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AREA INTEGRAL ESTIMATES FOR HIGHER ORDER ELLIPTIC EQUATIONS AND SYSTEMS

by B.E.J. DAHLBERG, C.E. KENIG, J. PIPHER and G.C. VERCHOTA

Introduction.

The classical formulation of boundary value problems for constant coefficient elliptic operators, or systems of operators, involves continuous data on the boundary of a (smooth) domain and leads to the existence and uniqueness of solutions continuous up to the boundary. If the given data is not continuous but exists only in some L^p space, it may still be possible to obtain unique solutions with this data. The data would be taken on in the sense of nontangential limits and one obtains nontangential L^p estimates on the solution which guarantee uniqueness in this class.

Let us recall the situation for harmonic functions in, say, the upper half space $\mathbb{R}^n_+$ [30]. Given data $f(x) \in L^p(\mathbb{R}^{n-1})$, $1 < p \leq \infty$, the function $u(x,y) = P_y * f(x)$ (where $P_y(x)$ denotes the Poisson kernel) is the unique solution to the Dirichlet problem : $\Delta u = 0$, $u|_{y=0} = f(x)$ with appropriate decay at ∞, and for which the nontangial maximal function of u is also in $L^p(\mathbb{R}^{n-1})$. The nontangential maximal function of u is $Nu(x) = \sup\{|u(x',y)| : |x'-x| < cy\}$, the supremum of values of u taken in the cone $\Gamma(x) = \{(x',y) : |x'-x| < cy\}$ with aperture determined by c. The nontangential maximal function plays a

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rule analogous to the role of the Hardy-Littlewood maximal function in the context of the Lebesgue differentiation theorem – via this maximal function, nontangential limits are controlled. (In fact, for the Poisson integral of an L^p function $u(x,y) = P_y * f(x)$, $Nu(x)$ and the Hardy-Littlewood maximal function of f are comparable.) In the setting of harmonic functions, the L^p-Dirichlet problem is uniquely solvable if, for every $f \in L^p(\mathbb{R}^{n-1})$, there exists a harmonic u which converges (nontangentially, i.e. through sequences $(x',y) \to (x,0)$ restricted to the cone at x) to the data, and which is in the class $Nu \in L^p$. Then as a consequence of linearity one has the estimate $\|N(u)\|_{L^p(R^{n-1})} \leq C\|f\|_{L^p(R^{n-1})}$.

The L^p-behavior of the nontangential maximal function gives very precise control over the growth of solutions to this elliptic boundary value problem, but it is not the only means of obtaining such control. One can also measure the L^p norm of the square function of solutions u, defined for $x \in \mathbb{R}^{n-1}$ by $S(u)(x) = \{\int_{\Gamma(x)} |\nabla u(x',y)|^2 y^{2-n} dx'dy\}^{\frac{1}{2}}$. There are several reasons one might prefer L^p estimates on the square function of a solution : the 'geometric' content of this quadratic expression, the connection with measuring Sobolev/Besov norms and the invariance of the square function under certain important linear operators that may not always be singular integral operators (e.g. Riesz transforms).

Our objective in this paper is to demonstrate the L^p equivalence $(0 < p < \infty)$ between the nontangential maximum function and the square function of solutions to the homogeneous equation for higher order elliptic systems on Lipschitz domains. Such results, in special cases, have been proven in earlier works [10], [25], [5]. Here we make no restriction on the order of differentiation or the size of the (determined) systems. The systems considered consist of only a principle part, satisfy the Legendre-Hadamard condition and are real symmetric.

In extending the square function estimates from the situation of harmonic functions in $\mathbb{R}^n_+$, there are three essential difficulties to be overcome. First, the boundary of our domain Ω is not smooth and the quantity which replaces the factor of $y = \text{dist}((x,y);\mathbb{R}^{n-1})$ in the definition, namely $\delta(X) = \text{dist}(X, \partial\Omega)$, is not more than once differentiable. Second, the elliptic operators are of higher order and may not have a quadratic form which is coercive (see [2] for example). In particular this means that the quadratic form associated to the operator is not in an obvious way related to the quadratic expression used in the definition of the square function. (Such relations exists when the operator can be written as a sum of squares however.) Finally, the situation of higher order systems is yet more complicated, and we develop an argument for reducing the case of systems of equations to single equations.

We can illustrate the elementary new arguments used to prove the square function estimates on nonsmooth domains just in the case of functions harmonic in a Lipschitz domain Ω. The idea is to use a variant of the 'adapted' distance function of Dahlberg [11] invented by C. Kenig and E. Stein [2]. The adapted distance function has been used to prove quadratic estimates in other settings – [22] for Clifford valued monogenic functions and [18] for solutions to parabolic equations.

Let $\Omega \subset \mathbb{R}^n$ be the domain above the graph of a Lipschitz function and let $\delta(X)$ for $X \in \Omega$ be the adapted distance function as discussed in §1 below. The properties of δ we need here are (i) $\delta(X) \approx \operatorname{dist}(X, \partial\Omega)$, (ii) $\dfrac{\partial}{\partial X_n}\delta(X) = D_n\delta(X) > c' > 0$, (iii) $|\nabla\delta(X)| \leq c''$, (iv) $\delta(X)|\nabla\nabla\delta(X)|^2 dX$ is a Carleson measure, and (v) $\nabla D_n\delta = (D_n\delta)^2 \sum_j \nabla D_n\tilde{\delta}j$ where each $\tilde{\delta}_j$ has the property that $|D_n\tilde{\delta}_j(X)|^2\delta^{-1}(X)dX$ is a Carleson measure and $|D_n\tilde{\delta}_j(X)| \leq c''$.

Let us now give the argument which proves

$$\int_{\partial\Omega} S^2(u)\, d\sigma \approx \int_{\partial\Omega} N^2(u)\, d\sigma$$

for u harmonic in Ω, as above, and which illustrates the main ideas needed to prove the analogous result for higher order homogeneous elliptic equations possessing coercive bilinear forms.

We first note that to dominate the nontangential maximal function of a harmonic function u in Ω by the square function of u, it suffices to dominate $\|u\|_{L^2(\partial\Omega, d\sigma)}$ by $\|S(u)\|_{L^2(\partial\Omega, d\sigma)}$. For harmonic functions, this follows from Dahlberg's theorem on the L^2-solvability of the Dirichlet problem ([9]). For biharmonic and polyharmonic functions, one needs the L^2 solvability results of [14] and [32], and for the general case of higher order elliptic equations or systems of equations, one invokes the results of [27] and [33]. With this in mind, let us take a function u, harmonic in $\Omega \subset \mathbb{R}^n$, and show that

$$\int_{\partial\Omega} u^2\, d\sigma \leq C \int_{\partial\Omega} S^2(u)\, d\sigma = C \int_{X\in\Omega} \delta(X)|\nabla u(X)|^2\, dX$$

where $\delta(X)$ is the adapted distance function. If $N = (N_1, \dots, N_n)$ denotes the unit normal vector to the boundary of Ω, then $N_n > c > 0$ for some c

that depends only on the Lipschitz constant of Ω. Thus,

$$\begin{aligned}\int_{\partial\Omega} u^2\,d\sigma \le C\int_{\partial\Omega} u^2 N_n\,d\sigma &= -\int_\Omega D_n(u^2)\,dX\\ &= -2\int_\Omega uD_nu\,dX = -2\int_\Omega uD_nu\frac{D_n\delta}{D_n\delta}\,dX\\ &= -2\int_\Omega D_n\Big(\frac{\delta}{D_n\delta}uD_nu\Big) - \delta D_n\Big(\frac{uD_nu}{D_n\delta}\Big)\,dX.\end{aligned}$$

The first integral above is zero, since $\delta \equiv 0$ on $\partial\Omega$, and the second integral above gives rise to three terms, upon distributing the derivative.

$$\begin{aligned}&\int_\Omega \delta D_n\Big(\frac{uD_nu}{D_n\delta}\Big)\,dX\\ &= \int_\Omega \delta\frac{|D_nu|^2}{D_n\delta}\,dX + \int_\Omega \frac{\delta u}{D_n\delta}D_nD_nu\,dX + \int_\Omega \delta uD_nu\Big(\frac{-D_nD_n\delta}{(D_n\delta)^2}\Big)\,dX\\ &= \mathbf{1}+\mathbf{2}+\mathbf{3}.\end{aligned}$$

Term $\mathbf{3}$ is bounded by $C\Big(\int_\Omega \delta|D_nu|^2\Big)^{\frac12}\cdot\Big(\int_\Omega u^2\,\delta|\nabla\nabla\delta|^2\Big)^{\frac12}$, since $D_n\delta > c' > 0$. By the Carleson measure property of $\delta|\nabla\nabla\delta|^2\,dX$, this is in turn controlled by $\Big(\int_\Omega \delta|\nabla u|^2 dX\Big)^{\frac12}\cdot\Big(\int_{\partial\Omega} N^2(u)d\sigma\Big)^{\frac12}$. Term $\mathbf{1}$ is equivalent to $\int_{\partial\Omega} S^2(u)\,d\sigma$ and it remains to handle term $\mathbf{2}$. Here we shall introduce the quantity $1 = \sum_j N_j^2$, where at each X $N = \dfrac{\nabla\delta}{|\nabla\delta|}$ and so $N_j = \dfrac{D_j\delta}{|\nabla\delta|}$. Hence

$$\begin{aligned}\mathbf{2} &= \int_\Omega \frac{\delta u}{D_n\delta}D_nD_nu\sum_j N_j^2\,dX\\ &= \int_\Omega \frac{\delta u}{D_n\delta}\Big[N_n^2D_nD_nu + \sum_{j=1}^{n-1} N_jD_nD_nuN_j\Big]\,dX.\end{aligned}$$

Now $N_jD_nD_nu = (N_jD_n - N_nD_j)D_nu + N_nD_jD_nu$. The expression $N_jD_n - N_nD_j$ for $1 \le j \le n-1$ is a tangential derivative to the level sets of δ, and we shall integrate by parts noting that tangential derivatives of δ are zero. This may be done by using the co-area formula. So, for $1 \le j \le n-1$,

$$\begin{aligned}&\int_\Omega \frac{\delta uN_j}{D_n\delta}(N_jD_n - N_nD_j)D_nu\,dX\\ &\quad= \int_0^\infty t\,dt\int_{\delta=t}\frac{uN_j}{|\nabla\delta|D_n\delta}(N_jD_n - N_nD_j)D_nu\,d\sigma_t\\ &\quad= \int_\Omega \delta D_nu|\nabla\delta|(N_nD_j - N_jD_n)\Big(\frac{uN_j}{D_n\delta|\nabla\delta|}\Big)\,dX\end{aligned}$$

where $d\sigma_t$ denotes surface measure along $\delta = t$. It is by this application of the co-area formula that the integration by parts in the proof of the main lemma of §2 is done. When the tangential derivative falls on u, we get a term equivalent to $\int_\Omega \delta|\nabla u|^2$ and when the derivative falls on $(D_n\delta)^{-1}$ or $|\nabla\delta|^{-1}$ we again use the Carleson measure property and Cauchy-Schwartz to get a 'mixed' term of the form $\big(\int_{\partial\Omega} N^2(u)\big)^{\frac{1}{2}} \cdot \big(\int_{\partial\Omega} S^2(u)\big)^{\frac{1}{2}}$. The term where the derivative falls on N_j is also controlled by a product of this form. For $1 \leq j \leq n$, it remains to evaluate $\int_\Omega \frac{\partial u}{D_n\delta} N_j N_n D_j D_n u$. Again, we introduce tangential derivatives by expressing $N_j D_n D_j u = [N_j D_n - N_n D_j u] D_j u + N_n D_j D_j u$. The expression $\sum_{j=1}^{n-1} \frac{\delta u}{D_u\delta} N_n^2 D_j D_j u$ cancels with $\frac{\delta u}{D_n\delta} N_n^2 D_n D_n u$ by using harmonicity of u and the term with the tangential derivatives is handled exactly as before, integrating by parts. This settles one half of the L^2 equivalence. See Theorem 1 of §2 below.

