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HANKEL DETERMINANTS OF THE THUE-MORSE SEQUENCE

by J.-P. ALLOUCHE, J. PEYRIÈRE, Z.-X. WEN (*), Z.-Y. WEN (**)

0. Introduction.

Let $S = \{a, b\}$ be a two-letter alphabet and S^* the free monoid generated by S. Consider the endomorphism θ defined on S^* by

$$\theta: \ a \mapsto ab, \ \ b \mapsto ba.$$

Since the word $\theta^n(a)$ is the left half of the word $\theta^{n+1}(a)$, it has a limit as n goes to infinity: the infinite sequence $\epsilon = \epsilon_0\epsilon_1 \cdots \epsilon_n \cdots \in \{a, b\}^{\mathbb{N}}$ which is called the Thue-Morse (or sometimes the Prouhet-Thue-Morse) sequence.

In this article, except in Section 4, we take $a = 1, \ b = 0$. Then the sequence ϵ satisfies the following relations: $\epsilon_0 = 1, \ \epsilon_{2n} = \epsilon_n, \ \epsilon_{2n+1} = 1-\epsilon_n$.

The study of the Thue-Morse sequence has been initiated by Thue (1906, [14]; 1912, [15]), who proved that it does not contain three consecutive identical blocks. A few years later, Morse (1921, [10]) studied the topological dynamical system generated by this sequence, and Gottschalk (1963, [9]) studied this sequence in the framework of minimal sets. In the last ten years, it occurred in many different fields of mathematics — ergodic theory, finite automata theory, formal language theory, number theory, algebraic

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formal power series over $\mathrm{GF}_2(X)$ — and also in physics in relation to quasicrystals (see, for instance, [1], [7], [8], [11], [16], [17], [18]).

In this article, we discuss some new properties of the Thue-Morse sequence.

Let $u = (u_k)_{k\geq 0}$ be a sequence of complex numbers; then the (p,n)-order Hankel matrix associated with the sequence u is defined to be

$$H_n^p = \begin{pmatrix} u_p & u_{p+1} & \cdots & u_{p+n-1} \\ u_{p+1} & u_{p+2} & \cdots & u_{p+n} \\ \cdots & \cdots & \cdots & \cdots \\ u_{p+n-1} & u_{p+n} & \cdots & u_{p+2n-2} \end{pmatrix},$$

where $n \geq 1$ and $p \geq 0$. The determinant of this matrix, denoted by $|H_n^p|$, is called the (p,n)-order Hankel determinant of the sequence u. The properties of Hankel determinants associated with a sequence are closely connected to the study of the moment problem, to Padé approximation, and to combinatorial properties of the sequence.

Here we consider $\mathcal{E}_n^p$, the (p,n)-order Hankel matrix of the Thue-Morse sequence. We denote by $|\mathcal{E}_n^p|$ its (p,n)-order Hankel determinant. Our purpose is to study the properties of the double sequence $(|\mathcal{E}_n^p|)_{n\geq 1, p\geq 0}$. Figure 1 on next page shows $|\mathcal{E}_n^p|$ modulo 2 (0's are replaced by a dot, 1's by nothing) for $1 \leq n \leq 96$ and $0 \leq p \leq 127$.

This article is organized as follows. Definitions and preliminaries are given in Section 1. Section 2 is mainly devoted to establishing recurrence formulae for the sequence modulo 2 of Hankel determinants associated with the Thue-Morse sequence. Automaticity properties of the sequence of these determinants modulo 2 are established in Section 3. Further properties and applications (non-repetition in the Thue-Morse sequence and existence of some Padé approximants) are given in Section 4.

1. Preliminaries.

Let $\epsilon = \epsilon_0\epsilon_1 \cdots \epsilon_n \cdots \in \{0,1\}^{\mathbb{N}}$ be the Thue-Morse sequence, defined by the following recurrence equations:

$$\epsilon_0 = 1, \quad \epsilon_{2n} = \epsilon_n, \quad \epsilon_{2n+1} = 1 - \epsilon_n, \quad \text{for } n \geq 0. \tag{1}$$

As we shall see below, in order to determine the Hankel determinants associated with ϵ, we have to calculate simultaneously those associated with another sequence $\delta = \delta_0\delta_1 \cdots \delta_n \cdots$ which is defined by $\delta_n = \epsilon_{n+1} - \epsilon_n$.

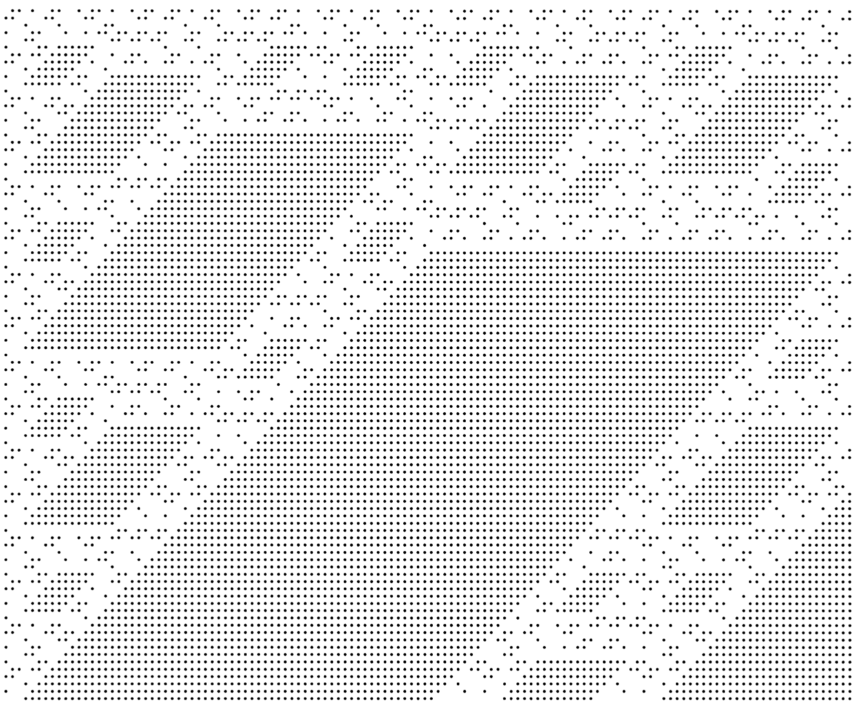

Figure 1. The set $|\mathcal{E}_n^p|$ modulo 2 for $1 \leq n \leq 96$ and $0 \leq p \leq 127$.

By (1) and the definition of δ, we have, for $n \geq 1$,

$$\begin{cases} \delta_{2n} + \delta_{2n+1} = \delta_n, \\ \delta_{2n} = 1 - 2\epsilon_n. \end{cases} \tag{2}$$

Remark 1.1. — The Thue-Morse sequence can be generated by the endomorphism of $\{0,1\}^*$ defined by $1 \mapsto 10$, $0 \mapsto 01$. The above sequence δ reduced modulo 2 is called the period-doubling sequence (some authors also call it the Toeplitz sequence). Like the Thue-Morse sequence, it can be generated by an endomorphism of $\{0,1\}^*$: $1 \mapsto 10$, $0 \mapsto 11$. Furthermore, it can be "induced" from the Thue-Morse sequence in the following way. Define the map $\Phi\colon \{0,1\}^2 \to \{0,1\}$ by $\Phi(00) = \Phi(11) = 0$, $\Phi(01) = \Phi(10) = 1$. Let $(\Psi_k)_{k\geq 0}$ be the sequence defined by $\Psi_k = \Phi(\epsilon_k\epsilon_{k+1})$, where $\epsilon_k\epsilon_{k+1}$ runs through the blocks of two consecutive letters occurring in the Thue-Morse sequence. Then the sequence $(\Psi_k)_{k\geq 0}$ is nothing but the period-doubling sequence.

Notations.

Throughout this paper, we adopt the following definitions and notations:

- The Thue-Morse sequence, the period-doubling sequence, and their corresponding (p,n)-order Hankel matrices (where $p \geq 0$, $n \geq 1$) are denoted respectively by ϵ, δ, $\mathcal{E}_n^p$, and Δ_n^p.

- For a square matrix A, let $|A|$ and A^t stand respectively for its determinant and the transposed matrix.

- $\mathbf{1}_{m,n}$ (resp. $\mathbb{O}_{m,n}$) is the $m \times n$ matrix with all its entries equal to 1 (resp. 0).

- If A is a square matrix of order n, $\overline{A}$ stands for the matrix

$$\begin{pmatrix} A & \mathbf{1}_{n,1} \\ \mathbf{1}_{1,n} & 0 \end{pmatrix},$$

and $A^{(j)}$ for the $n \times (n-1)$ matrix obtained by deleting the j-th column of A.

- The symbol $\equiv$, unless otherwise stated, means equality modulo 2 throughout this article.

- $P_1(n) = (e_1, e_3, \dots, e_{2[\frac{n+1}{2}]-1}, e_2, e_4, \dots, e_{2[\frac{n}{2}]})$, where e_j is the j-th unit column vector of order n, *i.e.*, the column vector with 1 as its j-th entry and zeros elsewhere. If no confusion can occur, we simply write P_1.

- $P_2(2n) = \begin{pmatrix} I_n & I_n \\ \mathbb{O}_{n,n} & I_n \end{pmatrix}$, where I_n denotes the $n \times n$ unit matrix.

- $P_2(2n+1) = \begin{pmatrix} I_n & \mathbb{O}_{n,1} & I_n \\ \mathbb{O}_{1,n} & 1 & \mathbb{O}_{1,n} \\ \mathbb{O}_{n,n} & \mathbb{O}_{n,1} & I_n \end{pmatrix}$.

- $P(n) = P_2(n)P_1(n)$.

We clearly have

$$|P_1| = |P_2| = |P| = \pm 1. \tag{3}$$

Some lemmas on matrices.

