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XIANG-YU ZHOU

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ON INVARIANT DOMAINS IN CERTAIN COMPLEX HOMOGENEOUS SPACES

by Xiang-Yu ZHOU (*)

Given a Reinhardt domain D which is relatively compact in $(\mathbb{C}^*)^n$ one has the following classical results:

(1) the automorphism group $\mathrm{Aut}(D)$ of D is compact;

(2) the proper holomorphic self-map of D is biholomorphic (for a proof, see [3]).

G. Fels and L. Geatti recently in [7] generalized the result (1) to the complex symmetric spaces case. Their result says that: let K be a connected compact Lie group, L a closed subgroup of K such that (K, L) is a compact symmetric pair, let $D \subset K^{\mathbb{C}}/L^{\mathbb{C}}$ be a relatively compact K-invariant domain (here the action is given by the left translations), then $\mathrm{Aut}(D)$ is compact.

Throughout the present paper, a domain means a connected open set.

In the present paper, we obtain the same conclusion without assuming that (K, L) is a compact symmetric pair. Our first main result is the following:

THEOREM 1. — *Let K be a connected compact Lie group, L a closed subgroup of K, $D \subset\subset K^{\mathbb{C}}/L^{\mathbb{C}}$ a K-invariant domain, then* $\mathrm{Aut}(D)$ *is compact.*

In order to extend result (2), we get our second main result:

THEOREM 2. — *Let K, L, D be as in Theorem* 1, *$f : D \to D$ be a proper holomorphic mapping, assume that K is semisimple, then f is biholomorphic.*

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Our proof of these theorems is heavily relied on topology. We use some known facts from fibre bundles theory, CW-complexes, covering spaces, homotopy and homology theory. In fact, our proof is a mixture of parts of several complex variables, topology, and Lie groups theory.

1. Preliminaries.

In this section, we collect some notations and known facts which will be used later.

1.1. Let $\xi = (E, p, B, G)$ be a principal G-bundle in topological category, if every space is a smooth (respectively, complex) manifold, every mapping is smooth (respectively, holomorphic), G is a Lie group (respectively, complex Lie group), we call ξ a smooth (respectively, holomorphic) principal fiber G-bundle. Given a topological left G-space F, associated to the principal G-fibre bundle ξ, one has a topological fibre bundle $\xi[F] = (E \times_G F, p_F, B, F, G)$. If F is a smooth G-manifold (respectively, holomorphic G-manifold for a complex Lie group G) associated to a smooth (respectively, holomorphic) principal fibre G-bundle, one has a smooth (respectively, holomorphic) fibre bundle.

As in Hirzebruch's book [16], we use $H^1(B;G_c)$ (or $H^1(B;G_\delta)$, $H^1(B;G_\omega)$) to denote the set of the topological (or smooth, or holomorphic) isomorphism classes of topological (or smooth, or holomorphic) principal fibre G-bundles.

Note that, theses sets are also the respectively isomorphism classes of the respective fibre G-bundles with the given fibre F.

We also use $k_G(B)$ as in Husemoller's book [18] to denote the set of the topological isomorphism classes of topological principal fibre G-bundles over B, *i.e.*, $k_G(B) = H^1(B;G_c)$. It is known that if G is discrete and Abelian, G_c is a constant sheaf, and $H^1(B;G_c)$ is the usual singular cohomology $H^1(B;G)$.

When G is discrete, a fibre G-bundle is nothing but a covering space; a principal G-bundle is a normal covering space. However, the category of principal G-bundles is not the same as the category of normal covering spaces since the morphisms in the categories are different.

1.2. *Examples.* — Let G be a Lie group, H a closed subgroup of G (it is known that H is then a closed Lie subgroup), H_1 a closed subgroup of H, then $p: G/H_1 \to G/H$, $p(gH_1) = gH$, $g \in G$ is a smooth fibre bundle with fibre H/H_1, this fibre bundle has, as its smooth principal fibre bundle, the bundle $G/H_0 \to G/H$ where H_0 is the largest normal subgroup in H of H_1. As a special case, $G/L_0 \to G/L$ is a principal fibre L/L_0-bundle (*i.e.* a normal covering) when L_0 is the identity component of L.

If all groups involved are complex, then the bundles are holomorphic.