We now give the argument for the converse inequality, which will require using the additional Carleson measure property (v) above

$$
\begin{aligned}
2\int_\Omega \sum_{j=1}^{n} |D_j u(X)|^2\, \delta(X) dX &= \int_\Omega \Delta(u^2)\delta \\
&= \int_\Omega \operatorname{div}[\delta\nabla(u^2)] - \nabla\delta\nabla(u^2) \\
&= -\int_\Omega \operatorname{div}(u^2\nabla\delta) + u^2\Delta\delta \\
&= \int_{\partial\Omega} u^2|\nabla\delta|\, d\sigma + \int_\Omega u^2\Delta\delta.
\end{aligned}
\tag{1}
$$

The first integral above is bounded by the desired term $\int N^2(u)\, d\sigma$ and to handle the second integral, we introduce the quantity $\frac{D_n\delta}{D_n\delta}$ and integrate by parts using D_n. Thus

$$
\int_\Omega u^2 \Delta\delta D_n\delta \frac{dX}{D_n\delta} = -\int_\Omega \delta D_n\Big[u^2\Delta\delta\frac{1}{D_n\delta}\Big]\, dX,
$$

and there are three terms arising from distributing the differentiation:

$$
\begin{aligned}
I &= -\int_\Omega \delta \cdot (2uD_n u)\Delta\delta (D_n\delta)^{-1} \\
&\leq C\Big(\int_\Omega \delta|D_n u|^2\Big)^{\frac{1}{2}} \cdot \Big(\int_\Omega u^2|\Delta\delta|^2 \frac{\delta}{D_n\delta}\Big)^{\frac{1}{2}} \\
&\leq \|S(u)\|_{L^2(d\sigma)} \cdot \|N(u)\|_{L^2(d\sigma)}.
\end{aligned}
$$

$$II = \int_\Omega \delta u^2 \Delta\delta \frac{D_n D_n \delta}{D_n \delta},$$

which is bounded by $\int_{\partial\Omega} N^2(u)\, d\sigma$.

$$\begin{aligned} III &= -\int_\Omega \delta u^2 \Delta D_n \delta \frac{1}{D_n\delta} \\ &= -\int_\Omega \operatorname{div}\left[\delta u^2 \frac{1}{D_n\delta}\nabla D_n\delta\right] - \nabla D_n\delta \cdot \nabla\Big(\frac{\delta u^2}{D_n\delta}\Big) \\ &= \int_\Omega \left[2\nabla D_n\delta\cdot\nabla u \frac{u\delta}{D_n\delta} - \nabla D_n\delta\cdot\Big(\frac{\nabla D_n\delta}{(D_n\delta)^2}\Big)u^2\delta + \nabla D_n\delta\cdot\nabla\delta\frac{u^2}{D_n\delta}\right]. \end{aligned}$$

The first integral is bounded by $(\|N(u)\|_{L^2})\cdot(\|S(u)\|_{L^2})$ and the second is bounded by $\|N(u)\|^2_{L^2}$. For the third term, we use the extra fact about $D_n\delta$, namely that $\nabla D_n\delta = (D_n\delta)^2 \sum_j \nabla D_n\tilde{\delta}_j$, where each $|D_n\tilde{\delta}_j|^2\delta^{-1}$ is Carleson. Let us replace $\nabla D_n\delta$ by the expression $(D_n\delta)^2\nabla D_n\tilde{\delta}$ and integrate by parts once again :

$$\begin{aligned} \int_\Omega \nabla D_n\tilde{\delta}\cdot\nabla\delta u^2 D_n\delta = &-\int_\Omega D_n\tilde{\delta}\Delta\delta u^2 D_n\delta - 2\int_\Omega D_n\tilde{\delta}\nabla\delta\cdot\nabla u\, u D_n\delta \\ &- \int_\Omega D_n\tilde{\delta} u^2\nabla\delta\cdot\nabla D_n\delta + \int_{\partial\Omega} D_n\tilde{\delta}|\nabla\delta|u^2 D_n\delta\, d\sigma. \end{aligned}$$

The first integral is dominated by

$$\Big(\int_\Omega |D_n\tilde{\delta}|^2\delta^{-1}u^2\Big)^{\frac12}\cdot\Big(\int_\Omega u^2|\Delta\delta|^2\delta\Big)^{\frac12} \le C\int_{\partial\Omega} N^2(u)\, d\sigma,$$

using both of the Carleson measure properties of δ. The remaining integrals are handled similarly. See Theorem 2 of §3 below.

We note that the first equality in (1), which relates the square function to the differential operator and pertains to the coercivity problem mentioned above, is generalized by line (14) of §3 below. We also note that the L^p equivalence ($p \neq 2$) between nontangential maximal functions and square functions of solutions can be obtained via the powerful technique of good-λ inequalities ([6]). We briefly sketch the arguments in the higher order situation in §4.

We shall now point out some important applications of Theorems 1 and 2. The first concerns the weak maximum principle in non-smooth domains. It was shown in [26] that a weak maximum exists for gradients of biharmonic functions in Lipschitz domains in $\mathbb{R}^2$ or $\mathbb{R}^3$ or in C^1 domains in $\mathbb{R}^n$, any n. That is, if $\Delta\Delta u = 0$ in Ω and $N(\nabla u) \in L^2(d\sigma)$ then in fact $||\nabla||_{L^\infty(\Omega)} \le C||\nabla u||_{L^\infty(\Omega)}$. (This was later proved in [28] for $\nabla^{m-1}u$ for $\Delta^m u = 0$ in such domains.) As noted in [26] such an estimate has new consequences for the classical Dirichlet problem posed for continuous data. The main ingredients necessary to derive weak maximum principle are the following. First, one requires L^2 solvability of the Dirichlet and Regularity problems (see [27] and [33] for higher order equations and systems). Second, and most important, one must extend solvability of the Regularity problem to p near 1. In three dimensions, for second order elliptic systems, this is first done in [13]. These ideas were used in [26] for the biharmonic equation. Finally, one needs an L^p relationship between solutions and their Riesz transforms for p near 1. This can be obtained from the L^p equivalence between nontangential maximal functions and square functions. The observation that these three ingredients lead to weak maximum principles was first made in [26], but see also [13], [4] and [29] for other applications. For a sketch of the proof of the weak maximum principle for higher order systems see [33].

Precise Sobolev/Besov space estimates on solutions are a second important application of square function estimates. These in turn lead to the existence and uniqueness of inhomogeneous Dirichlet problems for these higher order equations and systems of equations. Such a program has been carried out in the case of Laplace's equation in Lipschitz and C^1 domains in [19] and for the biharmonic operator in [1]. These inhomogeneous problems with Sobolev space data reduce to homogeneous problems with data in the appropriate trace spaces. One of the key ingredients in carrying out this program is determining precisely which Sobolev spaces tie the solution to Dirichlet or regularity problems with L^p boundary data. Thus, for example, the estimate

$$\int_\Omega \delta(X)|\nabla^m u(X)|^2\, dX < +\infty$$

for solutions u to $2m$-order elliptic operators, together with interior estimates implies that $\nabla^{m-1}u$ belongs to the Besov space $\Lambda^{2,2}_{\frac{1}{2}}(\Omega)$.

1. Preliminaries.

We define an elliptic symmetric K-system as follows. By D we denote the vector of first partial derivatives $\left(\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}\right)$ with $X = (X_1, \dots, X_n) \in \mathbb{R}^n$. Let $L^{kl} = \sum_{|\alpha|=|\beta|=m} D^\alpha a^{kl}_{\alpha\beta} D^\beta$, for m, k, and l positive integers, $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ multi-indices, and $a^{kl}_{\alpha\beta}$ real constants. Let $L^{lk}(\xi) = \sum_{|\alpha|=|\beta|=m} \xi^\alpha a^{kl}_{\alpha\beta}\xi^\beta$ for $\xi \in \mathbb{R}^n$. Then L is a symmetric K-system if $Lu = \sum_{l=1}^{K} L^{kl}u^l$ for $u = (u^1, \cdots, u^K)$, $k = 1, \cdots K$ and $L^{kl} = L^{lk}$. If in addition, we assume that the Legendre-Hadamard condition holds : $\operatorname{Re} \sum_{k,l=1}^{K} L^{kl}(\xi)\zeta_k\overline{\zeta}_l \geq E|\xi|^{2m}|\zeta|^2$ for some $E > 0$, $\zeta = (\zeta_1, \cdots, \zeta_n) \in \mathbb{C}^n$, $\xi \in \mathbb{R}^n$, then L is a (strongly) elliptic symmetric K-system. When $K = 1$, L is in general a single higher order elliptic equation.

$\Omega \subset \mathbb{R}^n$ is Lipschitz if there are a finite number of neighborhoods that cover the boundary $\partial\Omega$ so that within each neighborhood, $\partial\Omega$ may be given as the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$. The maximum of the Lipschitz norms of each φ together with the number of neighborhoods essentially describe the Lipschitz nature of Ω. See, for example, [9], [27]. Dirichlet data will be taken in $L^p(\partial\Omega)$ spaces with respect to surface measure $d\sigma$ where in general $1 < p \leq \infty$. Thus boundary values will be taken in the sense of nontangential convergence a.e. $d\sigma$. To do this we define nontangential approach regions for each $Q \in \partial D$

$$\Gamma(Q) = \Gamma_a(Q) = \{X \in \Omega : |X - Q| \leq (1 + a)\operatorname{dist}(X, \partial\Omega)\}$$

where $a > 0$ is taken large enough depending on the Lipschitz constant associated with Ω. Then if $\lim_{\substack{X \to Q \\ X \in \Gamma(Q)}} u(X) = g(Q)$ we say that u has nontangential limit $g(Q)$ at Q.

The nontangential maximal function of a function u in Ω is defined by

$$N(u)(Q) = \sup_{X \in \Gamma(Q)} |u(X)|.$$

Let Ω be the domain above the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$. We will need to recall the *adapted distance function* associated to Ω introduced in [11] as well as a version of this defined by [21]. If $F(x,t) = ct + \eta_t * \varphi(x)$, where $\eta_t(x) = \frac{1}{t^{n-1}}\eta(\frac{x}{t})$ is a smooth compactly supported bump function, and if c is sufficiently large (depending only on $\|\nabla\varphi\|_\infty$), then $\frac{\partial F}{\partial t} > c' > 0$ for $c' = c'(\|\nabla\varphi\|\infty)$. Thus, every $X \in \Omega$ can be represented as $(x, F(x,t))$ for $x \in \mathbb{R}^{n-1}$, $t > 0$, and one defines $\delta(x, F(x,t)) = t$. Then $\delta(X) \approx \operatorname{dist}(X, \partial\Omega)$, $D_n\delta(X) > C'' > 0$, $|\nabla\delta(X)| \leq C'''$, and $\delta(X)|\nabla\nabla\delta(X)|^2 dX$ is a Carleson measure. It is easy to see that $t|\nabla\nabla F(x,t)|^2 dxdt$ is a Carleson measure and the Carleson property of δ follows from the relationships between the derivatives of δ and of F.

Transverse differentiation of F or δ has an additional important property. If we define $\Psi^i(y) = y_i\eta(y)$, then $D_t\eta_t(y) = -\sum_i \frac{\partial}{\partial y_i}\Psi^i_t(y)$, so that if η is chosen appropriately to be radial and even, $\int \Psi^i(y) = 0$ for each i and so $\frac{1}{t}|\Psi^i_t * b(x)|^2 dxdt$ is also a Carleson measure for any $b \in L^\infty$. In terms of δ this means that $\nabla D_n\delta$ may be replaced by terms of the form $\nabla D_n\overline{\delta}$ where $\overline{\delta}$ has the properties that $|D_n\overline{\delta}(X)|^2\delta^{-1}(X)dX$ is a Carleson measure and $|D_n\overline{\delta}|$ is uniformly bounded depending only on $\|\nabla_\varphi\|_\infty$. In fact we may take $D_n\overline{\delta}(X) = D_n\overline{\delta}_j(X) = \Psi^j_t * \frac{\partial}{\partial y_j}\varphi(x)$, $(1 \le j \le n-1)$. These facts will prove useful in proving Theorem 2 of §3. In general the Carleson measure properties of δ that will be used take the form

$$\int_\Omega |u|^p|\nabla\nabla\delta|^2\delta dX \le C\int_{\partial\Omega} N(u)^p d\sigma$$

and

$$\int_\Omega |u|^p|D_n\tilde{\delta}|^2\delta^{-1}dX \le C\int_{\partial\Omega} N(u)^p d\sigma, \quad 0 < p < \infty.$$

The square function for a function u in $\Omega \subset \mathbb{R}^n$ is defined by

$$S(u)(Q) = \left(\int_{\Gamma(Q)} \frac{|\nabla u(X)|^2}{|X-Q|^{n-2}}\,dX\right)^{1/2}.$$

For solutions to elliptic symmetric K-systems of order $2m$ $Lu = 0$ it will be established in §4 that for any $0 < p < \infty$

$$\int_{\partial\Omega} N(\nabla^{m-1}u)^p\,d\sigma$$

is equivalent to

$$\int_{\partial\Omega} S(\nabla^{m-1}u)^p\,d\sigma.$$

When $p = 2$, by Fubini's theorem, this equivalence is the same as the equivalence between

$$\int_{\partial\Omega} N(\nabla^{m-1}u)^2\,d\sigma$$

and

$$\int_\Omega |\nabla^m u(X)|^2\,\mathrm{dist}(X)\,dX.$$

This latter equivalence is proved in §§2 and 3.

