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# THE BOCHNER-HARTOGS DICHOTOMY FOR WEAKLY 1-COMPLETE KÄHLER MANIFOLDS

by T. NAPIER (*) & M. RAMACHANDRAN (**)

## Introduction.

A complex manifold $M$ for which $H^1_c(M, \mathcal{O}) = 0$ is said to have the *Bochner-Hartogs property* (see Hartogs [H], Bochner [B], and Harvey and Lawson [HL]). Equivalently, for every $C^\infty$ compactly supported form $\alpha$ of type $(0,1)$ with $\bar{\partial}\alpha = 0$ on $M$, there is a $C^\infty$ compactly supported function $\beta$ on $M$ such that $\bar{\partial}\beta = \alpha$. Andreotti and Vesentini [AV] proved that a strongly $(n-1)$-complete complex manifold of dimension $n > 1$ has the Bochner-Hartogs property, and Grauert and Riemenschneider [GR], that a strongly hyper-$(n-1)$-convex Kähler manifold of dimension $n > 1$ has the Bochner-Hartogs property (see Section 1). In [R], the second author proved that if the universal covering $M$ of a compact Kähler manifold (or a Galois covering $M$ with infinite covering group of more than quadratic growth) admits a nonconstant holomorphic function, then $M$ satisfies the following dichotomy:

*(BHD) Either $M$ has the Bochner-Hartogs property or there exists a proper holomorphic mapping of $M$ onto a Riemann surface.*

A complex manifold which admits a continuous plurisubharmonic exhaustion function is said to be *weakly* 1*-complete.* The main result of this paper (Theorem 2.5) is the following:

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THEOREM. — *If $(M,g)$ is a connected noncompact weakly 1-complete Kähler manifold which has exactly one end, then $M$ satisfies (BHD).*

If a complex manifold $M$ has the Bochner-Hartogs property, then every holomorphic function $f$ on a neighborhood of infinity with no relatively compact connected components extends to a holomorphic function $f_0$ on $M$ as follows. Cutting off away from infinity, one obtains a $C^\infty$ function $\lambda$ on $M$. For $\alpha = \bar{\partial}\lambda$, there is a function $\beta$ as above and the function $f_0 = \lambda - \beta$ is the desired extension. In particular, a Riemann surface cannot have the Bochner-Hartogs property and a complex manifold cannot satisfy both of the conditions in (BHD). Moreover, a manifold with the Bochner-Hartogs property has only one end because, on a manifold with more than one end, there exists a function which is locally constant, but not constant, near infinity. A related result due to Arapura, Bressler, and the second author [ABR] is that the universal covering of a compact Kähler manifold has at most one end. In fact, as shown in [NR], a complete noncompact connected Kähler manifold $M$ which satisfies $H^1(M,\mathbb{R}) = 0$ and which has bounded geometry or is weakly 1-complete has exactly one end. It was also proved in [NR] that a complete Kähler manifold with at least three ends which has bounded geometry or is weakly 1-complete admits a proper holomorphic mapping onto a Riemann surface.

Facts concerning strictly $q$-plurisubharmonic functions and Green's functions are collected in Section 1. Section 2 contains the proof of the theorem. The main step is to prove Proposition 2.3 that (BHD) holds for a weakly 1-complete Kähler manifold on which there exists, outside a compact subset, a pair of pluriharmonic functions with linearly independent differentials (this may be thought of as a generalization of a theorem of Ohsawa [O1]). One may assume that $M$ is complete and admits a positive Green's function since one can exhaust $M$ by domains with these properties. Hence, if $[\alpha] \in H^1_c(M,\mathcal{O})$ is a nonzero element, then one can form the $L^2$ harmonic projection $\gamma$ and a function $\beta$ such that $\gamma = \alpha - \bar{\partial}\beta$. In particular, $\beta$ is pluriharmonic outside a compact set $K$. One then forms a pluriharmonic function on a covering space, pushes down to obtain a second pluriharmonic function on $M \setminus K$, and applies Proposition 2.3.

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## 1. Preliminaries on $q$-plurisubharmonic functions.

Most of the facts discussed in this section are known, so the proofs are only sketched. Throughout this section, $(M, g)$ denotes a Kähler manifold of dimension $n$ and $q$ denotes a positive integer.

Let $\varphi$ be a real-valued continuous function on $M$. We will say that $\varphi$ is *strictly q-plurisubharmonic* if $\varphi$ is an element of the class $\Psi(q)$ defined by Wu [W]. We will call $\varphi$ *q-plurisubharmonic* if the function $\varphi + \psi$ is strictly $q$-plurisubharmonic for every continuous strictly $q$-plurisubharmonic function $\psi$ on $M$.

*Remarks.*

1. If $\varphi$ of class $C^2$, then $\varphi$ is $q$-plurisubharmonic (strictly $q$-plurisubharmonic) if and only if, for each point $x_0 \in M$, the trace of the restriction of the *Levi form*

$$\mathcal{L}(\varphi) = \sum_{i,j} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j$$

of $\varphi$ to any complex subspace of $T^{1,0}_{x_0} M$ of dimension $q$ is nonnegative (respectively, positive).

2. If $\varphi$ is $q$-plurisubharmonic (strictly $q$-plurisubharmonic), then $\varphi$ is $(q+1)$-plurisubharmonic (respectively, strictly $(q+1)$-plurisubharmonic).

3. A real-valued function of class $C^2$ on a complex manifold is said to be *strictly q-convex* if its Levi form has at most $q-1$ nonpositive eigenvalues at each point. A function $\psi$ on a complex space $X$ is said to be *strictly q-convex* if, for each point $x \in X$, there is a proper embedding of a neighborhood $U$ of $x$ into an open subset $V$ of some complex Euclidean space and an extension of $\psi \restriction_U$ to a strictly $q$-convex function on $V$. It follows that if $\varphi$ is of class $C^2$ and strictly $q$-plurisubharmonic on $M$, then $\varphi$ is strictly $q$-convex on $M$ and on any analytic subset of $M$.

4. The set of smooth elements of $\Psi(q)$ is dense in the following sense:

PROPOSITION 1.1 (Wu [W], Proposition 1). — *If $\varphi$ is a continuous strictly q-plurisubharmonic function on a Kähler manifold $M$ and $\alpha$ is a positive continuous function on $M$, then there exists a $C^\infty$ strictly q-plurisubharmonic function $\psi$ such that $|\varphi - \psi| < \alpha$ on $M$.*

In particular, it follows that the restriction of a continuous $q$-plurisubharmonic (strictly $q$-plurisubharmonic) function to a complex submanifold of dimension $q$ is subharmonic (respectively, strictly subharmonic).

5. The Kähler manifold $(M, g)$ is said to be *hyper-q-complete* if $M$ admits a $C^\infty$ strictly $q$-plurisubharmonic exhaustion function. If there exists a $C^\infty$ $q$-plurisubharmonic exhaustion function which is strictly $q$-plurisubharmonic on the complement of some compact subset of $M$, then $(M, g)$ is said to be *strongly hyper-q-convex.*

6. Standard arguments show that if $\varphi$ and $\varphi'$ are continuous $q$-plurisubharmonic functions on $M$, then $\varphi + \varphi'$, $\max(\varphi, \varphi')$, and the composition $\chi(\varphi)$ of any nondecreasing convex function $\chi$ with $\varphi$, are all $q$-plurisubharmonic.

7. Hunt and Murray [HM] and Kalka [K] studied functions which satisfy a condition which they called $q$-plurisubharmonicity but which is weaker than the above notion.

The following result is contained implicitly in the work of Greene and Wu [GW], Ohsawa [O2], and Demailly [D2]:

THEOREM 1.2 (Demailly, Greene-Wu, Ohsawa). — *Let $X$ be an analytic subset of dimension $m \leq q$ in the Kähler manifold $M$ and let $Y$ be the union of the singular set $X_{\mathrm{sing}}$ with all irreducible components of $X$ which are noncompact or which have dimension strictly less than $q$. Then there exist neighborhoods $V$ of $X$ and $W$ of $Y$ in $M$ and a $C^\infty$ strictly $(q+1)$-plurisubharmonic function $\varphi$ on $V$ such that $\varphi\restriction_X$ exhausts $X$, $W \subset V$, and $\varphi\restriction_W$ is strictly $q$-plurisubharmonic.*

The proof is an easy modification of Demailly's [D2] proof of the analogous result for strictly $q$-convex functions, but we include a sketch here for completeness. Similarly, as in [D2] and in the work of Colţoiu [C], a hyper-$q$-complete submanifold admits a hyper-$q$-complete neighborhood, but we won't use this fact and the proof will not be sketched.