1.3. THEOREM (*cf.* Steenrod's book [31]). — *Let X be arcwise connected, arcwise locally connected, and semi locally 1-connected. Let G be a totally disconnected group, then the equivalent class of principal G-bundles over X are in one-one correspondence with the equivalent classes (under inner automorphisms of G) of homomorphisms of $\pi_1(X)$ into G, i.e., $k_G(X) = \mathrm{Hom}(\pi_1(X), G)/G$, where $(g \cdot \rho)(y) = g\rho(y)g^{-1}$, $g \in G$, $\rho \in \mathrm{Hom}(\pi_1(X), G)$, $y \in \pi_1(X)$.*

1.4 . It is known that over a smooth manifold B, one has

$$H^1(B;G_c) = H^1(B;G_\delta).$$

In general, over a complex manifold, $H^1(B;G_c) \neq H^1(B;G_\omega)$. However, a deep result of H. Grauert asserts that one has an equality over a Stein manifold.

GRAUERT'S OKA PRINCIPLE. — *Let B be a Stein manifold, then the map $H^1(B; G_\omega) \to H^1(B; G_c)$ induced by $G_\omega \hookrightarrow G_c$ is a bijection.*

1.5. Let K be a connected compact Lie group, $L \subset K$ be a closed subgroup. Denote by $K^{\mathbb{C}}$ and $L^{\mathbb{C}}$ the universal complexification of K and L. Then $K^{\mathbb{C}}$ acts holomorphically on $K^{\mathbb{C}}/L^{\mathbb{C}}$ by left translations.

A result of Matsushima-Onishchik asserts that $K^{\mathbb{C}}/L^{\mathbb{C}}$ is a Stein manifold.

1.6. Denote by $\mathfrak{k}$, $\mathfrak{l}$ the Lie algebras of K and L, by $\mathrm{Ad} : K \to \mathrm{GL}(\mathfrak{k})$ the adjoint representation, and by Ad_L its restriction to L. The representation of L is equivalent to the isotropy representation of L on the tangent space of K/L.

One can choose an Ad_L-invariant vector space $\mathfrak{p} \subset \mathfrak{k}$ such that $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{p}$.

MOSTOW'S DECOMPOSITION THEOREM. — *The map $K \times_L i\mathfrak{p} \to K^{\mathbb{C}}/L^{\mathbb{C}}$ given by $[k, v] \mapsto (k \cdot \exp(v))L^{\mathbb{C}}$ is a K-equivalent diffeomorphism.*

1.7. In this paragraph, we recall some concepts and results of P. Heinzner [11] and Zhou [34], retaining the notations in 1.5.

Let X be a $K^{\mathbb{C}}$-space, denote by b_z: $K^{\mathbb{C}} \to X$ the map $g \mapsto g \cdot z$ for a given $z \in X$. A K-set D in X is called orbit connected if $b_z^{-1}(D)$ is connected for all $z \in D$. A K-set D is called orbit convex if for each $x \in D$ and $v \in i\mathfrak{k}$ such that $\exp(v) \cdot x \in D$, it follows that $\exp(tv) \cdot x \in D$ for all $t \in [0, 1]$.

For a Stein K-domain in a holomorphical $K^{\mathbb{C}}$-space, orbit connectedness is the same as orbit convexity (for a simple proof, *cf.* [34]). A K-domain D in $K^{\mathbb{C}}/L^{\mathbb{C}}$ is orbit connected when L is connected, and a Stein K-domain $D \subset K^{\mathbb{C}}/L^{\mathbb{C}}$ is orbit convex when L is connected.

When (K, L) is a compact symmetric pair, any K-domain is orbit connected, and a K-domain is Stein if and only if it is orbit convex.

For an orbit convex K-domain $D \subset K^{\mathbb{C}}/L^{\mathbb{C}}$ and a strictly p.s.h. K-invariant function on D, if the minimum set of this function is not empty, then this set is nothing but a minimal K-orbit (*cf.* 4.2 in [11]).

1.8. *Example.* — In [11], P. Heinzner gave the following example: let U be a sufficiently small Stein open orbit-connected K-subset in $K^{\mathbb{C}}$ which contains K. One can choose a finite subgroup Γ in $K^{\mathbb{C}}$ such that $K \cap g\Gamma g^{-1} = 1$ for all $g \in U$. The image X of U in $K^{\mathbb{C}}/\Gamma$ is Stein but is not orbit-convex.

