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EXTREMAL PROJECTORS IN THE SEMI-CLASSICAL CASE

by Sophie CHEMLA

1. Introduction.

Let $\mathfrak{g}$ be a complex semi-simple finite dimensional Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and Δ the root system associated to $\mathfrak{h}$. We will write Δ^+ (respectively Δ^-) for the set of positive (respectively negative) roots of Δ and put $\rho = \frac{1}{2} \sum_{\gamma \in \Delta^+} \gamma$. We will denote by $B = (\alpha_1, \dots, \alpha_l)$ the set of simple roots. Let $\mathfrak{g}_\gamma$ be the root space associated to the root γ. We put

$$\mathfrak{n} = \bigoplus_{\gamma \in \Delta^+} \mathfrak{g}_\gamma, \quad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \ , \mathfrak{n}_- = \bigoplus_{\gamma \in \Delta^+} \mathfrak{g}_{-\gamma}.$$

Let $R(\mathfrak{h})$ be the field of rational functions on $\mathfrak{h}^*$. One introduces the algebra $U'(\mathfrak{g}) = U(\mathfrak{g}) \otimes_{S(\mathfrak{h})} R(\mathfrak{h})$. Let us consider the generic Verma module $V = \dfrac{U'(\mathfrak{g})}{U'(\mathfrak{g})\mathfrak{n}}$. Zhelobenko ([Z1]) showed that $V^{\mathfrak{n}} = R(\mathfrak{h})1_+$ (where $1_+ = 1 + U'(\mathfrak{g})\mathfrak{n}$). The decomposition $V = \mathfrak{n}^- V \oplus R(\mathfrak{h})1_+$ defines a projector p onto $R(\mathfrak{h})1_+$ called the extremal projector. Inspired by a work of Asherova, Smirnov and Tolstoy ([AST]), Zhelobenko ([Z1]) showed that p factorizes into elementary projectors. Let $(\gamma_1, \dots, \gamma_m)$ be a normal ordering on the positive roots. Introduce the following notations:

$$p_\alpha = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! f_{\alpha,k}} e_{-\alpha}^k e_\alpha^k$$

$$f_{\alpha,0} = 1,$$

$$\text{if } k > 0, \ f_{\alpha,k} = (h_\alpha + \rho(h_\alpha) + 1) \dots (h_\alpha + \rho(h_\alpha) + k)$$

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(e_δ being the root vector associated to the root δ and h_δ the coroot). We have $p = p_{\gamma_1} \dots p_{\gamma_m}$ ([Z1]). Let $w = s_1 \dots s_j$ be a reduced decomposition of $w \in W$ (with $s_k = s_{\beta_k}$, β_k a simple root). Put $w_i = s_1 \dots s_i$. The roots $\gamma_i = w_{i-1}(\beta_i)$ $(w_0 = 1)$ are pairwise distinct and

$$\Delta_w = \{\alpha \in \Delta_+ \mid w^{-1}(\alpha) < 0\} = \{\gamma_1, \dots, \gamma_j\}.$$

Put $\mathfrak{n}_w = \bigoplus_{\alpha \in \Delta_w} \mathfrak{g}_\alpha$. In [Z2], Zhelobenko gives an explicit description of $V^{\mathfrak{n}_w}$. We will establish similar results for the symmetric algebra (the so-called semi-classical case).

Let us consider the analytic manifold $(\mathfrak{g}/\mathfrak{n})^*$. We will endow it with the following coordinate system $\left((e_{-\alpha})_{\alpha \in \Delta_+}, (h_{\alpha_i})_{i \in [1,l]}\right)$. We will call U_δ the open subset of $(\mathfrak{g}/\mathfrak{n})^*$ defined by the equation $h_\delta \neq 0$. We define Φ_δ to be the following rational map of U_δ:

$$\forall \lambda \in U_\delta,\ \Phi_\delta(\lambda) = \exp\left(\frac{e_{-\delta}(\lambda)}{h_\delta(\lambda)} e_\delta\right) \cdot \lambda$$

