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COMPONENT GROUPS OF ABELIAN VARIETIES AND GROTHENDIECK'S DUALITY CONJECTURE

by Siegfried BOSCH

Let A_K be an abelian variety over a field K which is the field of fractions of a complete discrete valuation ring R. We write A'_K for its dual, A, A' for the associated Néron models, and ϕ_A, $\phi_{A'}$ for the corresponding component groups. Grothendieck [12] has constructed a pairing $\phi_A \times \phi_{A'} \longrightarrow \mathbb{Q}/\mathbb{Z}$ which represents the obstruction of extending the Poincaré bundle on $A_K \times A'_K$ as a biextension of $A \times A'$ by $\mathbb{G}_{\mathrm{m}}$. Thus, the pairing reflects certain arithmetic properties of the duality between A_K and A'_K; it is conjecturally perfect, a fact which has been established in the case of semi-stable reduction and in the case where the residue field k of K is perfect, except for infinite k if char$K > 0$.

Recently, the structure of the component groups ϕ_A, $\phi_{A'}$ has been investigated more intensively: Lorenzini [15] has introduced two filtrations on prime-to-p parts, p the residue characteristic, which then, in [8], were generalized to all of ϕ_A, $\phi_{A'}$, describing them more intrinsically by means of uniformization theory in the sense of rigid geometry. In the present article we show that, over a perfect residue field, the two filtrations are dual to each other under Grothendieck's pairing, provided this pairing is perfect.

Using the same methods, we can also achieve a small progress on the perfectness of Grothendieck's pairing itself. Again, in the case of a perfect residue field k, we show that it is perfect at least for abelian varieties with potentially multiplicative reduction. More generally, one would like to know that the pairing $\phi_A \times \phi_{A'} \longrightarrow \mathbb{Q}/\mathbb{Z}$ is perfect as soon as the pairing $\phi_B \times \phi_{B'} \longrightarrow \mathbb{Q}/\mathbb{Z}$ on the corresponding abelian parts with potentially good reduction is perfect. However, as it seems, our methods do not yield

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this result in full generality. We can verify it only in the case where the toric parts T_K, T'_K of A_K, A'_K have multiplicative reduction (without any restriction on the residue field k), or where the component groups ϕ_T, $\phi_{T'}$ are torsion-free.

Let us give some indications on the techniques we are using. The actual discussion of Grothendieck's pairing is started in Section 4. Working in terms of sheaves for the étale or smooth topology, we look at the isomorphism $A'_K \xrightarrow{\sim} \underline{\mathrm{Ext}}^1(A_K, \mathbb{G}_{\mathrm{m},K})$ given by the duality between A_K and A'_K. There is an associated isomorphism $A' \xrightarrow{\sim} \underline{\mathrm{Ext}}^1(A, j_*\mathbb{G}_{\mathrm{m},K})$ on the level of Néron models ($j\colon \mathrm{Spec}K \longrightarrow \mathrm{Spec}R$ is the canonical morphism), as well as an induced monomorphism $A'^0 \longrightarrow \underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},R})$, where A'^0 is the identity component of A'. The latter (as well as the corresponding morphism with A and A' interchanged) is an isomorphism if and only if Grothendieck's pairing is perfect for A_K and A'_K; see 5.1. Thus, in order to access the perfectness of the pairing, we must show that $\underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},R})$ is reduced to its identity component.

To do this, we use rigid uniformization and write A_K as a quotient E_K/M_K (in the sense of rigid K-groups), where E_K is an extension of an abelian variety B_K with potentially good reduction by a torus T_K, and where $M_K \subset E_K$ is a lattice; see [19] and [8], 1.2. Thus, the uniformization of A_K consists of short exact sequences

$$0 \longrightarrow M_K \longrightarrow E_K \longrightarrow A_K \longrightarrow 0, \qquad 0 \longrightarrow T_K \longrightarrow E_K \longrightarrow B_K \longrightarrow 0,$$

where all objects and morphisms exist in the algebraic category, except for the morphism $E_K \longrightarrow A_K$ which is rigid analytic and not algebraic (unless T_K is trivial). Switching to associated Néron models, it is our strategy to relate the sheaf $\underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},R})$ to $\underline{\mathrm{Ext}}^1(E, \mathbb{G}_{\mathrm{m},R})$, and the latter to $\underline{\mathrm{Ext}}^1(B, \mathbb{G}_{\mathrm{m},R})$. Assuming that $\underline{\mathrm{Ext}}^1(B, \mathbb{G}_{\mathrm{m},R})$ coincides with its identity component, we show in 5.3 that the same is true for $\underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},R})$, at least in the cases we are interested in. A similar approach is used in order to establish the duality result 6.1 of the filtrations of component groups, as mentioned above.

We do not want to hide the fact that there are several technical problems, which have to be solved, before one can proceed as indicated. In order to associate a morphism of Néron models $E \longrightarrow A$ to the rigid morphism $E_K \longrightarrow A_K$, we must change from ordinary Néron models of K-schemes to formal ones of rigid K-groups, as introduced in [7]. For the groups we are interested in, this is done by formal completion of ordinary

Néron models along their special fibres, a process which, in general, will destroy parts of the generic fibre. As we cannot afford such a defect in view of the isomorphism $A' \xrightarrow{\sim} \underline{\mathrm{Ext}}^1(A, j_*\mathbb{G}_{\mathrm{m},K})$, which we want to maintain, we always add the original generic fibre to a formal Néron model. Writing $\mathfrak{R} = (K, R)$, we arrive at the notion of $\mathfrak{R}$-models. These are pairs of type $X = (X_K, X_R)$, where X_K is a rigid K-space and X_R an admissible formal R-scheme, together with an open immersion $X_{R,K} \hookrightarrow X_K$ from the generic fibre $X_{R,K}$ of X_R into X_K.

The theory of $\mathfrak{R}$-models and of abelian sheaves on them is developed in Section 1. Furthermore, in Section 2, we study torsors and extensions in this setting, with the aim to transfer the isomorphism $A' \xrightarrow{\sim} \underline{\mathrm{Ext}}^1(A, j_*\mathbb{G}_{\mathrm{m},K})$ from the algebraic to the $\mathfrak{R}$-model category. Then, indeed, the morphism $E_K \longrightarrow A_K$ yields a morphism of associated Néron-$\mathfrak{R}$-models $E \longrightarrow A$, and thereby an induced morphism $\underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) \longrightarrow \underline{\mathrm{Ext}}^1(E, \mathbb{G}_{\mathrm{m},\mathfrak{R}})$ which is meaningful. Likewise, we get a morphism $\underline{\mathrm{Ext}}^1(B, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) \longrightarrow \underline{\mathrm{Ext}}^1(E, \mathbb{G}_{\mathrm{m},\mathfrak{R}})$ which, due to vanishing results on $\underline{\mathrm{Hom}}$ and $\underline{\mathrm{Ext}}^1$ sheaves of tori in Section 3, is formally an isomorphism with respect to the étale topology, at least in the cases we are interested in. We have then a formal morphism $\underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) \longrightarrow \underline{\mathrm{Ext}}^1(B, \mathbb{G}_{\mathrm{m},\mathfrak{R}})$, which we can compare with a similar one coming from uniformization data of the dual abelian variety A'_K. This will settle the proof of 5.3.

1. Formal models and abelian sheaves.

If X_K is a rigid K-space with Néron model X, as introduced in [7], 1.1, then the generic fibre of X is an open subspace of X_K which, in general, is different from X_K. Thus, in most cases, it is impossible to reconstruct X_K from X. To remedy this, we set $\mathfrak{R} = (K, R)$ and define a new category $\mathrm{Mod}_{\mathfrak{R}}$ of so-called $\mathfrak{R}$-models as follows: The objects of $\mathrm{Mod}_{\mathfrak{R}}$ consist of all pairs $X = (X_K, X_R)$, where X_K is a rigid K-space and X_R an admissible formal R-scheme, together with an open immersion of rigid K-spaces $X_{R,K} \hookrightarrow X_K$; the rigid K-space $X_{R,K}$ is meant to be the generic fibre of X_R. A morphism $X \longrightarrow Y$ between two such objects is a pair $\varphi = (\varphi_K, \varphi_R)$ of morphisms $\varphi_K\colon X_K \to Y_K$, $\varphi_R\colon X_R \to Y_R$ such that the diagram

$$\begin{array}{ccc} X_{R,K} & \xrightarrow{\varphi_{R,K}} & Y_{R,K} \\ \downarrow & & \downarrow \\ X_K & \xrightarrow{\varphi_K} & Y_K \end{array}$$

is commutative. We call $\varphi = (\varphi_K, \varphi_R)$ smooth (resp. étale) if both φ_K and φ_R are smooth (resp. étale). In particular, we can consider the small model smooth site over $\mathfrak{R}$ (more precisely, over $(\mathrm{Sp}K, \mathrm{Spf}R)$) or over any other $\mathfrak{R}$-model S, and we can look at abelian sheaves on such a site. The category $\mathrm{Mod}_{\mathfrak{R}}$ is quite similar to the category of R-models in the scheme setting. For example, fibred products exist in $\mathrm{Mod}_{\mathfrak{R}}$; they are constructed by taking the fibred product in each "component".

There is a canonical functor $i\colon \mathrm{For}_R \longrightarrow \mathrm{Mod}_{\mathfrak{R}}$ from the category of admissible formal R-schemes to the category of $\mathfrak{R}$-models, which associates to any admissible formal R-scheme X_R the $\mathfrak{R}$-model $(X_{R,K}, X_R)$. Writing $i^{-1}(X_K, X_R) = X_R$ for any $\mathfrak{R}$-model (X_K, X_R), we may interpret i as a morphism of sites, for example, from the small formal smooth site over R to the small model smooth site over $\mathfrak{R}$. Similarly, we can consider the functor $j\colon \mathrm{Rig}_K \longrightarrow \mathrm{Mod}_{\mathfrak{R}}$ from the category of rigid K-spaces to the category of $\mathfrak{R}$-models, which associates to any rigid K-space X_K the $\mathfrak{R}$-model $(X_K, \varnothing)$. Again, we can set $j^{-1}(X_K, X_R) = X_K$ for any $\mathfrak{R}$-model (X_K, X_R) and thereby interpret j as a morphism of sites, for example, from the small rigid smooth site over K to the small model smooth site over $\mathfrak{R}$. By definition, both functors i and j are fully faithful.

In the present paper, we consider exclusively, unless stated otherwise, the *small smooth sites* over $\mathfrak{R}$, R, or K, referring to them simply as the *model site* over $\mathfrak{R}$, the *formal site* over R, or the *rigid site* over K. Dealing with abelian sheaves, we use [1] as a general reference in order to define the functors i_*, i^*, j_*, j^*.

LEMMA 1.1. — (i) *Both, i_* and i^*, are exact, and we have $i^*i_*\mathcal{F} = \mathcal{F}$ for any sheaf $\mathcal{F}$ on the formal site over R.*

(ii) *j_* is left exact, j^* is exact, and we have $j^*j_*(\mathcal{F}) = \mathcal{F}$ for any sheaf $\mathcal{F}$ on the rigid site over K.*

Proof. — Let us start with assertion (i). As always, i_* is left exact. It is right exact, because for any $\mathfrak{R}$-model (U_K, U_R) in the model site over $\mathfrak{R}$ and any covering $(U_R^n)_n$ of U_R in the formal site over R, the pairs $(U_{R,K}^n, U_R^n)_n$, together with $(U_K, \varnothing)$ form a covering of (U_K, U_R). On the other hand, i^* is right exact by general reasons. That it is also left exact, follows from the equation $i^*\mathcal{G}(U_R) = \mathcal{G}(U_{R,K}, U_R)$, where U_R belongs to the formal site over R and $\mathcal{G}$ is any sheaf on the model site over $\mathfrak{R}$. Since

$$i^*i_*\mathcal{F}(U_R) = i_*\mathcal{F}(U_{R,K}, U_R) = \mathcal{F}(U_R),$$

for any sheaf $\mathcal{F}$ on the formal site over R, we see $i^*i_*\mathcal{F} = \mathcal{F}$.