1.9. One can check the following useful fact: let φ: $K_1 \to K_2$ be a covering (or surjective) holomorphism between two compact Lie group, it induces a holomorphic covering (or surjective) holomorphism $\varphi^{\mathbb{C}}$: $K_1^{\mathbb{C}} \to K_2^{\mathbb{C}}$ such that the diagram

$$\begin{tikzcd} K_1 \arrow[r, "\varphi"] \arrow[d, "i_1"'] & K_2 \arrow[d, "i_2"] \\ K_1^{\mathbb{C}} \arrow[r, "\varphi^{\mathbb{C}}"] & K_2^{\mathbb{C}} \end{tikzcd}$$

is commutative.

Let f: $X \to Y$ be an equivariant homeomorphism with respect to $\varphi^{\mathbb{C}}$ between a $K_1^{\mathbb{C}}$-space X and a $K_2^{\mathbb{C}}$-space Y, then a K_1-set $D \subset X$ is orbit convex if and only if $f(D)$ is an orbit convex K_2-set, a K_2-set $D' \subset Y$ is orbit convex if and only if $f^{-1}(D')$ is an orbit convex K_1-set.

And if $Y = K_2^{\mathbb{C}}/L^{\mathbb{C}}$, then one has a natural biholomorphic equivariant map with respect to $\varphi^{\mathbb{C}} : K_1^{\mathbb{C}}/(\varphi^{\mathbb{C}})^{-1}(L^{\mathbb{C}}) \to K_2^{\mathbb{C}}/L^{\mathbb{C}}$.

2. Proof of Theorem 1.

2.1. We need the following:

H. Cartan's theorem. — *Let X be a Stein manifold, $D \subset X$ be a relatively compact Stein domain, let $f_v \in \mathrm{Aut}(D)$, $f_v \to f$ is uniformly convergent on compacta, then either $f \in \mathrm{Aut}(D)$ or $f(D) \subset \partial D$.*

When $X = \mathbb{C}^n$, the above theorem was stated and proved in Narasimhan's book [27]. However, the proof there is also suitable for proving this general version.

Since one can choose $\widetilde{D} \subset\subset X$ such that $D \subset\subset \widetilde{D}$, and then $\widetilde{D}$ is hyperbolic, by Montel's theorem in the general case, $\mathrm{Aut}(D) \subset \mathrm{Hol}(D, \widetilde{D})$ is relatively compact, *i.e.*, given any sequence $f_v \in \mathrm{Aut}(D)$, one can choose a convergent subsequence which is convergent uniformly on compacta to a holomorphic mapping $f : D \to \overline{D}$.

2.2. Lemma. — *Let X be a Stein manifold with $\dim_{\mathbb{C}} X = n$, $D \subset\subset X$ be a Stein domain, let $f_v \in \mathrm{Aut}(D)$ such that $f_v \to f$ is uniformly convergent on compacta to a holomorphic mapping from D to $\overline{D}$, if one can choose $D \subset\subset \widetilde{D} \subset\subset X$ such that $\imath : D \hookrightarrow \widetilde{D}$ induces a nontrivial homomorphism $\imath^* : H^n(\widetilde{D};\mathbb{R}) \to H^n(D;\mathbb{R})$ and if $0 < \dim_{\mathbb{R}} H_n(D;\mathbb{R}) < +\infty$, then $f \in \mathrm{Aut}(D)$, and then $\mathrm{Aut}(D)$ is compact.*

Proof. — If $f \notin \mathrm{Aut}(D)$, by Cartan's theorem, one has $f : D \to \partial D \subset \widetilde{D}$, so f is degenerate everywhere, and then for each holomorphic $(n,0)$-form ω on $\widetilde{D}$, $f^*\omega = 0$.

By Andreotti-Frankel's theorem, $H_n(D;\mathbb{Z})$ is torsion free. Combining with the assumption, $H_n(D;\mathbb{Z})$ is then a finitely generated free Abelian group.