where the dot denotes natural action of $\mathfrak{n}$ on $(\mathfrak{g}/\mathfrak{n})^*$. By composition, Φ_δ defines an algebra morphism of $\mathcal{A}(U_\delta)$ which we call π_δ. We put

$$U_w = U_{\gamma_1} \cap \dots \cap U_{\gamma_j}.$$

We will denote by $\mathcal{P}(U_w)$ (respectively $\mathcal{A}(U_w)$) the set of regular functions (respectively analytic functions) on U_w and we will write $\mathcal{P}(U_w)^{\mathfrak{n}_w}$ (respectively $\mathcal{A}(U_w)^{\mathfrak{n}_w}$) the set of invariant functions of $\mathcal{P}(U_w)$ (respectively $\mathcal{A}(U_w)$) under the action of $\mathfrak{n}_w$. We prove the following result:

THEOREM. — *The algebra morphism $\pi_w = \pi_{\gamma_1} \circ \dots \circ \pi_{\gamma_j}$ does not depend on the reduced expression of w. It establishes an isomorphism between*

$$\mathcal{C}_w = \left\{ f \in \mathcal{A}(U_w) \ \Big| \ \frac{\partial f}{\partial e_{-\gamma_1}} = \dots = \frac{\partial f}{\partial e_{-\gamma_j}} = 0 \right\}$$

and $\mathcal{A}(U_w)^{\mathfrak{n}_w}$. Moreover π_w sends $\mathcal{C}_w \cap \mathcal{P}(U_w)$ onto $\mathcal{P}(U_w)^{\mathfrak{n}_w}$.

Let N_w be the connected simply connected group whose Lie algebra is $\mathfrak{n}_w$. The main ingredient of the proof will be the choice of a point in each N_w-orbit lying in U_w in accordance with the following proposition:

PROPOSITION. — *Let λ be in U_w. The point $\Phi_{\gamma_j} \Phi_{\gamma_{j-1}} \dots \Phi_{\gamma_1}(\lambda)$ is the unique point of the orbit $N_w \cdot \lambda$ whose coordinates $e_{-\gamma_1}, \dots, e_{-\gamma_j}$ vanish.*

In the appendix, we shall give a factorization for the extremal projector of the Virasoro algebra in the semi-classical case. Note that the non commutative case is still open. It is very different from the semi-simple case because the Virasoro algebra does not admit any normal ordering.

Notations. — Along all this article $\mathfrak{g}$ will denote a complex semi-simple finite dimensional Lie algebra and $\mathfrak{h}, \Delta, \Delta_+, \Delta_-, \mathfrak{n}, \mathfrak{n}_-$, $B = (\alpha_1, \ldots, \alpha_l)$ will be as above. Denote by W the Weyl group associated to these choices and $\overline{w}$ its longest element. Let γ be an element of Δ^+ and let h_γ be the unique element of $[\mathfrak{g}_\gamma, \mathfrak{g}_{-\gamma}]$ such that $\gamma(h_\gamma) = 2$. If e_γ is in $\mathfrak{g}_\gamma$, then there exists a unique $e_{-\gamma}$ such that $(h_\gamma, e_\gamma, e_{-\gamma})$ is a sl(2)-triple. If α and β are two roots, we set $[e_\alpha, e_\beta] = C_{\alpha,\beta} e_{\alpha+\beta}$ with the convention that $C_{\alpha,\beta}$ is zero if $\alpha + \beta$ is not a root.

The ordering $(\gamma_1, \ldots, \gamma_m)$ on the positive roots is normal if any composite root is located between its components. Thus for all positive roots $\gamma_i, \gamma_j, \gamma_k$, the equality $\gamma_k = \gamma_i + \gamma_j$ implies $i \leqslant k \leqslant j$ or $j \leqslant k \leqslant i$. There is a one to one correspondence between normal orderings and reduced expression of $\overline{w}$ ([Z2]). Let us recall it. Denote by s_i the reflexion with respect to a simple root β_i. If $\overline{w} = s_1 \ldots s_m$, then $(\beta_1, s_1(\beta_2), \ldots, s_1 \ldots s_{i-1}(\beta_i), \ldots, s_1 \ldots s_{m-1}(\beta_m))$ are in normal ordering.