Concerning assertion (ii), j_* is left exact and j^* is right exact by general reasons. Furthermore, j^* is right exact since we have $j^*\mathcal{G}(U_K) = \mathcal{G}(U_K, \varnothing)$ for any U_K in the rigid site over K and any abelian sheaf $\mathcal{G}$ on the model site over $\mathfrak{R}$. Finally, if $\mathcal{F}$ is a sheaf on the rigid site over K, we get

$$j^*j_*\mathcal{F}(U_K) = j_*\mathcal{F}(U_K, \varnothing) = \mathcal{F}(U_K)$$

and, thus, $j^*j_*\mathcal{F} = \mathcal{F}$. □

The functors i^* and j^* will be referred to as *restriction to the formal*, resp. *rigid parts*. Since we have, in a compatible way, descent of morphisms on the rigid part (see [8], 3.3) as well as on the formal part, it follows that any object $X = (X_K, X_R)$ in $\mathrm{Mod}_{\mathfrak{R}}$ with X_K being quasi-separated gives rise to a sheaf on the model site over $\mathfrak{R}$, namely to the sheaf $S \longmapsto \mathrm{Hom}_{\mathfrak{R}}(S, X)$. Thus, Néron models can be defined in the usual way using the functor j_*. However, in this setting, the formation of Néron models does not abandon the original generic fibre. If X_K is a rigid K-space with Néron model X_R in the sense of [7], 1.1, then (X_K, X_R) is the Néron model of X_K in the sense of models in $\mathrm{Mod}_{\mathfrak{R}}$.

As an example we may consider the Néron model $j_*\mathbb{G}_{\mathrm{m},K}$ of the multiplicative group $\mathbb{G}_{\mathrm{m},K}$. It inserts into an exact sequence

$$0 \longrightarrow \mathbb{G}_{\mathrm{m},\mathfrak{R}} \longrightarrow j_*\mathbb{G}_{\mathrm{m},K} \longrightarrow i_*\mathbb{Z} \longrightarrow 0,$$

where $\mathbb{G}_{\mathrm{m},\mathfrak{R}} = (\mathbb{G}_{\mathrm{m},K}, \mathbb{G}_{\mathrm{m},R})$ is the "multiplicative group" over $\mathfrak{R}$, consisting of the rigid multiplicative group $\mathbb{G}_{\mathrm{m},K}$ over K and the formal multiplicative group $\mathbb{G}_{\mathrm{m},R}$ over R.

LEMMA 1.2. — *Let $\mathcal{F}$ be an abelian sheaf on the model site over $\mathfrak{R}$. Then $\mathcal{F} = 0$ is equivalent to $i^*\mathcal{F} = 0$ and $j^*\mathcal{F} = 0$.*

Proof. — The only-if-part is trivial. So assume $i^*\mathcal{F} = 0$ as well as $j^*\mathcal{F} = 0$, and fix an object (U_K, U_R) in the model site over $\mathfrak{R}$. Then $\mathcal{F}(U_{R,K}, U_R) = 0$ and $\mathcal{F}(U_K, \varnothing) = 0$. Since (U_K, U_R) is covered by $(U_{R,K}, U_R)$ and $(U_K, \varnothing)$, we get $\mathcal{F}(U_K, U_R) = 0$. □

2. Torsors and extensions.

We begin with a lemma on torsors in the rigid category; torsors are always meant with respect to the fppf-topology.

Lemma 2.1. — *X_K be a rigid K-space and E_K a $\mathbb{G}_{m,K}$-torsor on it. Then E_K is trivial locally with respect to the Zariski topology on X_K* [1] *and, thus, representable by a rigid K-space.*

Proof. — By its definition, the torsor E_K is locally trivial with respect to the fppf-topology and, hence, becomes trivial after applying a local fppf-cover $Z_K \to X_K$ as base change. As we are dealing with $\mathbb{G}_m$-torsors, we may view $E_K \times_{X_K} Z_K \to Z_K$ as an invertible sheaf on Z_K, which is equipped with a descent datum with respect to $Z_K \to X_K$. Due to O. Gabber's version of faithfully flat descent for coherent modules on rigid spaces, see [18], 1.9, this module descends to a coherent X_K-module which, by the usual argument, is an invertible sheaf again. The latter can be interpreted as a line bundle on X_K and, as such, must be isomorphic to E_K. In particular, E_K is representable. □

Lemma 2.2. — *Let X be an $\mathfrak{R}$-model, and let i, j be as in Section 1.*

(i) *Any $\mathbb{G}_{m,\mathfrak{R}}$-torsor over X is locally trivial with respect to the Zariski topology.*

(ii) *Any $i_*\mathbb{Z}$-torsor over X is locally trivial with respect to the étale topology. Moreover, any such torsor is trivial if X is smooth over $\mathfrak{R}$.*

(iii) *Any $j_*\mathbb{G}_{m,K}$-torsor over X is locally trivial with respect to the Zariski topology, provided X is smooth over $\mathfrak{R}$.*

In particular, in cases (i) and (iii) the corresponding torsors are representable[2].

Proof. — We start with assertion (i). If E is a $\mathbb{G}_{m,\mathfrak{R}}$-torsor over X, we can restrict it to the rigid part as well as to the formal part, thereby obtaining a $\mathbb{G}_{m,K}$-torsor E_K over X_K and a $\mathbb{G}_{m,R}$-torsor E_R over X_R. Both are locally trivial with respect to the Zariski topology and, thus, representable; cf. 2.1 as far as E_K is concerned. Thinking in terms of coherent (locally free) modules, it follows from descent arguments that E_K is compatible with the restriction $E_{R,K}$. From this we see that E is locally trivial with respect to the Zariski topology on X, and we are done.

Next, considering case (ii), let us look at an $i_*\mathbb{Z}$-torsor E. Since the restriction of $i_*\mathbb{Z}$ to the rigid part is trivial, E_K must coincide with X_K. On

(1) To unify our language, we talk about the Zariski topology on X_K and thereby mean the classical rigid Grothendieck topology.

(2) As $i_*\mathbb{Z}$ is not representable by an $\mathfrak{R}$-model, we cannot expect the representability of $i_*\mathbb{Z}$-torsors in case (ii).

the other hand, using fppf-descent on the formal level, E_R, as a $\mathbb{Z}$-torsor, is locally trivial with respect to the étale topology, and it follows that E is locally trivial with respect to the étale topology on X. Now, assume that X is smooth over $\mathfrak{R}$. Then the small étale site on X_R is equivalent to the small étale site of its reduction (use for example [4], 1.4), and we see by [12], VIII, 5.1, that the restriction of E to the formal level is trivial. Hence E is trivial already over X.

To verify assertion (iii), consider a $j_*\mathbb{G}_{\mathrm{m},K}$-torsor E over X. Then the quotient $E/\mathbb{G}_{\mathrm{m},\mathfrak{R}}$ is an $i_*\mathbb{Z}$-torsor over X and, hence, trivial by (ii). Thus, $E/\mathbb{G}_{\mathrm{m},\mathfrak{R}}$ decomposes into different "components" isomorphic to X, and each part of E, lying over such a component may be viewed as a $\mathbb{G}_{\mathrm{m},\mathfrak{R}}$-torsor. The latter is locally trivial with respect to the Zariski topology by (i), and we see that E itself is locally trivial with respect to the Zariski topology. □

The preceding results say that, for torsors of the above type, the choice of the topology is not critical; the fppf-topology can be replaced by the Zariski or, at worst, the étale topology.

In the following we fix a *smooth* $\mathfrak{R}$-model $S = (S_K, S_R)$ in the sense of Section 1, which we will use as a base object. By an S-model X we mean a relative $\mathfrak{R}$-model over S; i. e., a morphism $X \longrightarrow S$ in the sense of $\mathfrak{R}$-models. Furthermore, an S-group is meant to be an S-model with a group structure relatively over S.

PROPOSITION 2.3. — *Let B be a smooth S-group. Then the functor j^* induces an equivalence between extensions of B by $j_*\mathbb{G}_{\mathrm{m},K}$ and extensions of B_K by $\mathbb{G}_{\mathrm{m},K}$.*

Proof. — We follow the argumentation in [12], VIII, 6.5 and 6.6. First, let us verify that the functor induced by j^* on $j_*\mathbb{G}_{\mathrm{m},K}$-torsors is fully faithful. So we have to show for $j_*\mathbb{G}_{\mathrm{m},K}$-torsors E and F on B that the restriction $\mathrm{Hom}(E,F) \longrightarrow \mathrm{Hom}(E_K, F_K)$ is bijective. Forgetting about the group structure of B and applying 2.2, it is enough to consider the case where E and F are trivial. Then we must know that the restriction map $\Gamma(B, j_*\mathbb{G}_{\mathrm{m},K}) \longrightarrow \Gamma(B_K, \mathbb{G}_{\mathrm{m},K})$ is bijective. However, this is a consequence of the Néron mapping property for $j_*\mathbb{G}_{\mathrm{m},K}$.

It remains to show that each $\mathbb{G}_{\mathrm{m},K}$-torsor over B_K is induced by a $j_*\mathbb{G}_{\mathrm{m},K}$-torsor over B or, what is enough up to push-out, by a $\mathbb{G}_{\mathrm{m},\mathfrak{R}}$-torsor over B. Again, we forget about the group structure of B. As we may assume $B_{R,K} = B_K$, we may identify B as well as torsors over it with their underlying admissible formal R-schemes. Now start with a $\mathbb{G}_{\mathrm{m},K}$-torsor E_K over B_K, and interpret it as a coherent locally free B_K-module. If B is affine, we can apply [10], 21.6.11. Namely, since B is smooth over R, its ring of global sections is regular by [13], 4.4, and we see that E_K extends to a locally free B-module E. From this it follows that, in the general case, there exists an open cover (B^i) of B such that E_K trivializes with respect to the cover (B^i_K) of B_K. As $H^1_{\mathrm{Zar}}(B, \mathbb{Z}) = 0$ due to the smoothness of B, we can find generators f_i of $E_K|_{B^i_K}$ in such a way that $f_i f_j^{-1}$ extends to a section on $B_i \cap B_j$, for all i, j; use the Néron mapping property for $j_*\mathbb{G}_{\mathrm{m},K}$. Consequently, E_K extends to a coherent locally free B-module E, as claimed. □

Proposition 2.4. — *For a smooth rigid K-group B_K we write* $\mathrm{Ext}(B_K, \mathbb{G}_{\mathrm{m},K})$ *for the group of extensions, in the sense of rigid K-groups, of B_K by $\mathbb{G}_{\mathrm{m},K}$, as well as* $\mathrm{Ext}^1(B_K, \mathbb{G}_{\mathrm{m},K})$ *for the corresponding* Ext^1 *group in the setting of sheaves on the small rigid smooth site over K. Then the canonical map*

$$\mathrm{Ext}(B_K, \mathbb{G}_{\mathrm{m},K}) \longrightarrow \mathrm{Ext}^1(B_K, \mathbb{G}_{\mathrm{m},K}),$$

which associates to any extension

$$0 \longrightarrow \mathbb{G}_{\mathrm{m},K} \longrightarrow E_K \longrightarrow B_K \longrightarrow 0$$

the class of

$$\begin{array}{ccc} & & B_K \\ & & \downarrow \\ E_K & \longrightarrow & B_K \end{array}$$

in $\mathrm{Ext}^1(B_K, \mathbb{G}_{\mathrm{m},K}) = \mathrm{Hom}^{\bullet}_{\mathrm{derived}}(B_K, \mathbb{G}_{\mathrm{m},K}[1])$, *is an isomorphism. The same is true for smooth rigid groups over any smooth rigid base S_K instead of K.*

Similarly, for any smooth S-group B in $\mathrm{Mod}_{\mathfrak{R}}$, *the canonical map*

$$\mathrm{Ext}(B, j_*\mathbb{G}_{\mathrm{m},K}) \longrightarrow \mathrm{Ext}^1(B, j_*\mathbb{G}_{\mathrm{m},K})$$

is bijective.

