | | 1 | | 1 | | 1 | | |
| 3 | | | | | | | 1 | | 1 | | 1 | | 1 | | | |
| 2 | | | | 1 | | 1 | | 1 | | 1 | | | | | | |
| 1 | | | 0 | | 1 | | 1 | | | | | | | | | |
| 0 | | 1 | | 1 | | | | | | | | | | | | |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | $m$ |

$k = 3$

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | | | | | | | | | | | | | | | | |
| 8 | | | | | | | | | | | | | | | | |
| 7 | | | | | | | | | | | | | | | | |
| 6 | | | | | | | | | | | | | | | 1 | |
| 5 | | | | | | | | | | | | 1 | | 1 | | |
| 4 | | | | | | | | | 1 | | 1 | | 1 | | 1 | |
| 3 | | | | | | 1 | | 1 | | 1 | | 1 | | 1 | | |
| 2 | | | 1 | | 1 | | 1 | | 1 | | 1 | | | | | |
| 1 | | 1 | | 1 | | 1 | | 1 | | | | | | | | |
| 0 | 0 | | 0 | | 1 | | | | | | | | | | | |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | $m$ |

$k = 4$

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | | | | | | | | | | | | | | | | |
| 8 | | | | | | | | | | | | | | | | |
| 7 | | | | | | | | | | | | | | | | |
| 6 | | | | | | | | | | | | | | 1 | | |
| 5 | | | | | | | | | | | 1 | | 1 | | 1 | |
| 4 | | | | | | | | 1 | | 1 | | 1 | | 1 | | |
| 3 | | | | | 1 | | 1 | | 1 | | 1 | | 1 | | 1 | |
| 2 | | | | 1 | | 2 | | 1 | | 1 | | 1 | | | | |
| 1 | | | 1 | | 1 | | 1 | | 1 | | | | | | | |
| 0 | | 0 | | 1 | | 1 | | | | | | | | | | |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | $m$ |

$k = 5$

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | | | | | | | | | | | | | | | | |
| 8 | | | | | | | | | | | | | | | | |
| 7 | | | | | | | | | | | | | | | | |
| 6 | | | | | | | | | | | | | 1 | | 1 | |
| 5 | | | | | | | | | | 1 | | 1 | | 1 | | |
| 4 | | | | | | | 1 | | 1 | | 1 | | 1 | | 1 | |
| 3 | | | | 1 | | 1 | | 1 | | 1 | | 1 | | 1 | | |
| 2 | | | 0 | | 2 | | 2 | | 1 | | 1 | | 1 | | | |
| 1 | | 0 | | 1 | | 1 | | 1 | | 1 | | | | | | |
| 0 | 0 | | 1 | | 1 | | 1 | | | | | | | | | |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | $m$ |

$k = 6$

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | | | | | | | | | | | | | | | | |
| 8 | | | | | | | | | | | | | | | | |
| 7 | | | | | | | | | | | | | | | 1 | |
| 6 | | | | | | | | | | | | 1 | | 1 | | |
| 5 | | | | | | | | | 1 | | 1 | | 1 | | 1 | |
| 4 | | | | | | 1 | | 1 | | 1 | | 1 | | 1 | | |
| 3 | | | | | 1 | | 2 | | 1 | | 1 | | 1 | | 1 | |
| 2 | | | | 1 | | 2 | | 2 | | 1 | | 1 | | 1 | | |
| 1 | | | 1 | | 2 | | 1 | | 1 | | 1 | | | | | |
| 0 | | 0 | | 0 | | 1 | | 1 | | | | | | | | |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | $m$ |

$k = 7$

Since multiplication by (0,6,2) translates a covariant of weight $k$ in $\sigma_4^{(m,n)}$ to one of the same weight in $\sigma_4^{(m+6,n+2)}$, it is easy to propose a system of $\mathcal{A}$-module generators in each case.

**3.4.5.** — If $H(k,m,n) - H(k,m-4,n) = 1$, since (0,4,0) is not a zero-divisor, then the complement of the weight $k$ covariants in $\sigma_4^{(m,n)}$ to those in $(0,4,0)\sigma_4^{(m-4,n)}$ has dimension 1. Hence, if the subspace of weight $k$ covariants not divisible by (0,4,0) in $\sigma_4^{(m-6,n-2)}$ is nonzero, $(0,6,2)\sigma_4^{(m-6,n-2)}$ must span the complement of $(0,4,0)\sigma_4^{(m-4,n)} \subseteq \sigma_4^{(m,n)}$ inside $\mathbb{V}_k$. The analogue holds for the complement of $(0,3,3)\sigma_4^{(m-3,n-3)} \subseteq \sigma_4^{(m,n)}$ inside $\mathbb{V}_k$ in case $V(k,m,n) - V(k,m-3,n-3) = 1$ and the subspace of weight $k$ covariants in $\sigma_4^{(m-6,n-2)}$ not divisible by (0,3,3) is not identically zero. This implies that if we are looking for additional generators for $V_k$ as an $\mathcal{A}$-module, we may disregard components $(k,m,n)$ satisfying the above conditions.