LEMMA 1.1. — *Let A and B be two square matrices of respective orders m and n, X a $1 \times m$ matrix, and Y a $1 \times n$ matrix. Then, we have*

$$\begin{vmatrix} A & Y \otimes \mathbf{1}_{m,1} \\ X \otimes \mathbf{1}_{n,1} & B \end{vmatrix} = |A| \cdot |B| - \begin{vmatrix} A & \mathbf{1}_{m,1} \\ X & 0 \end{vmatrix} \cdot \begin{vmatrix} 0 & Y \\ \mathbf{1}_{n,1} & B \end{vmatrix}.$$

Proof. — Set $D = \begin{vmatrix} A & Y\otimes \mathbf{1}_{m,1} \\ X\otimes \mathbf{1}_{n,1} & B \end{vmatrix}$. As a function of A and X, this is a multilinear alternating form of the columns of the matrix $\begin{pmatrix} A \\ X \end{pmatrix}$. Therefore it is of the form $\begin{vmatrix} A & V \\ X & \alpha \end{vmatrix}$, where V is a $m\times 1$ matrix. But, since permuting two rows of A only changes the sign of D, it follows that $V = \beta\mathbf{1}_{m,1}$. Therefore we have

$$D = \alpha|A| + \beta\begin{vmatrix} A & \mathbf{1}_{m,1} \\ X & 0 \end{vmatrix}.$$

By taking $X = 0$, we get $\alpha = |B|$. To show the equality $\beta = -\begin{vmatrix} 0 & Y \\ \mathbf{1}_{n,1} & B \end{vmatrix}$, it suffices to take $A = \begin{pmatrix} \mathbb{O}_{m-1,1} & I_{m-1} \\ 0 & \mathbb{O}_{1,m-1} \end{pmatrix}$ and $X = (1\ 0\ 0\ldots)$. □

As a corollary we have the following lemma.

Lemma 1.2. — *Let A and B be two square matrices of order m and n respectively, and a, b, x and y four numbers. One has*

$$\begin{vmatrix} aA & y\mathbf{1}_{m,n} \\ x\mathbf{1}_{n,m} & bB \end{vmatrix} = a^m b^n|A|\cdot|B| - xya^{m-1}b^{m-1}|\overline{A}|\cdot|\overline{B}|.$$

Lemma 1.3. — *Let A, B, and C be three square matrices of order m, n, and p respectively, and three numbers a, b, and c. One has*

$$\begin{vmatrix} A & c\mathbf{1}_{m,n} & b\mathbf{1}_{m,p} \\ c\mathbf{1}_{m,n} & B & a\mathbf{1}_{n,p} \\ b\mathbf{1}_{m,p} & a\mathbf{1}_{n,p} & C \end{vmatrix} = |A|\cdot|B|\cdot|C| - a^2|A|\cdot|\overline{B}|\cdot|\overline{C}| - b^2|\overline{A}|\cdot|B|\cdot|\overline{C}|$$
$$- c^2|\overline{A}|\cdot|\overline{B}|\cdot|C| - 2abc\,|\overline{A}|\cdot|\overline{B}|\cdot|\overline{C}|.$$

Proof. — We have, provided that $b \neq 0$,

$$\begin{vmatrix} A & c\mathbf{1}_{m,n} & b\mathbf{1}_{m,p} \\ c\mathbf{1}_{m,n} & B & a\mathbf{1}_{n,p} \\ b\mathbf{1}_{m,p} & a\mathbf{1}_{n,p} & C \end{vmatrix} = b^{2p}c^{-2p}\begin{vmatrix} A & c\mathbf{1}_{m,n} & c\mathbf{1}_{m,p} \\ c\mathbf{1}_{m,n} & B & acb^{-1}\mathbf{1}_{n,p} \\ c\mathbf{1}_{m,p} & acb^{-1}\mathbf{1}_{n,p} & c^2b^{-2}C \end{vmatrix}$$

$$= b^{2p}c^{-2p}|A|\cdot\begin{vmatrix} B & acb^{-1}\mathbf{1}_{n,p} \\ acb^{-1}\mathbf{1}_{n,p} & c^2b^{-2}C \end{vmatrix}$$

$$- b^{2p}c^{2-2p}|\overline{A}|\cdot\begin{vmatrix} B & acb^{-1}\mathbf{1}_{n,p} & \mathbf{1}_{n,1} \\ acb^{-1}\mathbf{1}_{n,p} & c^2b^{-2}C & \mathbf{1}_{p,1} \\ \mathbf{1}_{1,n} & \mathbf{1}_{1,p} & 0 \end{vmatrix},$$

(the second equality results from Lemma 1.2), from which we deduce the formula

$$
(4)\quad \begin{vmatrix} A & c\mathbf{1}_{m,n} & b\mathbf{1}_{m,p} \\ c\mathbf{1}_{m,n} & B & a\mathbf{1}_{n,p} \\ b\mathbf{1}_{m,p} & a\mathbf{1}_{n,p} & C \end{vmatrix} = |A| \cdot \begin{vmatrix} B & a\mathbf{1}_{n,p} \\ a\mathbf{1}_{n,p} & C \end{vmatrix} - |\overline{A}| \cdot \begin{vmatrix} B & a\mathbf{1}_{n,p} & c\mathbf{1}_{n,1} \\ a\mathbf{1}_{n,p} & C & b\mathbf{1}_{p,1} \\ c\mathbf{1}_{n,1} & b\mathbf{1}_{p,1} & 0 \end{vmatrix}
$$

which is valid without any restriction on b.

The last determinant of Formula (4) above can be itself computed by using (4) two more times. □

Lemma 1.3 can be extended in the following way. Although we shall not use this extension, we mention it because it could be of interest in similar situations. Let $\{a_{i,j}\}_{1\le i,j\le n}$ be a collection of numbers such that $a_{i,i} = 0$, and $\{A_i\}_{1\le i\le n}$ a sequence of square matrices of respective dimensions m_i.

Define a matrix M by blocks: put the A's on the diagonal, and the block $a_{i,j}\mathbf{1}_{m_i,m_j}$ at position (i,j) for $i \neq j$. Then

$$
|M| = \sum_{\sigma} \varepsilon_\sigma \Big[\prod_{i=\sigma(i)} |A_i| \Big] \prod_{i\neq\sigma(i)} \big(-|\overline{A}_i| a_{i,\sigma(i)}\big).
$$

LEMMA 1.4. — *Let $x \in \mathbb{R}$, and let A be an $m \times m$ matrix, then*

(i) $|x\mathbb{1}_{m,m} + A| = |A| - x|\overline{A}|$,

(ii) $|\overline{x\mathbb{1}_{m,m} + A}| = |\overline{A}|$,

(iii) $|\overline{-A}| = (-1)^{m+1}|\overline{A}|$.

Proof. — To prove (i), write

$$
|x\mathbb{1}_{m,m} + A| = \begin{vmatrix} x\mathbb{1}_{m,m} + A & \mathbb{O}_{m,1} \\ \mathbb{1}_{1,m} & 1 \end{vmatrix} = \begin{vmatrix} A & -x\mathbb{1}_{m,1} \\ \mathbb{1}_{1,m} & 1 \end{vmatrix},
$$

and conclude by using Lemma 1.2.

Assertions (ii) and (iii) are obvious. □

Remark 1.2. — Because they express identities between polynomials with integer coefficients, the preceding lemmas are valid for matrices with entries in any commutative ring.

2. Fundamental recurrence equations.

The aim of this section is to determine recurrence formulae for the sequence $|\mathcal{E}_n^p|$ $(n \geq 1, p \geq 0)$, which will play an essential rôle in this paper. We find that, in order to establish such formulae, we need to distinguish different cases according to the parities of n and p. These formulae involve the quantities $|\overline{\mathcal{E}_n^p}|$, $|\Delta_n^p|$ and $|\overline{\Delta_n^p}|$. Hence, we shall simultaneously establish recurrence formulae for all these sixteen quantities, thus obtaining the fundamental results of this section.

THEOREM 2.1. — *For $p \geq 0$ and $n \geq 1$, one has*

$$\text{1)}\quad \begin{aligned} |\mathcal{E}_{2n}^{2p}| &= |\mathcal{E}_n^p| \cdot |\Delta_n^p| - |\overline{\mathcal{E}_n^p}| \cdot |\overline{\Delta_n^p}| - 2|\mathcal{E}_n^p| \cdot |\overline{\Delta_n^p}| \\ &\equiv |\mathcal{E}_n^p| \cdot |\Delta_n^p| + |\overline{\mathcal{E}_n^p}| \cdot |\overline{\Delta_n^p}|, \end{aligned}$$

$$\text{2)}\quad \begin{aligned} |\overline{\mathcal{E}_{2n}^{2p}}| &= 4|\mathcal{E}_n^p| \cdot |\overline{\Delta_n^p}| + |\overline{\mathcal{E}_n^p}| \cdot |\Delta_n^p| + 2|\overline{\mathcal{E}_n^p}| \cdot |\overline{\Delta_n^p}| \\ &\equiv |\overline{\mathcal{E}_n^p}| \cdot |\Delta_n^p|, \end{aligned}$$

$$\text{3)}\quad \begin{aligned} |\mathcal{E}_{2n+1}^{2p}| &= |\mathcal{E}_{n+1}^p| \cdot |\Delta_n^p| - |\overline{\mathcal{E}_{n+1}^p}| \cdot |\overline{\Delta_n^p}| - 2|\mathcal{E}_{n+1}^p| \cdot |\overline{\Delta_n^p}| \\ &\equiv |\mathcal{E}_{n+1}^p| \cdot |\Delta_n^p| + |\overline{\mathcal{E}_{n+1}^p}| \cdot |\overline{\Delta_n^p}|, \end{aligned}$$

$$\text{4)}\quad \begin{aligned} |\overline{\mathcal{E}_{2n+1}^{2p}}| &= 4|\mathcal{E}_{n+1}^p| \cdot |\overline{\Delta_n^p}| + |\overline{\mathcal{E}_{n+1}^p}| \cdot |\Delta_n^p| + 2|\overline{\mathcal{E}_{n+1}^p}| \cdot |\overline{\Delta_n^p}| \\ &\equiv |\overline{\mathcal{E}_{n+1}^p}| \cdot |\Delta_n^p|, \end{aligned}$$

$$\text{5)}\quad \begin{aligned} |\mathcal{E}_{2n}^{2p+1}| &= (-1)^n \bigl\{ |\mathcal{E}_n^{p+1}| \cdot |\Delta_n^p| - 2|\mathcal{E}_n^{p+1}| \cdot |\overline{\Delta_n^p}| \\ &\qquad\qquad + |\overline{\mathcal{E}_n^{p+1}}| \cdot |\Delta_n^p| - |\overline{\mathcal{E}_n^{p+1}}| \cdot |\overline{\Delta_n^p}| \bigr\} \\ &\equiv |\mathcal{E}_n^{p+1}| \cdot |\Delta_n^p| + |\overline{\mathcal{E}_n^{p+1}}| \cdot |\Delta_n^p| + |\overline{\mathcal{E}_n^{p+1}}| \cdot |\overline{\Delta_n^p}|, \end{aligned}$$

$$\text{6)}\quad \begin{aligned} |\overline{\mathcal{E}_{2n}^{2p+1}}| &= (-1)^n \bigl\{ 4|\mathcal{E}_n^{p+1}| \cdot |\overline{\Delta_n^p}| - |\overline{\mathcal{E}_n^{p+1}}| \cdot |\Delta_n^p| + 2|\overline{\mathcal{E}_n^{p+1}}| \cdot |\overline{\Delta_n^p}| \bigr\} \\ &\equiv |\overline{\mathcal{E}_n^{p+1}}| \cdot |\Delta_n^p|, \end{aligned}$$