Proof. — Use [12], VII, 3.2.5, in conjunction with the representability results 2.1 and 2.2. □

The proposition says that we need not make a difference between extensions in the sense of group objects and of homological algebra.

Therefore we will switch between both notions in the following without mentioning this explicitly.

We want to derive another consequence from 2.2.

PROPOSITION 2.5. — *Let B be a smooth S-group. Then, by restriction to special fibres (which is indicated by an index k), we get isomorphisms between groups of extensions*

$$\mathrm{Ext}(B, i_*\mathbb{Z}) \xrightarrow{\sim} \mathrm{Ext}(B_R, \mathbb{Z}_R) \xrightarrow{\sim} \mathrm{Ext}(B_k, \mathbb{Z}_k) \xleftarrow{\sim} \mathrm{Ext}(\phi_B, \mathbb{Z})$$

where ϕ_B is the group of components of B_R.

Proof. — For any $\mathfrak{R}$-model U there are canonical bijections

$$\mathrm{Hom}(U, i_*\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}(U_R, \mathbb{Z}_R) \xrightarrow{\sim} \mathrm{Hom}(U_k, \mathbb{Z}_k).$$

Since $i_*\mathbb{Z}$-torsors on B are trivial by 2.2 (ii) and the same is true for $\mathbb{Z}_k$-torsors on B_k by [12], VIII, 5.1, it follows that the map $\mathrm{Ext}(B, i_*\mathbb{Z}) \xrightarrow{\sim} \mathrm{Ext}(B_k, \mathbb{Z}_k)$ is bijective. Furthermore, $\mathrm{Ext}(B_k, \mathbb{Z}_k) \xleftarrow{\sim} \mathrm{Ext}(\phi_B, \mathbb{Z})$ is bijective by [12], VIII, 5.5. □

Finally, let us point out that, again due to [12], VII, 3.2.5, the Ext groups in 2.5 can canonically be identified with the corresponding Ext^1 groups. In the subsequent sections we will also use the sheaf $\underline{\mathrm{Ext}}$; note that for abelian sheaves $\mathcal{F}$, $\mathcal{G}$, the presheaf $U \longmapsto \mathrm{Ext}^i(\mathcal{F}|_U, \mathcal{G}|_U)$ induces the sheaf $\underline{\mathrm{Ext}}^i(\mathcal{F}, \mathcal{G})$, since injectives are preserved under restriction.

3. Computation of some Hom and Ext groups.

We start by a technical lemma, which frequently allows us to reduce problems on free Galois modules M_K to those satisfying $H^1(I, M_K) = 0$, where I means the inertia subgroup of the absolute Galois group of K.

LEMMA 3.1. — *Let M_K be an étale K-group scheme which becomes isomorphic to a free $\mathbb{Z}$-module of finite rank after separable extension of K. Then there is an exact sequence*

$$0 \longrightarrow M_K \longrightarrow M_K^+ \longrightarrow M_K^- \longrightarrow 0$$

of K-groups of the same type where, in addition, $H^1(I, M_K^+) = R^1 j_* M_K^+ = 0$ and M_K^- is invariant under the inertia group I.

Proof. — We follow the ideas of [22], 2.13, and [8], 4.4. Thinking in terms of Galois modules, assume that M_K becomes constant over a Galois extension L/K with Galois group G, and let $M_K^{++} = \mathrm{Ind}^G(M_L)$ be the induced module of M_L; it satisfies $R^1 j_* M_K^{++} = H^1(I, M_K^{++}) = 0$ by [8], 4.4 and its proof. Furthermore, there is a canonical injection $M_K \hookrightarrow M_K^{++}$. Now set $M_K^{--} = M_K^{++}/M_K$, as well as $M_K^{-} = (M_K^{--})^I$, and let $M_K^{+} \subset M_K^{++}$ be the inverse image of M_K^{-}. Then we have an exact sequence $0 \longrightarrow M_K \longrightarrow M_K^{+} \longrightarrow N_K^{-} \longrightarrow 0$, where M_K^{-} is as required and satisfies $H^1(I, M_K^{-}) = 0$, since M_K^{-} is invariant under I. Now look at the commutative diagram

$$\begin{array}{ccccccc}
(M_K^{-})^I & \longrightarrow & H^1(I, M_K) & \longrightarrow & H^1(I, M_K^{+}) & \longrightarrow & H^1(I, M_K^{-}) = 0 \\
\| & & \| & & \downarrow & & \\
(M_K^{--})^I & \longrightarrow & H^1(I, M_K) & \longrightarrow & H^1(I, M_K^{++}) = 0 & &
\end{array}$$

whose rows are parts of long exact cohomology sequences with respect to the I-cohomology. It implies $H^1(I, M_K^{+}) = 0$ and thus, by [8], 4.4, $R^1 j_* M_K^{+} = 0$ so that we are done. □

LEMMA 3.2. — *Consider an étale K-group scheme M_K as in 3.1, and let M be its Néron model. Then:*

(i) $T = \underline{\mathrm{Hom}}(M, j_*\mathbb{G}_{\mathrm{m},K})$ *is the Néron model of* $T_K = \underline{\mathrm{Hom}}(M_K, \mathbb{G}_{\mathrm{m},K})$, *the torus with group of characters M_K.*

(ii) *The canonical map* $\underline{\mathrm{Hom}}(M, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) \longrightarrow \underline{\mathrm{Hom}}(M, j_*\mathbb{G}_{\mathrm{m},K})$ *is a monomorphism and identifies* $\underline{\mathrm{Hom}}(M, \mathbb{G}_{\mathrm{m},\mathfrak{R}})$ *with the subgroup* $T_{\mathrm{tor}} \subset T$ *corresponding to the torsion subgroup of* ϕ_T.

Proof. — The first assertion is obvious from the mapping property of Néron models. To derive the second one, let us use the abbreviation $T^{00} = \underline{\mathrm{Hom}}(M, \mathbb{G}_{\mathrm{m},\mathfrak{R}})$. By its definition, T^{00} equals the part of T' where all characters in M_K take integral invertible values. In particular, we have $T^0 \subset T^{00}$, and T^{00} contains the subgroup T_{tor} of T. As we might interpret i^*T^{00} as the formal Néron model of a quasi-compact open subgroup of T_K and as such Néron models are quasi-compact themselves, see [7], 1.2, we can conclude that T^{00} must coincide with T_{tor}. □

LEMMA 3.3. — *Let N_K be a torsion-free étale K-group scheme of finite type, which is invariant under the inertia group I. Then* $\underline{\mathrm{Ext}}^1(N, \mathcal{F}) = 0$ *for the Néron model N of N_K and any abelian sheaf $\mathcal{F}$ on the small smooth site over $\mathfrak{R}$.*

Proof. — We may assume that R is strictly henselian and, hence, that N_K and N are constant. So we need only to consider the case where $N = \mathbb{Z}_{\mathfrak{R}}$. As the functor $\mathcal{F} \longmapsto \underline{\mathrm{Hom}}(\mathbb{Z}_{\mathfrak{R}}, \mathcal{F})$ is an equivalence of sheaves on the site we are considering, this functor is exact. Consequently, $\underline{\mathrm{Ext}}^1(\mathbb{Z}_{\mathfrak{R}}, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) = 0$. □

LEMMA 3.4. — *Let T_K be a torus with Néron model T. Then the following hold:*

(i) $i^*\underline{\mathrm{Hom}}(T, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) = 0$.

(ii) $r^*\underline{\mathrm{Ext}}^1(T, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) = 0$, *provided we know either that T_K has multiplicative reduction or that the residue field k of K is perfect; r^* means restriction to the étale topology on $\mathfrak{R}$.*

Proof. — The group N_K of characters of T_K is an étale K-group scheme. Thus, it is enough to compute $i^*\underline{\mathrm{Hom}}(T, \mathbb{G}_{\mathrm{m},\mathfrak{R}})$ with respect to the étale topology on $\mathfrak{R}$. Assuming the valuation ring R strictly henselian, we just have to show $\mathrm{Hom}(T, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) = 0$. To do this, let us first consider the case where T_K is *split* over K. Each character $\chi_K \in N_K(K)$ defines a morphism $\chi: T_K \longrightarrow \mathbb{G}_{\mathrm{m},K}$ and, hence, a morphism of Néron models $\chi: T \longrightarrow j_*\mathbb{G}_{\mathrm{m},K}$. To the latter we can associate the morphism $\phi_\chi: \phi_T \longrightarrow \mathbb{Z}$ between component groups. It is more or less trivial that the resulting map $N_K(K) \longrightarrow \mathrm{Hom}(\phi_T, \mathbb{Z})$ is an isomorphism.

If we start with a morphism $\chi: T \longrightarrow \mathbb{G}_{\mathrm{m},\mathfrak{R}}$, its rigid part χ_K is in $N_K(K)$, and the associated morphism $\chi: T \longrightarrow j_*\mathbb{G}_{\mathrm{m},K}$ satisfies $\phi_\chi = 0$. But then χ itself must be trivial. So $\mathrm{Hom}(T, \mathbb{G}_{\mathrm{m},\mathfrak{R}})$ is trivial, which settles the case of a split torus T_K.

Assume now that T_K is not necessarily split. Viewing its group of characters N_K as an étale K-group scheme, we write $N_{K,I}$ for the maximal $\mathbb{Z}$-free quotient which is invariant under the inertia group I which, in our case, coincides with the absolute Galois group of K. Then there is an exact sequence

$$0 \longrightarrow \tilde{N}_K \longrightarrow N_K \longrightarrow N_{K,I} \longrightarrow 0$$

of torsion-free étale K-group schemes such that $\tilde{N}_K^I \subset \tilde{N}_K$, the subgroup of I-invariants, is trivial. Looking at the associated sequence of tori

$$0 \longrightarrow T_{K,I} \longrightarrow T_K \longrightarrow \tilde{T}_K \longrightarrow 0,$$

we get an exact sequence of Néron models

$$0 \longrightarrow T_I \longrightarrow T \longrightarrow \tilde{T} \longrightarrow 0;$$

note that $R^1 j_* T_{K,I} = 0$ due to [8], 4.2, as $T_{K,I}$ is split over K. Then we apply $\mathrm{Hom}(\cdot, \mathbb{G}_{m,\mathfrak{R}})$ to the latter sequence and obtain an isomorphism

$$\mathrm{Hom}(\tilde{T}, \mathbb{G}_{m,\mathfrak{R}}) \xrightarrow{\sim} \mathrm{Hom}(T, \mathbb{G}_{m,\mathfrak{R}}),$$

since $\mathrm{Hom}(T_I, \mathbb{G}_{m,\mathfrak{R}})$ is trivial by what we have shown above. However, the torus $\tilde{T}_K$ does not admit a non-trivial character over K, since $\tilde{N}_K^I$ is trivial. Consequently, $\mathrm{Hom}(\tilde{T}, \mathbb{G}_{m,\mathfrak{R}})$ and, hence, $\mathrm{Hom}(T, \mathbb{G}_{m,\mathfrak{R}})$ must be trivial, which had to be shown.