The following lemma is a standard interior estimate for solutions v of K-systems $Lv = f$ ([15], p. 517, [3]) using L^2 averages. To express it in this way requires an interior estimate assumption on f itself, which hypothesis will be met in the cases of interest to us. The distance of X to $\partial\Omega$ will be denoted $\operatorname{dist}(X)$.

LEMMA I (Interior estimates). — *Let L be a $2m$-order K-system as above and let $v \in C^{2m}(\overline{\Omega})$ satisfy $Lv = f$ on $\Omega \subseteq \mathbb{R}^n$, a Lipschitz domain. Suppose further that $\sup_{B_r}|\nabla^k f| \leq Cr^{-k} \fint_{B_{2r}}|g|$ where g satisfies an interior estimate : $\exists\, C_1 > 0$, so that*

$$\sup_{B_r}|g| \leq C_1\Big(\fint_{B_{2r}}|g|^2\Big)^{\frac{1}{2}} \quad \text{for } B_r \subseteq \Omega$$

with $\operatorname{dist}(B_{3r}, \partial\Omega) \approx r$. Then, for some $c_2 > 0$ independent of v, but depending on the Lipschitz character of Ω,

$$\int_\Omega \operatorname{dist}^{2i+1}(X)|\nabla^{m+i}v(X)|^2 dX \leq c_2\Big\{\int_\Omega |\nabla^m v(X)|^2 \operatorname{dist}(X)dX + \int_\Omega \operatorname{dist}^{2m+1}(X)|g(X)|^2 dX\Big\}.$$

LEMMA II. — *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and fix a point $P^* \in \Omega$. Let $F \in C^1(\overline{\Omega})$ with $F(P^*) = 0$. Then for any $\varepsilon > 0$ there exists $C_\varepsilon < \infty$ depending only ε, P^* and the Lipschitz nature of Ω so that*

$$\int_\Omega |F(X)|^2 \operatorname{dist}(X)dX \leq \varepsilon \int_{\partial\Omega} N(F)^2 d\sigma + C_\varepsilon \int_\Omega |\nabla F|^2 \operatorname{dist}(X)dX.$$

Proof. — By [24] (see for example [31]) Ω may be approximated by domains $\Omega' \subset \Omega$ with Lipschitz nature the same as that of Ω so that there is a homeomorphism $\Lambda : \partial\Omega \to \partial\Omega'$ with $\max_{Q\in\partial\Omega}|Q - \Lambda(Q)|$ as small as we want. Choosing an appropriate Ω' the first term on the right dominates $\int_{\Omega\setminus\Omega'}|F|^2 \operatorname{dist}(X)dX$ and by the standard Poincaré inequality the remainder of the left side may be dominated by the second term on the right. □

LEMMA III. — *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $F \in C^1(\overline{\Omega})$. Then there exists a Lipschitz domain $\overline{\Omega}'' \subset \Omega$ and a constant C with C and $\operatorname{dist}(\Omega'', \partial\Omega) > 0$*

$$(1) \qquad \int_{\partial\Omega} N(F)^2 d\sigma \leq C\int_{\partial\Omega} N(\nabla F)^2 d\sigma + C\int_{\Omega''} F^2 dX.$$

Proof. — By the fundamental theorem of calculus we have for each $Q \in \partial\Omega$

$$N(F)(Q) \leq |F(Q)| + CN(\nabla F)(Q).$$

Similarly with a domain Ω' and homeomorphism Λ as in the proof of the last lemma we get

$$|F(Q)| \leq |F(\Lambda(Q))| + CN(\nabla F)(Q).$$

Thus,

$$\int_{\partial\Omega} N(F)^2 d\sigma \leq C \int_{\partial\Omega} N(\nabla F)^2 d\sigma + C \int_{\partial\Omega'} F^2 d\sigma'.$$

Now one can average with respect to a continuum of such domains Ω' and obtain (1). □

2. Square function dominates maximal function in L^2.

The essence of the argument showing that the L^2 norm of the nontangential maximal function is dominated by the L^2 norm of the square function is in the proofs of the next two lemmas.

MAIN LEMMA 1. — *Let Ω be the domain above a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ and let $v \in C^{2m}(\overline{\Omega})$ be compactly supported in $\mathbb{R}^n$ and satisfy $Lv = f$ for any elliptic K-system L of order $2m$ as in §1. Let $\delta(X)$ be the adapted distance function of §1. Then there is a constant C depending only on $\|\nabla\varphi\|_\infty, m, n, K$ and E so that*

$$\begin{aligned}\int_{\partial\Omega} |\nabla^{m-1}v|^2 N_n d\sigma \leq C\Big[&\int_\Omega |\nabla^m v|^2 \delta(X) dX \\ &+ \int_\Omega |g|^2 \delta^{2m+1}(X) dX \\ &+ \|N(\nabla^{m-1}v)\|_{L^2(\partial\Omega)} \Big\{ \Big(\int_\Omega |\nabla^m v|^2 \operatorname{dist}(X) dX \Big)^{\frac{1}{2}} \\ &+ \Big(\int_\Omega |g|^2 \delta^{2m+1} dX \Big)^{\frac{1}{2}} \Big\} + \int_\Omega |f||\nabla^{m-1}v| \delta^m(X) dX \Big]\end{aligned} \tag{1}$$

where g is as in Lemma I of §1.

To prove this we will repeatedly apply the inequality established below.

LEMMA 2. — *With the hypotheses of Lemma 1 and multi-indices $|\alpha| = m-1$ and $|\beta| \geq m$,*

$$
\begin{aligned}
\Big| \int_\Omega D^\alpha v\ D^\beta v (D_n\delta)^{m-|\beta|}\delta^{|\beta|-m} dX \Big| \\
\leq c \Big[\int_\Omega |\nabla^m v|^2 \operatorname{dist}(X) dX + \int_\Omega |g|^2 \delta^{2m+1} dX \\
+ \|N(\nabla^{m-1}v)\|_{L^2(\partial\Omega)} \cdot \Big(\int_\Omega |\nabla^m v(X)|^2 \operatorname{dist}(X) dX \Big)^{\frac{1}{2}} \\
+ \|N(\nabla^{m-1}v)\|_{L^2(\partial\Omega)} \cdot \Big(\int_\Omega |g|^2 \delta^{2m+1} dX \Big)^{\frac{1}{2}} \\
+ \Big| \int_\Omega D^\alpha v D^\beta D_n v (D_n\delta)^{m-1-|\beta|}\delta^{|\beta|+1-m} dX \Big| \Big]
\end{aligned}
\tag{2}
$$

where g is the upper bound for $Lv = f$ satisfying the assumptions of Lemma I of §1.

Proof. — We write the integral on the left side of (2) as

$$
\begin{aligned}
&(1+|\beta|-m)^{-1} \int_\Omega D^\alpha v D^\beta v (D_n\delta)^{m-1-|\beta|} D_n(\delta^{1+|\beta|-m}) dX \\
&= -(1+|\beta|-m)^{-1} \int_\Omega D^\alpha D_n v D^\beta v (D_n\delta)^{m-1-|\beta|}\delta^{1+|\beta|-m} dX \\
&+ \int_\Omega D^\alpha v D^\beta v (D_n\delta)^{m-2-|\beta|} D_n^2\delta \delta^{1+|\beta|-m} dX \\
&- (1+|\beta|-m)^{-1} \int_\Omega D^\alpha v D^\beta D_n v (D_n\delta)^{m-1-|\beta|}\delta^{1+|\beta|-m} dX.
\end{aligned}
$$

By the Schwarz inequality the first integral is bounded by

$$
\Big(\int_\Omega |\nabla^m v|^2 \delta dX \Big)^{\frac{1}{2}} \Big(\int_\Omega |\nabla^{|\beta|} v|^2 \delta^{2|\beta|-2m+1} dX \Big)^{\frac{1}{2}}. \tag{3}
$$

By the lemma on interior estimates (3) is bounded by

$$
C \int_\Omega |\nabla^m v|^2 \delta dX + C \int_\Omega |g|^2 \delta^{2m+1} dX. \tag{4}
$$

By the Schwarz inequality, interior estimates and the Carleson measure property of δ the second integral is bounded by $\|N(\nabla^{m-1}v)\|_{L^2(\partial\Omega)}$ times the square root of (4). □

Proof of Main Lemma. — By the Gauss divergence theorem the left side of (1) is equal to a sum of terms of the form

$$
\int_\Omega D^\gamma v D^\gamma D_n v\, dX \qquad \text{where } |\gamma| = m-1.
$$

We apply Lemma 2 a total of m times in order to repeatedly bound the last term of Lemma 2. The result is (1) with

$$\int_\Omega D^\gamma v D^\gamma D_n^{m+1} v (D_n\delta)^{-m}\delta^m \, dX \tag{5}$$

in place of the last term in (1).

We next remark that on the boundary of a domain one may interchange the indices of components of the unit normal vector with the indices on spacial derivatives by introducing tangential derivatives on the boundary as the following typical calculation shows:

$$N^i N^\nu D_j D^\mu v = N^j N^\nu D_i D^\mu v + N^\nu (N^i D_j - N^j D_i) D^\mu v$$

(see the remark before Theorem 3.3 in [27]).

Hence define the symmetric K by K matrix $A = A(X)$ on all of Ω to have entries $\sum_{|\alpha|=|\beta|=m} N^\alpha(X) a_{\alpha\beta}^{kl} N^\beta(X)$ $1 \le k, l \le K$ where we write $L^{kl} = \sum_{|\alpha|=|\beta|=m} D^\alpha a_{\alpha\beta}^{kl} D^\beta$, the L^{kl} as in §1. Here $N(X)$ is the normal at X, viz. $\frac{\nabla\delta(X)}{|\nabla\delta(X)|}$, on the level surface $\{\delta = \delta(X)\}$. By the Legendre-Hadamard condition A has a uniformly bounded inverse for all X and we rewrite (5) as

$$\int_\Omega A^{-1}(D^\gamma v) \cdot A(D^\gamma D_n^{m+1} v)(D_n\delta)^{-m}\delta^m \, dX. \tag{6}$$

Next for each $1 \le k \le K$ the components of the vector obtained from operating with the matrix A may be analyzed by

$$\sum_{l=1}^{K} \sum_{|\alpha|=|\beta|=m} N^\alpha a_{\alpha\beta}^{kl} N^\beta D^\gamma D_n^{m+1} v^l \tag{7}$$
$$= \sum_{l=1}^{K} \sum_{|\alpha|=|\beta|=m} D^\alpha a_{\alpha\beta}^{kl} D^\beta v^l N^\gamma (N_n)^{m+1} +$$

terms with one tangential derivative on $2m-1$ spacial derivatives of v.

This follows by $2m$ applications of the "typical calculation" mentioned above. The last term in (1) is now obtained from the first term on the right of (7) when inserted in (6).