$$\text{7)}\quad \begin{aligned} |\mathcal{E}_{2n+1}^{2p+1}| &= (-1)^{n+1} \bigl\{ |\mathcal{E}_{n+1}^p| \cdot |\Delta_n^{p+1}| - 2|\mathcal{E}_{n+1}^p| \cdot |\overline{\Delta_n^{p+1}}| \\ &\qquad\qquad + |\overline{\mathcal{E}_{n+1}^p}| \cdot |\Delta_n^{p+1}| - |\overline{\mathcal{E}_{n+1}^p}| \cdot |\overline{\Delta_n^{p+1}}| \bigr\} \\ &\equiv |\mathcal{E}_{n+1}^p| \cdot |\Delta_n^{p+1}| + |\overline{\mathcal{E}_{n+1}^p}| \cdot |\Delta_n^{p+1}| + |\overline{\mathcal{E}_{n+1}^p}| \cdot |\overline{\Delta_n^{p+1}}|, \end{aligned}$$

$$\text{8)}\quad \begin{aligned} |\overline{\mathcal{E}_{2n+1}^{2p+1}}| &= (-1)^{n+1} \bigl\{ 4|\mathcal{E}_{n+1}^p| \cdot |\overline{\Delta_n^{p+1}}| - |\overline{\mathcal{E}_{n+1}^p}| \cdot |\Delta_n^{p+1}| \\ &\qquad + 2|\overline{\mathcal{E}_{n+1}^p}| \cdot |\overline{\Delta_n^{p+1}}| \bigr\} \\ &\equiv |\overline{\mathcal{E}_{n+1}^p}| \cdot |\Delta_n^{p+1}|, \end{aligned}$$

9) $$|\Delta_{2n}^{2p}| = (-1)^n |\Delta_n^p|^2 \equiv |\Delta_n^p|,$$

10) $$|\overline{\Delta_{2n}^{2p}}| \equiv 0,$$

11) $$|\Delta_{2n+1}^{2p}| = (-1)^n \{|\overline{\mathcal{E}_{n+1}^p}|^2 + 2|\mathcal{E}_{n+1}^p| \cdot |\overline{\mathcal{E}_{n+1}^p}|\} \equiv |\overline{\mathcal{E}_{n+1}^p}|,$$

12) $$|\overline{\Delta_{2n+1}^{2p}}| \equiv |\overline{\mathcal{E}_{n+1}^p}|,$$

13) $$|\Delta_{2n}^{2p+1}| \equiv |\Delta_n^p| \cdot |\Delta_n^{p+1}| + |\Delta_n^p| \cdot |\overline{\Delta_n^{p+1}}| + |\overline{\Delta_n^p}| \cdot |\Delta_n^{p+1}|,$$

14) $$|\overline{\Delta_{2n}^{2p+1}}| \equiv |\Delta_n^p| \cdot |\overline{\Delta_n^{p+1}}| + |\overline{\Delta_n^p}| \cdot |\Delta_n^{p+1}|,$$

15) $$|\Delta_{2n+1}^{2p+1}| \equiv |\Delta_{n+1}^p| \cdot |\Delta_n^{p+1}| + |\Delta_{n+1}^p| \cdot |\overline{\Delta_n^{p+1}}| + |\overline{\Delta_{n+1}^p}| \cdot |\Delta_n^{p+1}|,$$

16) $$|\overline{\Delta_{2n+1}^{2p+1}}| \equiv |\Delta_{n+1}^p| \cdot |\overline{\Delta_n^{p+1}}| + |\overline{\Delta_{n+1}^p}| \cdot |\Delta_n^{p+1}|.$$

Proof. — First we are going to establish a few general properties of Hankel matrices associated with a sequence $\{u_j\}_{j\geq 0}$. For $n \geq 1$ and $p \geq 0$ we consider the Hankel matrix $H_n^p = \big(u_{p+i+j-2}\big)_{1\leq i,j\leq n}$, together with the matrix $K_n^p = \big(u_{p+2(i+j-2)}\big)_{1\leq i,j\leq n}$.

When $u = \varepsilon$ is the Thue-Morse sequence, one has

$$K_n^{2p} = \mathcal{E}_n^p \quad\text{and}\quad K_n^{2p+1} = \mathbf{1}_{n,n} - \mathcal{E}_n^p. \tag{5}$$

When $u_n = \delta_n$ $(= \varepsilon_{n+1} - \varepsilon_n)$, one has

$$K_n^{2p} = \mathbf{1}_{n,n} - 2\mathcal{E}_n^p \quad\text{and}\quad K_n^{2p+1} = \Delta_n^p + 2\mathcal{E}_n^p - \mathbf{1}_{n.n}. \tag{6}$$

Let $M = \big(m_{i,j}\big)_{1\leq i,j\leq n}$ be any $n \times n$-matrix. Let $\nu = \left[\frac{1}{2}(n+1)\right]$ and $\mu = \left[\frac{1}{2}n\right]$. One can easily check the following formula

$$P_1^t M P_1 = \begin{pmatrix} \big(m_{2i-1,2j-1}\big)_{\substack{1\leq i\leq \nu\\ 1\leq j\leq \nu}} & \big(m_{2i-1,2j}\big)_{\substack{1\leq i\leq \nu\\ 1\leq j\leq \mu}} \\ \big(m_{2i,2j-1}\big)_{\substack{1\leq i\leq \mu\\ 1\leq j\leq \nu}} & \big(m_{2i,2j}\big)_{\substack{1\leq i\leq \mu\\ 1\leq j\leq \mu}} \end{pmatrix} \tag{7}$$

from which one can get

$$P_1^t H_{2n}^p P_1 = \begin{pmatrix} K_n^p & K_n^{p+1} \\ K_n^{p+1} & K_n^{p+2} \end{pmatrix} \tag{8}$$

and

$$P_1^t H_{2n+1}^p P_1 = \begin{pmatrix} K_{n+1}^p & (K_{n+1}^{p+1})^{(n+1)} \\ (K_{n+1}^{p+1})^{(n+1)t} & K_n^{p+2} \end{pmatrix} \tag{9}$$
$$= \begin{pmatrix} K_{n+1}^p & (K_{n+1}^p)^{(1)} \\ (K_{n+1}^p)^{(1)t} & K_n^{p+2} \end{pmatrix}.$$

In other words, $P_1^t H_{2n+1}^p P_1$ is obtained by removing the last row and the last column from the matrix $P_1^t H_{2(n+1)} P_1$.

Proof of 1). — By using (8) and (5), we have

$$P_1^t \mathcal{E}_{2n}^{2p} P_1 = \begin{pmatrix} \mathcal{E}_n^p & \mathbf{1}_{n,n} - \mathcal{E}_n^p \\ \mathbf{1}_{n,n} - \mathcal{E}_n^p & \mathcal{E}_n^{p+1} \end{pmatrix}.$$

Then by the definition of P_2 and Δ_n^p, we have

$$P_2^t P_1^t \mathcal{E}_{2n}^{2p} P_1 P_2 = \begin{pmatrix} \mathcal{E}_n^p & \mathbf{1}_{n,n} \\ \mathbf{1}_{n,n} & 2\mathbf{1}_{n,n} + \Delta_n^p \end{pmatrix}. \tag{10}$$

Hence by (3) and Lemmas 1.2 and 1.4, we obtain

$$\begin{aligned} |\mathcal{E}_{2n}^{2p}| &= \begin{vmatrix} \mathcal{E}_n^p & \mathbf{1}_{n,n} \\ \mathbf{1}_{n,n} & 2\mathbf{1}_{n,n} + \Delta_n^p \end{vmatrix} \\ &= |\mathcal{E}_n^p| \cdot |2\mathbf{1}_{n,n} + \Delta_n^p| - |\overline{\mathcal{E}_n^p}| \cdot |\overline{2\mathbf{1}_{n,n} + \Delta_n^p}| \\ &= |\mathcal{E}_n^p|\big(|\Delta_n^p| - 2|\overline{\Delta_n^p}|\big) - |\overline{\mathcal{E}_n^p}| \cdot |\overline{\Delta_n^p}| \\ &= |\mathcal{E}_n^p| \cdot |\Delta_n^p| - |\overline{\mathcal{E}_n^p}| \cdot |\overline{\Delta_n^p}| - 2|\mathcal{E}_n^p| \cdot |\overline{\Delta_n^p}|. \end{aligned}$$

The proofs of the other assertions follow the same lines: to compute the determinant of a matrix A, we find a matrix Q such that the determinant of QAQ^t is computable, for instance by using Lemmas 1.2, 1.3, and 1.4.

In the sequel, we only write the transformation matrices and omit the details of the calculations.

Proof of 2). — One has

$$\begin{pmatrix} P^t & \mathbb{O}_{2n,1} \\ \mathbb{O}_{1,2n} & 1 \end{pmatrix} \begin{pmatrix} \mathcal{E}_{2n}^{2p} & \mathbf{1}_{2n,1} \\ \mathbf{1}_{1,2n} & 0 \end{pmatrix} \begin{pmatrix} P & \mathbb{O}_{2n,1} \\ \mathbb{O}_{1,2n} & 1 \end{pmatrix} = \begin{pmatrix} P^t \mathcal{E}_{2n}^{2p} P & P^t \mathbf{1}_{2n,1} \\ \mathbf{1}_{1,2n} P & 0 \end{pmatrix}.$$

Therefore, taking into account (10),

$$|\overline{D}| = \begin{vmatrix} \mathcal{E}_n^p & \mathbf{1}_{n,n} & \mathbf{1}_{n,1} \\ \mathbf{1}_{n,n} & 2\mathbf{1}_{n,n} + \Delta_n^p & 2\mathbf{1}_{n,1} \\ \mathbf{1}_{1,n} & 2\mathbf{1}_{1,n} & 0 \end{vmatrix}.$$

Proof of 3). — Computing as in 1), we have successively

$$P_1^t \mathcal{E}_{2n+1}^{2p} P_1 = \begin{pmatrix} \mathcal{E}_{n+1}^p & \mathbf{1}_{n+1,n} - (\mathcal{E}_{n+1}^p)^{(n+1)} \\ \left(\mathbf{1}_{n+1,n} - (\mathcal{E}_{n+1}^p)^{(n+1)}\right)^t & \mathcal{E}_n^{p+1} \end{pmatrix},$$

and

$$P_2^t P_1^t \mathcal{E}_{2n+1}^{2p} P_1 P_2 = \begin{pmatrix} \mathcal{E}_{n+1}^p & \mathbf{1}_{n+1,n} \\ \mathbf{1}_{n,n+1} & 2\mathbf{1}_{n,n} + \Delta_n^p \end{pmatrix}. \tag{11}$$