By a theorem of Richberg [Ri], a $C^\infty$ strictly plurisubharmonic function on an analytic subset of a complex space extends to a $C^\infty$ strictly plurisubharmonic function on a neighborhood. Demailly proved a version of this theorem for strictly $q$-convex functions in which the function is approximated by a strictly $q$-convex function on a neighborhood (see also [P]). A natural modification of Richberg's proof shows that if a function on an analytic subset of a Kähler manifold admits local $C^\infty$ strictly $q$-plurisubharmonic extensions, then it admits a $C^\infty$ strictly $q$-plurisubharmonic extension to a neighborhood. We will only need this fact for submanifolds and the proof is simple in this case.

PROPOSITION 1.3 (Richberg). — *If $\varphi$ is a $C^\infty$ strictly $q$-plurisubharmonic function on a complex submanifold $N$ of $M$ (relative to the Kähler metric $g\restriction_N$), then there exists a $C^\infty$ strictly $q$-plurisubharmonic function $\psi$ on a neighborhood of $N$ in $M$ such that $\psi\restriction_N = \varphi$.*

*Sketch of the proof.* — Let $m = \dim N$ and suppose $(U, (z_1, \dots, z_n))$ is a holomorphic coordinate neighborhood in which $N$ is the zero set of $w = (z_1, \dots, z_{n-m})$. If $\varphi'$ is a function on a relatively compact polydisk $D$ in $U$ obtained by composing $\varphi$ with the associated projection mapping and $C > 0$ is sufficiently large, then the function $\varphi' + C|w|^2$ is strictly $q$-plurisubharmonic on a neighborhood of $N \cap D$. By using a partition of unity, one may patch these local extensions to obtain the desired function $\psi$. Q. E. D.

The main point of Demailly's [D2] proof of Ohsawa's theorem [O2] is essentially the following version of the theorem of Greene and Wu [GW] on the existence of subharmonic exhaustion functions (see [D2], Proof of Theorem 2):

PROPOSITION 1.4 (Demailly, Greene-Wu). — *Suppose $X$ is a complex space with no compact irreducible components, $Y$ is an analytic subset which contains the singular set $X_{\text{sing}}$, $U$ is a neighborhood of $Y$ in $X$, and $h$ is a Hermitian metric on $X \setminus Y$. Then there exists a $C^\infty$ nonnegative function $\varphi$ on $X$ such that*

(i) *$\varphi$ is positive and exhaustive on $X \setminus U$,*

(ii) *$\varphi$ vanishes on a neighborhood of $Y$, and*

(iii) *$\varphi$ is subharmonic (with respect to $h$) on $X \setminus Y$ and strictly subharmonic on the subset $\{x \in X \mid \varphi(x) > 0\}$ of $X \setminus Y$.*

*Proof of Theorem 1.2* ([D2]). — The statement of the theorem makes sense and is trivial for $q = 0$. We proceed by induction on $q$. Assume that $q > 0$ and that the theorem holds for nonnegative integers less than $q$. Let $A$ be the union of $X_{\text{sing}}$ and all of the irreducible components of $X$ of dimension less than $q$, let $B$ be the union of all of the noncompact irreducible components of $X$ of dimension $q$, and let $C$ be the closure of $X \setminus (A \cup B)$.

Applying the induction hypothesis to $A$ and cutting off away from $A$, we obtain a $C^\infty$ nonnegative function $\alpha$ on $M$ and a neighborhood $U$ of $A$ in $M$ such that $\alpha$ is strictly $q$-plurisubharmonic on $U$ and exhaustive on $\overline{U}$.

Next, by Proposition 1.4, there exists a nonnegative $C^\infty$ function $\psi$ on $X$ such that $\operatorname{supp}\psi \subset B \setminus A = B \setminus (A \cup C)$, $\psi > 2$ on $B \setminus U$, $\psi$ exhausts $B \setminus U$, and $\psi$ is strictly $q$-plurisubharmonic on the subset $N \equiv \psi^{-1}((0,\infty))$ of $B \setminus (A \cup C)$. By Richberg's theorem (Proposition 1.3), there exists a $C^\infty$ strictly $q$-plurisubharmonic function $\tau$ on a neighborhood $V_1$ of $N$ in $M$ such that $\tau = \psi$ on $N$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ nondecreasing convex function such that $\chi(t) = 0$ if $t < 1$, $\chi(t) > t$ if $t > 2$, and $\chi'(t) > 0$ if $t > 1$. Since $\psi^{-1}([0,1))$ is a neighborhood of the boundary of $N$ in $X$, we may assume (shrinking $V_1$ if necessary) that there is a neighborhood $V_2$ of $X \setminus N$ in $M$ such that $\tau < 1$ on $V_1 \cap V_2$. Thus the function $\chi(\tau)$ may be extended to a nonnegative $C^\infty$ $q$-plurisubharmonic function $\beta$ which is defined on the neighborhood $V_1 \cup V_2$ of $X$ and vanishes on a neighborhood of $A \cup C$. Moreover, since $\psi > 2$ on $B \setminus U$, $\beta$ is strictly $q$-plurisubharmonic and positive on a neighborhood of $B \setminus U$ in $M$ and $\beta$ exhausts $B \setminus U$.

Since $\dim C < q{+}1$, every $C^\infty$ function on the subset $C \setminus Y$ of $C \setminus X_{\text{sing}}$ is strictly $(q{+}1)$-plurisubharmonic. As in the construction of $\beta$, we may form a nonnegative $C^\infty$ $(q{+}1)$-plurisubharmonic function $\gamma$ on a neighborhood of $X$ such that $\gamma$ is positive and strictly $(q{+}1)$-plurisubharmonic on a neighborhood of $C \setminus U$ in $M$, $\gamma$ exhausts $C \setminus U$, and $\gamma$ vanishes on a neighborhood of $A \cup B = Y$.

It is now easy to check that if $\lambda$ is a $C^\infty$ increasing convex function on $\mathbb{R}$ and $\lambda'(t) \to \infty$ sufficiently fast as $t \to \infty$, then the function $\varphi \equiv \alpha + \lambda(\beta) + \lambda(\gamma)$ has the required properties on some neighborhood $V$ of $X$ and some neighborhood $W$ of $Y = A \cup B$. Q. E. D.

A $C^\infty$ strictly $q$-convex function $\varphi$ on a complex space $X$ of pure dimension $q$ has no local maximum points. Wu's approximation theorem (Proposition 1.1) implies that the same is true of the restriction of a continuous strictly $q$-plurisubharmonic function on $M$ to an analytic subset $X$ of pure dimension $q$. Similarly, we have the following:

Proposition 1.5 (Maximum principle). — *If the restriction $\varphi\restriction_X$ of a continuous $q$-plurisubharmonic function $\varphi$ on $M$ to a connected analytic subset $X$ of pure dimension $q$ assumes its maximum value $m = \varphi(x_0)$ at some point $x_0 \in X$, then $\varphi\restriction_X$ is constant.*

*Proof.* — Since $\varphi \restriction_{X_{\text{reg}}}$ is subharmonic, it suffices to show that $\varphi(x) = m$ at some point $x \in X_{\text{reg}}$. Assume that this is not the case. Taking successive singular sets $X_{\text{sing}}$, $(X_{\text{sing}})_{\text{sing}}, \dots$, we may assume that, for some nowhere dense analytic subset $Y$ of $X$, we have $x_0 \in Y_{\text{reg}}$ and $\varphi < m$ on $X \setminus Y$. Every $C^\infty$ function on $Y_{\text{reg}}$ is strictly $q$-plurisubharmonic since $\dim Y < q$. So, by applying Proposition 1.3, we may form a relatively compact neighborhood $U$ of $x_0$ in $M$ and a $C^\infty$ strictly $q$-plurisubharmonic function $\psi$ on a neighborhood of $\overline{U}$ such that $\psi(x_0) = 0$ and $\psi < 0$ on $(\overline{U} \setminus \{x_0\}) \cap Y$. In particular, on some neighborhood $V$ of $Y \cap \partial U$, we have $\psi < 0$ and hence $\varphi + \epsilon\psi < m$ for every $\epsilon > 0$. We also have $\varphi + \epsilon\psi < m$ on $(X \cap \partial U) \setminus V$ provided $\epsilon$ is sufficiently small. Therefore $\varphi + \epsilon\psi < m = (\varphi + \epsilon\psi)(x_0)$ on $X \cap \partial U$. This contradicts the maximum principle for continuous strictly $q$-plurisubharmonic functions, so the proposition follows. Q. E. D.