Next we want to show that $r^* \underline{\mathrm{Ext}}^1(T, \mathbb{G}_{m,\mathfrak{R}})$ is trivial. To do this, we assume R to be strictly henselian again and show first that $\mathrm{Ext}^1(T, \mathbb{G}_{m,\mathfrak{R}})$ as the formal part of this group is trivial. Applying 3.1, we choose an exact sequence of étale K-group schemes without torsion

$$0 \longrightarrow N_K \longrightarrow N_K^+ \longrightarrow N_K^- \longrightarrow 0,$$

where $H^1(I, M_K^+) = 0$ and N_K^- is invariant under I. Attached to it is an exact sequence of tori

$$0 \longrightarrow T_K^\ominus \longrightarrow T_K^\oplus \longrightarrow T_K \longrightarrow 0,$$

which gives rise to an exact sequence of associated Néron models

$$0 \longrightarrow T^\ominus \longrightarrow T^\oplus \longrightarrow T \longrightarrow 0$$

by [8], 4.2, since $T_K^\ominus$ is split. So we get an exact sequence

$$\mathrm{Hom}(T^\ominus, \mathbb{G}_{m,\mathfrak{R}}) \longrightarrow \mathrm{Ext}^1(T, \mathbb{G}_{m,\mathfrak{R}}) \longrightarrow \mathrm{Ext}^1(T^\oplus, \mathbb{G}_{m,\mathfrak{R}}),$$

where $\mathrm{Hom}(T^\ominus, \mathbb{G}_{m,\mathfrak{R}}) = 0$ as we have seen. Hence, it is enough to show $\mathrm{Ext}^1(T^\oplus, \mathbb{G}_{m,\mathfrak{R}}) = 0$ or, more specifically, that any extension of smooth $\mathfrak{R}$-groups

$$0 \longrightarrow \mathbb{G}_{m,\mathfrak{R}} \longrightarrow H \longrightarrow T^\oplus \longrightarrow 0$$

is split.

Let us fix such an extension and look at its rigid part

$$(*) \qquad 0 \longrightarrow \mathbb{G}_{m,K} \longrightarrow H_K \longrightarrow T_K^\oplus \longrightarrow 0,$$

which is an extension of smooth rigid K-groups. We claim that it is algebraizable and that H_K is a torus. To verify this, we might replace K by a finite separable extension field and thereby assume that $T_K^\oplus$ is split. Furthermore, we might choose a split lattice V_K of maximal rank in $T_K^\oplus$

and lift it to a V_K-linearization of H_K, viewing the latter as a line bundle on $T_K^{\oplus}$. Such a line bundle is trivial, as is shown in the proof of [3], 4.5. Hence, there is an isomorphism of rigid K-spaces $H_K \xrightarrow{\sim} \mathbb{G}_{m,K} \times T_K^{\oplus}$. Since invertible sections on tori are the same in the rigid and the algebraic sense, it follows that the group structure on H_K is algebraic. But then, as an extension of a torus by a torus, H_K must be a torus itself; cf. [11], exp. IX, prop. 8.2.

We want to show that, in fact, the sequence of tori $(*)$ splits over the field we started with. To do this, look at the associated sequence of groups of characters, viewed as étale group schemes over K:

$$0 \longrightarrow N_K^+ \longrightarrow \tilde{N}_K \longrightarrow \mathbb{Z}_K \longrightarrow 0.$$

Taking invariants under the inertia group I, we get an exact sequence

$$0 \longrightarrow N_K^{+I} \longrightarrow \tilde{N}_K^I \longrightarrow \mathbb{Z}_K \longrightarrow 0,$$

due to the fact that $H^1(I, N_K^+) = 0$. But then $\tilde{N}_K \longrightarrow \mathbb{Z}_K$ admits a section which is compatible with the action of I and, consequently, the sequence $(*)$ above is split.

Now let us look at the following commutative diagram of $\mathfrak{R}$-groups:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{G}_{m,\mathfrak{R}} & \longrightarrow & H & \xrightarrow{\varphi} & T^{\oplus} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \iota & & \| & & \\
0 & \longrightarrow & j_*\mathbb{G}_{m,K} & \longrightarrow & j_*H_K & \xrightarrow{\psi} & T^{\oplus} & \longrightarrow & 0.
\end{array}$$

The first row is just the extension we started with and whose triviality has to be shown, whereas the second one is the sequence of Néron models associated to $(*)$; i. e., the sequence of Néron models associated to the rigid part of the first row. The vertical maps exist due to the Néron mapping property, and ι is a monomorphism. The second row is split exact, since the same is true for the sequence $(*)$. So there is a section $\varepsilon: T^{\oplus} \longrightarrow j_*H_K$ of ψ, which is unique up to a character $T^{\oplus} \longrightarrow j_*\mathbb{G}_{m,K}$. Switching to component groups, we get the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \phi_H & \xrightarrow{\phi_\varphi} & \phi_{T^{\oplus}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \phi_\iota & & \| & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \phi_{j_*H_K} & \xrightarrow{\phi_\psi} & \phi_{T^{\oplus}} & \longrightarrow & 0
\end{array}$$

where ϕ_φ is an isomorphism and, hence, ϕ_ι an injection. There are now two sections of $\phi_\psi: \phi_{j_*H_K} \longrightarrow \phi_{T^{\oplus}}$, namely the section ϕ_ε induced from ε and the map $\phi_\iota \circ \phi_\varphi^{-1}$. Both differ by a homomorphism $\phi_\lambda : \phi_{T^\oplus} \longrightarrow \mathbb{Z}$. If the canonical map

$$N_K^{+I} \longrightarrow \operatorname{Hom}(\phi_{T^\oplus}, \mathbb{Z})$$

is bijective, we can conclude that ϕ_λ is induced from a character $\lambda \in N_K^{+I}$. Changing the section ε by means of λ, we may assume $\phi_\varepsilon = \phi_\iota \circ \phi_\varphi^{-1}$. Then ϕ_ε maps $\phi_{T^\oplus}$ into $\phi_\iota(\phi_H)$, and we see that ε factors through H. This means that H, as a $\mathbb{G}_{m,\mathfrak{R}}$-extension of $T^\oplus$, is split and, hence, trivial. So the formal part of $r^*\underline{\operatorname{Ext}}^1(T^\oplus, \mathbb{G}_{m,\mathfrak{R}})$ is trivial, provided the canonical map $N_K^{+I} \longrightarrow \operatorname{Hom}(\phi_{T^\oplus}, \mathbb{Z})$ is bijective. In fact, the bijectivity of this map follows from [22], 2.4, if the residue field k of K is perfect. If, on the other hand, T_K is known to have multiplicative reduction, we certainly can set $T_K^\oplus = T_K$ and get the same assertion for trivial reasons.

Finally, as $\mathbb{G}_{m,K}$-extensions of $T_K^\oplus$ split always over finite separable extensions of K, as shown above, we see that the rigid part of $r^*\underline{\operatorname{Ext}}^1(T^\oplus, \mathbb{G}_{m,\mathfrak{R}})$ is trivial and, hence, that $r^*\underline{\operatorname{Ext}}^1(T^\oplus, \mathbb{G}_{m,\mathfrak{R}}) = 0$ by 1.2. □

4. Review of Grothendieck's pairing.

We start by looking at a scheme situation, writing $i: \operatorname{Spec} k \longrightarrow \operatorname{Spec} R$ and $j: \operatorname{Spec} K \longrightarrow \operatorname{Spec} R$ for the obvious morphisms. Let A_K be an abelian variety over K with dual A'_K. Denote by A, A' the corresponding Néron models and by $\phi_A, \phi_{A'}$ their component groups. There is a canonical pairing

$$(*) \qquad \phi_A \otimes_{\mathbb{Z}} \phi_{A'} \longrightarrow \mathbb{Q}/\mathbb{Z},$$

introduced by Grothendieck in [12], IX, 1.2, which represents the obstruction of extending the Poincaré bundle on $A_K \times A'_K$ to a biextension of $A \times A'$ by $\mathbb{G}_{m,R}$. Conjecturally, the pairing is perfect, and this conjecture has been established in many important cases by contributions of Grothendieck [12], Artin-Mazur (unpublished), Bégueri [2], and McCallum [16]: if A_K has semi-stable reduction or, otherwise, if the residue field k is perfect, except for infinite k in the equal characteristic $p > 0$ case. A state of art proof in the semi-stable reduction case has recently been given by Werner [21]. In conjunction with [9], it settles the compatibility between Grothendieck's pairing and the monodromy pairing which was left to the reader in [12].

To recall the definition of Grothendieck's pairing, observe that, due to [12], VII, 3.7.5, the canonical exact sequence

$$0 \longrightarrow \mathbb{G}_{m,R} \longrightarrow j_*\mathbb{G}_{m,K} \longrightarrow i_*\mathbb{Z} \longrightarrow 0,$$

with $j_*\mathbb{G}_{m,K}$ denoting the classical Néron model of the multiplicative group $\mathbb{G}_{m,K}$, gives rise to an exact sequence

$$\mathrm{Biext}^1(A, A'; \mathbb{G}_{m,R}) \longrightarrow \mathrm{Biext}^1(A, A'; j_*\mathbb{G}_{m,K}) \longrightarrow \mathrm{Biext}^1(A, A'; i_*\mathbb{Z}).$$

Due to [12], VIII, 6.7, restriction to the generic fibre yields an isomorphism

$$\mathrm{Biext}^1(A, A'; j_*\mathbb{G}_{m,K}) \xrightarrow{\sim} \mathrm{Biext}^1(A_K, A'_K; \mathbb{G}_{m,K}),$$

and there is a canonical isomorphism

$$\mathrm{Biext}^1(A, A'; i_*\mathbb{Z}) \xrightarrow{\sim} \mathrm{Biext}^1(\phi_A, \phi_{A'}; \mathbb{Z})$$

by [12], VIII, 5.6 and 5.10. Furthermore, using the exact sequence obtained from dividing $\mathbb{Q}$ by $\mathbb{Z}$, we get an isomorphism

$$\mathrm{Biext}^1(\phi_A, \phi_{A'}; \mathbb{Z}) \xleftarrow{\sim} \mathrm{Biext}^0(\phi_A, \phi_{A'}; \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(\phi_A \otimes_{\mathbb{Z}} \phi_{A'}, \mathbb{Q}/\mathbb{Z}).$$

Thus, starting with the element of $\mathrm{Biext}^1(A, A'; j_*\mathbb{G}_{m,K})$ which corresponds to the Poincaré bundle on $A_K \times A'_K$, its image in $\mathrm{Biext}^1(A, A'; i_*\mathbb{Z})$ corresponds to a morphism $\phi_A \otimes_{\mathbb{Z}} \phi_{A'} \longrightarrow \mathbb{Q}/\mathbb{Z}$ which, by definition, is Grothendieck's pairing of component groups.

Of course, the pairing may also be written in the form of a homomorphism $\phi_{A'} \longrightarrow \underline{\mathrm{Hom}}_{\mathbb{Z}}(\phi_A, \mathbb{Q}/\mathbb{Z})$ of sheaves with respect to the smooth (or étale) topology on R. We claim that there is a commutative diagram of sheaves with respect to the smooth topology,

$$\begin{array}{ccccc} A' & \xrightarrow{\sim} & \underline{\mathrm{Ext}}^1(A, j_*\mathbb{G}_{m,K}) & & \\ \downarrow & & \downarrow & & \\ i_*\phi_{A'} & \longrightarrow & i_*\underline{\mathrm{Ext}}^1(\phi_A, \mathbb{Z}) & \xleftarrow{\sim} & i_*\underline{\mathrm{Hom}}(\phi_A, \mathbb{Q}/\mathbb{Z}) \end{array}$$

with the pairing homomorphism occurring in the lower row. To define the map in the first row, we look at the isomorphism $A'_K \xrightarrow{\sim} \underline{\mathrm{Ext}}^1(A_K, \mathbb{G}_{m,K})$, given by the duality between A_K and A'_K, and take its direct image under j; it is of the desired type since $j_*\underline{\mathrm{Ext}}^1(A_K, \mathbb{G}_{m,K}) = \underline{\mathrm{Ext}}^1(A, j_*\mathbb{G}_{m,K})$ by [12], VIII, 6.6. The first vertical map is, of course, the projection of A' onto its component group, whereas the second one is induced from the projection $j_*\mathbb{G}_{m,K} \longrightarrow i_*\mathbb{Z}$, using the fact that $\underline{\mathrm{Ext}}^1(A, i_*\mathbb{Z}) = i_*\underline{\mathrm{Ext}}^1(\phi_A, \mathbb{Z})$ by [12], VIII, 5.5 and 5.9 (these results extend to the smooth topology). Finally, that the diagram is commutative, at least with respect to the étale topology, follows from [12], VIII, 7.3.4. But then, since the points with values in étale extensions of R are schematically dense in A', the diagram is commutative also with respect to the smooth topology.