If V is a vector space, $S(V)$ will be the symmetric algebra of V. Lastly, if P is in $S(V)$, $S(V)_P$ will be the localization of $S(V)$ with respect to $\{P^n \mid n \in \mathbb{N}\}$.

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2. Extremal equations in $(\mathfrak{g}/\mathfrak{n})^*$.

We consider $(\mathfrak{g}/\mathfrak{n})^*$ as an analytic manifold. We endow it with the following coordinate system $\left((e_{-\alpha})_{\alpha\in\Delta_+}, (h_{\alpha_i})_{i\in[1,l]}\right)$. If δ is a positive root, we will denote by U_δ the open subset of $(\mathfrak{g}/\mathfrak{n})^*$ defined by the equation $h_\delta \neq 0$. If U is an open subset for the Zariski topology, we will write $\mathcal{A}(U)$ for the algebra of analytic functions on U and $\mathcal{P}(U)$ for the algebra of regular functions on U. We will define Φ_δ to be the following rational map of U_δ

$$\forall\lambda \in U_\delta,\quad \Phi_\delta(\lambda) = \exp\left(\frac{e_{-\delta}(\lambda)}{h_\delta(\lambda)} e_\delta\right)\cdot\lambda.$$

By composition, Φ_δ defines an algebra morphism of $\mathcal{A}(U_\delta)$ which we call π_δ. We will denote by X_δ the natural action of e_δ on $\mathcal{A}(U_\delta)$. Remark that X_δ is a derivation. If f is in $\mathcal{P}(U_\delta)$, we have

$$(*)\quad \pi_\delta(f) = \sum_{k=0}^{\infty}(-1)^k \frac{e_{-\delta}^k}{k! h_\delta^k} X_\delta^k \cdot f$$

where $e_{-\delta}$ denotes the multiplication by $e_{-\delta}$. The operator π_δ is the commutative analog of the Zhelobenko's elementary projector.

Let $w = s_1 \dots s_j$ be a reduced decomposition of $w \in W$ (with $s_k = s_{\beta_k}$, $\beta_k \in B$). Put $w_i = s_1 \dots s_i$. The roots $\gamma_i = w_{i-1}(\beta_i)$ ($w_0 = 1$) are pairwise distinct and

$$\Delta_w = \{\alpha \in \Delta_+ \mid w^{-1}(\alpha) < 0\} = \{\gamma_1, \dots, \gamma_j\}.$$

An ordering in Δ_w is called normal if it coincides with the initial segment of some normal ordering in Δ^+ (that is compatible with one of the reduced expression of $\overline{w}$). Note that $(\gamma_1, \dots, \gamma_j)$ is a normal ordering of Δ_w. Put

$$U_w = \bigcap_{\delta\in\Delta_w} U_\delta.$$

We have

$$\mathcal{P}(U_w) = \left(\frac{S(\mathfrak{g})}{S(\mathfrak{g})\mathfrak{n}}\right)_{h_{\gamma_1}\dots h_{\gamma_j}} = S\left(\frac{\mathfrak{g}}{\mathfrak{n}}\right)_{h_{\gamma_1}\dots h_{\gamma_j}}.$$

We will denote by N_w the connected and simply connected group whose Lie algebra is $\mathfrak{n}_w = \bigoplus_{\alpha\in\Delta_w} \mathfrak{g}_\alpha$. We will start by proving the following proposition.