In this paper, the main tool for obtaining the Bochner-Hartogs property is the following result of Grauert and Riemenschneider [GR] (who proved a version of the vanishing theorem for higher cohomology groups) and of Siu [S], Lemma 5.10 (who proved a version in the more general setting of harmonic maps into manifolds satisfying a certain curvature condition).

THEOREM 1.6 (Grauert-Riemenschneider, Siu). — *Suppose $\Omega$ is a relatively compact domain in a Kähler manifold $M$ and $\Omega$ has a $C^\infty$ $(n-1)$-plurisubharmonic defining function $\varphi$ whose differential is nonzero at every point in $\partial\Omega$.*

(a) *If $\beta$ is a $C^\infty$ function on $\overline{\Omega}$ which is harmonic on $\Omega$ and which satisfies the tangential Cauchy-Riemann equations $\bar{\partial}_b \beta = 0$ on $\partial\Omega$, then $\beta$ is pluriharmonic on $\Omega$.*

(b) *If $\varphi$ is strictly $(n-1)$-plurisubharmonic on a neighborhood of some point in $\partial\Omega$, then $H^1_c(\Omega, \mathcal{O}) = 0$.*

*Sketch of the proof.* — Suppose first that $\beta$ is a function as in (a). Let $\gamma = \bar{\partial}\beta$ and let $\eta = \overline{*\gamma}$, where $*$ denotes the Hodge star operator. Then $\eta$ may be thought of as a $C^\infty$ form of type $(n, n-1)$ or as a form of type $(0, n-1)$ with values in the canonical bundle $K_M$ on $\overline{\Omega}$. A computation in normal coordinates then shows that $\eta$ lies in the domain of the adjoint operator $\bar{\partial}^*$. The Bochner-Kodaira formula is then (see [GR] or [S], formula 2.1.4)

$$\|\bar{\partial}\eta\|^2_{L^2(\Omega)} + \|\bar{\partial}^*\eta\|^2_{L^2(\Omega)} = \|\overline{\nabla}\eta\|^2_{L^2(\Omega)} + \int_{\partial\Omega} |\gamma|^2 \cdot \tau \, d\sigma,$$

where $d\sigma$ is the volume element on $\partial\Omega$ and $\tau$ is the trace of the restriction of the Levi form $\mathcal{L}(\varphi)$ of $\varphi$ to $T^{1,0}(\partial\Omega)$. Here, the curvature terms drop out because $\eta$ is of type $(n, n-1)$ and we have used the fact that $\bar{\partial}\beta \wedge \bar{\partial}\varphi$ vanishes at each point of $\partial\Omega$. Since $\beta$ is harmonic, the left-hand side is equal to zero. Since $\varphi$ is $(n-1)$-plurisubharmonic, we have $\tau \geq 0$ and hence $\overline{\nabla}\eta = 0$. A computation in normal coordinates now shows that $\beta$ is pluriharmonic (see [S], Proof of Lemma 5.6(d)) and (a) is proved.

Suppose now that $\alpha$ is a $C^\infty$ compactly supported form of type $(0,1)$ with $\bar{\partial}\alpha = 0$. Let $\beta$ be the $C^\infty$ function on $\overline{\Omega}$ which vanishes on $\partial\Omega$ and which satisfies $-\frac{1}{2}\Delta\beta = \bar{\partial}^*\bar{\partial}\beta = \bar{\partial}^*\alpha$ on $\Omega$ (where $\Delta = -(d^*d + dd^*)$ is the Laplacian), let $\gamma = \alpha - \bar{\partial}\beta$, and let $\eta = \overline{*\gamma}$. Then

$$\bar{\partial}^*\eta = - * \partial * (*\bar{\gamma}) = *\overline{\bar{\partial}\gamma} = 0 \qquad \text{and} \qquad \bar{\partial}\eta = - * [\overline{\bar{\partial}^*\alpha} - \overline{\bar{\partial}^*\bar{\partial}\beta}] = 0.$$

Moreover, since $\alpha$ has compact support, $\gamma = -\bar{\partial}\beta$ near $\partial\Omega$; and, since $\beta$ vanishes on $\partial\Omega$, $\bar{\partial}_b\beta \equiv 0$. Applying the Bochner-Kodaira formula to $\eta$ as in the proof of (a), we get $\overline{\nabla}\eta = 0$ and it follows that $\partial\gamma = 0$ (and hence $d\gamma = 0$). Thus $\bar{\gamma}$ is a holomorphic 1-form on $\Omega$.

On the other hand, since $\tau > 0$ at some point, the Bochner-Kodaira formula implies that $|\gamma| = |\bar{\partial}\beta| = 0$ on some nonempty open subset in $\partial\Omega$. If $U$ is a connected neighborhood of a boundary point and $f$ is a $C^\infty$ function on $U \cap \overline{\Omega}$ which is holomorphic on $U \cap \Omega$ and which vanishes on $U \cap \partial\Omega$, then one may extend $f$ to a continuous function $h$ on $U$ which vanishes outside $\Omega$. But then $\bar{\partial}h = 0$ in the weak sense, so $h$ is holomorphic. It follows that $h$, and therefore $f$, must vanish identically. Letting $f$ be a coefficient of the holomorphic 1-form $\bar{\gamma}$ with respect to some local holomorphic frame, we see that $\gamma$ vanishes on a nonempty open subset of $\Omega$ and hence on all of $\Omega$. Thus $\beta$ is holomorphic outside the support $K$ of $\alpha$. Since $\beta$ vanishes on $\partial\Omega$, the above discussion (with $f = \beta$) implies that $\beta$ vanishes on each connected component of $\Omega \backslash K$ which is not relatively compact in $\Omega$. In other words, $\beta$ has compact support and $\alpha = \bar{\partial}\beta$. Thus (b) is proved. Q. E. D.

Since the proof of the main theorem involves related arguments on a complete Kähler manifold, we close this section with a discussion of Green's functions on Riemannian manifolds. A connected noncompact Riemannian manifold $N$ which admits a positive symmetric Green's function $G(x, y)$ is said to be *hyperbolic* (otherwise, $N$ is called *parabolic*). We normalize $G$ so that, for each point $x_0 \in N$,

$$\Delta_{\mathrm{distr}} G(\cdot, x_0) = -\delta_{x_0}$$

where $\delta_{x_0}$ is the Dirac function at $x_0$ and $\Delta = -(d^*d + dd^*)$ is the Laplacian. We will use the same notation for the corresponding integral operator $G$ given by

$$(G\alpha)(x) = \int_N G(x,y)\alpha(y)\, dV(y) \quad \forall x \in N$$

for each suitable function $\alpha$ on $N$. If $\alpha$ is a $C^\infty$ compactly supported function, then $\beta \equiv -G\alpha$ is a $C^\infty$ bounded function with finite energy (i.e. $\int_N |\nabla\beta|^2\, dV < \infty$) and $\Delta\beta = \alpha$. Moreover, $\beta(x_\nu) \to 0$ if $\{x_\nu\}$ is a sequence in $N$ with $x_\nu \to \infty$ and $G(\cdot, x_\nu) \to 0$. Such a sequence $\{x_\nu\}$ always exists and will be called a *regular sequence.*

## 2. Proof of the main result.

We begin with two lemmas. The first is a special case of a result of Nishino [Ni] who proved it without the assumption that $M$ is Kähler. In the Kähler case, one may prove it using arguments contained in the proof of [NR], Theorem 4.6.

LEMMA 2.1. — *Suppose $(M, g)$ is a connected weakly 1-complete Kähler manifold and there exists a proper holomorphic mapping of some nonempty open subset of $M$ onto a Riemann surface. Then $M$ also admits a proper holomorphic mapping onto a Riemann surface.*

The second lemma often helps one obtain a holomorphic mapping to a Riemann surface. An elementary proof is given here. One may also prove this fact by using holomorphic equivalence relations (see, for example, [Ka]).