Proof of 4). — One has, due to (11),

$$\begin{pmatrix} P^t & \mathbb{O}_{2n+1,1} \\ \mathbb{O}_{1,2n+1} & 1 \end{pmatrix} \begin{pmatrix} \mathcal{E}_{2n+1}^{2p} & \mathbf{1}_{2n+1,1} \\ \mathbf{1}_{1,2n+1} & 0 \end{pmatrix} \begin{pmatrix} P & \mathbb{O}_{2n+1,1} \\ \mathbb{O}_{1,2n+1} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \mathcal{E}_{n+1}^p & \mathbf{1}_{n+1,n} & \mathbf{1}_{n+1,1} \\ \mathbf{1}_{n,n+1} & 2\mathbf{1}_{n,n} + \Delta_n^p & 2\mathbf{1}_{n,1} \\ \mathbf{1}_{1,n+1} & 2\mathbf{1}_{1,n} & 0 \end{pmatrix}.$$

Proof of 5). — One has

$$\begin{aligned} P_2 P_1^t \mathcal{E}_{2n}^{2p+1} P_1 P_2^t &= P_2 \begin{pmatrix} \mathbf{1}_{n,n} - \mathcal{E}_n^p & \mathcal{E}_n^{p+1} \\ \mathcal{E}_n^{p+1} & \mathbf{1}_{n,n} - \mathcal{E}_n^{p+1} \end{pmatrix} P_2^t \\ &= \begin{pmatrix} 2\mathbf{1}_{n,n} + \Delta_n^p & \mathbf{1}_{n,n} \\ \mathbf{1}_{n,n} & \mathbf{1}_{n,n} - \mathcal{E}_n^{p+1} \end{pmatrix}. \end{aligned} \tag{12}$$

Proof of 6). — Due to (12), one has

$$\begin{pmatrix} P_2 P_1^t & \mathbb{O}_{2n,1} \\ \mathbb{O}_{1,2n} & 1 \end{pmatrix} \begin{pmatrix} \mathcal{E}_{2n}^{2p+1} & \mathbf{1}_{2n,1} \\ \mathbf{1}_{1,2n} & 0 \end{pmatrix} \begin{pmatrix} P_1 P_2^t & \mathbb{O}_{2n,1} \\ \mathbb{O}_{1,2n} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2\mathbf{1}_{n,n} + \Delta_n^p & \mathbf{1}_{n,n} & 2\mathbf{1}_{n,1} \\ \mathbf{1}_{n,n} & \mathbf{1}_{n,n} - \mathcal{E}_n^{p+1} & \mathbf{1}_{n,1} \\ 2\mathbf{1}_{1,n} & \mathbf{1}_{1,n} & 0 \end{pmatrix}.$$

Proof of 7). — Let

$$Q_1 = \begin{pmatrix} 1 & \mathbb{O}_{2n,1} & \\ \mathbb{O}_{1,2n} & I_n & I_n \\ & \mathbb{O}_{n,n} & I_n \end{pmatrix},$$

then

$$(13)\quad Q_1^t P_1^t \mathcal{E}_{2n+1}^{2p+1} P_1 Q_1 = Q_1^t \begin{pmatrix} \mathbf{1}_{n+1,n+1} - \mathcal{E}_{n+1}^p & (\mathcal{E}_{n+1}^{p+1})^{(1)} \\ ((\mathcal{E}_{n+1}^{p+1})^{(1)})^t & \mathbf{1}_{n,n} - \mathcal{E}_n^{p+1} \end{pmatrix} Q_1$$
$$= \begin{pmatrix} \mathbf{1}_{n+1,n+1} - \mathcal{E}_{n+1}^p & \mathbf{1}_{n+1,n} \\ \mathbf{1}_{n,n+1} & 2\mathbf{1}_{n,n} + \Delta_n^{p+1} \end{pmatrix},$$

and Formula 2.1.7 follows as above.

Proof of 8). — Due to (13), one has

$$\begin{pmatrix} Q_1^t P_1^t & \mathbb{O}_{2n+1,1} \\ \mathbb{O}_{1,2n+1} & 1 \end{pmatrix} \begin{pmatrix} \mathcal{E}_{2n+1}^{2p+1} & \mathbf{1}_{2n+1,1} \\ \mathbf{1}_{1,2n+1} & 0 \end{pmatrix} \begin{pmatrix} P_1 Q_1 & \mathbb{O}_{2n+1,1} \\ \mathbb{O}_{1,2n+1} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{1}_{n+1,n+1} - \mathcal{E}_{n+1}^p & \mathbf{1}_{n+1,n} & \mathbf{1}_{n+1,1} \\ \mathbf{1}_{n,n+1} & 2\mathbf{1}_{n,n} + \Delta_n^{p+1} & 2\mathbf{1}_{n,1} \\ \mathbf{1}_{1,n+1} & 2\mathbf{1}_{1,n} & 0 \end{pmatrix}.$$

Proof of 9). — Write

$$(14)\quad P_2^t P_1^t \Delta_{2n}^{2p} P_1 P_2 = P_2^t \begin{pmatrix} \mathbf{1}_{n,n} - 2\mathcal{E}_n^p & \Delta_n^p - \mathbf{1}_{n,n} + 2\mathcal{E}_n^p \\ \Delta_n^p - \mathbf{1}_{n,n} + 2\mathcal{E}_n^p & \mathbf{1}_{n,n} - 2\mathcal{E}_n^{p+1} \end{pmatrix} P_2$$
$$= \begin{pmatrix} \mathbf{1}_{n,n} - 2\mathcal{E}_n^p & \Delta_n^p \\ \Delta_n^p & \mathbb{O}_{n,n} \end{pmatrix},$$

hence

$$|\Delta_{2n}^{2p}| = (-1)^n |\Delta_n^p|^2.$$

Proof of 10). — Due to (14)

$$\begin{pmatrix} P_1^t & \mathbb{O}_{2n,1} \\ \mathbb{O}_{1,2n} & 1 \end{pmatrix} \begin{pmatrix} \Delta_{2n}^{2p} & \mathbf{1}_{2n,1} \\ \mathbf{1}_{1,2n} & 0 \end{pmatrix} \begin{pmatrix} P_1 & \mathbb{O}_{2n,1} \\ \mathbb{O}_{1,2n} & 1 \end{pmatrix} \equiv \begin{pmatrix} \mathbf{1}_{n,n} & \Delta_n^p & \mathbf{1}_{n,1} \\ \Delta_n^p & \mathbb{O}_{n,n} & \mathbb{O}_{n,1} \\ \mathbf{1}_{1,n} & \mathbb{O}_{1,n} & 0 \end{pmatrix},$$

hence $|\overline{\Delta_{2n}^{2p}}| \equiv 0$.

Proof of 11). — We write

$$P_2^t P_1^t \Delta_{2n+1}^{2p} P_1 P_2 = P_2^t \begin{pmatrix} \mathbf{1}_{n+1,n+1} - 2\mathcal{E}_{n+1}^p & B^{(n+1)} \\ (B^{(n+1)})^t & \mathbf{1}_{n,n} - 2\mathcal{E}_n^{p+1} \end{pmatrix} P_2$$

$$= \begin{pmatrix} \mathbf{1}_{n+1,n+1} - 2\mathcal{E}_{n+1}^p & D \\ D^t & \mathbb{O}_{n,n} \end{pmatrix},$$

where $B = \Delta_{n+1}^p - \mathbf{1}_{n+1,n+1} + 2\mathcal{E}_{n+1}^p$ and $D = (\Delta_{n+1}^p)^{(n+1)}$, hence

$$|\Delta_{2n+1}^{2p}| = \begin{vmatrix} \mathbf{1}_{n+1,n+1} - 2\mathcal{E}_{n+1}^p & D \\ D^t & \mathbb{O}_{n,n} \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{1}_{n+1,n+1} - 2\mathcal{E}_{n+1}^p & D & -v \\ D^t & \mathbb{O}_{n,n} & \mathbb{O}_{n,1} \\ \mathbb{O}_{1,n+1} & \mathbb{O}_{1,n} & 1 \end{vmatrix},$$

where $v = (\epsilon_{p+n}, \epsilon_{p+n+1}, \dots, \epsilon_{p+2n})^t$. Now, we add the $(2n+2)$-th column of the above matrix to the $(2n+1)$-th column, we then add the resulting $(2n+1)$-th column to the $(2n)$-th column, and we continue this procedure until the $(n+2)$-th column. Then, from the definition of Δ_n^p and noticing that v is precisely the last column of $\mathcal{E}_{n+1}^p$, we have

$$|\Delta_{2n+1}^{2p}| = \begin{vmatrix} \mathbf{1}_{n+1,n+1} - 2\mathcal{E}_{n+1}^p & -\mathcal{E}_{n+1}^p \\ D^t & \mathbb{O}_{n,n+1} \\ \mathbb{O}_{1,n+1} & \mathbf{1}_{1,n+1} \end{vmatrix}$$

$$= - \begin{vmatrix} \mathbf{1}_{n+1,n+1} - 2\mathcal{E}_{n+1}^p & -\mathcal{E}_{n+1}^p & \mathbb{O}_{n+1,1} \\ D^t & \mathbb{O}_{n,n+1} & \mathbb{O}_{n,1} \\ -v^t & \mathbb{O}_{1,n+1} & 1 \\ \mathbb{O}_{1,n+1} & \mathbf{1}_{1,n+1} & 0 \end{vmatrix}$$

$$= - \begin{vmatrix} \mathbf{1}_{n+1,n+1} - 2\mathcal{E}_{n+1}^p & -\mathcal{E}_{n+1}^p & \mathbb{O}_{n+1,1} \\ -\mathcal{E}_{n+1}^p & \mathbb{O}_{n+1,n+1} & \mathbf{1}_{n+1,1} \\ \mathbb{O}_{1,n+1} & \mathbf{1}_{1,n+1} & 0 \end{vmatrix}$$

$$= - \begin{vmatrix} \mathbf{1}_{n+1,n+1} & -\mathcal{E}_{n+1}^p & -\mathbf{1}_{n+1,1} \\ -\mathcal{E}_{n+1}^p & \mathbb{O}_{n+1,n+1} & \mathbf{1}_{n+1,1} \\ -\mathbf{1}_{1,n+1} & \mathbf{1}_{1,n+1} & 0 \end{vmatrix}$$

$$= (-1)^n \begin{vmatrix} -\mathcal{E}_{n+1}^p & \mathbf{1}_{n+1,n+1} & -\mathbf{1}_{n+1,1} \\ \mathbb{O}_{n+1,n+1} & -\mathcal{E}_{n+1}^p & \mathbf{1}_{n+1,1} \\ \mathbf{1}_{1,n+1} & -\mathbf{1}_{1,n+1} & 0 \end{vmatrix}$$

$$= (-1)^n \begin{vmatrix} \mathcal{E}_{n+1}^p & -\mathbf{1}_{n+1,n+1} & -\mathbf{1}_{n+1,1} \\ -\mathbf{1}_{n+1,n+1} & \mathbf{1}_{n+1,n+1} + \mathcal{E}_{n+1}^p & \mathbf{1}_{n+1,1} \\ -\mathbf{1}_{1,n+1} & -\mathbf{1}_{1,n+1} & 0 \end{vmatrix}.$$