The tangential derivative terms of (7) all typically look like the last term of the "typical calculation" with $|\nu| = |\mu| = 2m-1$. Since the tangential derivative is along the level sets of δ, transferring it by integration by parts to the other functions in (6) results in either one more derivative on $D^\gamma v$ or a second derivative on δ when an N or $D_n\delta$ or coefficient from A^{-1} is differentiated. Since $D_n\delta$ is uniformly bounded away from zero and $|\nabla\delta|$ is uniformly bounded from above, (6) now yields two more types of integrals which may be bounded by

$$\int_\Omega |\nabla^m v|\ |\nabla^{2m-1}v|\delta^m dX + \int_\Omega |\nabla^{m-1}v|\ |\nabla^{2m-1}v|\ |\nabla\nabla\delta|\delta^m dX.$$

An application of Young's inequality and the interior estimate of Lemma I with $i = m-1$ to the first integral yields the first two terms on the right of (1). By the Schwarz inequality the second integral is bounded by

$$\Big(\int_\Omega |\nabla^{m-1}v|^2\ |\nabla\nabla\delta|^2\delta dX\Big)^{\frac{1}{2}}\Big(\int_\Omega |\nabla^{2m-1}v|^2\delta^{2m-1}dX\Big)^{\frac{1}{2}}$$

which by the Carleson measure property of δ and Lemma I again yields the third summand on the right of (1). □

We may now prove the theorem of this section

THEOREM 1. — *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with connected boundary. Fix a point $P^* \in \Omega$. Let L be any elliptic symmetric K-system of order $2m$ from §1. Let u be any solution to $Lu = 0$ so that $u(P^*), \nabla u(P^*), \ldots, \nabla^{m-1}u(P^*)$ vanish. Then there exists a constant C depending only on the Lipschitz character of Ω, n, m, K and E so that*

$$\int_{\partial\Omega} N(\nabla^{m-1}u)^2 d\sigma \le C\int_\Omega |\nabla^m u(X)|^2 \operatorname{dist}(X)dX. \tag{8}$$

Proof. — Since it suffices to establish (8) on domains approximating Ω from the inside as in Lemma II with a constant independent of the approximation we may assume that all nontangential limits for u and its derivatives exists on $\partial\Omega$. Consequently by [27] (see [33] for systems) the left side of (8) is dominated by $C\int_{\partial\Omega}|\nabla^{m-1}u|^2 d\sigma$ with C depending only on the Lipschitz nature of Ω.

By definition of bounded Lipschitz domain, $\partial\Omega$ may be covered by a *finite* collection of open right circular truncated cylinders $\{Z_j\}$ where each Z_j is centered at a boundary point, has a lateral diameter $\operatorname{diam} Z_j$, the concentric double $2Z_j$ is associated with a rectangular coordinate system and Lipschitz function $\varphi_j : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $2Z_j \cap \Omega = \{(x,s) : \varphi_j(x) < s < 20\|\nabla\varphi_j\|_\infty \operatorname{diam} Z_j, |x| < \operatorname{diam} Z_j\}$, and $2Z_j \cap \overline{\Omega}^c = \{(x,s) : s < \varphi_j(x)\} \cap 2Z_j$.

For each cylinder let χ_j be a smooth cut-off function supported in $2Z_j$, identically equal to 1 in Z_j, and so that $|\nabla^k \chi_j(X)| \le C(\operatorname{diam} Z_j)^{-k}$ for all $k \ge 0$ and $X \in \mathbb{R}^n$. Putting $u_j = \chi_j u$ we will prove for any $\varepsilon > 0$ and each j

$$\int_{\partial\Omega \cap 2Z_j} |\nabla^{m-1} u_j|^2 d\sigma \le C_\varepsilon \int_\Omega |\nabla^m u(X)|^2 \operatorname{dist}(X) dX + \varepsilon \int_{\partial\Omega} N(\nabla^{m-1} u)^2 d\sigma \tag{9}$$

where $C_\varepsilon < \infty$ depends only on ε and the Lipschitz nature of Ω. By choosing ε small enough depending only on the Lipschitz nature of Ω (8) follows. We have $\partial\Omega \cap 2Z_j \subset \Lambda_j = \{(x, \varphi_j(x)) : x \in \mathbb{R}^{n-1}\}$ with $\|\nabla \varphi_j\|_\infty \le M$ where M depends only on the Lipschitz nature of Ω and $\operatorname{supp} u_j \subset \{(x,s) : s > \varphi_j(x)\}$. Let $N = (N^1, \dots, N^n)$ denote the <u>inner</u> unit normal vector on Λ_j.

Then the left side of (9) may be dominated by

$$\int_{\partial\Omega \cap 2Z_j} |\nabla^{m-1} u_j|^2 d\sigma \le C \int_{\Lambda_j} |\nabla^{m-1} u_j|^2 N^n d\sigma$$

where C depends only on M. Applying the main Lemma, with $f = Lu_j$ and $|f| \le |g_j|$ since $Lu = 0$ implies we may take

$$g_j = C \sum_{k=1}^{2m} (\operatorname{diam} Z_j)^{-k} |\nabla^{2m-k} u|,$$

we obtain

$$\begin{aligned}\int_{\partial\Omega \cap 2Z_j} & |\nabla^{m-1} u_j|^2 d\sigma \\ \le C\Big[& \int_\Omega |\nabla^m u_j|^2 \operatorname{dist}(X) dX + \int_\Omega |g_j|^2 \operatorname{dist}^{2m+1}(X) dX \\ & + \|N(\nabla^{m-1} u_j)\|_{L^2(\partial\Omega)} \Big(\Big(\int_\Omega |\nabla^m u_j|^2 \operatorname{dist}(X) dX \Big)^{\frac{1}{2}} \\ & + \Big(\int_\Omega |g_j|^2 \operatorname{dist}(X)^{2m+1} dX \Big)^{\frac{1}{2}} \Big) \\ & + \int_\Omega |Lu_j|\, |\nabla^{m-1} u_j| \operatorname{dist}^m(X) dX \Big].\end{aligned} \tag{10}$$

That g_j satisfies the interior estimates in the hypothesis of Lemma I is standard (see e.g. Lemma 4.1 of [27]). The first term on the right of (10) contains terms dominated by

$$\int_\Omega |\nabla^k u|^2 \operatorname{dist}(X) dX \quad k = 0, \dots, m.$$

By repeated application of Lemma II these may all be dominated by

$$\varepsilon \sum_{k=0}^{m-1} \int_{\partial\Omega} N(\nabla^k u)^2 d\sigma + C_\varepsilon \int_\Omega |\nabla^m u|^2 \operatorname{dist}(X) dX \tag{11}$$

for any $\varepsilon > 0$ and C_ε as in the statement of Lemma II. This in turn by repeated use of Lemma III may be dominated by

$$\varepsilon \int_{\partial\Omega} N(\nabla^{m-1} u)^2 d\sigma + \varepsilon \sum_{k=0}^{m-1} \int_{\Omega''} |\nabla^k u|^2 dX + C_\varepsilon \int_\Omega |\nabla^m u|^2 \operatorname{dist}(X) dX$$

for any $\varepsilon > 0$ where $\overline{\Omega''} \subset \Omega$ is as in Lemma III. Now the vanishing at P^* justifies the Poincaré inequality and we get for any $\varepsilon > 0$

$$C \int_\Omega |\nabla^m u_j| \operatorname{dist}(X) dX \leq \varepsilon \int_{\partial\Omega} N(\nabla^{m-1} u)^2 d\sigma + C_\varepsilon \int_\Omega |\nabla^m u|^2 \operatorname{dist}(X) dX.$$

Standard interior estimates [20] may be applied to the first $m-1$ terms in the definition of g_j to reduce the second term on the right of (10) to the first.

The $\|N(\nabla^{m-1} u_j)\|_{L^2(\partial\Omega)}$ factor in the third term on the right of (10) may after repeated application of Lemma III be dominated by

$$C\|N(\nabla^{m-1} u)\|_{L^2(\partial\Omega)} + C \sum_{k=0}^{m-1} \Big(\int_\Omega |\nabla^k u|^2 dX \Big)^{\frac{1}{2}}$$

and then via the Poincaré inequality by just $\|N(\nabla^{m-1} u)\|_{L^2(\partial\Omega)}$. The second factor is dominated by the square root of the right side of (11).

Applying Young's inequality to the last term on the right of (10), it may be dominated by, for example,

$$C \int_\Omega |g_j|^2 \operatorname{dist}(X)^{2m-1} dX + C \int_\Omega |\nabla^{m-1} u_j|^2 \operatorname{dist}(X) dX.$$

Now there are just enough powers on the distance function so that the g_j integral can still be dealt with as described above and the other term is lower order.

All together an inequality of the type (9) follows. □

3. Maximal function dominates square function in L^2 - Part 1.

Let $L = (L^{kl})_{k,l=1}^{K}$ be an elliptic symmetric K-system homogeneous of order $2m$. When $K \geq 2$ one may define adj (L) to be the K-system of cofactors of (L^{kl}). Then $\mathrm{adj}(L)$ is symmetric and homogeneous of order $2(K-1)m$. The Legendre=Hadamard condition for $\mathrm{adj}(L)$ follows from that for L and elementary properties of positive definite symmetric matrices. Likewise define $\det(L)$ to be the single elliptic operator homogeneous of order $2Km$ obtained by taking the determinant of (L^{kl}). Then

$$L\,\mathrm{adj}(L) = \mathrm{adj}(L)L = \det(L)I$$

where I is the $K \times K$ identity operator.

If $Lu = 0$ in Ω we want to show that

$$\int_{\Omega} |\nabla^m u|^2 \,\mathrm{dist}(X) dX \leq C \int_{\partial\Omega} N(\nabla^{m-1}u)^2 d\sigma \tag{1}$$

with C depending only on the Lipschitz nature of Ω. We will show in this section that (1) follows for K-systems, $K \geq 2$, under the assumption that it holds, for scalar solutions to single homogeneous equations of order $2m$ for any m. In particular we will assume that (1) holds for the operators $\det(L)$ with m replaced by Km.

Denote the fundamental solution matrix for L by $\Gamma(X)$ [20], pp. 75–76. Put $\mathcal{N} = \mathrm{adj}(L)$ and $\mathcal{M} = \det(L)I$ and denote the fundamental solution matrices for $\mathcal{N}$ and $\mathcal{M}$ by $\Gamma_{\mathcal{N}}$ and $\Gamma_{\mathcal{M}}$ respectively. Then $L\Gamma_{\mathcal{M}} = \Gamma\mathcal{N}$. Given a solution $Lu = 0$ in Ω and a multi-index $|\alpha| = m$ define

$$w(X) = \int_{\Omega} \Gamma_{\mathcal{N}}(X - Y) D^{\alpha} u(Y) dY. \tag{2}$$

Then

$$\mathcal{N}w = D^{\alpha}u$$

and

$$\mathcal{M}w = 0$$

in Ω.

Since the components of w and all derivatives of those components each satisfy a single elliptic homogeneous equation of order $2Km$, our assumption yields

$$\int_{\Omega} |\nabla^{2(K-1)m} w|^2 \,\mathrm{dist}(X) dX \leq C \int_{\partial\Omega} N(\nabla^{2(K-1)m-1} w)^2 d\sigma.$$

Since $\mathcal{N}$ is of order $2(K-1)m$,

$$\int_\Omega |D^\alpha u|^2 \operatorname{dist}(X)dX = \int_\Omega |\mathcal{N}w|^2 \operatorname{dist}(X)dX$$
$$\leq C \int_{\partial\Omega} N(\nabla^{2(K-1)m-1}w)^2 d\sigma$$

and (1) follows from the following lemma by letting $W = D^\beta w$ for any $|\beta| = 2(K-1)m-1$ and w defined by (2).

LEMMA. — *Let L be an elliptic symmetric K-system homogeneous of degree $2m$ and $Lu = 0$ in Ω. Let B be a K by K matrix of functions such that $\nabla^{2m}B(-X) = -\nabla^{2m}B(X)$ and $\nabla^{2m}B(tX) = t^{1-n}\nabla^{2m}B(X)$ for any $t > 0$ and all $0 \neq X \neq \mathbb{R}^n$. Let $|\alpha| = m$ and put*

$$W(X) = \int_\Omega LB(X-Y)D^\alpha u(Y)dY, \quad X \in \Omega.$$

Then there is a $C < \infty$ depending only on L, B and the Lipschitz nature of Ω so that

$$\|N(W)\|_{L^2(\partial\Omega)} \leq C\|N(\nabla^{m-1}u)\|_{L^2(\partial\Omega)}. \tag{3}$$

Proof. — Let χ be smooth cut-off function supported in a covering cylinder $2Z$ such that $\chi \equiv 1$ in Z. Define $\tilde{u} = \chi u$ and

$$\tilde{W}(X) = \int_{2Z\cap\Omega} LB(X-Y)D^\alpha \tilde{u}(Y)dY. \tag{4}$$

It suffices to obtain (3) with $\frac{1}{2}Z \cap \partial\Omega$ in place of $\partial\Omega$ on the left side.