In terms of the above diagram, Grothendieck's conjecture on the pairing $(*)$ being perfect is equivalent to the bijectivity of the map

$$(**) \qquad \phi_{A'} \longrightarrow \underline{\mathrm{Ext}}^1(\phi_A, \mathbb{Z}),$$

and of the corresponding one with A and A' interchanged. Furthermore, for any subgroup $\vartheta \subset \phi_A$, the kernel of

$$\phi_{A'} \longrightarrow \underline{\mathrm{Ext}}^1(\phi_A, \mathbb{Z}) \longrightarrow \underline{\mathrm{Ext}}^1(\vartheta, \mathbb{Z})$$

is the orthogonal complement of ϑ under the pairing $(*)$.

Let us switch now to rigid K-spaces and their $\mathfrak{R}$-models; the morphisms i and j are as in Section 1. Then, due to the well-known properties of the rigid GAGA-functor, see [14], due to the representability of $\mathbb{G}_{\mathrm{m},K}$-torsors, see 2.1, and due to the cohomological characterization of group extensions, see 2.4, the duality isomorphism $A'_K \xrightarrow{\sim} \underline{\mathrm{Ext}}^1(A_K, \mathbb{G}_{\mathrm{m},K})$ carries over to an isomorphism of sheaves in the rigid category. In fact, if we restrict this map to the étale topology on K, both isomorphisms, in the algebraic and the rigid sense, coincide. Using 2.3, it follows that the above isomorphism extends to an isomorphism $A' \xrightarrow{\sim} \underline{\mathrm{Ext}}^1(A, j_*\mathbb{G}_{\mathrm{m},K})$, where now A and A', as well as $j_*\mathbb{G}_{\mathrm{m},K}$ are the Néron $\mathfrak{R}$-models of A_K, A'_K, and $\mathbb{G}_{\mathrm{m},K}$. Furthermore, just as in the scheme case, there is a canonical isomorphism $\underline{\mathrm{Ext}}^1(A, i_*\mathbb{Z}) \xrightarrow{\sim} i_*\underline{\mathrm{Ext}}^1(\phi_A, \mathbb{Z})$, see 2.5, which we also might write in the form of an isomorphism $i^*\underline{\mathrm{Ext}}^1(A, i_*\mathbb{Z}) \xrightarrow{\sim} \underline{\mathrm{Ext}}^1(\phi_A, \mathbb{Z})$ by 1.1. In the following we will simplify our notation by not explicitly mentioning i_* on the level of component groups and their $\underline{\mathrm{Ext}}^1$-groups.

PROPOSITION 4.1. — *There is a canonical commutative diagram*

$$\begin{array}{ccc} A' & \xrightarrow{\sim} & \underline{\mathrm{Ext}}^1(A, j_*\mathbb{G}_{\mathrm{m},K}) \\ \downarrow & & \downarrow \\ \phi_{A'} & \longrightarrow & \underline{\mathrm{Ext}}^1(\phi_A, \mathbb{Z}) \end{array}$$

of sheaves on the small model smooth site over $\mathfrak{R}$, with the pairing morphism $(**)$ occurring at the bottom place. It is commutative and, if restricted to the small étale site over $\mathfrak{R}$, coincides with the corresponding one in the scheme case.

Proof. — The diagram coincides with the one we have in the scheme case if we restrict to the étale topology. In particular, the diagram is commutative with respect to the étale topology. As the points with values in étale extensions of $\mathfrak{R}$ are schematically dense in A', the diagram is commutative also with respect to the smooth topology. □

5. Criteria for perfectness of the pairing.

We can enlarge the diagram of 4.1 to get a diagram with exact columns as follows:

$$\begin{array}{ccccc}
0 & & \underline{\mathrm{Hom}}(A, i_*\mathbb{Z}) & = & 0 \\
\downarrow & & \downarrow & & \\
A'^0 & \longrightarrow & \underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) & & \\
\downarrow & & \downarrow & & \\
A' & \xrightarrow{\sim} & \underline{\mathrm{Ext}}^1(A, j_*\mathbb{G}_{\mathrm{m},K}) & & \\
\downarrow & & \downarrow & & \\
\phi_{A'} & \longrightarrow & \underline{\mathrm{Ext}}^1(\phi_A, \mathbb{Z}) & \xleftarrow{\sim} & \underline{\mathrm{Hom}}(\phi_A, \mathbb{Q}/\mathbb{Z}) \\
\downarrow & & & & \\
0 & & & &
\end{array}$$

Of course, A'^0 is the identity component of A', and we have $\underline{\mathrm{Hom}}(A, i_*\mathbb{Z}) = 0$, since the formal part of A consists of only finitely many connected components. Thus, the map $A'^0 \longrightarrow \underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},\mathfrak{R}})$ exists and is a monomorphism. There is the following criterion:

PROPOSITION 5.1. — *Grothendieck's pairing $\phi_A \times \phi_{A'} \longrightarrow \mathbb{Q}/\mathbb{Z}$ is perfect if and only if $A'^0 \longrightarrow \underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},\mathfrak{R}})$ and the corresponding morphism with A and A' interchanged are isomorphisms.*

Proof. — The morphisms

$$A'^0 \longrightarrow \underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},\mathfrak{R}}), \qquad A^0 \longrightarrow \underline{\mathrm{Ext}}^1(A', \mathbb{G}_{\mathrm{m},\mathfrak{R}})$$

are bijective if and only if the maps

$$\phi_{A'} \longrightarrow \underline{\mathrm{Hom}}(\phi_A, \mathbb{Q}/\mathbb{Z}), \qquad \phi_A \longrightarrow \underline{\mathrm{Hom}}(\phi_{A'}, \mathbb{Q}/\mathbb{Z})$$

obtained from Grothendieck's pairing are injective; thus, if and only if this pairing is non-degenerate on both sides and, hence, perfect. □

Next, let

$$0 \longrightarrow M_K \longrightarrow E_K \longrightarrow A_K \longrightarrow 0, \qquad 0 \longrightarrow M'_K \longrightarrow E'_K \longrightarrow A'_K \longrightarrow 0$$

be the uniformizations of A_K, A'_K in the sense of [8], 1.2, and

$$0 \longrightarrow T_K \longrightarrow E_K \longrightarrow B_K \longrightarrow 0, \qquad 0 \longrightarrow T'_K \longrightarrow E'_K \longrightarrow B'_K \longrightarrow 0$$

the associated Raynaud extensions exhibiting E_K, E'_K as semi-abelian group schemes.

PROPOSITION 5.2. — *Assume that the pairings $\phi_A \times \phi_{A'} \longrightarrow \mathbb{Q}/\mathbb{Z}$ and $\phi_B \times \phi_{B'} \longrightarrow \mathbb{Q}/\mathbb{Z}$ are perfect, and consider the following canonical maps:*

$$\underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) \longrightarrow \underline{\mathrm{Ext}}^1(E, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) \longleftarrow \underline{\mathrm{Ext}}^1(B, \mathbb{G}_{\mathrm{m},\mathfrak{R}}).$$

Then, restricting sheaves to formal parts and to the étale topology and assuming that k is perfect or T_K has multiplicative reduction, the left map is an epimorphism and the right map is an isomorphism. More precisely, there is a canonical commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & T' \cap E'^0 & \longrightarrow & E'^0 = A'^0 & \longrightarrow & B'^0 & \longrightarrow 0 \\ & & & \downarrow & & \downarrow & \\ & & & \underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) & \longrightarrow & \underline{\mathrm{Ext}}^1(E, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) & \end{array}$$

with an exact upper row and the vertical maps being canonical identifications in the sense above.

Proof. — Using the perfectness of the pairings, there are canonical maps

$$A'^0 \xrightarrow{\sim} \underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) \longrightarrow \underline{\mathrm{Ext}}^1(E, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) \longleftarrow \underline{\mathrm{Ext}}^1(B, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) \xleftarrow{\sim} B'^0.$$

Furthermore, the Raynaud extension of A_K gives rise to an exact sequence of Néron models

$$0 \longrightarrow T \longrightarrow E \longrightarrow B \longrightarrow 0,$$

since $R^1 j_* T_K = 0$ by [8], 4.2, and, hence, to an exact sequence

$$\underline{\mathrm{Hom}}(T, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) \longrightarrow \underline{\mathrm{Ext}}^1(B, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) \longrightarrow \underline{\mathrm{Ext}}^1(E, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) \longrightarrow \underline{\mathrm{Ext}}^1(T, \mathbb{G}_{\mathrm{m},\mathfrak{R}}).$$

So, using 3.4, it follows that $\underline{\mathrm{Ext}}^1(B, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) \longrightarrow \underline{\mathrm{Ext}}^1(E, \mathbb{G}_{\mathrm{m},\mathfrak{R}})$ is an isomorphism, at least if restricted to the étale topology on formal parts. Thus, all in all, the above sequence of maps yields a morphism $i^* A'^0 \longrightarrow i^* B'^0$, if we restrict to the étale topology on formal parts.

On the other hand, the uniformization theory of A'_K yields canonical maps

$$A'^0 \longleftarrow E'^0 \longrightarrow B'^0,$$

where the left one is bijective on formal parts by [8], 2.3. As $E' \longrightarrow B'$ and, hence by [8], 4.8, also $E'^0 \longrightarrow B'^0$ are epimorphisms, the above sequence of maps yields an epimorphism $i^*A'^0 \longrightarrow i^*B'^0$ with kernel $T' \cap A'^0$. But then, going through the duality theory of [3], Sect. 6, one can realize that the map coincides with the preceding one. Thus, we are done. □

Using the idea of 5.2, we can derive the perfectness of Grothendieck's pairing in certain situations.

THEOREM 5.3. — *Assume that the pairing $\phi_B \times \phi_{B'} \longrightarrow \mathbb{Q}/\mathbb{Z}$ is perfect and, furthermore, that one of the following conditions is satisfied:*

(i) *T_K and T'_K have multiplicative reduction.*

(ii) *k is perfect, and the component groups ϕ_T, $\phi_{T'}$ are torsion-free.*

(iii) *k is perfect, and the abelian parts with potentially good reduction B_K, B'_K are trivial.*

Then the pairing $\phi_A \times \phi_{A'} \longrightarrow \mathbb{Q}/\mathbb{Z}$ is perfect.

To prepare the proof, let us consider an exact sequence

$$0 \longrightarrow M_K \longrightarrow M_K^+ \longrightarrow M_K^- \longrightarrow 0$$

as in 3.1. Then we define E_K^+ via the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_K & \longrightarrow & E_K & \longrightarrow & A_K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_K^+ & \longrightarrow & E_K^+ & \longrightarrow & A_K & \longrightarrow & 0 \end{array}$$

where the lower row is the push-out of the upper one with respect to the injection $M_K \longrightarrow M_K^+$. Since $R^1 j_* M_K^+ = 0$, the lower row gives rise to an exact sequence of associated Néron models

$$0 \longrightarrow M^+ \longrightarrow E^+ \longrightarrow A \longrightarrow 0$$

which we will use.