We summarize for each $k = 1,\ldots,7$, for which $(m,n)$ we have

$$H(k,m,n) - H(k,m-4,n) = 1, \tag{3.4.5.1}$$

$$V(k,m,n) - V(k,m-3,n-3) = 1, \tag{3.4.5.2}$$

respectively. The proofs of these identities are tedious calculations of the kind encountered in 3.4.4.

Equality (3.4.5.1) holds in the following cases:

| $k =$ | $m =$ | $n \geq$ |
|---|---|---|
| 1 | $3n+1$ | 1 |
| 2 | $3n+2$ | 0 |
| | $3n$ | 1 |
| 3 | $3n+3$ | 0 |
| | $3n+1$ | 0 |
| 4 | $3n+4$ | 0 |
| | $3n+2$ | 1 |
| | $3n$ | 1 |

| $k =$ | $m =$ | $n \geq$ |
|---|---|---|
| 5 | $3n+5$ | 0 |
| | $3n+3$ | 0 |
| | $3n+1$ | 1 |
| 6 | $3n+6$ | 0 |
| | $3n+4$ | 0 |
| | $3n+2$ | 1 |
| 7 | $3n+7$ | 0 |
| | $3n+5$ | 0 |
| | $3n+3$ | 1 |

Equality (3.4.5.2) is valid for the following values of $k, m, n$:

<table>
<tr><td>$k =$</td><td>$m =$</td><td>$n \geq$</td></tr>
<tr><td>1</td><td>$3n-1$</td><td>1</td></tr>
<tr><td>2</td><td>$3n-2$</td><td>2</td></tr>
<tr><td rowspan="2">3</td><td>$3n-1$</td><td>2</td></tr>
<tr><td>$3n-3$</td><td>2</td></tr>
<tr><td rowspan="2">4</td><td>$3n-2$</td><td>1</td></tr>
<tr><td>$3n-4$</td><td>2</td></tr>
</table>

<table>
<tr><td>$k =$</td><td>$m =$</td><td>$n \geq$</td></tr>
<tr><td rowspan="2">5</td><td>$3n-3$</td><td>2</td></tr>
<tr><td>$3n-5$</td><td>3</td></tr>
<tr><td rowspan="2">6</td><td>$3n-4$</td><td>3</td></tr>
<tr><td>$3n-6$</td><td>3</td></tr>
<tr><td rowspan="2">7</td><td>$3n-5$</td><td>3</td></tr>
<tr><td>$3n-5$</td><td>4</td></tr>
</table>

**3.4.6.** — It is not hard to see that in the regions in which $\kappa = 0$, multiplication by $(0,4,0)$ or by $(0,3,3)$ (appropriately) is in fact surjective. This means that if there were additional $\mathcal{A}$-module generators, they would be situated in the region in which $\kappa > 0$.

For $k = 1, \ldots, 4$, the argument of 3.4.5 clearly eliminates the possibility of any additional generators. Hence these are all free $\mathcal{A}$-modules. For completeness, we list their generators:

<table>
<tr><td>$k$</td><td>$(m, n)$</td></tr>
<tr><td>1</td><td>$(2,1), (4,1), (5,2), (7,2)$</td></tr>
<tr><td>2</td><td>$(2,0), (3,1), (5,1), (4,2), (6,2), (7,3)$</td></tr>
<tr><td>3</td><td>$(1,0), (3,0), (4,1), (6,1)$<br>$(3,2), (5,2), (6,3), (8,3)$</td></tr>
<tr><td>4</td><td>$(4,0), (1,1), (3,1), (5,1), (7,1)$<br>$(2,2), (4,2), (6,2), (8,2), (5,3)$</td></tr>
</table>

**3.4.7.** — Unfortunately, the argument of 3.4.5 does not apply uniformly in the cases $k = 5, 6, 7$. This is because (3.4.5.1) and (3.4.5.2) do not hold in the whole region $\kappa > 0$ for such $k$. More specifically, there are triply homogeneous components in which the images both of multiplication by (0,4,0) and by (0,3,3) have codimension 2, and so 3.4.5 does not apply ... As we may see from the tables in 3.4.5, in the case $k = 5$, these components lie on the line $m = 3n - 1$, for $k = 6$, on $m = 3n + 1, 3n - 1$ and $3n - 3$.

Ultimately, however, we are interested in algebra generators for $\mathcal{R}(N\setminus \mathrm{Sp}_4/U)$, not for $\mathcal{A}$-module generators, so we consider the tables of 1.3 to see where algebra generators may be situated. We remind the reader that since we are considering $\sigma_4^{(m,n)} \subseteq \mathcal{R}(\mathrm{Sp}_4/U)$, in terms of $\mathcal{P}(\mathbb{C}^4 \oplus \mathbb{C}^5)$, we may think of this as the $\mathrm{Sp}_4$-module with highest highest weight in the homogeneous component of bidegree $(m-n,n)$.