Hence for $Q \in \frac{1}{2}Z\cap\partial\Omega$, $N(W)(Q) \leq N(\tilde{W})(Q) + N(W-\tilde{W})(Q)$ and the last term is easily controlled by

$$C\sum_{k=0}^{m-1}(\|\nabla^k u\|_{L^2(\partial\Omega)} + \|\nabla^k u\|_{L^2(\Omega)})$$

since a single application of the Gauss divergence theorem yields

$$|(W-\widetilde{W})(X)| \leq C\int_{\partial\Omega\setminus Z} \frac{1}{|X-P|^{n-1}}|\nabla^{m-1}(u-\tilde{u})(P)|dP$$
$$+ C\int_{\Omega\setminus Z} \frac{1}{|X-Y|^n}|\nabla^{m-1}(u-\tilde{u})(Y)|dY$$

and $(1-\chi)u$ is supported uniformly away from X when X is in the truncated cones at Q.

To control $N(\tilde{W})$ define, as on p. 16 of [27], the *mth* primitive $\tilde{u}_{-m}$ of $\tilde{u}$ so that $D_n^m\tilde{u}_{-m} = \tilde{u}$ and $\tilde{u}_{-m}$ is also supported in $2Z\cap\Omega$. Here a rectangular coordinate system has been chosen so that the derivative D_n is transverse to $\partial\Omega \cap 2Z$. Then integrating (4) by parts to transfer the operator L onto $\tilde{u}_{-m}$ and the derivatives $D^\alpha D_n^m$ yields as in line (3) p. 15 of [27]

$$(5) \qquad N(\tilde{W})(Q) \le N\Big(\mathcal{K}(\nabla^{2m-1}\tilde{u}_{-m})\Big)(Q) + \sup_{X\in\Gamma(Q)} \Big| \int_{2Z} D^\alpha D_n^m B(X-Y) L\tilde{u}_{-m}(Y)dY \Big|$$

where $\mathcal{K}$ represents potential operators on $\partial\Omega$ that by [8] have the property

$$\|N\Big(\mathcal{K}(f)\Big)\|_{L^p(\partial\Omega)} \le C_p \|f\|_{L^p(\partial\Omega)}, \quad 1 < p < \infty.$$

But now the right side of (5) may be dealt with as in the proof of Theorem 4.6 of [27] by using Theorem 3.2 and Lemma 4.5 of that paper to control the first term and Lemma 4.4 [27] to control the second. (See [33] for a discussion of the extension of these results to symmetric K-systems.)

Altogether and summing over all covering cylinders

$$\|N(W)\|_{L^2(\partial\Omega)} \le C \sum_{k=0}^{m-1} \big(\|\nabla^k u\|_{L^2(\partial\Omega)} + \|\nabla^k u\|_{L^2(\Omega)}\big).$$

Since the definition of W is unchanged upon subtracting any polynomial of degree $m-1$ from u, Poincaré inequalities on Ω and $\partial\Omega$ may be justified to yield (3). $\square$

We remark that it is the proof of this lemma with its reliance on the Riesz transform results of [27] and [33] where symmetry of our real systems plays a role. For some of the difficulties encountered with nonsymmetric or complex coefficient systems see [34] and [35].

3. Maximal function dominates square function in L^2 - Part 2.

Part 1 above showed that the inequality (1) holds for solutions to K-systems if it is known that (1) holds for solutions to single homogeneous equations of order $2m, m \in \mathbb{Z}_+$. Now we will show that (1) must hold for these latter solutions if it is known that (1) holds for solutions to homogeneous equations $\mathcal{M}u = 0$ where $\mathcal{M}$ is elliptic and of the specialized form

$$(6) \qquad \mathcal{M} = \sum_{|\alpha|=p} a_\alpha (D^\alpha)^2, \quad a_\alpha > 0 \text{ for all } |\alpha| = p.$$

The pointwise quadratic form over the Sobolev space $H^p(\Omega)$ associated with $\mathcal{M}$ is then

$$\sum_{|\alpha|=p} a_\alpha (D^\alpha u(X))^2, \quad a_\alpha > 0 \text{ for all } |\alpha| = p$$

which is clearly equivalent to $|\nabla^p u(X)|^2$ so that trivially

$$\int |\nabla^p u(X)|^2 \operatorname{dist}(X) dX \leq C \int_\Omega \sum_{|\alpha|=p} a_\alpha (D^\alpha u(X))^2 \operatorname{dist}(X) dX. \tag{7}$$

In part 3 below it will be shown how the latter quantity can be dominated by the right side of (1) plus a term that can be hidden on the left.

As an example of the type of elliptic operators we need to analyze here, consider in $\mathbb{R}^4$ the 4th order operator

$$L = D_4^4 + (D_1 D_2)^2 + (D_2 D_3)^2 + (D_3 D_1)^2 - 4 D_1 D_2 D_3 D_4.$$

Given any $\varepsilon > 0$ $L_\varepsilon = L + \varepsilon(D_1^4 + D_2^4 + D_3^4)$ is an elliptic operator. L is derived from a corresponding semipositive definite polynomial known as a Motzkin polynomial [23]. It has been shown (see also [7]) that L does not admit a form of the type required by the first hypothesis of the Aronszajn-Smith coerciveness Theorem [2], p. 161, i.e. that the form be a sum of squares

$$\int_\Omega \sum_{j=1}^k |P_j(D) u|^2 dX$$

where the $P_j(D)$ are homogeneous constant coefficient polynomials. It is easy to show that the same is true for L_ε when ε is small.

The Aronszajn-Smith result is apparently still the best available for obtaining inequalities somewhat more general than (7) with C depending only on the Lipschitz nature of Ω (see comments [2], p. 167), but clearly cannot be used for operators such as the L_ε.

Instead we rely on a theorem due to Habicht [17], pp. 300–302 on strictly positive homogeneous polynomials with real coefficients, i.e. any of our elliptic operators L. By examination of the formula (11.3.2) of [17], p. 302 one can assert that any elliptic homogeneous operator L with real coefficients will satisfy

$$\mathcal{N} L = L \mathcal{N} = \mathcal{M}$$

where $\mathcal{M}$ is as in (6) and $\mathcal{N}$ is another elliptic homogeneous operator like L.

But now $L, \mathcal{M}$ and $\mathcal{N}$ have precisely the same formal relationship as they did in Part 1 above and one can argue in the same way. We conclude that

(1) holds for solutions to $Lu = 0$ if it is known to hold for solutions to $\mathcal{M}u = 0$ with $\mathcal{M}$ as in (6).

3. Maximal function dominates square functions in L^2 - Part 3.

Consider now elliptic operators $L = \sum_{|\alpha|=m} a_\alpha D^{2\alpha}$ where $a_\alpha > 0$ for all $|\alpha| = m$. We will establish the inequality

$$\sum |\alpha| = m a_\alpha \int_\Omega (D^\alpha u(X))^2 \operatorname{dist}(X)\, dX \leq C \int_{\partial\Omega} N(\nabla^{m-1} u)^2\, d\sigma \tag{8}$$

for solutions to $Lu = 0$. As explained in Part 2 above this suffices to establish (1) for all elliptic K-systems.

As with the model case of harmonic functions, after replacing the distance function with the adapted distance function $\delta(X)$, our immediate goal is to use integration by parts twice in order to rewrite the left side of (8) as boundary integrals plus solid integrals with the latter integrals having two derivatives on δ. In order to do this we will use the following combinatorial lemma and a modification of it, Lemma 4, below. Denote the permutation group on m elements by $\mathcal{P}_m$ and members of $\mathcal{P}_m$ by $(i) = (i_1, \ldots, i_m)$.

LEMMA 1. — *Let each of $D_1, \ldots, D_m$ denote a differential monomial $\frac{\partial}{\partial X_1}, \ldots,$ or $\frac{\partial}{\partial X_n}$. Let $u \in C^{2m}$. Then*

$$\begin{aligned}(D_1 \ldots D_m u)^2 = {}& \frac{1}{2(m-1)!} \sum_{(i)\in\mathcal{P}_m} D_{i_1}^2 (D_{i_2} \ldots D_{i_m} u)^2 \\ & - \frac{1}{m!} \sum_{k=0}^{m-2} (-1)^k (m-1-k) \\ & \times \sum_{(i)\in\mathcal{P}_m} D_{i_1} D_{i_2} \\ & \quad (D_{i_3} \ldots D_{i_{m-k}} u D_{i_1} \ldots D_{i_{m-k}} D_{i_{m-k+1}}^2 \ldots D_{i_m}^2 u) \\ & + (-1)^m u D_1^2 \ldots D_m^2 u.\end{aligned} \tag{9}$$

Here when $m = 1$ this can be read as

$$(D_1 u)^2 = \frac{1}{2} D_1^2 u^2 - u D_1^2 u.$$

The proof of Lemma 1 is by induction on m and uses the next lemma to take care of an intermediate term.

LEMMA 2. — *With the hypotheses of Lemma 1*

$$
\begin{aligned}
(10)\quad & m \sum_{(i)\in\mathcal{P}_m} D_{i_1}uD_{i_1}D_{i_2}^2\dots D_{i_m}^2u \\
&= (-1)^m\sum_{k=0}^{m-2}(-1)^k(k+1) \\
&\quad \times \sum_{(i)\in\mathcal{P}_m} D_{i_1}D_{i_2}(D_{i_3}\dots D_{i_{m-k}}uD_{i_1}\dots D_{i_{m-k+1}}^2\dots D_{i_m}^2u) \\
&\quad + (-1)^{m-1}\sum_{(i)\in\mathcal{P}_m}(D_{i_1}\dots D_{i_m}u)^2 - (m-1)\sum_{(i)\in\mathcal{P}_m} uD_{i_1}^2\dots D_{i_m}^2u.
\end{aligned}
$$

Proof. — When $m=2$ this reads

$$2(D_1uD_1D_2^2u+D_2uD_2D_1^2u) = 2D_1D_2(uD_1D_2u)-2(D_1D_2u)^2-2uD_1^2D_2^2u$$

which is true.

For the inductive step assume (10) holds for m. Let $(j)\in\mathcal{P}_{m+1}$. When summed in (j) the identity

$$
\begin{aligned}
&D_{j_1}uD_{j_1}D_{j_2}^2\dots D_{j_{m+1}}^2u + D_{j_2}uD_{j_1}^2D_{j_2}D_{j_3}^2\dots D_{j_{m+1}}^2u \\
&\quad = D_{j_1}D_{j_2}(uD_{j_1}D_{j_2}D_{j_3}^2\dots D_{j_{m+1}}^2u) - D_{j_1}D_{j_2}uD_{j_1}D_{j_2}D_{j_3}^2\dots D_{j_{m+1}}^2u \\
&\qquad - uD_{j_1}^2\dots D_{j_{m+1}}^2u
\end{aligned}
$$

becomes

$$
\begin{aligned}
(11)\quad & 2\sum_{(j)\in\mathcal{P}_{m+1}} D_{j_1}uD_{j_1}D_{j_2}^2\dots D_{j_{m+1}}^2u \\
&= \sum_{(j)\in\mathcal{P}_{m+1}}\big[D_{j_1}D_{j_2}(uD_{j_1}D_{j_2}D_{j_3}^2\dots D_{j_{m+1}}^2u) \\
&\quad - D_{j_1}D_{j_2}uD_{j_1}D_{j_2}D_{j_3}^2\dots D_{j_{m+1}}^2u\big] - (m+1)!uD_1^2\dots D_{m+1}^2u.
\end{aligned}
$$