LEMMA 5.4. — *There is a commutative diagram*

$$\begin{array}{ccccc} T^{\oplus 0} & \xrightarrow{\varepsilon^0} & T'^0 & \xrightarrow{\alpha^0} & A^0 \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}(M^+, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) & \xrightarrow{\varepsilon_{\mathrm{tor}}} & \underline{\mathrm{Hom}}(M, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) & & \underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}(M^+, j_*\mathbb{G}_{\mathrm{m},K}) & \xrightarrow{\varepsilon} & \underline{\mathrm{Hom}}(M, j_*\mathbb{G}_{\mathrm{m},K}) & \xrightarrow{\alpha} & \underline{\mathrm{Ext}}^1(A, j_*\mathbb{G}_{\mathrm{m},K}) \\ \| & & \| & & \| \\ T^{\oplus} & & T' & & A' \end{array}$$

with the following properties:

(i) *Writing* $T^{\oplus}$ *for the Néron model of the torus* $T_K^{\oplus} = \underline{\mathrm{Hom}}(M_K^+, \mathbb{G}_{\mathrm{m},K})$, *all vertical maps are canonical inclusions. In fact,* $\underline{\mathrm{Hom}}(M, \mathbb{G}_{\mathrm{m},\mathfrak{R}})$ *equals the subgroup of* T' *corresponding to the torsion part of* $\phi_{T'}$.

(ii) ε *and* $\varepsilon_{\mathrm{tor}}$ *are induced from the monomorphism* $M \hookrightarrow M^+$, *whereas* ε^0 *is the map between identity components;* ε *is an epimorphism if the residue field* k *of* K *is perfect.*

(iii) α *is the map obtained from the uniformization theory of* A'_K, *and* α^0 *is its restriction to identity components.*

(iv) *The composition* $\alpha^+ = \alpha \circ \varepsilon$ *is the map occurring in the long* $\underline{\mathrm{Ext}}$ *sequence associated to the above short exact sequence involving* M^+, E^+, *and* A. *The same holds for* $\alpha^+_{\mathrm{tor}} = \alpha \circ \varepsilon_{\mathrm{tor}}$, *viewed as a map to* $\underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},\mathfrak{R}})$.

Proof. — For assertion (i) we refer to 3.2. So let us look at the remaining ones. The map

$$\alpha^+ : \underline{\mathrm{Hom}}(M^+, j_*\mathbb{G}_{\mathrm{m},K}) \longrightarrow \underline{\mathrm{Ext}}^1(A, j_*\mathbb{G}_{\mathrm{m},K})$$

induced from the long $\underline{\mathrm{Ext}}$ sequence associated the short exact sequence involving M^+, E^+, and A, has as rigid part a map

$$\alpha_K^+ : \underline{\mathrm{Hom}}(M_K^+, \mathbb{G}_{\mathrm{m},K}) \longrightarrow \underline{\mathrm{Ext}}^1(A_K, \mathbb{G}_{\mathrm{m},K})$$

so that we can write $\alpha^+ = j_*\alpha_K^+$; use 2.3. As the formation of long exact cohomology sequences is compatible with restriction to rigid parts, we may interpret α_K^+ as being obtained from the long exact cohomology sequence associated to

$$0 \longrightarrow M_K^+ \longrightarrow E_K^+ \longrightarrow A_K \longrightarrow 0.$$

In a similar way, writing A_K as a quotient of E_K by M_K, we get a morphism

$$\alpha_K : \underline{\mathrm{Hom}}(M_K, \mathbb{G}_{\mathrm{m},K}) \longrightarrow \underline{\mathrm{Ext}}^1(A_K, \mathbb{G}_{\mathrm{m},K}).$$

Comparing both long exact cohomology sequences, it follows that α_K^+ must factor through α_K via ε_K. However, α_K is well-known. Namely, using the identifications $\underline{\mathrm{Hom}}(M_K, \mathbb{G}_{\mathrm{m},K}) = T'_K$ and $\underline{\mathrm{Ext}}^1(A_K, \mathbb{G}_{\mathrm{m},K}) = A'_K$, we can view it as the canonical map $T'_K \longrightarrow A'_K$ obtained from the uniformization theory of A'_K. To justify this, one has to realize that the connecting homomorphism α_K^+ associates to any homomorphism $x_K : M_K \longrightarrow \mathbb{G}_{\mathrm{m},K}$, say over any rigid base S_K, the push-out of

$$0 \longrightarrow M_K \longrightarrow E_K \longrightarrow A_K \longrightarrow 0$$

under x_K and that we may interpret the latter as an S_K-valued point of A'_K. On the other hand, using the duality theory of [3], in particular, 4.12 and 6.8, one has to observe that the same push-out is obtained, when we take the image of x_K under the map $T'_K \longrightarrow A'_K$.

The map α_K has a Néron model $\alpha: T' \longrightarrow A'$, and it follows that α^+ factors through α via ε. The latter map is surjective. Namely, the exact sequence

$$0 \longrightarrow M_K \longrightarrow M_K^+ \longrightarrow M_K^- \longrightarrow 0$$

gives rise to an exact sequence of tori

$$0 \longrightarrow T_K^{\ominus} \longrightarrow T_K^{\oplus} \longrightarrow T'_K \longrightarrow 0$$

and to an exact sequence of Néron models

$$0 \longrightarrow T^{\ominus} \longrightarrow T^{\oplus} \longrightarrow T' \longrightarrow 0,$$

since $R^1 j_* T_K^{\ominus} = 0$ by [8], 4.2. Thus, ε, which coincides with $T^{\oplus} \longrightarrow T'$, is surjective. This settles the assertions of the lemma. □

Now let us do the *proof* of 5.3. Starting out from the uniformization

$$0 \longrightarrow M_K \longrightarrow E_K \longrightarrow A_K \longrightarrow 0$$

of A_K, we choose an exact sequence

$$0 \longrightarrow M_K \longrightarrow M_K^+ \longrightarrow M_K^- \longrightarrow 0$$

as in 3.1 and look at the exact sequence

$$0 \longrightarrow M_K^+ \longrightarrow E_K^+ \longrightarrow A_K \longrightarrow 0$$

obtained from the above one via push-out with respect to $M_K \longrightarrow M_K^+$. In the case, where the tori T_K, T'_K have multiplicative reduction, we set $M_K^+ = M_K$ and $E_K^+ = E_K$. This works, as M_K, the group of characters of T'_K, becomes constant over an unramified extension of K, so that we must have $H^1(I, M_K) = 0$; cf. [8], 4.4. Switching to Néron models, the above exact sequence yields an exact sequence

$$0 \longrightarrow M^+ \longrightarrow E^+ \longrightarrow A \longrightarrow 0,$$

as well as the following part of the corresponding long $\underline{\mathrm{Ext}}$ sequence:

$$\underline{\mathrm{Hom}}(M^+, \mathbb{G}_{m,\mathfrak{R}}) \longrightarrow \underline{\mathrm{Ext}}^1(A, \mathbb{G}_{m,\mathfrak{R}}) \longrightarrow \underline{\mathrm{Ext}}^1(E^+, \mathbb{G}_{m,\mathfrak{R}}).$$

Writing $T^{\oplus}$ for the Néron model of the torus $T^{\oplus}_K = \underline{\mathrm{Hom}}(M^+_K, \mathbb{G}_{\mathrm{m},K})$, we know from [22], 2.7, that the component group $\phi_{T^{\oplus}}$ is torsion-free. Hence, using 3.2, it follows that $\underline{\mathrm{Hom}}(M^+, \mathbb{G}_{\mathrm{m},\mathfrak{R}})$ can be identified with the identity component $T^{\oplus 0}$ of $T^{\oplus}$. Moreover, if T'_K is the torus part of E'_K, we know from 5.4 that $T^{\oplus 0} \longrightarrow A'^0 \hookrightarrow \underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},\mathfrak{R}})$ factors through the canonical map $T'^0 \longrightarrow A'^0$ via a map $\varepsilon^0 \colon T^{\oplus 0} \longrightarrow T'^0$; the latter is an epimorphism by [8], 4.8, since the corresponding map $\varepsilon \colon T^{\oplus} \longrightarrow T'$ is an epimorphism. Thus, writing T'^0 again for the common image of $T^{\oplus 0}$ and T'^0 in A^0, we arrive at canonical monomorphisms

$$A'^0/T'^0 \hookrightarrow \underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},\mathfrak{R}})/T'^0 \hookrightarrow \underline{\mathrm{Ext}}^1(E^+, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) \longrightarrow \underline{\mathrm{Ext}}^1(E, \mathbb{G}_{\mathrm{m},\mathfrak{R}}),$$

which, with respect to the étale topology on formal parts, we may continue by the inverse of the isomorphism $\underline{\mathrm{Ext}}^1(E, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) \xleftarrow{\sim} B'^0$ of 5.2. The composition is surjective by 5.2.

The main problem is now to get information on the map

$$\tau \colon \underline{\mathrm{Ext}}^1(E^+, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) \longrightarrow \underline{\mathrm{Ext}}^1(E, \mathbb{G}_{\mathrm{m},\mathfrak{R}}).$$

If it is injective on the image of $\underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},\mathfrak{R}})/T'^0$, the map $A'^0/T'^0 \hookrightarrow \underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},\mathfrak{R}})/T'^0$ must be bijective, and we see that $A'^0 \hookrightarrow \underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},\mathfrak{R}})$ is bijective so that we are done. So let us go through the cases listed in 5.3. First, if T_K and T'_K have multiplicative reduction, we can set $M^+_K = M_K$ and $E^+ = E$. Then τ is the identity, and the desired assertion follows.

Next, let us assume that k is perfect and that the torsion parts of ϕ_T and $\phi_{T'}$ are trivial. Then the cohomology groups $H^1(I, M_K)$ and $H^1(I, M'_K)$ are trivial by [22], 2.19, and we can set $M^+_K = M_K$ and $E^+_K = E_K$ again. A more direct argument works as follows. Choose a finite Galois extension L/K such that A_K and A'_K acquire semi-stable reduction over L. Then the epimorphism $E'_L \longrightarrow A'_L$ induces an isomorphism ${E'_L}^0 \xrightarrow{\sim} {A'_L}^0$ on the parts corresponding to identity components of Néron models over the valuation ring of L; cf. [8], 2.3. By Galois descent on the rigid level, the epimorphism $E'_K \longrightarrow A'_K$ induces an isomorphism ${E'_K}^{00} \xrightarrow{\sim} {A'_K}^{00}$ between quasi-compact open subgroups which occur as descended forms of ${E'_L}^0$ and ${A'_L}^0$. In particular, the morphism $E' \longrightarrow A'$ between Néron models restricts to an isomorphism $E'^{00} \longrightarrow A'^{00}$ of open subgroups, which can be interpreted as the Néron models of ${E'_K}^{00}$ and ${A'_K}^{00}$. Furthermore, it follows that $E'^{00} \subset E'$ is the subgroup corresponding to the torsion subgroup of $\phi_{E'}$. Now, using an argument of base change, we see that $\underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},\mathfrak{R}})$, as a subgroup of A'^0, must be contained in

A'^{00}. Since the kernel of $E'^{00} \longrightarrow B'$ is precisely $T' \cap E'^{00}$, and since $\phi_{T'}$ is torsion-free, this kernel reduces to T'^{0}. From this it follows that the map $\underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},\mathfrak{R}})/T'^{0} \longrightarrow B'^{0}$ is injective, and we see that $\underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},\mathfrak{R}})$ must coincide with A'^{0}.