PROPOSITION 2.1. — *Let λ be in U_w. The point $\Phi_{\gamma_j}\Phi_{\gamma_{j-1}}\dots\Phi_{\gamma_1}(\lambda)$ is the unique point of the orbit $N_w\cdot\lambda$ whose coordinates $e_{-\gamma_1}, \dots, e_{-\gamma_j}$ vanish. In particular $\Phi_{\gamma_j}\Phi_{\gamma_{j-1}}\dots\Phi_{\gamma_1}$ does not depend on the normal ordering on Δ_w.*

Proof of Proposition 2.1. — Complete $(\gamma_1, \dots, \gamma_j)$ into a normal ordering on the positive roots $(\gamma_1, \dots, \gamma_m)$. $\mathfrak{g}/\mathfrak{n}$ is endowed with the basis $(e_{-\gamma_1}, \dots, e_{-\gamma_m}, h_{\alpha_1}, \dots, h_{\alpha_l})$. Let $(e^*_{-\gamma_1}, \dots, e^*_{-\gamma_m}, h^*_{\alpha_1}, \dots, h^*_{\alpha_l})$ be the dual basis. We will often identify the point $a_{\gamma_1}e^*_{-\gamma_1} + \dots + a_{\gamma_m}e^*_{-\gamma_m} + b_1 h^*_{\alpha_1} +$

$\ldots + b_l h^*_{\alpha_l}$ with its coordinates $(a_{\gamma_1}, \ldots, a_{\gamma_m}, b_1, \ldots, b_l)$. Let us see that there is a unique point in $N_w \cdot \lambda$ whose coordinates $e_{-\gamma_1}, \ldots, e_{-\gamma_j}$ vanish. Assume that there are two such points $f = (0, \ldots, 0, a_{\gamma_{j+1}}, \ldots, a_{\gamma_m}, b_1, \ldots, b_l)$ and $f' = (0, \ldots, 0, a'_{\gamma_{j+1}}, \ldots, a'_{\gamma_m}, b'_1, \ldots, b'_l)$. Then there exist complex numbers $(t_1, \ldots, t_j)$ such that $\exp(t_1 e_{\gamma_1} + \ldots + t_j e_{\gamma_j}) \cdot f = f'$. One can show easily the following equalities:

$$\begin{aligned} e_{\gamma_l} \cdot e^*_{-\gamma_k} &= -C_{\gamma_l, -\gamma_k} e^*_{-\gamma_l - \gamma_k} \\ e_{\gamma_l} \cdot h^*_{-\alpha_i} &= -h^*_{-\alpha_i}(h_{\gamma_l}) e^*_{-\gamma_l}. \end{aligned}$$

From these equalities, one deduces easily that the term in $e^*_{-\gamma_1}$ of $\exp(t_1 e_{\gamma_1} + \ldots + t_j e_{\gamma_j}) \cdot (0, \ldots, 0, a_{\gamma_{j+1}}, \ldots, a_{\gamma_m}, b_1, \ldots, b_l)$ is $-t_1 f(h_{\gamma_1})$. As f is in U_w, we get $t_1 = 0$. We reproduce the same reasoning to show that $t_2, t_3, \ldots, t_j$ are zero. So that we have proved that the two points f and f' coincide. It is not difficult to deduce from the normal ordering property that Φ_{γ_i} sends the point $(x_{\gamma_1}, \ldots, x_{\gamma_m}, y_1, \ldots, y_l)$ to a point $(x'_{\gamma_1}, \ldots, x'_{\gamma_{i-1}}, 0, x'_{\gamma_{i+1}}, \ldots, x'_{\gamma_m}, y_1, \ldots, y_l)$ and that it sends the point $(0, \ldots, 0, x_{\gamma_i}, \ldots, x_{\gamma_m}, y_1, \ldots, y_l)$ to a point $(0, \ldots, 0, x'_{\gamma_{i+1}}, \ldots, x'_{\gamma_m}, y_1, \ldots, y_l)$. So that $\Phi_{\gamma_j} \Phi_{\gamma_{j-1}} \ldots \Phi_{\gamma_1}(\lambda)$ is the unique point of $N_w \cdot \lambda$ whose coordinates $e_{-\gamma_1}, \ldots, e_{-\gamma_j}$ vanish. This finishes the proof of Proposition 2.1.