Since the argument of 3.4.5 eliminates potential additional $\mathcal{A}$-module generators (and therefore potential additional algebra generators), applying it to the relevant cases for $k=5$ leaves us with one remaining case: that in which $(m,n)=(8,3)$.

To get rid of this case, note that the image of the weight 4 covariants in $\sigma_4^{(6,2)}$ under multiplication by (1,2,1) has dimension 2 in the subspace of weight 5 covariants in $\sigma_4^{(8,3)}$. Since the weight 5 covariant in $\sigma_4^{(5,0)}$ is not divisible by (1,2,1), its image under multiplication by (0,3,3) in $\sigma_4^{(8,3)}$ is not divisible by (1,2,1) either, so unique factorization implies that these 3 polynomials are linearly independent. Thus, to locate algebra generators of weight 5, we only need to look for them in the proposed set of $\mathcal{A}$-module generators for $\mathbb{V}_5$.

For the cases $k=6,7$, comparison with the appropriate tables in 1.3 tells us that we only need to look in the proposed systems of $\mathcal{A}$-module generators. These are listed below for $k=5,6,7$.

| $k$ | $(m,n)$ |
|---|---|
| 5 | $(3,0),(5,0),(2,1),(4,1),(6,1),(8,1),$ $(3,2),(5,2)_1,(5,2)_2,(7,2),$ $(4,3),(6,3),(7,4)$ |
| 6 | $(2,0)(4,0),(6,0),(3,1),(5,1),(7,1),$ $(4,2),(4,2)_2,(6,2)_1,(6,2)_2,$ $(3,3),(5,3),(7,3),(6,4),(8,4)$ |
| 7 | $(5,0),(7,0),(2,1),(4,1)_1,(4,1)_2,(6,1),$ $(8,1),(10,1),(3,2),(5,2)_1,(5,2)_2,$ $(7,2)_1,(7,2)_2,(9,2),(4,3),(6,3)_1,(6,3)_2$ $(5,4),(7,4),(8,5)$ |

**3.4.8.** — Finally we are in a position to prove that the algebra $\mathcal{R}(N \setminus \mathrm{Sp}_4 /U)$ is generated as stated in 3.1.1.

Firstly, note that the generators of $\mathcal{A}$, (0,9,3), and the generators of $\mathbb{V}_1$ over $\mathcal{A}$ must all be contained in a generating set for $\mathcal{R}(N \setminus \mathrm{Sp}_4 /U)$. To reduce the above lists for $k \geq 2$, we must determine which covariants can be expressed as products of lower weight covariants. If $\mu(k,m,n) = 1$, then this is a straightforward matter since $\mathcal{R}(\mathrm{Sp}_4 /U)$ is an integral domain. Comparing the lists of 3.4.7 with those of §1.3, we see that the only cases which are more complicated have $\mu(k,m,n) = 2$, and these are

$$(3.4.8.1) \qquad (k,m,n) = \begin{cases} (3,6,3),(3,8,3), \\ (5,5,2),(5,6,3), \\ (6,5,3), \\ (7,6,3). \end{cases}$$

But these cases are also easily dealt with. Using the fact that $\mathcal{R}(\mathrm{Sp}_4 /U)$ is a unique factorization domain, one can easily show linear independence of the appropriate polynomials. For example, consider (7,6,3): unique factorization implies that any two of (3,1,0)(3,3,2)(1,2,1), $(1,2,1)^2(5,2,1)$ and (1,2,1)(2,3,1)(4,1,1) are linearly independent. The other cases are just as simple.

**3.4.9.** — So far we have avoided saying anything about the relations between these generators, but it is clear that the numerics and the structural properties of $\mathcal{R}(\mathrm{Sp}_4 /U)$ will yield information about these. There are three obvious sources for the relations. The first comes from the $\mathcal{A}$-module structures of the modules $\mathbb{V}_k$. Although we have not determined these structures explicitly for $k = 5,6,7$, it is clear that these modules cannot be free over $\mathcal{A}$ — looking at the tables in 3.4.4, we see that multiplication by (0,6,2) takes the components with $\kappa = 2$ to components in which $\kappa = 1$, so relations must occur. The second source of relations is the structure of the modules $\mathbb{V}_k$ over the full algebra of invariants $\mathbb{V}_0$. Here it is clear that none of the modules $\mathbb{V}_k$ is free for $k \geq 1$; in fact, multiplication of any covariant generator by (0,9,3) implies a relation. Finally, taking into consideration products of covariants of nonzero weights will also say something concerning the relations. For example, consider the homogeneous component (3,7,2); this has dimension 2 and it is easy to see that it is spanned by (0,4,0)(3,3,2) and (0,6,2)(3,1,0). However, (2,2,0)(1,5,2) has the same grading and so must be expressible as a linear combination of the previous two products... We do expect that most, if not all, the relations can be determined in these manners.

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