Proof of 12). — We write

$$\begin{pmatrix} P^t & \mathbb{O}_{2n+1,1} \\ \mathbb{O}_{1,2n+1} & 1 \end{pmatrix} \begin{pmatrix} \Delta_{2n+1}^{2p} & \mathbf{1}_{2n+1,1} \\ \mathbf{1}_{1,2n+1} & 0 \end{pmatrix} \begin{pmatrix} P & \mathbb{O}_{2n+1,1} \\ \mathbb{O}_{1,2n+1} & 1 \end{pmatrix}$$
$$\equiv \begin{pmatrix} \mathbf{1}_{n+1,n+1} & D & \mathbf{1}_{n+1,1} \\ D^t & \mathbb{O}_{n,n} & \mathbb{O}_{n,1} \\ \mathbf{1}_{1,n+1} & \mathbb{O}_{1,n} & 0 \end{pmatrix},$$

where D is defined as in 11). Therefore computing as in 11), we have

$$|\overline{\Delta_{2n+1}^{2p}}| \equiv |D\ \mathbf{1}_{n+1,1}| \cdot \begin{vmatrix} D^t \\ \mathbf{1}_{1,n+1} \end{vmatrix} \equiv |\overline{\mathcal{E}_{n+1}^{p}}|^2 \equiv |\overline{\mathcal{E}_{n+1}^{p}}|.$$

Proof of 13). — We have

$$P_1^t \Delta_{2n}^{2p+1} P_1 = \begin{pmatrix} \Delta_n^p - \mathbf{1}_{n,n} + 2\mathcal{E}_n^p & \mathbf{1}_{n,n} - 2\mathcal{E}_n^{p+1} \\ \mathbf{1}_{n,n} - 2\mathcal{E}_n^{p+1} & \Delta_n^{p+1} - \mathbf{1}_{n,n} + 2\mathcal{E}_n^{p+1} \end{pmatrix}$$
$$\equiv \begin{pmatrix} \Delta_n^p - \mathbf{1}_{n,n} & \mathbf{1}_{n,n} \\ \mathbf{1}_{n,n} & \Delta_n^{p+1} - \mathbf{1}_{n,n} \end{pmatrix}.$$

Proof of 14). — As in 13), we have

$$\begin{pmatrix} P_1^t & \mathbb{O}_{2n,1} \\ \mathbb{O}_{1,2n} & 1 \end{pmatrix} \begin{pmatrix} \Delta_{2n}^{2p+1} & \mathbf{1}_{2n,1} \\ \mathbf{1}_{1,2n} & 0 \end{pmatrix} \begin{pmatrix} P_1 & \mathbb{O}_{2n,1} \\ \mathbb{O}_{1,2n} & 1 \end{pmatrix}$$
$$\equiv \begin{pmatrix} \Delta_n^p - \mathbf{1}_{n,n} & \mathbf{1}_{n,n} & \mathbf{1}_{n,1} \\ \mathbf{1}_{n,n} & \Delta_n^{p+1} - \mathbf{1}_{n,n} & \mathbf{1}_{n,1} \\ \mathbf{1}_{1,n} & \mathbf{1}_{1,n} & 0 \end{pmatrix}.$$

Proof of 15). — As in 13), we have

$$Q_1^t \Delta_{2n+1}^{2p+1} Q_1 \equiv \begin{pmatrix} \Delta_{n+1}^p - \mathbf{1}_{n+1,n+1} & \mathbf{1}_{n+1,n} \\ \mathbf{1}_{n,n+1} & \Delta_n^{p+1} - \mathbf{1}_{n,n} \end{pmatrix}.$$

Proof of 16). — As in 15), we have

$$\begin{pmatrix} Q_1^t & \mathbb{O}_{2n+1,1} \\ \mathbb{O}_{1,2n+1} & 1 \end{pmatrix} \begin{pmatrix} \Delta_{2n+1}^{2p+1} & \mathbf{1}_{2n+1,1} \\ \mathbf{1}_{1,2n+1} & 0 \end{pmatrix} \begin{pmatrix} Q_1 & \mathbb{O}_{2n+1,1} \\ \mathbb{O}_{1,2n+1} & 1 \end{pmatrix}$$
$$\equiv \begin{pmatrix} \Delta_{n+1}^p - \mathbf{1}_{n+1,n+1} & \mathbf{1}_{n+1,n} & \mathbf{1}_{n+1,1} \\ \mathbf{1}_{n,n+1} & \Delta_n^{p+1} - \mathbf{1}_{n,n} & \mathbf{1}_{n,1} \\ \mathbf{1}_{1,n+1} & \mathbf{1}_{1,n} & 0 \end{pmatrix}. \qquad \square$$

Remark 2.1. — The Hankel determinants $(H_n^p)_{n\geq 1, p\geq 0}$ of a sequence of complex numbers satisfy the following recurrence equation (see for example [3], p. 96),

$$|H_n^p| \cdot |H_n^{p+2}| - |H_n^{p+1}|^2 = |H_{n-1}^{p+2}| \cdot |H_{n+1}^p|.$$

Now let ℓ, j and k be given; if $|H_n^\ell| = 0$ for $j \leq n \leq j+k$, then by the above formula, $|H_n^p| = 0$ on the rhombus whose vertices are (ℓ, j), $(\ell+k, j)$, $(\ell - k, j+k)$ and $(\ell, j+k)$. Similarly, if $|H_{j+1}^p| = 0$ for $\ell - 1 \leq p \leq \ell + k$, then, either $|H_{j-1}^p| = 0$ for $\ell + 1 \leq p \leq \ell + k + 1$, or $|H_{j+1}^p| = 0$ for $\ell - 1 \leq p \leq \ell + k - 1$. Therefore the set of zeros of the sequence $(|H_n^p|)_{n\geq 1, p\geq 0}$ is the union of rhombi which are separated by nonzero elements. This explains the patterns shown on Figure 1.

PROPOSITION 2.1. — *Define*

$$|\mathcal{E}_0^p| = \begin{cases} 0 & \textit{if } p = 0, \\ 1 & \textit{if } p \geq 1, \end{cases} \qquad |\overline{\mathcal{E}_0^p}| = 1 - |\mathcal{E}_0^p|,$$

$$|\Delta_0^p| = 1, \qquad |\overline{\Delta_0^p}| = 0 \quad \textit{for } p \geq 0.$$

Then formulae of Theorem 2.1 *hold for* $p \geq 0$ *and* $n \geq 0$.

Proof. — Formulae have to be checked one by one. The conclusion results from (1), (2), and the following facts: $|\overline{\mathcal{E}_1^p}| = |\overline{\Delta_1^p}| = -1$, $|\mathcal{E}_1^p| = \varepsilon_p$, and $|\Delta_1^p| = \delta_p$. □

From Proposition 2.1, we see that if we can determine the quantities involved in Theorem 2.1 for $p = 0, 1$, then we can determine these quantities for all $p \geq 2$ by the recurrence equations of Proposition 2.1. Propositions 2.2 and 2.3 below are devoted to this purpose.

PROPOSITION 2.2. — *With the above notations, we have*

$$|\mathcal{E}_n^0| \equiv n, \quad |\Delta_n^0| \equiv 1, \quad |\overline{\mathcal{E}_n^0}| \equiv 1, \quad |\overline{\Delta_n^0}| \equiv n.$$

Proof. — For $n = 1$, the above equalities can be checked directly. Assume that the proposition is true for $n \leq k$. Then, if $n = k + 1 = 2\ell$ is even, we have by 2.1.1 and the induction hypothesis,

$$|\mathcal{E}_n^0| \equiv |\mathcal{E}_\ell^0| \cdot |\Delta_\ell^0| + |\overline{\mathcal{E}_\ell^0}| \cdot |\overline{\Delta_\ell^0}| \equiv 2\ell.$$

If $n = k + 1 = 2\ell + 1$ is odd, then, by 2.1.3, we have

$$|\mathcal{E}_n^0| \equiv |\mathcal{E}_{\ell+1}^0| \cdot |\Delta_\ell^0| + |\overline{\mathcal{E}_{\ell+1}^0}| \cdot |\overline{\Delta_\ell^0}| \equiv \ell + 1 + \ell \equiv 2\ell + 1.$$

Thus we obtain the first assertion. The other ones can be obtained by the same method. □

PROPOSITION 2.3. — *For $p = 1$, we have the following relations :*

1) $|\mathcal{E}_n^1| \equiv \begin{cases} 0 & \textit{if } n \equiv 1,\, 2 \pmod 6, \\ 1 & \textit{otherwise}; \end{cases}$

2) $|\Delta_n^1| \equiv \begin{cases} 0 & \textit{if } n \equiv 1 \pmod 3, \\ 1 & \textit{otherwise}; \end{cases}$

3) $|\overline{\mathcal{E}_n^1}| \equiv \begin{cases} 0 & \textit{if } n \equiv 0 \pmod 3, \\ 1 & \textit{otherwise}; \end{cases}$

4) $|\overline{\Delta_n^1}| \equiv \begin{cases} 0 & \textit{if } n \equiv 0, 5 \pmod 6, \\ 1 & \textit{otherwise.} \end{cases}$

Proof. — Assertions 13), 15), 14) and 16) of Theorem 2.1 yield

$$|\Delta_{2n}^1| \equiv |\overline{\Delta_{2n+1}^1}| \equiv (n+1)|\Delta_n^1| + |\overline{\Delta_n^1}|, \tag{15}$$

$$|\Delta_{2n+1}^1| \equiv |\overline{\Delta_{2n}^1}| \equiv n|\Delta_n^1| + |\overline{\Delta_n^1}|, \tag{16}$$

from which one gets

$$|\Delta_{4n}^1| \equiv |\Delta_n^1|, \tag{17}$$

$$|\Delta_{4n+1}^1| \equiv |\Delta_{2n+1}^1|, \tag{18}$$

$$|\Delta_{4n+2}^1| \equiv |\Delta_{2n}^1|, \tag{19}$$

$$|\Delta_{4n+3}^1| \equiv |\Delta_n^1|. \tag{20}$$

It is easily checked that $|\Delta_1^1| = 0$, $|\Delta_2^1| = -1$, and $|\Delta_3^1| = 1$. We will prove Assertion 2 by induction. Suppose that it is true for $1 \le n < 4k$, and consider four cases.