Next let $\mathcal{P}_m^j$ denote permutations of the m elements $\{1,2,\dots,j-1.\ j+1,\dots,m+1\}$. Then

$$
\begin{aligned}
m & \sum_{(j)\in\mathcal{P}_{m+1}} D_{j_1}D_{j_2}uD_{j_1}D_{j_2}D^2_{j_3}\dots D^2_{j_{m+1}}u \\
&= m\sum_{j=1}^{m+1}\sum_{(i)\in\mathcal{P}^j_m} D_{i_1}D_juD_{i_1}D^2_{i_2}\dots D^2_{i_m}D_ju \\
&= \sum_{j=1}^{m+1}[(-1)^m\sum_{k=0}^{m-2}(-1)^k(k+1)\sum_{(i)\in\mathcal{P}^j_m} \\
&\quad D_{i_1}D_{i_2}(D_{i_3}\dots D_{i_{m-k}}D_juD_{i_1}\dots D_{i_{m-k}}D^2_{i_{m-k+1}}\dots D^2_{i_m}D_ju) \\
&\quad +(-1)^{m-1}\sum_{(i)\in\mathcal{P}^j_m}(D_{i_1}\dots D_{i_m}D_ju)^2-(m-1) \\
&\quad \sum_{(i)\in\mathcal{P}^j_m} D_juD^2_{i_1}\dots D^2_{i_m}D_ju]
\end{aligned}
$$

where the inductive hypothesis has been used. Thus

$$
\begin{aligned}
\text{(12)}\qquad & -m\sum_{(j)\in\mathcal{P}_{m+1}} D_{j_1}D_{j_2}uD_{j_1}D_{j_2}D^2_{j_3}\dots D^2_{j_{m+1}}u \\
&= (-1)^{m+1}\sum_{k=0}^{m-2}(-1)^k(k+1)\times\sum_{(j)\in\mathcal{P}_{m+1}} D_{j_1}D_{j_2} \\
&\quad (D_{j_3}\dots D_{j_{m-k+1}}uD_{j_1}\dots D_{j_{m-k+1}}D^2_{j_{m-k+2}}\dots D^2_{j_{m+1}}u) \\
&\quad +(-1)^m(m+1)!(D_1\dots D_{m+1}u)^2+(m-1) \\
&\quad \sum_{(j)\in\mathcal{P}_{m+1}} D_{j_1}uD_{j_1}D^2_{j_2}\dots D^2_{j_{m+1}}u.
\end{aligned}
$$

Multiplying (11) by m, substituting (12) for the second term under he summation on the right side, combining like terms, and noting that the first term on the right side of (11) becomes the $k = m-1$ term of (10) we obtain (10) with $m+1$ in place of m. □

Proof of Lemma 1. — The $m = 2$ case reads

$$(D_1D_2u)^2 = \frac{1}{2}(D^2_1(D_2u)^2 + D^2_2(D_1u)^2) - D_1D_2(uD_1D_2u) + uD^2_1D^2_2u.$$

Assume (9) holds for m. Then

$$\frac{m+1}{m}(m+1)!(D_1\dots D_{m+1}u)^2 = \frac{m+1}{m}\sum_{(j)\in\mathcal{P}_{m+1}}(D_{j_1}\dots D_{j_{m+1}}u)^2$$

$$= \frac{m+1}{m}\sum_{j=1}^{m+1}\sum_{(i)\in\mathcal{P}_m^j}(D_{i_1}\dots D_{i_m}D_ju)^2$$

$$= \frac{m+1}{m}\sum_{j=1}^{m+1}\Bigg[\frac{m}{2}\sum_{(i)\in\mathcal{P}_m^j}D_{i_1}^2(D_{i_2}\dots D_{i_m}D_ju)^2$$

$$-\sum_{k=0}^{m-2}(-1)^k(m-1-k)\sum_{(i)\in\mathcal{P}_m^j}D_{i_1}D_{i_2}(D_{i_3}\dots D_{i_{m-k}}D_juD_{i_1}\dots D_{i_{m-k}}$$

$$D_{i_{m-k+1}}^2\dots D_{i_m}^2D_ju)+(-1)^m\sum_{(i)\in\mathcal{P}_m^j}D_juD_{i_1}^2\dots D_{i_m}^2D_ju\Bigg]$$

$$= \frac{m+1}{2}\sum_{(j)\in\mathcal{P}_{m+1}}D_{j_1}^2(D_{j_2}\dots D_{j_{m+1}}u)^2$$

$$-\frac{m+1}{m}\sum_{k=0}^{m-2}(-1)^k(m-1-k)$$

$$\times\sum_{(j)\in\mathcal{P}_{m+1}}D_{j_1}D_{j_2}(D_{j_3}\dots D_{j_{m-k+1}}uD_{j_1}\dots D_{j_{m-k+1}}D_{j_{m-k+2}}^2\dots D_{j_{m+1}}^2u)$$

$$+(-1)^m\frac{m+1}{m}\sum_{(j)\in\mathcal{P}_{m+1}}D_{j_1}uD_{j_1}D_{j_2}^2\dots D_{j_{m+1}}^2u$$

where the third equality uses the inductive hypothesis. Dividing by $(m+1)!$ and using Lemma 2 on the last term we obtain

$$\frac{m+1}{m}(D_1\dots D_{m+1}u)^2 = \frac{1}{2(m!)}\sum_{(j)\in\mathcal{P}_{m+1}}D_{j_1}^2(D_{j_2}\dots D_{j_{m+1}}u)^2$$

$$-\frac{m+1}{m}\frac{1}{(m+1)!}\sum_{k=0}^{m-2}(-1)^k(m-1-k)$$

$$\times\sum_{(j)\in\mathcal{P}_{m+1}}D_{j_1}D_{j_2}(D_{j_3}\dots D_{j_{m-k+1}}uD_{j_1}\dots D_{j_{m-k+1}}D_{j_{m-k+2}}^2\dots D_{j_{m+1}}^2u)$$

$$-\frac{1}{m}\frac{1}{(m+1)!}\sum_{k=0}^{m-1}(-1)^k(k+1)$$

$$\times \sum_{(j)\in\mathcal{P}_{m+1}} D_{j_1}D_{j_2}(D_{j_3}\dots D_{j_{m-k+1}}uD_{j_1}\dots D_{j_{m-k+1}}D^2_{j_{m-k+2}}\dots D^2_{j_{m+1}}u)$$
$$+\frac{1}{m}(D_1\dots D_{m+1}u)^2+(-1)^{m+1}uD_1^2\dots D^2_{m+1}u.$$

The two summations in k combine to yield the summation in k of (9) when m is replaced with $m+1$ and the lemma follows. □

The equation $Lu = 0$ will be used to eliminate the last terms arising from the application of Lemma 1 to the left side of (8). The remaining terms allow us to transfer two derivatives to the (adapted) distance function. Unlike the first term, however, the middle summation does not directly yield boundary terms like the right side of (8). To remedy this we resort to primitives as in the proof of the lemma from part 1 above. Here v_{-k} will denote the kth primitive of v and ∂ will represent the derivative such that $\partial^k v_{-k}$. First a preliminary lemma.

LEMMA 3. — *With the hypothesis of Lemma 1 and $v \in C^{2m}$, $m \geq 2$,*

(13)

$$\sum_{k=1}^{m-1}(-1)^{k-1}\sum_{(i)\in\mathcal{P}_m}\partial D_{i_2}(D_{i_3}\dots D_{i_{m-k+1}}uD^2_{i_1}D_{i_2}\dots D_{i_{m-k+1}}$$
$$D^2_{i_{m-k+2}}\dots D^2_{i_m}v_{-1})$$
$$=\sum_{k=2}^{m}[(-1)^{m-k}\partial^2\sum_{(i)\in\mathcal{P}_m}\partial^{k-2}uD^2_{i_1}\dots D^2_{i_m}v_{-k}$$
$$+\partial^2\sum_{(i)\in\mathcal{P}_m}D_{i_2}\dots D_{i_{m+2-k}}\partial^{k-2}uD^2_{i_1}D_{i_2}\dots D_{i_{m+2-k}}D^2_{i_{m+3-k}}\dots D^2_{i_m}v_{-k}$$
$$-\sum_{(i)\in\mathcal{P}_m}\partial D_{i_2}(D_{i_3}\dots D_{i_{m+2-k}}\partial^{k-1}uD^2_{i_1}D_{i_2}\dots D_{i_{m+2-k}}$$
$$D^2_{i_{m+3-k}}\dots D^2_{i_m}v_{-k})].$$

Proof. — The $m = 2$ case reads

$$\partial D_2(uD_1^2D_2v_{-1})+\partial D_1(uD_2^2D_1v_{-1})=2\partial^2(uD_1^2D_2^2v_{-2})$$
$$+\partial^2(D_2uD_1^2D_2v_{-2}+D_1uD_2^2D_1v_{-2})$$
$$-\partial D_2(\partial uD_1^2D_2v_{-2})-\partial D_1(\partial uD_2^2D_1v_{-2}).$$

We will establish (13) with $m+1$ in place of m assumming (13) holds as written. Hence

$$\sum_{k=1}^{m}(-1)^{k-1}\sum_{(i)\in\mathcal{P}_{m+1}}$$
$$\partial D_{i_2}(D_{i_3}\dots D_{i_{m-k+2}}uD^2_{i_1}D_{i_2}\dots D_{i_{m-k+2}}D^2_{i_{m-k+3}}\dots D^2_{i_{m+1}}v_{-1})$$
$$=(-1)^{m-1}\sum_{(i)\in\mathcal{P}_{m+1}}\partial^2(uD^2_{i_1}\dots D^2_{i_{m+1}}v_{-2})$$
$$+\sum_{k=1}^{m-1}(-1)^{k-1}\sum_{(i)\in\mathcal{P}_{m+1}}$$
$$\partial^2(D_{i_3}\dots D_{i_{m-k+2}}uD^2_{i_1}D^2_{i_2}D_{i_3}\dots D_{i_{m-k+2}}D^2_{i_{m-k+3}}\dots D^2_{i_{m+1}}v_{-2})$$
$$+\sum_{(i)\in\mathcal{P}_{m+1}}\partial^2(D_{i_2}\dots D_{i_{m+1}}uD^2_{i_1}D_{i_2}\dots D_{i_{m+1}}v_{-2})$$
$$+\sum_{k=2}^{m}(-1)^{k-1}\sum_{(i)\in\mathcal{P}_{m+1}}$$
$$\partial^2(D_{i_2}\dots D_{i_{m-k+2}}uD^2_{i_1}D_{i_2}\dots D_{i_{m-k+2}}D^2_{i_{m-k+3}}\dots D^2_{i_{m+1}}v_{-2})$$
$$-\sum_{(i)\in\mathcal{P}_{m+1}}\partial D_{i_2}(D_{i_3}\dots D_{i_{m+1}}\partial uD^2_{i_1}D_{i_2}\dots D_{i_{m+1}}v_{-2})$$
$$+\sum_{k=2}^{m}(-1)^{k}\sum_{(i)\in\mathcal{P}_{m+1}}$$
$$\partial D_{i_2}(D_{i_3}\dots D_{i_{m-k+2}}\partial uD^2_{i_1}D_{i_2}\dots D_{i_{m-k+2}}D^2_{i_{m-k+3}}\dots D^2_{i_{m+1}}v_{-2}).$$

Here we have used the identity

$$\partial D(UV)=\partial^2(UDV_{-1})+\partial^2(DUV_{-1})-\partial D(\partial UV_{-1})$$

to each term on the left, obtaining three sums on the right from which we have isolated the mth term, the 1st term and the 1st term respectively.

Using the fact that we are summing over the permutation group, the first and second summations in k now add to zero.

Replacing $\sum\limits_{\mathcal{P}_{m+1}}$ with $\sum\limits_{l=1}^{m+1}\sum\limits_{\mathcal{P}^l_m}$ as in the proof of Lemma 2 and k with $k+1$, the third summation in k may be calculated using the inductive hypothesis with ∂u in place of u and $D^2_l v_{-2}$ in place of v_1. The third summation becomes

$$\sum_{k=2}^{m}\Big[(-1)^{m-k}\partial^2 \sum_{(i)\in\mathcal{P}_{m+1}} \partial^{k-1}uD_{i_1}^2\dots D_{i_{m+1}}^2 v_{-k-1}$$
$$+\,\partial^2 \sum_{(i)\in\mathcal{P}_{m+1}} D_{i_2}\dots D_{i_{m+2-k}}\partial^{k-1}uD_{i_1}^2 D_{i_2}\dots D_{i_{m+2-k}}D_{i_{m+3-k}}^2$$
$$\dots D_{i_{m+1}}^2 v_{-k-1}$$
$$-\sum_{(i)\in\mathcal{P}_{m+1}} \partial D_{i_2}(D_{i_3}\dots D_{i_{m+2-k}}\partial^{k}uD_{i_1}^2 D_{i_2}\dots D_{i_{m+2-k}}D_{i_{m+3-k}}^2$$
$$\dots D_{i_{m+1}}^2 v_{-k-1})\Big].$$

Now replace k with $k-1$ and the three isolated terms supply the missing $k=2$ terms. □

The next lemma gives us what we need when $u=v$.