LEMMA 2.2. — *If $\omega_1$ and $\omega_2$ are two linearly independent closed holomorphic 1-forms satisfying $\omega_1 \wedge \omega_2 \equiv 0$ on a connected complex manifold $M$, then the meromorphic function $h \equiv \omega_1/\omega_2$ has no points of indeterminacy in $M$ and is locally constant on the analytic set $S \equiv \{x \in M \mid (\omega_1)_x = 0\} \cup \{x \in M \mid (\omega_2)_x = 0\}$.*

*Proof.* — Let $I$ be the set of points of indeterminacy of $h$ and, for each $\zeta \in \mathbb{P}^1$, let

$$F_\zeta = \overline{(h{\restriction}_{(M\setminus I)})^{-1}(\zeta)} \supset I$$

be the fiber over $\zeta$ (see [Gu]). Since the problem is local, we may assume that there exist holomorphic functions $f_1$ and $f_2$ on $M$ such that $df_i = \omega_i$ for $i = 1, 2$. One may see that $f_1$ and $f_2$ are locally constant on $F_\zeta$ for each $\zeta \in \mathbb{P}^1$ as follows. Near each smooth point $x$ of $F_\zeta$ at which $df_2 \neq 0$, we may choose holomorphic coordinates $z = (z_1, \dots, z_n)$ in which $z_1 = f_2$ on a connected neighborhood of $x$. Since $df_1 \wedge dz_1 = 0$, we have then $f_1 = f_1(z_1)$ and hence $h = f_1'(z_1)$. In particular, $h$ is constant along each fiber of $z_1$ and hence each of these fibers (being of codimension 1) must be an open set in $F_\zeta$. Therefore, if $v$ is a vector tangent to $F_\zeta$ at $x$, then $\omega_2(v) = df_2(v) = 0$. This is also the case if $df_2 = 0$ at $x$. Thus $f_2$ is locally constant on $F_\zeta$ for each $\zeta \in \mathbb{P}^1$, and, by symmetry, the same is true of $f_1$. It follows that, if $x_0 \in I$, then $f_2^{-1}(f_2(x_0))$ contains the connected component $H_\zeta$ of $F_\zeta$ containing $x_0$ for each $\zeta \in \mathbb{P}^1$. But $\{H_\zeta\}_{\zeta \in \mathbb{P}^1}$ is an infinite collection of distinct analytic sets of pure dimension $n-1$, so this is impossible. Therefore $I = \emptyset$.

It remains to show that $h$ is locally constant on the analytic set $S$, which we may assume to be irreducible. Since $f_1$ or $f_2$ is then constant on $S$, we may assume that $S$ lies in some irreducible component $T$ of the zero set of $f_1 \cdot f_2$ and it suffices to show that $h$ is constant on $T$. By working near a generic point of $T$, we may assume that there exist holomorphic coordinates $z = (z_1, \dots, z_n)$ on $M$ in which $T$ is the zero set of $z_1$ and, for $j = 1, 2$, $f_j(z) = z_1^{m_j} g_j(z)$ where $g_j$ is a nonvanishing holomorphic function and $m_j \geq 0$. By replacing $z_1$ by $z_1 u$ for some (local) $m_2^{\text{th}}$ root $u$ of $g_j$, we may also assume that $g_2 \equiv 1$. Since $df_1 \wedge df_2 \equiv 0$, we then get $f_1 = f_1(z_1)$ and $g_1 = g_1(z_1)$. Hence the function $h(z) = f_1'(z_1)/f_2'(z_1)$ depends only on $z_1$ and is therefore constant on $T$. Thus the lemma is proved.

The main step in the proof of the theorem stated in the introduction is the following proposition, which is, in a sense, a generalization of a theorem of Ohsawa [O1] and a result in [NR].

PROPOSITION 2.3. — *Let $(M, g)$ be a connected noncompact Kähler manifold of dimension $n$ on which there exists a continuous $(n-1)$-plurisubharmonic exhaustion function $\varphi$. Suppose there is a compact subset $K$ of $M$ such that, on each connected component $E$ of $M \setminus K$, there exists a pair of real-valued pluriharmonic functions $\rho_1$ and $\rho_2$ with linearly independent differentials $d\rho_1$ and $d\rho_2$. Then $M$ satisfies (BHD).*

*Remarks.*

1. The above condition on $E$ holds if, for example, there exists a

nonconstant holomorphic function on $E$.

2. If $M$ satisfies the above conditions and has more than one end, then there exists a proper holomorphic mapping of $M$ onto a Riemann surface.

*Proof.* — Choosing $a \in \mathbb{R}$ sufficiently large, we may assume without loss of generality that

$$K = \{x \in M \mid \varphi(x) \leq a\} \neq \emptyset.$$

Let $E$ be a connected component of $M \setminus K$ with noncompact closure and let $\rho_1$ and $\rho_2$ be pluriharmonic functions on $E$ as in the statement of the proposition. We may assume that $\rho_1$ and $\rho_2$ extend to pluriharmonic functions on a neighborhood of $\overline{E}$.

There are three possibilities:

(a) The analytic subset $X = \{x \in E \mid (\partial\rho_1 \wedge \partial\rho_2)_x = 0\}$ of $E$ is nowhere dense,

(b) $X = E$ and the set $Q$ of points in $E$ which lie in a compact level of the holomorphic mapping

$$h = \frac{\partial\rho_1}{\partial\rho_2} : E \to \mathbb{P}^1$$

(see Lemma 2.2) is nonempty, or

(c) $X = E$ and $Q = \emptyset$.

We will show that, if (a) or (c) holds, then $M$ admits an exhaustion by $C^\infty$ domains which are strongly hyper-$(n-1)$-convex at each boundary point in $E$; and if (b) holds, then $E$ admits a proper holomorphic mapping onto a Riemann surface. Briefly, in the case (a), we will work on a relatively compact (weakly) hyper-$(n-1)$-convex domain $\Omega$ in $M$. Combining $\varphi$, $\rho_1^2 + \rho_2^2$, and a strictly $(n-1)$-plurisubharmonic function near $X$ (obtained from Theorem 1.2), we will obtain a strictly $(n-1)$-plurisubharmonic function near $E \cap \partial\Omega$. In the case $(b)$, we will show that $Q = E$ and hence, by Stein factorization, one obtains the desired mapping. Finally, in the case $(c)$, we will again work on a hyper-$(n-1)$-convex domain $\Omega$ and we will apply a standard patching argument to strictly $(n-1)$-plurisubharmonic functions on neighborhoods of fibers of $h$ (or of a suitable holomorphic function if $h$ is constant) to obtain a strictly $(n-1)$-plurisubharmonic function as in the case (a).

Suppose first that $E$ satisfies the condition (a) and let $Y$ be the union of all of the compact irreducible components of $X$ of dimension $n-1$. The maximum principle (Proposition 1.5) implies that $\varphi(Y)$ is a countable set, so we may choose $b \in (a, \infty) \setminus \varphi(Y)$ so large that $K$ is contained in some connected component $\Omega$ of $\{x \in M \mid \varphi(x) < b\}$. By Theorem 1.2 (Demailly, Greene-Wu, Ohsawa), there exists a positive $C^\infty$ function $\psi$ on $E \setminus Y$ which is strictly $(n-1)$-plurisubharmonic on a neighborhood of $X \setminus Y$ in $E$. Therefore, since $\rho_1^2 + \rho_2^2$ is strictly $(n-1)$-plurisubharmonic on $E \setminus X$ and since $\varphi^{-1}(b) \cap Y = \emptyset$, the function $\gamma \equiv r \cdot (\rho_1^2 + \rho_2^2) + \psi$, where $r$ is a sufficiently large positive constant, is strictly $(n-1)$-plurisubharmonic on a relatively compact neighborhood $U$ of $\varphi^{-1}(b) \cap E$ in $E \setminus Y$. Applying Wu's approximation theorem [W] (Proposition 1.1) to the function $\gamma - \log(b - \varphi)$ on $\Omega \cap U$, we get a $C^\infty$ strictly $(n-1)$-plurisubharmonic function $\lambda$ on $U \cap \Omega$ which approaches infinity at $\partial\Omega$. Finally, let $c > 0$ and let $\chi : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ nondecreasing convex function such that $\chi(t) = 0$ for $t \le c$, $\chi'(t) > 0$ for $t > c$, and $\chi(t) \to \infty$ as $t \to \infty$. If $c$ is sufficiently large, then the function $\chi(\lambda)$ extends to a $C^\infty$ nonnegative $(n-1)$-plurisubharmonic function on $\Omega \cup (M \setminus E)$ which vanishes on $M \setminus E$, which is strictly $(n-1)$-plurisubharmonic near $E \cap \partial\Omega$, and which exhausts $\Omega \cap \overline{E}$.

Assuming now that (b) holds, we show that $Q = E$. We first observe that $Q$ is open for point-set topological reasons. For if $x_0 \in Q$ and $L_0$ is the (compact) level of $h$ through $x_0$, then there is a relatively compact neighborhood $U$ of $L_0$ in $E \setminus (h^{-1}(h(x_0)) \setminus L_0)$. Since $\partial U$ is compact, there is a neighborhood $V$ of $h(x_0)$ in $\mathbb{P}^1$ such that $h^{-1}(V) \cap \partial U = \emptyset$. Hence the level through each point in the neighborhood $h^{-1}(V) \cap U$ does not meet $\partial U$ and must, therefore, be a compact subset of $U$.