(i) By (17), and since $4k \equiv k \pmod 3$, we have

$$|\Delta_{4k}^1| \equiv |\Delta_k^1| \equiv \begin{cases} 0 & \text{if } 4k \equiv 1 \pmod 3, \\ 1 & \text{otherwise.} \end{cases}$$

(ii) By (18), and since $4k+1 \equiv 1 \Leftrightarrow 2k+1 \equiv 1 \pmod 3$, we have

$$|\Delta^1_{4k+1}| \equiv |\Delta^1_{2k+1}| \equiv \begin{cases} 0 & \text{if } 4k+1 \equiv 1 \pmod 3, \\ 1 & \text{otherwise.} \end{cases}$$

(iii) By (19), and since $4k+2 \equiv 1 \Leftrightarrow 2k \equiv 1 \pmod 3$, we have

$$|\Delta^1_{4k+2}| \equiv |\Delta^1_{2k}| \equiv \begin{cases} 0 & \text{if } 4k+2 \equiv 1 \pmod 3, \\ 1 & \text{otherwise.} \end{cases}$$

(iv) The case $4k+3$ is the same as (i).

This proves Assertion 2.

By using Assertion 2 and Formulae (15) and (16), one can prove Assertion 4 by induction.

By Equalities 6) and 8) in Theorem 2.1 and Proposition 2.2 we have $|\overline{\mathcal{E}^1_{2n}}| = |\overline{\mathcal{E}^1_n}|$ and $|\overline{\mathcal{E}^1_{2n+1}}| = |\overline{\Delta^1_n}|$, then by the fourth congruence in Proposition 2.2 and induction, we prove 3).

Finally, by 3), 4) and Proposition 2.2, $|\mathcal{E}^1_{2n}| \equiv |\mathcal{E}^1_n| + (n+1)|\overline{\mathcal{E}^1_n}|$ and $|\mathcal{E}^1_{2n+1}| \equiv n|\Delta^1_n| + |\overline{\Delta^1_n}|$, then, by Propositions 2, 3, 4 and the same discussion as above, 1) is proved by induction. □

Generating series

Let $(u_n)_{n\geq 0}$ be a sequence with $u_n \in \mathbb{F}_2$, then the formal power series

$$u(x) = \sum_{n\geq 0} u_n x^n$$

is called the generating series of the sequence $(u_n)_{n\geq 0}$.

A sequence $\{u_n\}_{n\geq 0}$ is periodic of period s if and only if its generating series adds up to a rational fraction of the form $\frac{P(x)}{1+x^s}$, where P is a polynomial of degree less than s.

Let $A(x) = \sum_{n\geq 0} a_n x^n$, $B(x) = \sum_{n\geq 0} b_n x^n$ be two formal power series with $a_n, b_n \in \mathbb{F}_2$, then their Hadamard product is defined to be

$$A(x) \star B(x) = \sum_{n\geq 0} a_n b_n x^n.$$

The Hadamard product of generating series of periodic sequences is the generating series of a periodic sequence having as a period the lowest common multiple of the periods.

For $p = 0, 1, \ldots$, define

$$(21)\qquad \begin{cases} f^{(p)}(x) = \sum_{n\geq 0} |\mathcal{E}_n^p| x^n, & g^{(p)}(x) = \sum_{n\geq 0} |\Delta_n^p| x^n, \\ \overline{f^{(p)}}(x) = \sum_{n\geq 0} |\overline{\mathcal{E}_n^p}| x^n, & \overline{g^{(p)}}(x) = \sum_{n\geq 0} |\overline{\Delta_n^p}| x^n, \end{cases}$$

where coefficients are taken modulo 2, with the convention of Proposition 2.1 when $p = 0$.

By Propositions 2.2 and 2.3, we have

$$(22)\qquad \begin{cases} f^{(0)} = \dfrac{x}{1+x^2}, & \overline{f^{(0)}} = \dfrac{1}{1+x}, \\ g^{(0)} = \dfrac{1}{1+x}, & \overline{g^{(0)}} = \dfrac{x}{1+x^2}, \\ f^{(1)} = \dfrac{1+x^3+x^4+x^5}{1+x^6}, & \overline{f^{(1)}} = \dfrac{x+x^2}{1+x^3}, \\ g^{(1)} = \dfrac{1+x^2}{1+x^3}, & \overline{g^{(1)}} = \dfrac{x+x^2+x^3+x^4}{1+x^6}. \end{cases}$$

Recurrence formulae of Proposition 2.1 and Propositions 2.2 and 2.3 make it possible to compute the above quantities recursively. As an illustration, we compute $g^{(2)}$ and $f^{(2)}$:

$$\begin{aligned} g^{(2)} &= \sum_{n\geq 0} |\Delta_n^2| x^n = \sum_{n\geq 0} |\Delta_{2n}^2| x^{2n} + \sum_{n\geq 0} |\Delta_{2n+1}^2| x^{2n+1} \\ &= \sum_{n\geq 0} |\Delta_n^1| (x^2)^n + x \sum_{n\geq 0} |\overline{\mathcal{E}_{n+1}^1}| (x^2)^n \end{aligned}$$

(by Theorem 2.1, Assertions 9 and 11)

$$(23)\qquad = \frac{1+x^4}{1+x^6} + \frac{x(1+x^2)}{1+x^6} = \frac{1+x}{1+x^3}$$

(by (22));

$$\begin{aligned} f^{(2)} &= \sum_{n\geq 0} |\mathcal{E}_n^2| x^n = \sum_{n\geq 0} |\mathcal{E}_{2n}^2| x^{2n} + \sum_{n\geq 0} |\mathcal{E}_{2n+1}^2| x^{2n+1} \\ &= \sum_{n\geq 0} \big(|\mathcal{E}_n^1| \cdot |\Delta_n^1| + |\overline{\mathcal{E}_n^1}| \cdot |\overline{\Delta_n^1}| \big) x^{2n} + x \sum_{n\geq 0} \big(|\mathcal{E}_{n+1}^1| \cdot |\Delta_n^1| + |\overline{\mathcal{E}_{n+1}^1}| \cdot |\overline{\Delta_n^1}| \big) x^{2n} \end{aligned}$$

(by Theorem 2.1, Assertions 5 and 7)

$$= \left(f^{(1)} \star g^{(1)} + \overline{f^{(1)}} * \overline{g^{(1)}}\right)^2 + x\left(\widehat{f^{(1)}} \star g^{(1)} + \widehat{\overline{f^{(1)}}} \star \overline{g^{(1)}}\right)$$

where, by (22),

$$\widehat{f^{(1)}} = \sum_{n\geq 0} |\mathcal{E}^1_{n+1}| x^n = \frac{x^2+x^3+x^4+x^5}{1+x^6}, \tag{24}$$

and

$$\widehat{\overline{f^{(1)}}} = \sum_{n\geq 0} |\overline{\mathcal{E}^1_{n+1}}| x^n = \frac{1+x}{1+x^3}. \tag{25}$$

By using Equalities (22), (24), and (25), we obtain

$$f^{(1)} \star g^{(1)} = \frac{1+x^3+x^5}{1+x^6}, \qquad \overline{f^{(1)}} \star \overline{g^{(1)}} = \frac{x+x^2+x^4}{1+x^6},$$

$$\widehat{f^{(1)}} \star g^{(1)} = \frac{x^2+x^3+x^5}{1+x^6}, \qquad \widehat{\overline{f^{(1)}}} \star \overline{g^{(1)}} = \frac{x+x^3+x^4}{1+x^6},$$

and

$$f^{(2)} = \frac{1}{(1+x)^2} + x\left(\frac{x+x^2+x^4+x^5}{1+x^6}\right)^2 = \frac{1+x^2+x^3+x^4+x^5}{1+x^6}. \tag{26}$$

In the same way one can compute $\overline{f^{(2)}}$ and $\overline{g^{(2)}}$:

$$\overline{f^{(2)}} = \frac{x}{1+x^3}, \quad \overline{g^{(2)}} = \frac{x+x^3}{1+x^6}. \tag{27}$$

THEOREM 2.2. — *For any $p \geq 0$, the sequences (modulo 2)*

$$\{|\mathcal{E}^p_n|\}_{n\geq 0}, \quad \{|\overline{\mathcal{E}^p_n}|\}_{n\geq 0}, \quad \{|\Delta^p_n|\}_{n\geq 0}, \quad \{|\overline{\Delta^p_n}|\}_{n\geq 0}$$

are all periodic. Furthermore, $3\cdot 2^k$ is a period if $2^k+1 \leq p \leq 2^{k+1}$.

Proof. — If $p = 0,1,2$, by equalities (22), (23), (26), and (27), these four sequences are periodic. Now, suppose $p \geq 3$. We shall prove by induction that $3\cdot 2^k$ is a period if $2^k+1 \leq p \leq 2^{k+1}$. By (27), the conclusion is true for $k=1$. Suppose that the conclusion is true for $p \leq 2^k$.

Consider now $2^k + 1 \leq p \leq 2^{k+1}$. If $p = 2q$, then $2^{k-1} + 1 \leq q \leq 2^k$, thus by Theorems 2.1.1 and 2.1.3, we have

$$
\begin{aligned}
|\mathcal{E}^p_{2n}| &\equiv |\mathcal{E}^q_n| \cdot |\Delta^q_n| + |\overline{\mathcal{E}^q_n}| \cdot |\overline{\Delta^q_n}|,\\
|\mathcal{E}^p_{2n+1}| &\equiv |\mathcal{E}^q_{n+1}| \cdot |\Delta^q_n| + |\overline{\mathcal{E}^q_{n+1}}| \cdot |\overline{\Delta^q_n}|.
\end{aligned}
\tag{28}
$$

On the other hand, by the induction hypothesis, all sequences occurring on the right hand sides of the equalities (28) have period $3 \cdot 2^{k-1}$, and so do the product and the sum of these sequences. It follows that the sequences $|\mathcal{E}^p_{2n}|$ and $|\mathcal{E}^p_{2n+1}|$ are both $3 \cdot 2^k$-periodic and that this holds also for the sequence $|\mathcal{E}^p_n|$. The case p odd can be elucidated in the same way. Similar methods apply to the other three sequences. □

3. Automaticity properties.

In this section, we discuss further properties of the sequences introduced in Section 2. As the main result of this section, we prove that these sequences modulo 2 are all 2-automatic in the sense of Salon [12], [13]. As said in the introduction, automatic sequences have been widely and deeply studied in the recent years, as a general reference, one can read the survey by Dekking, Mendès France and van der Poorten [8] or the survey by Allouche [1]. For the two-dimensional automatic sequences, see [12], [13].