LEMMA 4. — *With the hypotheses of Lemma 3,*

$$\sum_{k=0}^{m-2}(-1)^k(m-1-k)\sum_{(i)\in\mathcal{P}_m} D_{i_1}D_{i_2}(D_{i_3}\dots D_{i_{m-k}}uD_{i_1}\dots D_{i_{m-k}}D_{i_{m-k+1}}^2$$
$$\dots D_{i_m}^2 v)$$
$$=\sum_{k=2}^{m}(k-1)\Big[(-1)^{m-k}\partial^2\sum_{(i)\in\mathcal{P}_m}\partial^{k-2}uD_{i_1}^2\dots D_{i_m}^2 v_{-k}$$
$$+\,\partial^2\sum_{(i)\in\mathcal{P}_m} D_{i_2}\dots D_{i_{m+2-k}}\partial^{k-2}uD_{i_1}^2 D_{i_2}\dots D_{i_{m+2-k}}D_{i_{m+3-k}}^2\dots D_{i_m}^2 v_{-k}$$
$$-\sum_{(i)\in\mathcal{P}_m}\partial D_{i_2}(D_{i_3}\dots D_{i_{m+2-k}}\partial^{k-1}uD_{i_1}^2 D_{i_2}\dots D_{i_{m+2-k}}D_{i_{m+3-k}}^2$$
$$\dots D_{i_m}^2 v_{-k})\Big]$$
$$+\sum_{k=1}^{m-1}(m-k)\sum_{(i)\in\mathcal{P}_m}\partial D_{i_2}(D_{i_1}D_{i_3}\dots D_{i_{m-k+1}}\partial^{k-1}uD_{i_1}\dots D_{i_{m-k+1}}$$
$$D_{i_{m-k+2}}^2\dots D_{i_m}^2 v_{-k})$$
$$-\sum_{k=1}^{m-1}(m-k)\sum_{(i)\in\mathcal{P}_m} D_{i_1}D_{i_2}(D_{i_3}\dots D_{i_{m-k+1}}\partial^{k}uD_{i_1}\dots D_{i_{m-k+1}}D_{i_{m-k+2}}^2$$
$$\dots D_{i_m}^2 v_{-k}).$$

Proof. — The $m = 2$ case reads

$$\begin{aligned} 2D_1D_2(uD_1D_2v) =& 2\partial^2(uD_1^2D_2^2v_{-2}) \\ &+ \partial^2(D_2uD_1^2D_2v_{-2} + D_1uD_2^2D_1v_{-2}) \\ &- \partial D_2(\partial uD_1^2D_2v_{-2}) - \partial D_1(\partial uD_2^2D_1v_{-2}) \\ &+ \partial D_2(D_1uD_1D_2v_{-1}) + \partial D_1(D_2uD_1D_2v_{-1}) \\ &- 2D_1D_2(\partial uD_1D_2v_{-1}). \end{aligned}$$

Using the identity

$$D_1D_2(UV) = \partial D_2(UD_1V_{-1} + D_1UV_{-1}) - D_1D_2(\partial UV_{-1})$$

and summation over the permutation group, the $m+1$ case can be written

$$\begin{aligned} &\sum_{k=0}^{m-1}(-1)^k(m-k) \sum_{(i)\in\mathcal{P}_{m+1}} D_{i_1}D_{i_2}\big(D_{i_3}\dots D_{i_{m+1-k}}uD_{i_1}\dots D_{i_{m+1-k}} \\ &\qquad\qquad D^2_{i_{m+2-k}}\dots D^2_{i_{m+1}}v\big) \\ &= \sum_{k=1}^{m}(-1)^{k-1} \sum_{(i)\in\mathcal{P}_{m+1}} \partial D_{i_2}\big(D_{i_3}\dots D_{i_{m+2-k}}uD^2_{i_1}D_{i_2}\dots D_{i_{m+2-k}} \\ &\qquad\qquad D^2_{i_{m+3-k}}\dots D^2_{i_{m+1}}v_{-1}\big) \\ &\quad + m \sum_{(i)\in\mathcal{P}_{m+1}} \partial D_{i_2}\big(D_{i_1}D_{i_3}\dots D_{i_{m+1}}uD_{i_1}\dots D_{i_{m+1}}v_{-1}\big) \\ &\quad - m \sum_{(i)\in\mathcal{P}_{m+1}} D_{i_1}D_{i_2}\big(D_{i_3}\dots D_{i_{m+1}}\partial uD_{i_1}\dots D_{i_{m+1}}v_{-1}\big) \\ &\quad + \sum_{k=0}^{m-2}(-1)^k(m-1-k) \sum_{(i)\in\mathcal{P}_{m+1}} D_{i_1}D_{i_2}\big(D_{i_3}\dots D_{i_{m-k}}\partial uD_{i_1}\dots D_{i_{m-k}} \\ &\qquad\qquad D^2_{i_{m+1-k}}\dots D^2_{i_{m+1}}v_{-1}\big). \end{aligned}$$

Now Lemma 3 is used on the first term and the inductive hypothesis as in the proof of Lemma 3 on the last term. □

Combining Lemmas 1 and 4 we obtain

$$\begin{aligned} (D^\alpha u)^2 =& -m\partial^{m-1}uD^{2\alpha}u_{-m+1} - (m-1)\partial^m uD^{2\alpha}u_{-m} \\ &+ \text{ terms of the form} \\ &\nabla^2(\nabla^{m-1}u\nabla^{m-1+k}u_{-k}) \text{ for } k = 0,\dots,m \end{aligned} \tag{14}$$

whenever $|\alpha| = m$. With this observation we can prove the main lemma of this section. Let Ω be a bounded Lipschitz domain and let Z denote a covering cylinder with associated cut-off function χ as in the proof of Theorem 1.

LEMMA 5. — *Let $L = \sum_{|\alpha|=m} a_\alpha D^{2\alpha}$ as described above and $Lu = 0$ in a neighborhood of $2Z \cap \Omega$. Let ψ be a cut-off function like χ except supported in $(1/4)Z$ and identically 1 in $(1/8)Z$. Then*

$$(15)\quad \sum_{|\alpha|=m} a_\alpha \int_\Omega (D^\alpha u(X))^2 \psi(X) \operatorname{dist}(X) dX$$
$$\leq C_\varepsilon \int_{\partial\Omega\cap 2Z} N(\nabla^{m-1}u)^2 d\sigma + \varepsilon \int_{2Z\cap\Omega} |\nabla^m u(X)|^2 \operatorname{dist}(X) dX$$

where C_ε depends only on the Lipschitz nature of Ω and $\varepsilon > 0$.

Proof. — As in §2 it suffices to obtain our estimate for approximating smooth domains Ω.

Define $\tilde{u} = \chi u$ and primitives $\tilde{u}_{-k}$ as the proof of the Lemma from Part 1 where a coordinate system has been chosen so that D_n is transverse to $\partial\Omega \cap 2Z$. Let $\delta(X)$ denote the adapted distance function of §1 with respect to the coordinate system. Then δ may be substituted for dist on the left of (15) and the result is by (14) equal to

$$-m \int_\Omega D_n^{m-1} u L \tilde{u}_{-m+1} \psi \delta dX - (m-1) \int_\Omega D_n^m u L \tilde{u}_{-m} \psi \delta dX$$
$$+ \text{ terms of the form } \int_\Omega \nabla^2 (\nabla^{m-1} u \nabla^{m-1+k} \tilde{u}_{-k}) \psi \delta dX, \quad k = 0, \dots, m.$$

Here because of the choice of coordinate systems D_n plays the role of ∂ from (14) while L is actually a rotation of the original L. That L is no longer in precisely the form (6) is of no consequence.

The first two terms are lower order by virtue of $Lu = 0$. They may be analyzed, for example, by line (7) on p. 18 of [27] together with interior estimates like Lemma 4.1 of that paper. The result is an upper bound of

$$(16)\qquad C \sum_{j=0}^{m-1} \int_{Z\cap\Omega} |\nabla^j u|^2 dX.$$

For the main terms, whenever a derivative lands on the cut-off function ψ when using the divergence theorem the result is a lower order term no worse than $\int_\Omega |\nabla^{m-1} u \nabla^{m+k} \tilde{u}_{-k}| \delta dX$.

Such a term is also bounded by (16) using the Schwarz inequality, Hardy inequalities like Lemma 4.2 of [27] and interior estimates like Lemma

4.1 of [27]. Thus in the following we will disregard any terms arising from derivatives falling on ψ.

Hence, applying the divergence theorem twice to the main terms leads to boundary integrals of the form

$$\int_{\partial\Omega\cap\frac{1}{4}Z} |\nabla^{m-1}u|\,|\nabla^{m-1+k}\tilde{u}_{-k}|\,|\nabla\delta|d\sigma$$

and solid integrals of the form

$$\int_{\Omega} \nabla^{m-1}u\nabla^{m-1+k}\tilde{u}_{-k}\psi\nabla^2\delta dX. \tag{17}$$

By the Schwarz inequality and Corollary 4.7 of [27] the boundary terms are bounded by (16) plus

$$C||\nabla^{m-1}\tilde{u}||^2_{L^2(2Z\cap\partial\Omega)}. \tag{18}$$

For the solid integrals we introduce the term $\dfrac{D_n\delta}{D_n\delta}$ into the integrals (17) and apply the divergence theorem removing the D_n derivative from the numerator of the introduced term. This results in three types of integrals

$$\mathrm{I} = \int_{\Omega} \nabla^{m-1}u\nabla^{m-1+k}\tilde{u}_{-k}\psi\nabla^2 D_n\delta\frac{\delta}{D_n\delta}dX,$$

$$\mathrm{II} = \int_{\Omega} D_n(\nabla^{m-1}u\nabla^{m-1+k}\tilde{u}_{-k})\psi\nabla^2\delta\frac{\delta}{D_n\delta}dX, \text{ and}$$

$$\mathrm{III} = \int_{\Omega} \nabla^{m-1}u\nabla^{m-1+k}\tilde{u}_{-k}\psi\nabla^2\delta D_n^2\delta\frac{\delta}{(D_n\delta)^2}dX.$$

The Schwarz inequality allows one to apply the Carleson measure property of δ in III to obtain a bound of the type

$$||N(\nabla^{m-1+k}\tilde{u}_{-k})||^2_{L^2(\frac{1}{2}Z\cap\partial\Omega)}, \quad k = 0,\dots,m$$

which in turn is bounded by (16) plus (18).

The product rule followed by applications of Young's inequality, $ab \le \varepsilon a^2 + \dfrac{b^2}{4\varepsilon}$, bounds II by the sum of C_ε times the bound for III plus integrals of the type

$$\varepsilon\int_{\Omega} |\nabla^{m+k}\tilde{u}_{-k}|^2\psi\delta dX, \quad k = 0,\dots,m$$

for $\varepsilon > 0$ to be chosen later.

By Hardy type inequalities (e.g. [27], p. 17) and interior estimates (e.g. [27], p. 16 these may in turn be bounded by

$$\varepsilon C\sum_{j=0}^{m}\int_{2Z\cap\Omega} |\nabla^j u|^2\delta dX.$$

For integrals of type I the divergence theorem is used again to remove a ∇ from $\nabla^2 D_n\delta$. Integrals of type II and III are again obtained together with the new integral type

$$\int_\Omega \nabla^{m-1}u\nabla^{m-1+k}\tilde{u}_{-k}\psi\nabla D_n\delta\frac{\nabla\delta}{D_n\delta}dX.$$

Now the crucial fact from §1 that $\nabla D_n\delta$ may be replaced by $\nabla D_n\tilde{\delta}$ where $\tilde{\delta}$ has the property that $|D_n\tilde{\delta}|^2\delta^{-1}dX$ is a Carleson measure is used. Using the divergence theorem to remove the ∇ from $\nabla D_n\tilde{\delta}$ now yields a boundary integral of the type already bound, a type II integral where $D_n\tilde{\delta}$ replaces $(\nabla^2\delta)\delta$, and type III integrals where $(D_n\tilde{\delta})(\nabla D_n\delta)$ or $(D_n\tilde{\delta})(\nabla^2\delta)$ replaces $(\nabla^2\delta)(D_n^2\delta)\delta$.

Summarizing we may write

$$\begin{aligned}(19)\qquad \sum_{|\alpha|=m} a_\alpha\int_\Omega(D^\alpha u)^2\psi\delta dX \le C_\varepsilon\Big(&\sum_{j=0}^{m-1}\int_{2Z\cap\Omega}|\nabla^j u|^2dX \\ &+\|\nabla^{m-1}\tilde{u}\|^2_{L^2(2Z\cap\partial\Omega)}\Big)+\varepsilon\int_{2Z\cap\Omega}|\nabla^m u|^2\delta dX\end{aligned}$$

where $C_\varepsilon<\infty$ and $\varepsilon>0$ may still be chosen depending only on the Lipschitz nature of Ω.