Next, let $b > a$ be so large that there is a connected component $\Omega$ of the set $\{x \in M \mid \varphi(x) < b\}$ such that $Q \cap \Omega \neq \emptyset$, $K \subset \Omega$, and $\Omega \cap E$ is connected. If $x_0 \in \overline{Q} \cap \Omega$, then the irreducible component $A$ of $h^{-1}(h(x_0))$ containing $x_0$ is a compact subset of $\Omega \cap E$. For if $\{x_\nu\}$ is a sequence in $Q \cap \Omega \setminus \{x_0\}$ converging to $x_0$ and, for each $\nu$, $L_\nu$ is the (compact) level of $h$ through $x_\nu$, then, by the Remmert-Stein-Thullen theorem (see [Gu]), $A$ lies in the closure of $\bigcup_\nu L_\nu$. On the other hand, $\varphi\restriction_{L_\nu}$ is constant for each $\nu$ and $\varphi(x_\nu) \to \varphi(x_0)$ where $a < \varphi(x_0) < b$. Therefore, since $\varphi = a$ or $b$ at each boundary point of $\Omega \cap E$, $\bigcup_\nu L_\nu$ must lie in some compact subset of $\Omega \cap E$ and the claim follows.

Since $h$ extends to a holomorphic mapping on a neighborhood of $\overline{E}$ ($\rho_1$ and $\rho_2$ extend by the choice of $a$), the set of critical values of $h\restriction_{(\Omega \cap E)}$ is finite and the inverse image of this finite set is an analytic subset $B$ of $\Omega \cap E$. The above discussion implies that, if $L_0$ is the level of $h$ through a point $x_0 \in \overline{Q} \cap \Omega$ and $L_0 \cap \Omega$ is smooth, then $L_0 \subset Q \cap \Omega$. Therefore $Q \cap \Omega \setminus B$ is a closed subset of the connected set $\Omega \cap E \setminus B$ and hence, since $Q$ is open, we must have equality. In particular, $Q \cap \Omega$ is dense in $\Omega \cap E$. Applying the above again, we see that if $L_0$ is any level of $h$ which meets $\Omega$, then every irreducible component of $L_0$ that meets $\Omega$ must be a compact subset of $\Omega \cap E$. It follows that $L_0$ is compact. Thus $\Omega \cap E \subset Q$ and, since the choice of $b$ was arbitrary, we get $Q = E$. Therefore every level of $h$ is compact and, by Stein factorization [St], we obtain a proper holomorphic mapping of $E$ onto a Riemann surface.

Finally, assuming that $X = E$ and $Q = \emptyset$ (i.e. that $E$ satisfies the condition (c)), we apply a modification of a construction due to Ohsawa [O1] for the case of a weakly 1-complete surface. We also assume for now that the mapping $h : E \to \mathbb{P}^1$ is nonconstant.

We first show that the union $C$ of the collection $\mathcal{C}$ of all compact irreducible components of fibers of $h$ is a nowhere dense analytic subset of $E$. For if $K_0$ is a compact subset of $E$ and $L_0$ is a level of $h$, then any compact irreducible component $C_0$ of $L_0$ which meets $K_0$ must lie in the compact subset $\varphi^{-1}(\varphi(K_0))$. Moreover, since $C_0 \neq L_0$, $C_0$ must meet some irreducible component $H_0$ of the analytic set $H \equiv \{x \in E \mid (h_*)_x = 0\}$ and, since $h$ is locally constant on $H$, we have $H_0 \subset L_0$. Only finitely many irreducible components of $H$ meet $\varphi^{-1}(\varphi(K_0))$, so the collection of all levels $L_0$ with such an irreducible component $C_0$ is finite. It follows that $\mathcal{C}$ is locally finite in $E$ and hence that $C$ is an analytic set.

The set $\varphi(C)$ is discrete, so we may choose a number $b \in (a, \infty) \setminus \varphi(C)$ so large that there is a connected component $\Omega$ of $\{x \in M \mid \varphi(x) < b\}$ such that $K \subset \Omega$ and $\Omega \cap E$ is connected. We may also choose a relatively compact neighborhood $W$ of $\varphi^{-1}(b) \cap E$ in $E \setminus C$. For each point $x \in \varphi^{-1}(b) \cap E$, the analytic set $h^{-1}(h(x)) \setminus C$ has no compact irreducible components and hence, by Theorem 1.2, there is a $C^\infty$ strictly $(n-1)$-plurisubharmonic function $\gamma_1$ on a neighborhood $V_1$ of $h^{-1}(h(x)) \setminus C$ in $E$. Moreover, we have $h^{-1}(D_1') \cap W \subset V_1$ for any sufficiently small neighborhood $D_1'$ of $h(x)$ in $\mathbb{P}^1$. There is also a nonnegative $C^\infty$ function $\lambda_1$ with $\operatorname{supp} \lambda_1 \subset D_1'$ and $\lambda_1 \equiv 1$ on some neighborhood $D_1$ of $h(x)$. We may, therefore, choose $C^\infty$ strictly $(n-1)$-plurisubharmonic functions $\gamma_1, \dots, \gamma_m$ on open sets $V_1, \dots, V_m$ in $E$, respectively; open sets $D_1, \dots, D_m$ and $D_1', \dots, D_m'$ in $\mathbb{P}^1$; and nonnegative $C^\infty$ functions $\lambda_1, \dots, \lambda_m$ on $\mathbb{P}^1$ such that, for each $j = 1, \dots, m$, we have $\lambda_j \equiv 1$ on $D_j$, $\operatorname{supp} \lambda_j \subset D_j'$, $h^{-1}(D_j') \cap W \subset V_j$, and $\varphi^{-1}(b) \cap E \subset h^{-1}(D_1) \cup \ldots \cup h^{-1}(D_m)$. Moreover, by Lemma 2.2, $h$ is locally constant on the analytic set $S \equiv \{x \in E \mid (\partial\rho_2)_x = 0\}$, so $h(S \cap \phi^{-1}(b))$ is a finite set and we may assume that, for each $j$, $\lambda_j(h)$ is constant near each point of $W \cap S$.

We now show that, for a sufficiently large positive constant $s$, the $C^\infty$ function

$$\gamma \equiv s \cdot (\rho_2)^2 + \sum_{j=1}^{m} \lambda_j(h)\gamma_j$$

is strictly $(n-1)$-plurisubharmonic on some neighborhood of $\varphi^{-1}(b) \cap E$. It is easy to see that $\gamma$ is strictly $(n-1)$-plurisubharmonic on a neighborhood of $S \cap \varphi^{-1}(b) \cap E$, since, near each point of this set, each of the nonnegative functions $\lambda_1(h), \dots, \lambda_m(h)$ is constant and at least one of the functions is positive. Given a point $x$ near $\varphi^{-1}(b) \cap E$ and a tangent vector $v \in T^{1,0}_x M$, we have

$$\begin{aligned}\mathcal{L}(\gamma)(v,v) =& 2s|\partial\rho_2(v)|^2 + \sum_{j=1}^{m} \gamma_j(x)\mathcal{L}(\lambda_j)(h_*v, h_*v) \\ &+ \sum_{j=1}^{m} 2\operatorname{Re}\big[\partial\lambda_j(h_*v)\overline{\partial\gamma_j(v)}\,\big] + \sum_{j=1}^{m} \lambda_j(h(x))\mathcal{L}(\gamma_j)(v,v).\end{aligned}$$

If $x$ is not near $S$, then, since $(\partial\rho_2)_x \neq 0$ and $\partial\rho_1 \wedge \partial\rho_2 \equiv 0$, we may choose holomorphic coordinates $(z_1, \dots, z_n)$ near $x$ in which $\rho_2 = 2\operatorname{Re} z_1$, $\rho_1 = \rho_1(z_1)$, and

$$h = \frac{\partial\rho_1}{\partial\rho_2} = \frac{\partial\rho_1}{\partial z_1} = \frac{\partial\rho_1}{\partial z_1}(z_1) \in \mathbb{P}^1 \setminus \{\infty\} = \mathbb{C}.$$