First of all, we recall one of the definitions of automatic sequences.

Let A be a finite alphabet and let A^* be the set of finite words. A substitution over A is a map $\sigma\colon\ A \to A^*$. If for any $a \in A$, the length of $\sigma(a)$ (*i.e.*, the number of letters of the word $\sigma(a)$) is equal to k, where $k > 1$ is an integer, then σ is called a k-substitution. Furthermore, if $a \in A$ is such that $\sigma(a) = aw$, $w \in A^*$, then

$$\sigma^\omega(a) \colon = \lim_{n\to\infty} \sigma^n(a)$$

defines an infinite word

$$x = x_1 x_2 \cdots x_n \cdots \in A^{\mathbb{N}}$$

(see [6], [4]), which is said to be generated by this k-substitution. Let B be another finite alphabet and let τ be a map $\tau\colon\ A \to B$, then the sequence

$$\tau(x) = \{\tau(x_n)\}_{n\geq 1}$$

is also called a k-substitutive sequence (Cobham [6] proved that a k-substitutive sequence can be produced by a k-automaton and vice versa). We saw in Section 1 that the Thue-Morse sequence is generated by the 2-substitution $\sigma : 1 \mapsto 10,\ 0 \mapsto 01$ and the period-doubling sequence by the substitution $\theta : 1 \mapsto 10,\ 0 \mapsto 11$.

The definition of a k-substitutive two-dimensional sequence [12], [13] is analogous, but a k-substitution in two dimensions associates with a single letter a "square" of letters of size k. For example define a two-dimensional 2-substitution as follows

$$0 \longmapsto \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}, \quad 1 \longmapsto \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}.$$

As previously, this operation can be iterated:

$$0 \longmapsto \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \longmapsto \begin{matrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{matrix} \longmapsto \cdots$$

Let $\{F_n^p\}_{n\geq 0,\, p\geq 0}$ be a double sequence. Its 2-kernel is the set of subsequences

$$\Big\{ \big(F_{2^k n+i}^{2^k p+j}\big)_{n\geq 0, p\geq 0},\ 0 \leq k,\ 0 \leq i, j \leq 2^k - 1 \Big\}.$$

It is known (see [12] , [13]) that a sequence is 2-automatic if and only if its 2-kernel is finite.

THEOREM 3.1.

(i) *The sequences (modulo 2)*

$$\{|\mathcal{E}_n^p|\}_{n\geq 1, p\geq 0}, \quad \{|\overline{\mathcal{E}_n^p}|\}_{n\geq 1, p\geq 0}, \quad \{|\Delta_n^p|\}_{n\geq 1, p\geq 0}, \quad \{|\overline{\Delta_n^p}|\}_{n\geq 1, p\geq 0}$$

are all 2-automatic.

(ii) *For any $n \geq 1$, the sequences (modulo 2)*

$$\{|\mathcal{E}_n^p|\}_{p\geq 0}, \quad \{|\overline{\mathcal{E}_n^p}|\}_{p\geq 0}, \quad \{|\Delta_n^p|\}_{p\geq 0}, \quad \{|\overline{\Delta_n^p}|\}_{p\geq 0}$$

are all 2-automatic.

Proof. — For α and β in $\{0,1\}$, define two operations

$$S_\alpha^\beta u = \left\{u_{n+\alpha}^{p+\beta}\right\}_{n\geq 0,\, p\geq 0}, \quad D_\alpha^\beta u = \left\{u_{2n+\alpha}^{2p+\beta}\right\}_{n\geq 0,\, p\geq 0}$$

on sequences $\left\{u_n^p\right\}_{n\geq 0,\, p\geq 0}$. One has

$$D_\alpha^\beta S_{\alpha'}^{\beta'} = \begin{cases} D_{\alpha+\alpha'}^{\beta+\beta'} & \text{if } \alpha+\alpha' \leq 1 \text{ and } \beta+\beta' \leq 1, \\ S_0^1 D_{\alpha+\alpha'}^0 & \text{if } \alpha+\alpha' \leq 1 \text{ and } \beta+\beta' = 2, \\ S_1^0 D_0^{\beta+\beta'} & \text{if } \alpha+\alpha' = 2 \text{ and } \beta+\beta' \leq 1, \\ S_1^1 D_0^0 & \text{if } \alpha+\alpha' = 2 \text{ and } \beta+\beta' = 2. \end{cases} \tag{29}$$

Let $\mathcal{E}$, $\overline{\mathcal{E}}$, Δ, and $\overline{\Delta}$ stand for the sequences $\left\{\mathcal{E}_n^p\right\}_{n\geq 0,\, p\geq 0}$, $\left\{\overline{\mathcal{E}_n^p}\right\}_{n\geq 0,\, p\geq 0}$, $\left\{\Delta_n^p\right\}_{n\geq 0,\, p\geq 0}$, and $\left\{\overline{\Delta_n^p}\right\}_{n\geq 0,\, p\geq 0}$ modulo 2.

Theorem 2.1 together with Proposition 2.1 can be reformulated in the following way:

$$\begin{cases} D_0^0\mathcal{E} \equiv \mathcal{E}\cdot\Delta + \overline{\mathcal{E}}\cdot\overline{\Delta}, & D_0^1\mathcal{E} \equiv S_0^1\mathcal{E}\cdot\Delta + S_0^1\overline{\mathcal{E}}\cdot\Delta + S_0^1\overline{\mathcal{E}}\cdot\overline{\Delta}, \\ D_0^0\overline{\mathcal{E}} \equiv \overline{\mathcal{E}}\cdot\Delta, & D_0^1\overline{\mathcal{E}} \equiv S_0^1\overline{\mathcal{E}}\cdot\Delta, \\ D_0^0\Delta \equiv \Delta, & D_0^1\Delta \equiv \Delta\cdot S_0^1\Delta + \Delta\cdot S_0^1\overline{\Delta} + \overline{\Delta}\cdot S_0^1\Delta, \\ D_0^0\overline{\Delta} \equiv 0, & D_0^1\overline{\Delta} \equiv \Delta\cdot S_0^1\overline{\Delta} + \overline{\Delta}\cdot S_0^1\Delta, \\ D_1^0\mathcal{E} \equiv S_1^0\mathcal{E}\cdot\Delta + S_1^0\overline{\mathcal{E}}\cdot\overline{\Delta}, & D_1^1\mathcal{E} \equiv S_1^0\mathcal{E}\cdot S_0^1\Delta + S_1^0\overline{\mathcal{E}}\cdot S_0^1\Delta \\ & \qquad + S_1^0\overline{\mathcal{E}}\cdot S_0^1\overline{\Delta}, \\ D_1^0\overline{\mathcal{E}} \equiv S_1^0\overline{\mathcal{E}}\cdot\Delta, & D_1^1\overline{\mathcal{E}} \equiv S_1^0\overline{\mathcal{E}}\cdot S_0^1\Delta, \\ D_1^0\Delta \equiv S_1^0\overline{\mathcal{E}}, & D_1^1\Delta \equiv S_1^0\Delta\cdot S_0^1\Delta + S_1^0\Delta\cdot S_0^1\overline{\Delta} \\ & \qquad + S_1^0\overline{\Delta}\cdot S_0^1\Delta, \\ D_1^0\overline{\Delta} \equiv S_1^0\overline{\mathcal{E}}, & D_1^1\overline{\Delta} \equiv S_1^0\Delta\cdot S_0^1\overline{\Delta} + S_1^0\overline{\Delta}\cdot S_0^1\Delta. \end{cases} \tag{30}$$

(i) Set $\mathcal{X} = \left\{\mathcal{E}, \overline{\mathcal{E}}, \Delta, \overline{\Delta}\right\}$ and $\mathcal{Y} = \left\{S_\alpha^\beta F \mid F\in\mathcal{X},\ \alpha=0,1,\ \beta=0,1\right\}$. It results from (29) and (30) that, for any $F\in\mathcal{Y}$ and α and β in $\{0,1\}$, $D_\alpha^\beta F$ can be expressed as a polynomial with coefficients in GF_2 of the elements of $\mathcal{Y}$. As the elements of the 2-kernel of a sequence are obtained by successive applications of operators D_α^β, it follows that the 2-kernels of the sequences $\mathcal{E}$, $\overline{\mathcal{E}}$, Δ, and $\overline{\Delta}$ are included in the set of sequences that are polynomials in sequences from $\mathcal{Y}$.

But, there is only a finite number of polynomial functions on GF_2 with sixteen variables. Therefore, these 2-kernels are finite. Then, it results from [12], [13] that the sequences $|\mathcal{E}_n^p|_{n\geq 1,p\geq 0}$, $|\Delta_n^p|_{n\geq 1,p\geq 0}$, $|\overline{\mathcal{E}_n^p}|_{n\geq 1,p\geq 0}$, and $|\overline{\Delta_n^p}|_{n\geq 1,p\geq 0}$ (modulo 2) are 2-automatic.

(ii) An immediate consequence of a result of Salon [12], [13] is that, if the double sequence $(F_n^p)_{n,p}$ is 2-automatic, then, for any fixed $n \geq 1$, the sequence $(F_n^p)_p$ is 2-automatic, which proves our claim.

Alternatively we give a direct proof. Let $\epsilon = \epsilon_0\epsilon_1 \cdots \epsilon_n \cdots \in \{0,1\}^{\mathbb{N}}$ be the Thue-Morse sequence. For $n \geq 1$, let Ω_{2n+1} be the set of all subwords of ϵ of length $2n+1$. Now the 2-substitution σ above induces a new 2-substitution σ_n on Ω_{2n+1} in the following way: let $\omega = \omega_0\omega_1 \cdots \omega_{2n}$ be an element of Ω_{2n+1}; if

$$\sigma(\omega) = \sigma(\omega_0\omega_1 \cdots \omega_{2n}) = \sigma(\omega_0) \cdots \sigma(\omega_{2n}) = \eta_0\eta_1 \cdots \eta_{4n+1},$$

then we set

$$\sigma_n(\omega) = (\eta_0\eta_1 \cdots \eta_{2n})(\eta_1\eta_2 \cdots \eta_{2n+1}) \in \Omega_{2n+1}^2.$$

It is easy to check that

$$\sigma_n^{\omega}(u_0) = \lim_{k\to\infty} \sigma_n^k(u_0) = u_0u_1 \cdots \in \Omega_{2n+1}^{\mathbb{N}},$$

where u_j is the block $\epsilon_j\epsilon_{j+1} \ldots \epsilon_{j+2n}$. This means that the sequence u is generated by the 2-substitution σ_n. Now define the map $\tau_n : \Omega_{2n+1} \to \{0,1\}$ by $\tau_n(u_p) := |\mathcal{E}_n^p|$ (mod 2). Then, the image of the sequence $(u_p)_{p\geq 0}$ under τ_n is equal to the sequence $\{|\mathcal{E}_n^p|\}_{p\geq 0}$ modulo 2. Hence the sequence $\{|\mathcal{E}_n^p|\}_{p\geq 0}$ modulo 2 is 2-automatic. It can be proved in the same way that the other three sequences are 2-automatic. □

Remark 3.1. — If $u = u_0u_1 \cdots u_n \cdots$ is an automatic sequence that can be generated by a primitive substitution, then the sequence is minimal: any factor of the sequence u occurs in u with a non-zero frequency [6]. Hence by Theorem 3.1, we have the following corollary.