Let $[\gamma_1], \ldots, [\gamma_k] \in H_n(D;\mathbb{Z}) \subset H_n(D;\mathbb{C})$ be a basis where the γ_i's are n-cycles, $[\gamma_1'], \ldots, [\gamma_\ell'] \in H_n(\widetilde{D};\mathbb{Z})$ a basis, $[\omega_1], \ldots, [\omega_\ell] \in H^n(\widetilde{D};\mathbb{C})$ be a dual basis where the ω_j's are closed n-forms, *i.e.*

$$\int_{\gamma_i'} \omega_j = \delta_{ij}.$$

By holomorphic version of de Rham's theorem for a Stein manifold, one can choose a holomorphic representative in $[\omega_j]$, so without loss of generality, we can assume that the ω_j's are holomorphic $(n,0)$-forms. Since

$$\imath_* \circ (f_v)_* : H_n(D;\mathbb{Z}) \longrightarrow H_n(\widetilde{D};\mathbb{Z})$$

is given by a nontrivial and integer valued $k \times \ell$ matrix (a_{ij}^v), one has

$$\sum_i \Big| \int_{\gamma_i} (\imath \circ f_v)^* \omega_j \Big| = \sum_i \Big| \int_{\imath_* \circ (f_v)_* \gamma_i} \omega_j \Big| = \sum_i |a_{ij}^v| \geq 1$$

for some j. Let $v \to \infty$, then $\sum_i \big| \int_{\gamma_i} f^* \omega_j \big| \geq 1$, but $\int_{\gamma_i} f^* \omega_j = 0$, a contradiction. □

Remark. — Since D is hyperbolic, $\mathrm{Aut}(D)$ is a Lie group which acts on D properly and real analytically.

2.3. Lemma. — *Let $D \subset\subset K^{\mathbb{C}}/L^{\mathbb{C}}$ be an orbit convex K-domain, $eL^{\mathbb{C}} \in D$, then $D \simeq K \times_L id$ (K-equivariant diffeomorphism), where $d = \{v \in \mathfrak{p} : \exp(iv) \cdot L^{\mathbb{C}} \in D\}$, and $0 \in d$, d is star-shaped with respect to the origin, and so $K \times_L 0 = K/L$ is a deformation retraction of D.*

This is an immediate consequence of Mostow's decomposition theorem.

Remark 1. — When (K, L) is a compact symmetric pair, without assuming $eL^{\mathbb{C}} \in D$ a result of Lassalle [22] asserts that d is convex.

Remark 2. — By Heinzner's example (*cf.* 1.8), in the above lemma, if D is only Stein but not orbit convex, then K/L and D maybe have not the same homotopy type. This shows that orbit convexity plays an important role in our proof.

2.4. Proposition. — *Let $D \subset\subset K^{\mathbb{C}}/L^{\mathbb{C}}$ be a Stein K-domain, L be connected, then* $\mathrm{Aut}(D)$ *is compact.*

Proof. — Let $gL^{\mathbb{C}} \in D$, $L_{g^{-1}} : K^{\mathbb{C}}/L^{\mathbb{C}} \to K^{\mathbb{C}}/L^{\mathbb{C}}$ be the left translation which is biholomorphic, so $eL^{\mathbb{C}} \in L_{g^{-1}}(D)$, and $L_{g^{-1}}(D)$ is Stein. Since L is connected, $L_{g^{-1}}(D)$ is orbit convex (*cf.* 1.7), and K/L is an oriented smooth compact manifold, so $H_n(K/L;\mathbb{Z}) = \mathbb{Z}$, and $H_n(L_{g^{-1}}(D);\mathbb{Z}) = H_n(K/L;\mathbb{Z})$ (by Lemma 2.3).

It is easy to find a K-domain $\widetilde{D}$ with $L_{g^{-1}}(D) \subset\subset \widetilde{D} \subset\subset K^{\mathbb{C}}/L^{\mathbb{C}}$ such that the inclusion $\imath: L_{g^{-1}}(D) \to \widetilde{D}$ is homotopic equivariant (using Lemma 2.3).

By Lemma 2.2, $\mathrm{Aut}(L_g^{-1}(D))$ is compact, $\mathrm{Aut}(D)$ is then also compact. □

2.5. Lemma. — *Let K, L, D be as in Proposition* 2.4, *then D contains a minimal K-orbit, and* $\mathrm{Aut}(D)$ *fixes this minimal K-orbit.*

This is an immediate consequence of the above proposition and a result of P. Heinzner (*cf.* 1.7). In fact, one can construct an $\mathrm{Aut}(D)$-invariant strictly p.s.h. exhaustion function on D since $\mathrm{Aut}(D)$ is compact, then Heinzner's result says that the minimal set of this function is a minimal K-orbit. It is easy to see that $g \cdot M \subset M$ for $g \in \mathrm{Aut}(D)$, so $\mathrm{Aut}(D) \cdot M = M$.