As a consequence of the previous proposition, we may write Φ_w for the operator $\Phi_{\gamma_j} \Phi_{\gamma_{j-1}} \ldots \Phi_{\gamma_1}$. The algebra homomorphism defined by Φ_w on $\mathcal{A}(U_w)$ will be denoted by π_w. Using Proposition 2.1, we will give a geometric proof of the following result.

THEOREM 2.2. — 1) *If $\bar{\mathfrak{n}}_w$ denotes the linear hull of $(e_{-\alpha})_{\alpha \in \Delta_w}$, one has* $\mathrm{Ker}\pi_w = \bar{\mathfrak{n}}_w \mathcal{A}(U_w)$.

2) *The operator π_w is the projector onto $\mathcal{A}(U_w)^{\mathfrak{n}_w}$ with kernel $\bar{\mathfrak{n}}_w \mathcal{A}(U_w)$ and its restriction to $\mathcal{P}(U_w)$ is the projector onto $\mathcal{P}(U_w)^{\mathfrak{n}_w}$ with kernel $\bar{\mathfrak{n}}_w \mathcal{P}(U_w)$.*

3) *The operator π_w establishes an isomorphism Π_w between*

$$\mathcal{C}_w = \left\{ f \in \mathcal{A}(U_w) \;\mid\; \frac{\partial f}{\partial e_{-\gamma_1}} = \ldots \frac{\partial f}{\partial e_{-\gamma_j}} = 0 \right\}$$

and $\mathcal{A}(U_w)^{\mathfrak{n}_w}$. Moreover Π_w sends $\mathcal{C}_w \cap \mathcal{P}(U_w)$ onto $\mathcal{P}(U_w)^{\mathfrak{n}_w}$. If f is in $\mathcal{A}(U_w)^{\mathfrak{n}_w}$, $\Pi_w^{-1}(f)$ is the restriction of f to the subvariety of equations $e_{-\gamma_1} = \ldots = e_{-\gamma_j} = 0$.

Proof of Theorem 2.2. — From the previous proposition, the inclusion $\overline{\mathfrak{n}}_w\mathcal{A}(U_w) \subset \mathrm{Ker}\pi_w$ is clear. Moreover, a standard reasoning shows that

$$\mathcal{A}(U_w) = \mathcal{C}_w \oplus \overline{\mathfrak{n}}_w\mathcal{A}(U_w).$$

Then one sees easily that $\mathrm{Ker}\pi_w \cap \mathcal{C}_w = \{0\}$. So that we have $\overline{\mathfrak{n}}_w\mathcal{A}(U_w) = \mathrm{Ker}\pi_w$.

Let us now show that $\mathrm{Im}\pi_w = \mathcal{A}(U_w)^{\mathfrak{n}_w}$ and that π_w is a projector. Let α be in Δ_w. For any f in $\mathcal{A}(U_w)$ and any λ in U_w, we have

$$(X_\alpha \circ \pi_w)(f)(\lambda) = \frac{d}{dt} f\left(\Phi_{\gamma_j} \ldots \Phi_{\gamma_1} \exp(-te_\alpha)\lambda\right)_{|t=0}.$$

But for any t, $\Phi_{\gamma_j} \ldots \Phi_{\gamma_1}\exp(-te_\alpha)\lambda$ is the unique point of $N_w \cdot \lambda$ whose coordinates $e_{-\gamma_1}, \ldots, e_{-\gamma_j}$ vanish. So that $X_\alpha \circ \pi_w = 0$. We have thus proved the inclusion $\mathrm{Im}\pi_w \subset \mathcal{A}(U_w)^{\mathfrak{n}_w}$. Now it is clear that π_w is a projector : check that $\pi_w \circ \pi_w = \pi_w$ on coordinates using the formula (*). The reverse inclusion $\mathcal{A}(U_w)^{\mathfrak{n}_w} \subset \mathrm{Im}\pi_w$ will be a consequence of the following lemma.