COROLLARY 3.1. — *For any $p \geq 0$ (resp. $n \geq 1$), there are infinitely many numbers n (resp. p), such that $|\mathcal{E}_n^p| \neq 0$. The same property holds also for the sequence $|\Delta_n^p|$.*

Proof. — For a fixed $n \geq 1$ consider the map $\Lambda : \{0,1\}^{2n-1} \to \{0,1\}$ defined by

$$\Lambda(a_1, \ldots, a_{2n-1}) = \det\{a_{i+j-1}\}_{1\leq i,\, j\leq n}.$$

One has $\Lambda(\epsilon_p, \ldots, \epsilon_{p+2n-2}) = |\mathcal{E}_n^p|$, for all $p \geq 0$. As the Thue-Morse sequence is minimal, each block $a_1 \cdots a_{2n-1}$ that occurs in the Thue-Morse sequence occurs an infinite number of times and with bounded gaps. Hence we have the same property for the "block" (actually the letter) $\Lambda(a_1, \ldots, a_{2n-1})$ in the sequence $(|\mathcal{E}_n^p|)_{p \geq 0}$. Hence the frequency of any letter in the sequence $(|\mathcal{E}_n^p|)_{p \geq 0}$ is strictly positive and this sequence is not 0 identically.

For a fixed $p \geq 0$ the sequence $n \mapsto (|\mathcal{E}_n^p|)_n$ is periodic and it is not identical to zero, (one has $|\mathcal{E}_1^p| = \epsilon_p$). Hence any letter occurs an infinite number of times in the sequence $(|\mathcal{E}_n^p|)_{n \geq 1}$. □

4. Applications.

Now we consider another form of the Thue-Morse sequence which will be used in the applications below. In the definition of the introduction, if we take $a = 1$, $b = -1$, then we obtain an infinite sequence $\eta = \eta_0 \eta_1 \cdots \eta_n \cdots \in \{1, -1\}^{\mathbb{N}}$ which satisfies the recurrence equations $\eta_0 = 1$, $\eta_{2n} = \eta_n$, $\eta_{2n+1} = -\eta_n$. We define $\pi_n = -\eta_{n+1}\eta_n$ (note that $\left\{\frac{1}{2}(1 + \pi_n)\right\}$ is nothing but the period doubling sequence). The Hankel determinants corresponding to η and π are denoted by $|A_n^p|$ and $|B_n^p|$.

We clearly have

$$\eta_n = 2\epsilon_n - 1, \quad \pi_n = 2|\delta_n| - 1. \tag{31}$$

The following proposition relates $|A_n^p|$ and $|B_n^p|$ to $|\mathcal{E}_n^p|$ and $|\Delta_n^p|$.

Proposition 4.1. — *We have*

$$2^{1-n}|A_n^p| = |\overline{\mathcal{E}_n^p}| + 2|\mathcal{E}_n^p| \equiv |\overline{\mathcal{E}_n^p}|,$$

$$2^{1-n}|B_n^p| = |\overline{\widetilde{\Delta}_n^p}| + 2|\widetilde{\Delta}_n^p| \equiv |\overline{\Delta_n^p}|,$$

where $\widetilde{\Delta}_n^p$ is the (n, p)-Hankel matrix of the sequence $|\delta_j|$.

Proof. — This proposition results from Lemma 1.4 and Formula (31). □

A direct consequence of Theorems 2.1, 2.2, 3.1 and Proposition 4.1 is the following.

THEOREM 4.1. — *The sequences modulo* 2

$$\{2^{1-n}|A_n^p|\}_{n\geq 1, p\geq 0}, \quad \{2^{1-n}|\overline{A_n^p}|\}_{n\geq 1, p\geq 0},$$
$$\{2^{1-n}|B_n^p|\}_{n\geq 1, p\geq 0}, \quad \{2^{1-n}|\overline{B_n^p}|\}_{n\geq 1, p\geq 0}$$

are 2-automatic.

For any $n \geq 1$, the sequences modulo 2

$$\{2^{1-n}|A_n^p|\}_{p\geq 0}, \quad \{2^{1-n}|\overline{A_n^p}|\}_{p\geq 0},$$
$$\{2^{1-n}|B_n^p|\}_{p\geq 0}, \quad \{2^{1-n}|\overline{B_n^p}|\}_{p\geq 0}$$

are 2-automatic.

For any $p \geq 0$, the sequences modulo 2

$$\{2^{1-n}|A_n^p|\}_{n\geq 1}, \quad \{2^{1-n}|\overline{A_n^p}|\}_{n\geq 1},$$
$$\{2^{1-n}|B_n^p|\}_{n\geq 1}, \quad \{2^{1-n}|\overline{B_n^p}|\}_{n\geq 1}$$

are periodic. □

From the theorem above and Propositions 2.2, 4.1, we get the following corollary.

COROLLARY 4.1. — *For $n \geq 1$, one has $2^{1-n}|A_n^0| \equiv 1$ (mod 2). In particular, $|A_n^0| \neq 0$ for $n \geq 1$.* □

By a remark similar to Remark 3.1, we have the following result.

COROLLARY 4.2. — *For any $n \geq 1$ (resp. $p \geq 0$) there are infinitely many integers p (resp. n) such that $|A_n^p| \neq 0$. The conclusion is also valid for the sequence $(|B_n^p|)$.* □

Now we discuss the strongly nonrepetitive structure of the Thue-Morse sequence.

Let $A = \{a, b\}$. If $w \in A^*$ is a finite word, we denote by $\widetilde{w}$ the word obtained by flipping a and b in it.

Let $u = u_0u_1 \ldots u_n \cdots \in A^{\mathbb{N}}$ and let $w_{p,n} = u_p \cdots u_{p+n}$ be a factor (american terminology: subword) of u. We say that a word $w_{p,2n}$ is nonrepetitive (resp. strongly nonrepetitive) if, for all k and ℓ such that $p \le k < \ell \le p+n$, we have $w_{k,n} \ne w_{\ell,n}$ (resp. $w_{k,n} \ne w_{\ell,n}$ and $w_{k,n} \ne \widetilde{w}_{\ell,n}$). It is known that any factor $w_{p,2n}$ of the Thue-Morse sequence is nonrepetitive [14], [15]. Notice now that if $w_{k,n} = \eta_k\eta_{k+1}\cdots\eta_{k+n}$, then $\widetilde{w}_{k,n}$ is just the word $(-\eta_k)(-\eta_{k+1})\ldots(-\eta_{k+n})$. Hence, if the (p,n)-order Hankel determinant $|A_n^p|$ is nonzero, then the word $w_{p,2n}$ is strongly nonrepetitive. Thus, by Corollaries 4.1 and 4.2, we obtain the following theorem.

THEOREM 4.2. — *Let $u = u_0u_1\ldots u_n\ldots \in A^{\mathbb{N}}$ be the Thue-Morse sequence, then, for any $n \ge 1$, the words $w_{0,2n}$ are strongly nonrepetitive. Furthermore, for any $p \ge 1$, there are infinitely many n such that the words $w_{p,2n}$ are strongly nonrepetitive.* □

Remark 4.1. — Notice that $w_{2,2} = bab$, $w_{2,1} = ba$, $w_{3,1} = ab$, so $w_{2,1} = \widetilde{w}_{3,1}$, that is, the word $w_{2,2}$ is not strongly nonrepetitive. Thus, in this sense, the theorem cannot be improved.

Remark 4.2.

(i) Let $v = v_0v_1\cdots v_n\cdots \in A^{\mathbb{N}}$ be the period-doubling sequence. For $p = 0$, we do not have the same result as for the Thue-Morse sequence. In fact, consider $w_{0,2} = aba$, then $w_{0,1} = \widetilde{w}_{1,2} = ab$. Nevertheless, by Corollary 4.2, we still have that, for any p, there are infinitely many n, such that the words $w_{p,2n}$ are strongly nonrepetitive.

(ii) From Proposition 2.2, $|\Delta_n^0| \ne 0$, hence for any $n \ge 1$, the subword $w_{0,2n}$ of w is nonrepetitive. Furthermore, this result cannot be improved, for example, $w_{4,4} = ababa$, but $w_{4,2} = w_{6,2} = aba$.

On the other hand, from Remark 1.1, we see that, nonrepetitivity for the period-doubling sequence is equivalent to strong nonrepetitivity for the Thue-Morse sequence, hence, the first conclusion of the above theorem can be derived from Remark 4.2 (ii).

Now consider again the Thue-Morse sequence $\eta = \eta_0\eta_1\cdots\eta_n\cdots$ in $\{1,-1\}^{\mathbb{N}}$. Set

$$f(x) = \sum_{n\ge 0} \eta_n x^n;$$

then $f(x)$ is a transcendental function, and Cobham [5] proved that $f(x)$ is the unique solution of the following functional equation:

$$F(x^2) = \frac{F(x)}{1+x} - \frac{x}{(1+x)(1+x^2)}, \quad |x| < 1,$$

such that $F(0) = 1$, see also [7], [8].

Thus, we are naturally led to consider approximating $f(x)$ by rational functions. Nice candidates are the Padé approximants (if they exist).

A (p,q)-order Padé approximant of f, noted $\left[\frac{p}{q}\right]_f$, is a rational function $P(x)/Q(x)$ whose denominator has degree q and whose numerator has degree p such that

$$f(x) - \frac{P(x)}{Q(x)} = O(x^{p+q+1}), \quad x \to 0.$$

In particular, the approximants $\left[\frac{k-1}{k}\right]_f$ for $k \geq 1$ play an important rôle in the study of Padé approximant theory (for a general reference, see [2]). By a classical result, if $|A_n| \neq 0$, then the Padé approximant $\left[\frac{n-1}{n}\right]_f$ exists (furthermore, it can be expressed explicitly, see [3], pp. 34–36). Hence by Corollary 4.1, we have the following theorem.

THEOREM 4.3. — *Let $f(x) = \sum_{n\geq 0} \eta_n x^n$; then for any $n \geq 1$, the $(n-1,n)$-order Padé approximant of f exists.* □

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