It remains to discuss the case where B_K, B'_K are trivial and, hence, E_K, E'_K coincide with the tori T_K, T'_K. From the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_K & \longrightarrow & T_K & \longrightarrow & A_K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_K^+ & \longrightarrow & T_K^+ & \longrightarrow & A_K & \longrightarrow & 0 \end{array}$$

we obtain an exact sequence

$$0 \longrightarrow T_K \longrightarrow T_K^+ \longrightarrow M_K^- \longrightarrow 0$$

and, using [8], 4.2, an exact sequence of Néron models

$$0 \longrightarrow T \longrightarrow T^+ \longrightarrow M^- \longrightarrow 0.$$

As $\underline{\mathrm{Ext}}^1(M^-, \mathbb{G}_{\mathrm{m},\mathfrak{R}})$ is trivial by 3.3, it follows that the canonical map

$$\underline{\mathrm{Ext}}^1(T^+, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) \longrightarrow \underline{\mathrm{Ext}}^1(T, \mathbb{G}_{\mathrm{m},\mathfrak{R}})$$

is injective. Thus, also in this case, the desired assertion follows. □

Remark 5.5. — The above proof, in particular, the second argument we have given for establishing 5.3 (ii), shows that the assertion of 5.3 extends to the case where the kernels of the maps $\phi_T \longrightarrow \phi_E$ and $\phi_{T'} \longrightarrow \phi_{E'}$ are as big as possible; i.e., contain the full torsion parts of ϕ_T, resp. $\phi_{T'}$. In other cases, a better knowledge of these kernels should make it possible to derive the assertion of 5.3 in more general situations.

6. Duality of the natural filtrations.

We continue to consider an abelian variety A_K over K and its dual A'_K, as well as the corresponding Néron models A, A' and their component groups ϕ_A, $\phi_{A'}$. As before, let

$$0 \longrightarrow M_K \longrightarrow E_K \longrightarrow A_K \longrightarrow 0, \qquad 0 \longrightarrow M'_K \longrightarrow E'_K \longrightarrow A'_K \longrightarrow 0$$

be the uniformizations of A_K, A'_K in the sense of [8], 1.2, and

$$0 \longrightarrow T_K \longrightarrow E_K \longrightarrow B_K \longrightarrow 0, \qquad 0 \longrightarrow T'_K \longrightarrow E'_K \longrightarrow B'_K \longrightarrow 0$$

the exact sequences exhibiting E_K, E'_K as semi-abelian group schemes. Also we will need the exact sequences

$$0 \longrightarrow M_K^r \longrightarrow M_K \longrightarrow M_{I,K} \longrightarrow 0,\ 0 \longrightarrow {M'_K}^r \longrightarrow M'_K \longrightarrow M'_{I,K} \longrightarrow 0,$$

where $M_{I,K}, M'_{I,K}$ are the biggest free quotients of M_K, M'_K which are fixed by the inertia subgroup I of the absolute Galois group of K. The groups $M_{I,K}, M'_{I,K}$ give rise to subtori $T_{I,K} \hookrightarrow T_K$ and $T'_{I,K} \hookrightarrow T'_K$, and we can consider the quotients $\tilde{E}_K = E_K/M_K^r$ and $\tilde{E}'_K = E'_K/{M'_K}^r$. Switching to Néron models and their component groups, we have then a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \phi_{T_I} & \longrightarrow & \phi_T & \longrightarrow & \phi_E & \longrightarrow & \phi_{\tilde{E}} & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & \longrightarrow & \phi_{T,\mathrm{tor}} & \longrightarrow & \phi_{E,\mathrm{tor}} & \longrightarrow & \phi_{\tilde{E},\mathrm{tor}} & \longrightarrow & \phi_{\tilde{E}}
\end{array}$$

where the vertical maps are inclusions of torsion subgroups. We may take now images in ϕ_A, which we indicate by a bar . As $\overline{\phi}_{\tilde{E}} = \phi_A$, see [8], 5.4, we thereby get the filtrations

$$\begin{array}{cccccccccccc}
\Sigma: & 0 & \longrightarrow & \overline{\phi}_{T_I} & \longrightarrow & \overline{\phi}_T & \longrightarrow & \overline{\phi}_E & \longrightarrow & \phi_A & & \\
 & & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\Theta: & & & 0 & \longrightarrow & \overline{\phi}_{T,\mathrm{tor}} & \longrightarrow & \overline{\phi}_{E,\mathrm{tor}} & \longrightarrow & \overline{\phi}_{\tilde{E},\mathrm{tor}} & \longrightarrow & \phi_A
\end{array}$$

of [8], section 5, which were called the Σ- and Θ-filtrations and which correspond to Lorenzini's filtrations [15] on prime-to-p parts (p the residue characteristic of K). Of course, there are corresponding $'$-filtrations for $\phi_{A'}$.

As in the previous section we require in the following that the residue field k is *perfect*, or that the tori T_K, T'_K have *multiplicative reduction.*

THEOREM 6.1. — *Assume that Grothendieck's pairings $\phi_A \times \phi_{A'} \longrightarrow \mathbb{Q}/\mathbb{Z}$ and $\phi_B \times \phi_{B'} \longrightarrow \mathbb{Q}/\mathbb{Z}$ are perfect. Then the Σ-filtration on ϕ_A is the orthogonal of the Θ-filtration on $\phi_{A'}$, and vice versa.*

We will split the proof of 6.1 into several pieces. As the dual of A'_K coincides with A_K, and the pairing is assumed to be perfect, it is enough to show the following assertions:

(6.1.1) $\overline{\phi}_{T'}$ is the orthogonal complement of $\overline{\phi}_{E,\mathrm{tor}}$.

(6.1.2) $\overline{\phi}_{T'_I}$ is the orthogonal complement of $\overline{\phi}_{\tilde{E},\mathrm{tor}}$.

(6.1.3) $\overline{\phi}_{T',\mathrm{tor}}$ is the orthogonal complement of $\overline{\phi}_E$.

Let us start with the verification of assertion (6.1.1). We choose an exact sequence

$$0 \longrightarrow M_K \longrightarrow M_K^+ \longrightarrow M_K^- \longrightarrow 0$$

as in 3.1 and define E_K^+ via the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_K & \longrightarrow & E_K & \longrightarrow & A_K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_K^+ & \longrightarrow & E_K^+ & \longrightarrow & A_K & \longrightarrow & 0 \end{array}$$

where the lower row is the push-out of the upper one via the injection $M_K \longrightarrow M_K^+$. Then, since $R^1 j_* M_K^+ = 0$, the lower row gives rise to an exact sequence of associated Néron models

$$0 \longrightarrow M^+ \longrightarrow E^+ \longrightarrow A \longrightarrow 0$$

which we will use in the following. From the exact sequence

$$0 \longrightarrow E_K \longrightarrow E_K^+ \longrightarrow M_K^- \longrightarrow 0$$

we get an exact sequence

$$0 \longrightarrow E \longrightarrow E^+ \longrightarrow M^< \longrightarrow 0,$$

where $M^<$ denotes the cokernel of the map between Néron models $E \longrightarrow E^+$; one can show that $i^* M^<$ is a subgroup of finite index in $i^* M^-$.

LEMMA 6.2. — *The group* $\underline{\mathrm{Ext}}^1(M^<, j_*\mathbb{G}_{\mathrm{m},K})$ *is trivial. Hence, the canonical map* $\underline{\mathrm{Ext}}^1(E^+, j_*\mathbb{G}_{\mathrm{m},K}) \longrightarrow \underline{\mathrm{Ext}}^1(E, j_*\mathbb{G}_{\mathrm{m},K})$ *is injective.*

Proof. — We show that the canonical map

$$\underline{\mathrm{Ext}}^1(M^-, j_*\mathbb{G}_{\mathrm{m},K}) \longrightarrow \underline{\mathrm{Ext}}^1(M^<, j_*\mathbb{G}_{\mathrm{m},K})$$

is an epimorphism. As $\underline{\mathrm{Ext}}^1(M^-, j_*\mathbb{G}_{\mathrm{m},K})$ is trivial by 3.3, this is enough. So consider a $j_*\mathbb{G}_{\mathrm{m},K}$-extension $H^<$ of $M^<$. Then, by 2.3, its rigid part gives rise to a $j_*\mathbb{G}_{\mathrm{m},K}$-extension H of M^- which prolongs $H^<$, and we are done. □

In order to verify assertion (6.1.1), let us look at the following commutative diagram whose maps we think to be restricted to formal parts and to the étale topology:

$$\begin{array}{ccccc} & & \underline{\mathrm{Ext}}^1(A, \mathbb{G}_{m,\mathfrak{R}}) & \longrightarrow & \underline{\mathrm{Ext}}^1(E, \mathbb{G}_{m,\mathfrak{R}}) \longrightarrow 0 \\ & & \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}(M, j_*\mathbb{G}_{m,K}) & \xrightarrow{\alpha} & \underline{\mathrm{Ext}}^1(A, j_*\mathbb{G}_{m,K}) & \xrightarrow{\beta} & \underline{\mathrm{Ext}}^1(E, j_*\mathbb{G}_{m,K}) \\ \downarrow & & \downarrow & & \downarrow \\ \phi_{T'} & \xrightarrow{\phi_\alpha} & \underline{\mathrm{Ext}}^1(\phi_A, \mathbb{Z}) & \xrightarrow{\phi_\beta} & \underline{\mathrm{Ext}}^1(\phi_E, \mathbb{Z}) \\ \downarrow & & \downarrow & & \\ 0 & & 0 & & \end{array}$$

All maps are the canonical ones, including α, which we have introduced in 5.4. The first row is exact due to 5.2. Using 5.4 and 6.2, we see from the exact sequence

$$\underline{\mathrm{Hom}}(M^+, j_*\mathbb{G}_{m,K}) \longrightarrow \underline{\mathrm{Ext}}^1(A, j_*\mathbb{G}_{m,K}) \longrightarrow \underline{\mathrm{Ext}}^1(E^+, j_*\mathbb{G}_{m,K})$$

that also the second row is exact. Furthermore, the columns are exact, and an easy diagram chase shows that also the third row is exact. But then assertion (6.1.1) is clear from the following statement:

LEMMA 6.3. — *$A_{E,\mathrm{tor}} \subset A$ be the open subgroup corresponding to the subgroup $\overline{\phi}_{E,\mathrm{tor}} \subset \phi_A$, and $E_{\mathrm{tor}} \subset E$ the open subgroup corresponding to the torsion subgroup $\phi_{E,\mathrm{tor}} \subset \phi_E$. Consider the commutative diagrams*

$$\begin{array}{ccc} A & \leftarrow & A_{E,\mathrm{tor}} \\ \uparrow & & \uparrow \\ E & \leftarrow & E_{\mathrm{tor}} \end{array} \qquad\qquad \begin{array}{ccc} \underline{\mathrm{Ext}}^1(\phi_A, \mathbb{Z}) & \xrightarrow{a} & \underline{\mathrm{Ext}}^1(\overline{\phi}_{E,\mathrm{tor}}, \mathbb{Z}) \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ \underline{\mathrm{Ext}}^1(\phi_E, \mathbb{Z}) & \xrightarrow{d} & \underline{\mathrm{Ext}}^1(\phi_{E,\mathrm{tor}}, \mathbb{Z}) \end{array}$$

where the right one is obtained from the left by applying $\underline{\mathrm{Ext}}^1(\cdot, i_*\mathbb{Z})$; use 2.5. Then c is an isomorphism, and d is a monomorphism. In particular, $\ker a = \ker b$.