LEMMA 2.3. — *Let k be in $[1, j]$ and let f be in $\mathcal{A}(U_w)$. If $X_{\gamma_k} f = 0$, then $\pi_{\gamma_k} f = f$.*

Proof of Lemma 2.3. — We first remark that $(\pi_{\gamma_k}(e_{-\gamma_1}), \ldots, \pi_{\gamma_k}(e_{-\gamma_{k-1}}), e_{-\gamma_k}, \pi_{\gamma_k}(e_{-\gamma_{k+1}}), \ldots, \pi_{\gamma_k}(e_{-\gamma_m}), h_{\alpha_1}, \ldots, h_{\alpha_l})$ is a coordinate system in U_w. Indeed, one may see by induction that for any $i \leqslant k-1$ (respectively $i \geqslant k+1$), $e_{-\gamma_i}$ may be expressed as a regular function of $(\pi_{\gamma_k}(e_{-\gamma_1}), \ldots, \pi_{\gamma_k}(e_{-\gamma_i}), h_{\alpha_1}, \ldots, h_{\alpha_l})$ (respectively $(\pi_{\gamma_k}(e_{-\gamma_i}), \ldots, \pi_{\gamma_k}(e_{-\gamma_m}), h_{\alpha_1}, \ldots, h_{\alpha_l})$). We put $(\epsilon_1, \ldots, \epsilon_{m+l}) = (\pi_{\gamma_k}(e_{-\gamma_1}), \ldots, \pi_{\gamma_k}(e_{-\gamma_{k-1}}), e_{-\gamma_k}, \pi_{\gamma_k}(e_{-\gamma_{k+1}}), \ldots, \pi_{\gamma_k}(e_{-\gamma_m}), h_{\alpha_1}, \ldots, h_{\alpha_l})$. In these coordinates, we have $X_{\gamma_k} = h_{\gamma_k}\dfrac{\partial}{\partial \epsilon_k}$. So that if $X_{\gamma_k} f = 0$, then f does not depend on ϵ_k and it becomes clear that there exists g such that $f = \pi_{\gamma_k} g$. As π_{γ_k} is a projector, we have $\pi_{\gamma_k} f = \pi_{\gamma_k}\pi_{\gamma_k} g = \pi_{\gamma_k} g$, which finishes the proof of the lemma.

It is clear from the proof that π_w sends $\mathcal{C}_w \cap \mathcal{P}(U_w)$ onto $\mathcal{P}(U_w)^{\mathfrak{n}_w}$.

In particular $\pi_{\overline{w}|\mathcal{P}(U_{\overline{w}})}$ is the projector onto $S(\mathfrak{h})_{h_{\gamma_1} \ldots h_{\gamma_m}}$ with kernel $\mathfrak{n}_-\mathcal{P}(U_{\overline{w}})$. By analogy to Asherova, Tolstoy, Smirnov and Zhelobenko's work, we will call it the extremal projector.

Proposition 2.1 gives a geometric interpretation of the projector π_w.

3. Appendix: Extremal projector for the Virasoro algebra in the semi-classical case.

In this section, we shall give a factorization of the Virasoro algebra extremal projector in the semi-classical case. Note that the non commutative case is still open. It is very different from the semi-simple case because the Virasoro algebra does not admit any normal ordering. Recall that the Virasoro algebra Vir is the infinite dimensional Lie algebra generated by $\{e_i \mid i \in \mathbb{Z}\} \cup \{c\}$ with commutation rules

$$[e_i, e_j] = (j-i)\, e_{i+j} + \frac{(j^3-j)}{12} \delta_{i+j,0} c, \quad [e_i, c] = 0.$$

Vir admits the following triangular decomposition:

$$\mathrm{Vir} = \mathrm{Vir}_+ \oplus \mathrm{Vir}_0 \oplus \mathrm{Vir}_-$$

where

$$\mathrm{Vir}_+ = \underset{i\geqslant 1}{\oplus} \mathbb{C}e_i, \ \mathrm{Vir}_0 = \mathbb{C}e_0 \oplus \mathbb{C}c, \ \mathrm{Vir}_- = \underset{i\leqslant -1}{\oplus} \mathbb{C}e_i.$$