Thus $\partial h = \dfrac{\partial h}{\partial z_1} dz_1$ and hence $|\partial\rho_2(v)|^2 = |dz_1(v)|^2 \geq q_0|\partial h(v)|^2$ for some positive constant $q_0$. Since $\lambda_1, \dots, \lambda_m \geq 0$, since $\max(\lambda_1(h), \dots, \lambda_m(h)) > 0$ near $\varphi^{-1}(b) \cap E$, and since, for each $j$, $\gamma_j$ is strictly $(n-1)$-plurisubharmonic on the neighborhood $V_j$ of $\operatorname{supp} \lambda_j(h) \cap \varphi^{-1}(b)$, there exist positive constants $q_1$ and $q_2$ (independent of $s$) such that, for every $\epsilon > 0$, for every point $x \in \varphi^{-1}(b) \cap E$, and for every collection of orthonormal tangent vectors $e_1, \dots, e_{n-1} \in T^{1,0}_x M$, we have

$$\sum_{i=1}^{n-1} \mathcal{L}(\gamma)(e_i, e_i) \geq (2sq_0 - q_1 \cdot (1 + (2\epsilon)^{-1})) \cdot \sum_{i=1}^{n-1} |\partial h(e_i)|^2 - \frac{1}{2}\epsilon q_1 + q_2.$$

Thus if we choose $\epsilon < 2q_2/q_1$ and $s > (2q_0)^{-1}q_1 \cdot (1 + (2\epsilon)^{-1})$, then $\gamma$ will be strictly $(n-1)$-plurisubharmonic near points in $\varphi^{-1}(b) \cap E$ which lie outside an (arbitrarily small) neighborhood of $S$ as well as those which lie near $S$.

Proceeding now as in the case (a), one gets a $C^\infty$ nonnegative $(n-1)$-plurisubharmonic function on $\Omega \cup (M \setminus E)$ which vanishes on $M \setminus E$, which exhausts $\overline{E} \cap \Omega$, and which is strictly $(n-1)$-plurisubharmonic near $E \cap \partial\Omega$.

If $h$ is constant, then $\partial(c_1\rho_1 + c_2\rho_2) \equiv 0$ for some pair of constants $c_1, c_2 \in \mathbb{C}$ which are not both zero. The function $f = \bar{c}_1\rho_1 + \bar{c}_2\rho_2$ on $E$ is holomorphic and nonconstant; because the functions 1, $\rho_1$, and $\rho_2$ are linearly independent. One may now proceed as above by using $f$ in place of the mapping $h$.

Thus for each of the (finitely many) connected components $E$ of $M \setminus K$ which have noncompact closure, either $E$ admits a proper holomorphic mapping onto a Riemann surface (and hence a $C^\infty$ plurisubharmonic exhaustion function), or there exists an arbitrarily large relatively compact domain $\Omega$ in $M$ and a $C^\infty$ nonnegative $(n-1)$-plurisubharmonic function on $\Omega \cup (M \setminus E)$ which vanishes on $M \setminus E$, which exhausts $\overline{E} \cap \Omega$, and which is strictly $(n-1)$-plurisubharmonic near $E \cap \partial\Omega$. If all of these connected components of $M \setminus K$ have the former property, then $M$ admits a $C^\infty$ plurisubharmonic exhaustion function and a complete Kähler metric and Lemma 2.1 implies that there is a proper holomorphic mapping of $M$ onto a Riemann surface. If at least one of these connected components has the latter property, then $H^1_c(M, \mathcal{O}) = 0$. For if $\alpha$ is a $C^\infty$ compactly supported form of type $(0,1)$ and $\bar{\partial}\alpha = 0$, then we may choose a $C^\infty$ relatively compact domain $\Omega$ which contains the support of $\alpha$ and which admits a $C^\infty$ $(n-1)$-plurisubharmonic defining function which is strictly $(n-1)$-plurisubharmonic near at least one connected component of $\partial\Omega$. The theorem of Grauert and Riemenschneider [GR] and Siu [S] (Theorem 1.6) then implies that $H^1_c(\Omega, \mathcal{O}) = 0$ and hence that there is a $C^\infty$ compactly supported function $\beta$ on $\Omega$, and therefore on $M$, such that $\bar{\partial}\beta = \alpha$. Thus $M$ has the property (BHD). Q. E. D.

To apply Proposition 2.3, it will be convenient to have the following lemma, which is essentially a special case of [NR], Theorem 2.6.

LEMMA 2.4. — *Let $(M, g)$ be a connected noncompact complete hyperbolic Kähler manifold, let $M_0$ be a $C^\infty$ relatively compact domain in $M$, and let $E$ be a connected component of $M \setminus \overline{M}_0$ with noncompact closure. Then there exists a pluriharmonic function $\tau$ on $M$ such that $0 \leq \tau \leq 1$, $\tau$ has finite energy, and, for every regular sequence $\{x_\nu\}$ in $M$ approaching $\infty$ (see Section 1), $\tau(x_\nu) \to 1$ if $x_\nu \in E$ for $\nu \gg 0$ and $\tau(x_\nu) \to 0$ if $x_\nu \in M \setminus E$ for $\nu \gg 0$.*

The construction of $\tau$ is contained within the proof of [NR], Theorem 2.6. That $\tau \leq 1$ is not proved explicitly, but one can easily verify this property by forming an exhaustion of $M$ by $C^\infty$ relatively compact domains and writing the Green's function on $M$ as the limit of the corresponding sequence of Green's functions.

We are now ready to prove the main result.

THEOREM 2.5. — *Let $(M, g)$ be a connected Kähler manifold of dimension $n$ which has exactly one end and which admits a continuous $(n-1)$-plurisubharmonic exhaustion function $\varphi$. Assume that at least one of the following conditions is satisfied:*

(i) *$\varphi$ plurisubharmonic,*

(ii) *$(M, g)$ is complete and hyperbolic, or*

(iii) *$\varphi$ is of class $C^\infty$.*

*Then $M$ satisfies (BHD).*

*Remark.* — An example of Cousin [Co] shows that one cannot remove the requirement that $M$ have only one end (see, for example, [NR], Example 3.9).

*Proof.* — Assuming that there exists a $C^\infty$ compactly supported form $\alpha$ of type $(0,1)$ such that $\bar{\partial}\alpha = 0$ and $[\alpha] \neq 0$ in $H^1_c(M, \mathcal{O})$, we will show that there is a proper holomorphic mapping of $M$ onto a Riemann surface.

We first assume that (i) or (ii) holds. Fix a $C^\infty$ relatively compact domain $M_0$ in $M$ such that $\operatorname{supp}\alpha \subset M_0$ and $M \setminus \overline{M}_0$ is connected. If $\varphi$ is plurisubharmonic (i.e. (i) holds), let $a > \sup_{M_0} \varphi$ and let $\Omega$ be the connected component of $\{x \in M \mid \varphi(x) < a\}$ containing $\overline{M}_0$. Then, by a theorem of Nakano [N] and Demailly [D1], the (weakly 1-complete) domain $\Omega$ admits a complete Kähler metric $g'$. Moreover, $(\Omega, g')$ is hyperbolic (in fact, the Green's function vanishes at the boundary) and if $a$ is sufficiently large, then $\Omega \setminus \overline{M}_0$ is connected. It suffices to show that $\Omega$ admits a proper holomorphic mapping onto a Riemann surface (for a suitable choice of $a$). For we may then form an exhaustion of $M$ by such domains. Applying a theorem of Narasimhan [Na], Corollary 1 to the Cartan-Remmert reductions, we get the required mapping on $M$. If $\varphi$ is not plurisubharmonic, we set $(\Omega, g') = (M, g)$.

Thus, in either case, $(\Omega, g')$ admits a continuous $(n-1)$-plurisubharmonic exhaustion function and a positive symmetric Green's function $G(x,y)$, and the Kähler metric $g'$ is complete. Therefore, since $\bar{\partial}^*\alpha$ is a $C^\infty$ function with compact support on $\Omega$, the function $\beta \equiv -2G(\bar{\partial}^*\alpha)$ is a $C^\infty$ bounded function with finite energy and $-\frac{1}{2}\Delta\beta = \bar{\partial}^*\bar{\partial}\beta = \bar{\partial}^*\alpha$. Moreover, $\beta(x_\nu) \to 0$ for any regular sequence $\{x_\nu\}$ in $\Omega$ approaching $\infty$ (see Section 1). The form $\gamma \equiv \alpha - \bar{\partial}\beta$ is a harmonic form in $L^2_{0,1}(\Omega, g')$; the harmonic projection of $\alpha$. By the Gaffney theorem [G], $\gamma$ is closed (and coclosed). In particular, $\bar{\gamma}$ is a holomorphic 1-form on $\Omega$ (since $\partial\gamma = 0$) and $\beta$ is pluriharmonic on the connected set $E \equiv \Omega \setminus \overline{M}_0$. Since $[\alpha] \neq 0$ in $H^1_c(M, \mathcal{O})$ and $\operatorname{supp}\alpha \subset M_0 \subset\subset \Omega$, we have $[\alpha] \neq 0$ in $H^1_c(\Omega, \mathcal{O})$.