By applying a Poincaré lemma like Lemma 4.3 of [27] as on page 20 of that paper, the lower order derivatives on u that appear in $2Z$ and on $2Z\cap 2\Omega$ may be eliminated and (15) follows. □

THEOREM 2. — *Let $\Omega\subset\mathbb{R}^n$ be a bounded Lipschitz domain with connected boundary. Let L be any elliptic symmetric K-system of order $2m$ from §1. Let $Lu=0$ in Ω. Then there exists a constant C depending only on the Lipschitz character of Ω, n, m, K and E so that*

$$(20)\qquad \int_\Omega|\nabla^m u(X)|^2\operatorname{dist}(X)dX\le C\int_{\partial\Omega}N(\nabla^{m-1}u)^2d\sigma.$$

Proof. — As explained in Parts 1 and 2 above it suffices to prove the result for L as in Lemma 5. By interior estimates only the portion of Ω near the boundary need be considered for the left side of (20). The boundary can be covered by a finite number of covering cylinders $\left\{\frac{1}{8}Z_j\right\}$ with the Z_j otherwise having the same properties as in the proof of Theorem 1. Since the left side of (8) dominates the left side of (20) with a multiplicative constant depending only on ellipticity, a finite number of applications of Lemma 5 yields

$$\int_\Omega |\nabla^m u|^2 \operatorname{dist}(X)dX \leq C_\varepsilon \int_{\partial\Omega} N(\nabla^{m-1}u)^2 d\sigma + \varepsilon \int_\Omega |\nabla^m u|^2 \operatorname{dist}(X)dX$$

where $\varepsilon > 0$ may be chosen depending only on the stated quantities, completing the proof. □

4. L^p equivalence of maximal function and square function.

From Theorems 1 and 2, which give the equivalence between the L^2 norms of the nontangential maximal function and the square function of solutions to $Lu = 0$ on every Lipschitz domain, a series of near-standard arguments give the L^p norm equivalence for all $0 < p < \infty$. In the two lemmas which follow, we provide the necessary details to prove the good-λ inequalities from which one obtains the next theorem ([6], [16], [10], 12).

THEOREM 3. — *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with constant M and suppose $Lu = 0$ in Ω, where L is a $2m$ order elliptic symmetric K-system. Then, if $0 < p < \infty$, $\exists C > 0$ depending only on m, n, N, M and p such that*

$$C^{-1} \int_{\partial\Omega} N(\nabla^{m-1}u)^p d\sigma \leq \int_{\partial\Omega} S(\nabla^{m-1}u)^p d\sigma \leq C \int_{\partial\Omega} N(\nabla^{m-1}u)^p d\sigma.$$

(For the left-hand inequality it is necessary to assume a normalization: $u(P^*) = \nabla u(P^*) = \ldots = \nabla^{m-1}u(P^*)$ for some $P^* \in \Omega$.)

LEMMA 1. — *If L is a $2m$-order elliptic symmetric K-system and $Lu = 0$ in $\Omega \subset \mathbb{R}^n$, a Lipschitz domain with Lipschitz constant M, then $\exists \gamma > 0$ sufficiently small and $c > 0$ both depending only on M such that for all $\lambda > 0$, $\sigma\{Q \in \partial\Omega : S(\nabla^{m-1}u)(Q) > 2\lambda,\ N(\nabla^{m-1}u)(Q) \leq \gamma\lambda\} \leq c\gamma^2\{Q \in \partial\Omega : S(\nabla^{m-1}u)(Q) > \lambda\}$.*

Proof. — Let $\Delta_j(Q_j, r_j)$ be a Whitney decomposition of

$$\{S(\nabla^{m-1}u) > \lambda\}.$$

That is, $\{Q_j\}$ is a sequence of points in $\{S(\nabla^{m-1}u) > \lambda\}$ such that

(a) $\{S(\nabla^{m-1}u) > \lambda\} = \cup\Delta_j$, where $\Delta_j(Q_j, r_j) = B(Q_j, r_j) \cap \partial\Omega$.

(b) $\sum\limits_j \chi_{B(Q_j,r_j)}(X) \leq C$, where C depends only on u and M.

(c) Each $B_j = B(Q_j, r_j)$ is such that $\partial\Omega \cap B_j$ is the graph of a Lipschitz function and,

(d) $\exists r_o = r_o(\Omega)$ such that if $r_j \leq r_o$, then there is a point Q_j^* in $B(Q_j, 2r_j) \cap \partial\Omega$ with $S(\nabla^{m-1}u)(Q_j^*) \leq \lambda$.

As in [12], a sawtooth region D_j associated to

$$E_j = \Delta_j \cap \{S(\nabla^{m-1}u) > 2\lambda, N(\nabla^{m-1}u) \leq \gamma\lambda\}$$

may be constructed so that

(i) $\bigcup_{Q\in E_j} (\Gamma(Q) \cap B(Q, c_1 r_j)) \subset D_j \subset \bigcup_{Q\in E_j} (\overline{\Gamma}(Q) \cap B(Q, c_2 r_j))$ for some fixed c_1, c_2 independent of j

(ii) $\partial D_j \cap \partial\Omega = E_j$

(iii) ∂D_j is c starlike Lipschitz domain with Lipschitz constant less than a fixed multiple of M and

(iv) $\operatorname{diam} D_j \approx r_j$.

We fix now a particular Δ_j with $r_j \leq r_o$. By interior estimates and property (d) above it can be shown that, for any $\tau > 0, \gamma$ may be chosen sufficiently small so that $S_{\tau r_j}(\nabla^{m-1}u)(Q) > \lambda/2$ for $Q \in E_j$ where $S_{\tau r_j}$ is the square function defined by integration over the truncated cone $\Gamma_{\tau r_j}(Q) = \Gamma(Q) \cap B(Q, \tau r_j)$. Suppressing the j-subscripts, we wish to shown that $\sigma(E) \leq c\gamma^2\sigma(\Delta)$. Summing, then on j gives the Lemma. Then we have

$$\sigma(E) \leq \frac{1}{\lambda^2}\int_E \int_{\Gamma_{\tau(Q)}} \delta(X)^{2-n}|\nabla^m u|^2 dX d\sigma(Q)$$

where $\delta(X) \approx |X - Q|$. If the aperture of the cones used to define N is chosen larger than those used to define S, this last term is bounded from above by

$$\frac{1}{\lambda^2}\int_{\partial D}\int_{\Gamma_{\tau(Q)}} d(X)^{2-n}|\nabla^m u(X)|^2 dX d\sigma.$$

Here τ has been chosen sufficiently small so that $\Gamma_\tau(Q) \subset D$ for all $Q \in \partial\Omega$, and the collection forms a family of cones for the square function with respect to the new domain D and $d(X)$ denotes $d_D(X) = (X, \partial D) \approx |X - Q|$. By Theorem 2, this is in turn dominated by $\frac{1}{\lambda^2}\int_{\partial D} N_\Omega(\nabla^{m-1}u)^2 d\sigma \leq \gamma^2\sigma(\Delta)$, by (iv) and $|\nabla^{m-1}u| \leq \gamma\lambda$ in D. □

LEMMA 2. — *Let L be a $2m$-order elliptic symmetric K-system and suppose $Lu = 0$ in a Lipschitz domain $\Omega \subset \mathbb{R}^n$ with Lipschitz constant*

M and that $u(P^*) = \nabla u(P^*) = \ldots = \nabla^{m-1}u(P^*) = 0$ for some $P^* \in \Omega$. Then, for sufficiently small $\gamma > 0$ and $\varepsilon > 0$, there is a constant $c > 0, c, \gamma, \varepsilon$ depending only on K, m, n and M, such that

$$\sigma\{N(\nabla^{m-1}u) > 4\lambda, S(\nabla^{m-1}u) \le \gamma\lambda\} \cap \{S(\nabla^{m-1}u) > \gamma\lambda\}^*_\varepsilon \le c\gamma^2\sigma\{N(\nabla^{m-1}u) > \lambda\},$$

(where, for any set $G \subset \partial\Omega$, the set G^*_ε is defined by

$$G^*_\varepsilon = \Big\{Q \in \partial\Omega : \sup_{Q\in\Delta} \frac{\sigma(\Delta \cap G)}{\sigma(\Delta)} \le \varepsilon\Big\}).$$

Proof. — We make the a priori assumption that $\|S(\nabla^{m-1}u)\|_{L^p(d\sigma)} = R < +\infty$ for some $0 < p < +\infty$, since our goal is to use the lemma to bound $\|N(\nabla^{m-1}u)\|_p$ by $\|S(\nabla^{m-1}u)\|_p$. This fact together with interior estimates on solutions and the normalization on u (the vanishing of $\nabla^{m-1}u(P^*)$) implies that for any compact subset K of Ω,

$$\sup_{X\in K}\{|D^\beta u| : |\beta| \le m\} \le C(K, R).$$

As in [10], this estimate can be used to handle the 'large' surface balls in the Whitney decomposition of $\{N(\nabla^{m-1}u) > \lambda\}$. Thus, let Δ be one of these surface balls from this Whitney decomposition (defined as in Lemma 1) whose radius is less than r_o. Set $E = \Delta \cap \{N(\nabla^{m-1}u) > 4\lambda,\ S(\nabla^{m-1}u) \le \gamma\lambda\} \cap \{S(\nabla^{m-1}u) > \gamma\lambda\}^*_\varepsilon$.

By choosing γ sufficiently small we can ensure that $N_{\tau r}(\nabla^{m-1}u)(Q) > 2\lambda$ when $Q \in E$, where $N_{\tau r}(v)(Q) = \sup_{X\in\Gamma(Q)\cap B(Q,\tau r)} |v(X)|$. This is straight forward using interior estimates and assuming that the square function is defined with respect to cones with a larger aperture than those used to define $N(\nabla^{m-1}u)$. (See for example [10] and then [25].)

Let D be the sawtooth region associated to E. Then, for ε sufficiently small, and $\Delta_X = \{Q \in \Delta : X \in \Gamma_\tau(Q)\}$, we have

$$\begin{aligned}
&\iint_D d_\Omega(X)|\nabla^m u(X)|^2 dX \\
&\quad\le C_\varepsilon \iint_D d_\Omega(X)|\nabla^m u(X)|^2 \Big\{\frac{\sigma(\Delta_X \cap \{S(\nabla^{m-1}u) < \gamma\lambda\})}{\sigma(\Delta_X)}\Big\} dX \\
&\quad= C_\varepsilon \int_{\Delta\cap\{S(\nabla^{m-1}u)<\gamma\lambda\}} \int_{\Gamma(Q)} d_\Omega^{2-n}(X)|\nabla^m u(X)|^2 dX\, d\sigma(Q) \\
&\quad\le C_\varepsilon \gamma^2\lambda^2\sigma(\Delta).
\end{aligned}$$

Recall that D is a starlike Lipschitz domain; let p_o be the star-center. We claim that we may assume the normalizations: $u(p_o) = \ldots = \nabla^{m-1}u(p_o) = 0$, by subtracting off a polynomial from u. This depends on the estimate $|\nabla^{m-1}u(p_o)| \leq \lambda + c\gamma\lambda \leq 2\lambda$ for γ small enough so that if u^* is the normalized u, then $N(\nabla^{m-1}u^*) > 2\lambda$ when $N(\nabla^{m-1}u) > 4\lambda$. Thus, for this normalized u, we may use the L^2 result of Theorem 1 to obtain

$$\begin{aligned}\sigma(E) &\leq \frac{c}{\lambda^2}\int_E N_\tau^2(\nabla^{m-1}u)d\sigma \\ &\leq \frac{c}{\lambda^2}\int_{\partial D} N_D^2(\nabla^{m-1}u)d\sigma \\ &\leq \frac{c}{\lambda^2}\int_{\partial D} S_D^2(\nabla^{m-1}u)d\sigma \\ &= \frac{c}{\lambda^2}\iint_D d_D(X)|\nabla^{m-1}u(X)|^2 dX \\ &\leq \frac{c}{\lambda^2}\iint_D d_\Omega(X)|\nabla^{m-1}u(X)|^2 dX \leq c\gamma^2\sigma(\Delta).\end{aligned}$$

Summing over the regions E and δ completes the proof. □

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