Proof. — First, c is an isomorphism since the canonical map $\phi_{E,\mathrm{tor}} \longrightarrow \overline{\phi}_{E,\mathrm{tor}}$ is an isomorphism. To show that d is a monomorphism, it is enough to mention that $\phi_E/\phi_{E,\mathrm{tor}}$ is an étale k-group scheme, torsion-free and finitely generated. □

Next, let us concentrate on assertion (6.1.2). The exact sequence

$$0 \longrightarrow M_{I,K} \longrightarrow \tilde{E}_K \longrightarrow A_K \longrightarrow 0$$

gives rise to an exact sequence of associated Néron models

$$0 \longrightarrow M_I \longrightarrow \tilde{E} \longrightarrow A \longrightarrow 0,$$

due to the fact that $M_{I,K}$ is invariant under the inertia group I and, hence, that $R^1 j_* M_{I,K} = 0$ by [8], 4.4. Similarly as before, we consider the commutative diagram

$$\begin{array}{ccccc} \underline{\mathrm{Hom}}(M_I, j_*\mathbb{G}_{\mathrm{m},K}) & \xrightarrow{\tilde{\alpha}} & \underline{\mathrm{Ext}}^1(A, j_*\mathbb{G}_{\mathrm{m},K}) & \xrightarrow{\tilde{\beta}} & \underline{\mathrm{Ext}}^1(\tilde{E}, j_*\mathbb{G}_{\mathrm{m},K}) \\ \downarrow & & \downarrow & & \downarrow \\ \phi_{T'_I} & \xrightarrow{\phi_{\tilde{\alpha}}} & \underline{\mathrm{Ext}}^1(\phi_A, \mathbb{Z}) & \xrightarrow{\phi_{\tilde{\beta}}} & \underline{\mathrm{Ext}}^1(\phi_{\tilde{E}}, \mathbb{Z}) \end{array}$$

where the first row is the long $\underline{\mathrm{Ext}}$ sequence for $j_*\mathbb{G}_{\mathrm{m},K}$ associated to the above short exact sequence. Again, we can identify the map $\tilde{\alpha}$ with the canonical map $T'_I \longrightarrow A'$ so that we obtain the map $\phi_{\tilde{\alpha}}$ in the second row. In order to show that the second row is exact, we look at the exact sequence

$$\underline{\mathrm{Ext}}^1(A, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) \longrightarrow \underline{\mathrm{Ext}}^1(\tilde{E}, \mathbb{G}_{\mathrm{m},\mathfrak{R}}) \longrightarrow \underline{\mathrm{Ext}}^1(M_I, \mathbb{G}_{\mathrm{m},\mathfrak{R}}).$$

As $\underline{\mathrm{Ext}}^1(M_I, \mathbb{G}_{\mathrm{m},\mathfrak{R}})$ is trivial by 3.3, the left map is an epimorphism, and we can conclude that the lower row of the above diagram is exact. But then, similarly as before, the assertion of (6.1.2) follows from the following statement:

LEMMA 6.4. — *Let $A_{\tilde{E},\mathrm{tor}} \subset A$ be the open subgroup corresponding to the subgroup $\overline{\phi}_{\tilde{E},\mathrm{tor}} \subset \phi_A$, and $\tilde{E}_{\mathrm{tor}} \subset \tilde{E}$ the open subgroup corresponding to the torsion subgroup $\phi_{\tilde{E},\mathrm{tor}} \subset \phi_{\tilde{E}}$. Consider the commutative diagrams*

$$\begin{array}{ccc} A & \leftarrow & A_{\tilde{E},\mathrm{tor}} \\ \uparrow & & \uparrow \\ \tilde{E} & \leftarrow & \tilde{E}_{\mathrm{tor}} \end{array} \qquad \begin{array}{ccc} \underline{\mathrm{Ext}}^1(\phi_A, \mathbb{Z}) & \xrightarrow{a} & \underline{\mathrm{Ext}}^1(\overline{\phi}_{\tilde{E},\mathrm{tor}}, \mathbb{Z}) \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ \underline{\mathrm{Ext}}^1(\phi_{\tilde{E}}, \mathbb{Z}) & \xrightarrow[d]{} & \underline{\mathrm{Ext}}^1(\phi_{\tilde{E},\mathrm{tor}}, \mathbb{Z}) \end{array}$$

where the right one is obtained from the left by applying $\underline{\mathrm{Ext}}^1(\cdot, i_*\mathbb{Z})$; use 2.5. Then c is an isomorphism, and d is a monomorphism. In particular, $\ker a = \ker b$.

The *proof* is the same as the one of 6.3. □

It remains to verify assertion (6.1.3). To do this we introduce the biggest submodule $M_K^I \subset M_K$, which is fixed by the inertia group I, and consider the corresponding exact sequence

$$0 \longrightarrow M_K^I \longrightarrow M_K \longrightarrow \check{M}_K \longrightarrow 0,$$

as well as the associated exact sequence of tori

$$0 \longrightarrow \check{T}'_K \longrightarrow T'_K \longrightarrow {T'_K}^I \longrightarrow 0.$$

We claim:

Lemma 6.5. — *The map of component groups $\phi_{\check{T}'} \longrightarrow \phi_{T'}$ corresponding to the injection $\check{T}'_K \longrightarrow T'_K$ maps $\phi_{\check{T}'}$ surjectively onto $\phi_{T',\mathrm{tor}}$, the torsion part of $\phi_{T'}$.*

Proof. — The problem is trivial if T'_K has multiplicative reduction. So we may assume that k is perfect. First note that the component group $\phi_{\check{T}'}$ is a torsion group, since $\check{T}'$ must have a quasi-compact Néron model by [6], 10.2.1. There is an exact sequence of Néron models

$$0 \longrightarrow \check{T}' \longrightarrow T' \longrightarrow T'^I \longrightarrow 0$$

by [8], 4.2, and, hence, an exact sequence of component groups

$$\phi_{\check{T}'} \longrightarrow \phi_{T'} \longrightarrow \phi_{T'^I} \longrightarrow 0$$

by [8], 4.9. As $\phi_{T'^I} = \mathrm{Hom}(M_K^I, \mathbb{Z})$ by [22], 1.1, this group is torsion-free, and the assertion of 6.5 follows. □

Now, fix an exact sequence

$$0 \longrightarrow \check{M}_K \longrightarrow \check{M}_K^+ \longrightarrow \check{M}_K^- \longrightarrow 0$$

as in 3.1 with $H^1(I, \check{M}_K^+) = 0$ and $\check{M}_K^-$ invariant under I. Writing $\check{E}_K = E_K/M_K^I$, we can consider the exact sequence

$$0 \longrightarrow \check{M}_K \longrightarrow \check{E}_K \longrightarrow A_K \longrightarrow 0$$

and its push-out

$$0 \longrightarrow \check{M}_K^+ \longrightarrow \check{E}_K^+ \longrightarrow A_K \longrightarrow 0$$

with respect to $\check{M}_K \longrightarrow \check{M}_K^+$. As in the proof of (6.1.2), we get an exact sequence of Néron models

$$0 \longrightarrow \check{M}^+ \longrightarrow \check{E}^+ \longrightarrow A \longrightarrow 0$$

and an associated long <u>Ext</u> sequence, part of which occurs as the first row of the following commutative diagram:

$$\begin{array}{ccccc}
\underline{\mathrm{Hom}}(\check{M}^+, j_*\mathbb{G}_{\mathrm{m},K}) & \xrightarrow{\check{\alpha}^+} & \underline{\mathrm{Ext}}^1(A, j_*\mathbb{G}_{\mathrm{m},K}) & \xrightarrow{\check{\beta}^+} & \underline{\mathrm{Ext}}^1(\check{E}^+, j_*\mathbb{G}_{\mathrm{m},K}) \\
\downarrow \check{\varepsilon} & & \| & & \downarrow \check{\tau} \\
\underline{\mathrm{Hom}}(\check{M}, j_*\mathbb{G}_{\mathrm{m},K}) & \xrightarrow{\check{\alpha}} & \underline{\mathrm{Ext}}^1(A, j_*\mathbb{G}_{\mathrm{m},K}) & \xrightarrow{\check{\beta}} & \underline{\mathrm{Ext}}^1(\check{E}, j_*\mathbb{G}_{\mathrm{m},K}) \\
\downarrow & & \downarrow & & \downarrow \\
\phi_{T',\mathrm{tor}} & \xrightarrow{\phi_{\check{\alpha}}} & \underline{\mathrm{Ext}}^1(\phi_A, \mathbb{Z}) & \xrightarrow{\phi_{\check{\beta}}} & \underline{\mathrm{Ext}}^1(\phi_{\check{E}}, \mathbb{Z})
\end{array}$$

The left upper square exists by an argument as the one used in the proof of 5.4.

LEMMA 6.6. — (i) $\check{\varepsilon}$ *is an epimorphism,* $\check{\tau}$ *is a monomorphism.*

(ii) *The rows of the diagram are exact.*

Proof. — That $\check{\varepsilon}$ is an epimorphism follows as in the proof of 5.4. To see that $\check{\tau}$ is a monomorphism, we can proceed as in 6.2. Then, as the first row of the diagram is exact, the same is true for the middle one. Furthermore, by the usual argumentation, we can show that also the last row is exact, provided we know that the canonical map

$$A'^0 = \underline{\mathrm{Ext}}^1(A, \mathbb{G}_{m,\mathfrak{R}}) \longrightarrow \underline{\mathrm{Ext}}^1(\check{E}, \mathbb{G}_{m,\mathfrak{R}})$$

is an epimorphism on formal parts, when restricted to the étale topology. To verify this fact, we use 5.2 and look at the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T' \cap A'^0 & \longrightarrow & A'^0 & \longrightarrow & \underline{\mathrm{Ext}}^1(E, \mathbb{G}_{m,\mathfrak{R}}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \check{\tau} & & \downarrow & & \\ & & \underline{\mathrm{Hom}}(M^I, \mathbb{G}_{m,\mathfrak{R}}) & \longrightarrow & \underline{\mathrm{Ext}}^1(\check{E}, \mathbb{G}_{m,\mathfrak{R}}) & \longrightarrow & \underline{\mathrm{Ext}}^1(E, \mathbb{G}_{m,\mathfrak{R}}). & & \end{array}$$

As the component group of $T'^I = \underline{\mathrm{Hom}}(M^I, j_*\mathbb{G}_{m,K})$ does not have torsion, the epimorphism $T' \longrightarrow T'^I$ must map $T' \cap A'^0$ into the identity component T'^{I0}, so that the left vertical map really exists. Using [8], 4.8, we see that it is, in fact, an epimorphism. Then a diagram chase shows that the middle vertical map is an epimorphism, too. □

Now similarly as before, we can finish the proof of (6.1.3) by establishing the following assertion:

LEMMA 6.7. — *Let* $A_E \subset A$ *be the open subgroup corresponding to the subgroup* $\overline{\phi}_E \subset \phi_A$. *Then* $\check{E} \longrightarrow A$ *factors through* A_E. *Consider the commutative diagrams*

$$\begin{array}{ccc} A & \longleftarrow & A_E \\ \uparrow & & \uparrow \\ \check{E} & = & \check{E} \end{array} \qquad \begin{array}{ccc} \underline{\mathrm{Ext}}^1(\phi_A, \mathbb{Z}) & \xrightarrow{a} & \underline{\mathrm{Ext}}^1(\overline{\phi}_E, \mathbb{Z}) \\ b \downarrow & & \downarrow c \\ \underline{\mathrm{Ext}}^1(\phi_{\check{E}}, \mathbb{Z}) & = & \underline{\mathrm{Ext}}^1(\phi_{\check{E}}, \mathbb{Z}) \end{array}$$

where the right one is obtained from the left one by applying $\underline{\mathrm{Ext}}^1(\cdot, i_*\mathbb{Z})$; *use 2.5. Then c is an isomorphism. In particular,* $\ker a = \ker b$.

Proof. — As we have an exact sequence $0 \longrightarrow M \longrightarrow E \longrightarrow A$ and as $M^I \longrightarrow M$ is a formal isomorphism, also $\check{E} = E/M^I \longrightarrow E/M = A_E$ is a formal isomorphism. Thus, the induced map on component groups is an isomorphism, and it follows that c is an isomorphism. □

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