It follows that $\beta$ is not constant on $E$. For, if $\beta$ were constant on $E$, then, since $\beta$ vanishes at infinity along any regular sequence, we would have $\beta \equiv 0$ on $\Omega \setminus \overline{M}_0$. Hence $\gamma$ would vanish on $E$ and, therefore, on $\Omega$, since $\gamma$ is harmonic. But we would then get $\alpha = \bar{\partial}\beta$, where $\beta$ is a $C^\infty$ function with compact support in $\Omega$; which contradicts the nonvanishing of $[\alpha]$. Thus $\beta\restriction_E$ is not constant (and hence is nowhere locally constant).

If $\gamma = 0$, then $\beta\restriction_E$ is a nonconstant holomorphic function on the end $E$. Proposition 2.3 then implies that $\Omega$ satisfies (BHD) and hence, since $H^1_c(\Omega, \mathcal{O}) \neq 0$, there is a proper holomorphic mapping onto a Riemann surface. Thus we may assume that $\gamma$ is not everywhere zero. Let $\rho = \operatorname{Re}\beta\restriction_E$ and $\rho' = \operatorname{Im}\beta\restriction_E$. If the functions 1, $\rho$, and $\rho'$ are linearly independent on $E$, then, again, Proposition 2.3 gives one the required mapping. If 1, $\rho$, and $\rho'$ are linearly dependent on $E$, then, since $\rho$ and $\rho'$ vanish at infinity along any regular sequence, the functions $\rho$ and $\rho'$ must be linearly dependent. Therefore, after multiplying $\alpha$ by a suitable nonzero complex constant, we may assume that $\beta\restriction_E = \rho$ (i.e. $\beta\restriction_E$ is real-valued) and hence that $\gamma\restriction_E = -\bar{\partial}\rho$. To obtain a second pluriharmonic function (i.e. to apply Proposition 2.3) we will use Lemma 2.4 to obtain a pluriharmonic function on a covering space of $\Omega$ with two ends and then push down to $E$.

We first observe that, for any point $x_0 \in E$, the mapping $\pi_1(E, x_0) \to \pi_1(\Omega, x_0)$ is *not surjective*. For if the mapping is surjective, then, since the $C^\infty$ closed real 1-form $\theta = -\gamma - \bar{\gamma}$ on $\Omega$ is equal to $d\rho$ on $E$, it follows that the function $\rho_0$ given by

$$\rho_0(x) = \rho(x_0) + \int_{x_0}^{x} \theta \quad \forall x \in \Omega$$

is a well-defined $C^\infty$ function with $d\rho_0 = \theta = d\rho$ on $E$ and $\rho_0(x_0) = \rho(x_0)$. Hence $\rho_0 = \rho$ on $E$, so $\beta - \rho_0$ is a $C^\infty$ function on $\Omega$ which vanishes on

$E = \Omega \setminus \overline{M}_0$ and which satisfies

$$\bar{\partial}(\beta - \rho_0) = \bar{\partial}\beta - \bar{\partial}\rho_0 = \alpha - \gamma + \gamma = \alpha.$$

But this contradicts the assumption that $[\alpha] \neq 0$ in $H^1_c(\Omega, \mathcal{O})$, so the mapping cannot be surjective.

Fix a point $x_0 \in E$, let $\Gamma$ be the image of $\pi_1(E, x_0)$ in $\pi_1(\Omega, x_0)$, and let $\pi : \widetilde{\Omega} \to \Omega$ be a connected covering space with $\pi_*(\pi_1(\widetilde{\Omega}, x_1)) = \Gamma$ for some point $x_1 \in \pi^{-1}(x_0)$. Since $\Gamma$ is a proper subgroup and $E$ is a $C^\infty$ domain, $\pi$ maps a neighborhood of the closure $\overline{E}_1$ of the connected component $E_1$ of $\widetilde{E} = \pi^{-1}(E)$ containing $x_1$ isomorphically onto a neighborhood of $\overline{E}$ and $\widetilde{\Omega} \setminus \overline{E}_1$ is not relatively compact in $\widetilde{\Omega}$ (i.e. $\widetilde{\Omega}$ has at least two ends). In particular, $E_1$ is a hyperbolic end of $\widetilde{\Omega}$ with respect to the complete Kähler metric $\pi^* g'$. Therefore, by Lemma 2.4, there exists a pluriharmonic function $\tau$ on $\widetilde{\Omega}$ such that $0 \leq \tau \leq 1$, $\tau$ has finite energy, and, for every regular sequence $\{x_\nu\}$ in $\widetilde{\Omega}$ approaching $\infty$,

$$\lim_{\nu \to \infty} \tau(x_\nu) = \begin{cases} 1 & \text{if } x_\nu \in E_1 \text{ for } \nu \gg 0 \\ 0 & \text{if } x_\nu \in \widetilde{\Omega} \setminus E_1 \text{ for } \nu \gg 0. \end{cases}$$

Since $\pi$ maps $E_1$ isomorphically onto $E$, the restriction $\tau\restriction_{E_1}$ determines a pluriharmonic function on $E$. If the differential of this function and the differential of the pluriharmonic function $\rho$ are linearly independent on $E$, then, by Proposition 2.3, $\Omega$ satisfies (BHD). Therefore, since $\rho$ is nonconstant, it now suffices to assume that there exist real constants $r$ and $s$ such that $\rho \circ \pi = r\tau + s$ on $E_1$ and to obtain a contradiction.

The first observation is that if $\tilde{\gamma} = \pi^*\gamma$, then $\tilde{\gamma}$ is a closed form of type $(0,1)$ on $\widetilde{\Omega}$ which is equal to the form $-\bar{\partial}(r\tau + s)$ on the nonempty open set $E_1$ and hence on the entire set $\widetilde{\Omega}$. Therefore, on the nonempty open set $\widetilde{E} \setminus \overline{E}_1$, we have

$$-\bar{\partial}(\rho \circ \pi) = \tilde{\gamma} = -\bar{\partial}(r\tau + s).$$

Hence the restriction of the function $(\rho \circ \pi) - (r\tau + s)$ to $\widetilde{E} \setminus \overline{E}_1$ is real-valued and holomorphic and is therefore locally constant. Thus if $E_2$ is a connected component of $\widetilde{E}$ which is not equal to $E_1$, then, for some real constant $s'$, we have $\rho \circ \pi = r\tau + s'$ on $E_2$. Now since $\pi(E_1) = \pi(E_2) = E$, we may choose a regular sequence $\{x_\nu\}$ in $E$ and sequences $\{y_\nu\}$ and $\{z_\nu\}$ in $E_1$ and $E_2$, respectively, such that $\pi(y_\nu) = \pi(z_\nu) = x_\nu$ for each $\nu$. The sequences $\{y_\nu\}$ and $\{z_\nu\}$ are then regular sequences in $\widetilde{\Omega}$, because the lifting of the function $v = -G(x_0, \cdot)$ to $\widetilde{\Omega}$ is a negative subharmonic function and $v(y_\nu), v(z_\nu) \to 0$. Therefore $\tau(y_\nu) \to 1$ and $\tau(z_\nu) \to 0$. Since $\rho$ vanishes at infinity along any regular sequence, we get

$$0 = \lim \rho(x_\nu) = \lim(r\tau(y_\nu) + s) = r + s$$

and

$$0 = \lim \rho(x_\nu) = \lim(r\tau(z_\nu) + s') = s'.$$

Therefore, for each point $x \in E$ and each pair of points $y \in E_1 \cap \pi^{-1}(x)$ and $z \in E_2 \cap \pi^{-1}(x)$, we have $\tau(y) - 1 = r^{-1}\rho(x) = \tau(z)$. Since $0 < \tau < 1$ on $\widetilde{\Omega}$, this is not possible. Thus the proof is complete for the cases (i) and (ii).