We will also use the notation

$$\mathrm{Vir}_{r,+} = \underset{i\geqslant r}{\oplus} \mathbb{C}e_i \ \text{ and } \ \mathrm{Vir}_{r,-} = \underset{i\leqslant -r}{\oplus} \mathbb{C}e_i.$$

$\mathrm{Vir}_{r,+}$ and $\mathrm{Vir}_{r,-}$ are Lie subalgebras of Vir.

Let $R(\mathrm{Vir}_0)$ be the field of fractions of $S(\mathrm{Vir}_0)$. We introduce the algebra

$$S'(\mathrm{Vir}) = S(\mathrm{Vir}) \underset{S(\mathrm{Vir}_0)}{\otimes} R(\mathrm{Vir}_0) = S'\left(\mathrm{Vir}/\mathrm{Vir}_-\right).$$

There is a natural action of Vir_- on $S'\left(\mathrm{Vir}/\mathrm{Vir}_-\right)$. Through this action, for any negative i, e_i defines a derivation X_i of $S'\left(\dfrac{\mathrm{Vir}}{\mathrm{Vir}_-}\right)$. Set

$$T_r = \left(\frac{S'(\mathrm{Vir})}{S'(\mathrm{Vir})\mathrm{Vir}_-}\right)^{\mathrm{Vir}_{r,-}}.$$

The result and the proof of the following lemma is left to the reader.

LEMMA 3.1.

$$T_r = \underset{k_1,\dots,k_{r-1}\in\mathbb{N}}{\oplus} R(\mathrm{Vir}_0) e_1^{k_1} \dots e_{r-1}^{k_{r-1}}.$$

As a consequence of Lemma 3.1, we have the following decomposition:

$$S'\left(\mathrm{Vir}/\mathrm{Vir}_-\right) = T_r \oplus \mathrm{Vir}_{r,+} S'\left(\mathrm{Vir}/\mathrm{Vir}_-\right).$$

The proof of the next lemma is an easy computation.

LEMMA 3.2. — *For any $i > 1$, the operator*

$$\pi_i = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \left(2ie_0 + \dfrac{(i^3 - i)c}{12}\right)^k} e_i^k X_{-i}^k$$

is an algebra morphism and satisfies the relations

$$X_{-i} \circ \pi_i = 0 \text{ and } \pi_i \circ e_i = 0$$

(where e_i denotes multiplication by e_i).

It is not hard to see that the operator $\Pi_r = \prod_{i=r}^{\infty} \pi_i$ is well defined. Actually $\prod_{i=r}^{\infty} \pi_i(e_1^{a_1} \dots e_k^{a_k}) = \prod_{r \leqslant i \leqslant k} \pi_i(e_1^{a_1} \dots e_k^{a_k}) = 0$ (by Lemma 3.2).

THEOREM 3.3. — *The operator Π_r satisfies the relations*

$$\forall i \geqslant r,\ X_{-i} \circ \Pi_k = 0,\ \Pi_k \circ e_i = 0.$$

It is the projector onto T_r with kernel $\mathrm{Vir}_{r,+} S'\left(\dfrac{\mathrm{Vir}}{\mathrm{Vir}_-}\right)$.

In particular, Π_1 is the extremal projector.

Proof of Theorem 3.3. — The relations of the theorem are easy to check and they prove that Π_r is a projector. To prove that the kernel of Π_r is $\mathrm{Vir}_{r,+}$, we proceed as in the semi-simple case. The inclusion $\mathrm{Im}\Pi_r \subset T_r$ is a consequence of the theorem. To prove the reverse inclusion, remark that if x is in T_r, then $\Pi_r x = x$, so that x is in $\mathrm{Im}\Pi_r$.

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