Finally, suppose $\varphi$ is of class $C^\infty$ (i.e. the condition (iii) holds). As in the case (i), we fix $M_0$, $a$, and $\Omega$. Here, we choose $a$ to be a regular value of $\varphi$ and we let $g'$ be the restriction of $g$ to $\Omega$. It suffices to show that there exists an arbitrarily large choice of $a$ for which $\Omega$ admits a proper holomorphic mapping $\Psi$ onto a Riemann surface. For if $\Psi$ is such a mapping, then $\varphi$ is constant on each level of $\Psi$. Hence, near a generic point of $\Omega$, there exist holomorphic coordinates $(z_1, \dots, z_n)$ in which $\varphi$ is a function of $z_1$ and we get

$$\mathcal{L}(\varphi) = \frac{\partial^2 \varphi}{\partial z_1 \partial \bar{z}_1} dz_1 d\bar{z}_1 = (2g^{1\bar{1}})^{-1} \Delta\varphi dz_1 d\bar{z}_1 \geq 0.$$

Thus it will follow that $\varphi$ is plurisubharmonic on each of the domains $\Omega$. Letting $a \to \infty$, it will then follow immediately that $\varphi$ is plurisubharmonic on $M$ as in the case (i) and the proof in the case (iii) will be complete.

As in the proof of Theorem 1.6 (Grauert-Riemenschneider and Siu), since the boundary of $\Omega$ is regular, we have $\gamma = \alpha - \bar{\partial}\beta$ on $\Omega$ where $\gamma$ is harmonic and $\beta$ vanishes on $\partial\Omega$. The metric $g'$ is not complete, but the proof of Theorem 1.6 shows that $\gamma$ is closed. In particular, $\beta$ is pluriharmonic on the connected set $E = \Omega \setminus \overline{M_0}$. Moreover, since $[\alpha] \neq 0$, $\beta$ is nonconstant on $E$.

If $\partial\Omega$ is not connected, then we may form a $C^\infty$ function $\tau$ on $\overline{\Omega}$ which is harmonic on $\Omega$ and locally constant, but not constant, on $\partial\Omega$. By Theorem 1.6 (a), $\tau$ is then pluriharmonic on $\Omega$ and, since $\beta$ vanishes on $\partial\Omega$, $d\tau$ and $d\beta$ are linearly independent on $E$. Therefore, by Proposition 2.3, $\Omega$ admits a proper holomorphic mapping onto a Riemann surface. Thus we may assume that $\partial\Omega$ is connected.

If $a'$ is a regular value in the interval $(\sup_{M_0} \varphi, a)$ which is sufficiently close to $a$ and $\Omega'$ is the connected component of $\{x \in M \mid \varphi(x) < a'\}$ containing $\overline{M_0}$, then $\Omega' \subset\subset \Omega$ and the set $E' = \Omega' \setminus \overline{M_0}$ is connected. As above, we may write $\gamma' = \alpha - \bar{\partial}\beta'$ on $\Omega'$ where $\gamma'$ harmonic and $\beta'$ vanishes on $\partial\Omega'$. If $d\beta$ and $d\beta'$ are linearly independent on $E'$, then Proposition 2.3 implies that $\Omega'$ satisfies (BHD). If $d\beta$ and $d\beta'$ are linearly dependent, then $\beta$ is constant on $\partial\Omega'$. Since $\beta$ is nowhere locally constant in $\Omega \setminus \overline{\Omega'}$ and $\beta \equiv 0$ on $\partial\Omega$, the maximum principle implies then that $\beta$ is equal to a nonzero constant on $\partial\Omega'$. Hence the restriction of the real part or the imaginary part of $\beta$ to $\Omega\backslash\overline{\Omega'}$ is a positive or negative pluriharmonic function which vanishes at points in $\partial\Omega$. It follows that $\Omega$ admits a plurisubharmonic exhaustion function and hence, a proper holomorphic mapping onto a Riemann surface. Therefore, by the above remarks, $M$ satisfies (BHD). Q. E. D.

## BIBLIOGRAPHY


[AV] A. ANDREOTTI and E. VESENTINI, Carlemann estimates for the Laplace-Beltrami equation on complex manifolds, Publ. Math. Inst. Hautes Études Sci., 25 (1965), 81–130.

[ABR] D. ARAPURA, P. BRESSLER, and M. RAMACHANDRAN, On the fundamental group of a compact Kähler manifold, Duke Math. J., 64 (1992), 477–488.

[B] S. BOCHNER, Analytic and meromorphic continuation by means of Green's formula, Ann. of Math., 44 (1943), 652–673.

[C] M. COLŢOIU, Complete locally pluripolar sets, J. reine angew. Math., 412 (1990), 108–112.

[Co] P. COUSIN, Sur les fonctions triplement périodiques de deux variables, Acta Math., 33 (1910), 105–232.

[D1] J.-P. DEMAILLY, Estimations $L^2$ pour l'operateur $\bar{\partial}$ d'un fibreé vectoriel holomorphe semi- positif au-dessus d'une variété Kählerienne complète, Ann. Scient. Éc. Norm. Sup., 15 (1982), 457–511.

[D2] J.-P. DEMAILLY, Cohomology of $q$-convex spaces in top degrees, Math. Z., 204 (1990), 283–295.

[G] M. GAFFNEY, A special Stokes theorem for Riemannian manifolds, Ann. of Math., 60 (1954), 140–145.

[GR] H. GRAUERT and O. RIEMENSCHNEIDER, Kählersche Mannigfältigkeiten mit hyper-$q$-konvexen Rand, Problems in analysis (A Symposium in Honor of S. Bochner, Princeton 1969), Princeton University Press, Princeton (1970), 61–79.

[GW] R. GREENE and H. WU, Embedding of open Riemannian manifolds by harmonic functions, Ann. Inst. Fourier, 25-1 (1975), 215–235.

[Gr] M. GROMOV, Kähler hyperbolicity and $L_2$-Hodge theory, J. Diff. Geom., 33 (1991), 263–292.

[Gu] R. GUNNING, Introduction to holomorphic functions of several variables, Vol. II, Wadsworth, Belmont, 1990.

[H] F. HARTOGS, Zur Theorie der analytischen Functionen mehrener unabhangiger Veränderlichen insbesondere über die Darstellung derselben durch Reihen, welche nach Potenzen einer Veränderlichen fortschreiten, Math. Ann., 62 (1906) 1–88.

[HL] F.R. HARVEY and H.B. LAWSON, Boundaries of complex analytic varieties I, Math. Ann., 102 (1975), 223–290.

[HM] L.R. HUNT and J.J. MURRAY, Plurisubharmonic functions and a generalized Dirichlet problem, Michigan Math. J., 25 (1978), 299–316.

[K] M. KALKA, On a conjecture of Hunt and Murray concerning $q$-plurisubharmonic functions, Proc. Amer. Math. Soc., 73 (1979), 30–34.

[Ka] B. KAUP, Über offene analytische Äquivalenzrelationen auf komplexen Räumen, Math. Ann., 183 (1969), 6–16.

[N] S. NAKANO, Vanishing theorems for weakly 1-complete manifolds II, Publ. R.I.M.S., Kyoto, 10 (1974), 101–110.

[NR] T. NAPIER and M. RAMACHANDRAN, Structure theorems for complete Kähler manifolds and applications to Lefschetz type theorems, Geom. and Func. Analysis, 5 (1995), 809–851.

[Na] R. NARASIMHAN, The Levi problem for complex spaces II, Math. Ann., 146 (1962), 195–216.

[Ni] T. NISHINO, L'existence d'une fonction analytique sur une variété analytique complexe à dimension quelconque, Publ. Res. Inst. Math. Sci., 19 (1983), 263–273.

[O1] T. OHSAWA, Weakly 1-complete manifold and Levi problem, Publ. R.I.M.S., Kyoto, 17 (1981), 153–164.

[O2] T. OHSAWA, Completeness of noncompact analytic spaces, Publ. R.I.M.S., Kyoto, 20 (1984), 683–692.

[P] M. PETERNELL, Algebraische Varietäten und $q$-vollständige komplexe Räume, Math. Z., 200 (1989), 547–581.

[R] M. RAMACHANDRAN, A Bochner-Hartogs type theorem for coverings of compact Kähler manifolds, Comm. Anal. Geom., 4 (1996), 333–337.

[Ri] R. RICHBERG, Stetige streng pseudokonvexe Funktionen, Math. Ann., 175 (1968), 257–286.

[S] Y.-T. SIU, Complex-analyticity of harmonic maps, vanishing and Lefschetz theorems, J. Diff. Geom., 17 (1982), 55–138.

[St] K. STEIN, Maximale holomorphe und meromorphe Abbildungen, I, Amer. J. Math., 85 (1963), 298–315.

[W] H. WU, On certain Kähler manifolds which are $q$-complete, Complex Analysis of Several Variables, Proceedings of Symposia in Pure Mathematics, 41, Amer. Math. Soc., Providence (1984), 253–276.

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