2.6. Proposition. — *Let K, L, D be as in Theorem* 1, *assume further that D is Stein and orbit convex, $eL^{\mathbb{C}} \in D$, then* $\mathrm{Aut}(D)$ *is compact.*

Proof. — Let L_0 be the identity component of L, so L/L_0 is a finite group.

Note that $(L_0)^{\mathbb{C}} = (L^{\mathbb{C}})_0$, and $L^{\mathbb{C}}/L_0^{\mathbb{C}} = L/L_0$, so $p: K^{\mathbb{C}}/L_0^{\mathbb{C}} \to K^{\mathbb{C}}/L^{\mathbb{C}}$ is a holomorphic principal L/L_0-bundle(*i.e.*, a finite normal covering), $p: p^{-1}(D) \to D$ is also a finite normal covering.

By K. Stein's theorem, a covering of a Stein space is still Stein, so $p^{-1}(D)$ is Stein.

Since D is orbit convex, $p^{-1}(D)$ is arcwise connected.

By Proposition 2.4 and Lemma 2.5, $\mathrm{Aut}(p^{-1}(D))$ is compact and fixes a minimal K-orbit M.

By Grauert's Oka principle, $k_{L/L_0}(D) = H^1(D;(L/L_0)_c) = H^1(D;(L/L_0)_\omega)$.

By Lemma 2.3, D and K/L are homotopic equivalent, so $k_{L/L_0}(D) = k_{L/L_0}(K/L)$.

We claim that $k_{L/L_0}(K/L)$ is a finite set, and postpone our proof to the next lemma.

Denote by the mapping $\rho: \mathrm{Aut}(D) \longrightarrow \mathrm{Per}(H^1(D;(L/L_0)_\omega))$, $\rho(g) = g^*$, where $\mathrm{Per}(H^1(D;(L/L_0)_\omega))$ is the permutation group (a finite group) of the set $H^1(D;(L/L_0)_\omega)$.

Let H be the image of $\mathrm{Aut}(D)$ under ρ, $G :=\rho^{-1}(H_{[p^{-1}(D)]})$ where $[p^{-1}(D)] \in H^1(D;(L/L_0)_\omega)$.

In other words,

$$\begin{aligned} G &= \{g \in \mathrm{Aut}(D) : [g^*(p^{-1}(D))] = [p^{-1}(D)]\} \\ &= \{g \in \mathrm{Aut}(D) : g^*(p^{-1}(D)) \\ &\qquad \text{and } p^{-1}(D) \text{ are holomorphic isomorphic principal} \\ &\qquad\qquad \text{fiber } L/L_0\text{-bundles over } D\}, \end{aligned}$$

so there exist biholomorphic mappings $\tilde{g}_2, \tilde{g}_1$ such that the following diagram

$$\begin{tikzcd} p^{-1}(D) \arrow[r, "\tilde{g}_2"] \arrow[dr, "p"'] & g^*(p^{-1}(D)) \arrow[r, "\tilde{g}_1"] \arrow[d, "p_1"] & p^{-1}(D) \arrow[d, "p"] \\ & D \arrow[r, "g"] & D \end{tikzcd}$$

is commutative.

And then for all $g \in G$, there exists $\tilde{g} \in \mathrm{Aut}(p^{-1}(D))$ such that $p \circ \tilde{g} = g \circ p$.

Since $\tilde{g} \cdot M \subset M$, $g \cdot (p(M)) \subset p(M), G \cdot p(M) = p(M)$.

Since D is hyperbolic, $\mathrm{Aut}(D)$ acts on D properly, G is then compact.

Note that $\mathrm{Aut}(D)/G = H/H_{[p^{-1}(D)]}$ is finite, so $\mathrm{Aut}(D)$ is compact. □

2.7. Lemma. — *Let M be a connected finite CW-complex, F be a finite group, then $k_F(M)$ is a finite set.*

Proof. — By Theorem 1.3, one has $k_F(M) = \mathrm{Hom}(\pi_1(M), F)/F$.

Since the fundamental group of a finite CW-complex is a finitely presented free group, we can let $\pi_1(M) = P/N$, where P is a finitely generated free group, N is a normal subgroup of P.

Let P is generated by $x_1, \ldots, x_n$, $F = \{y_i : 1 \leq i \leq m\}$, $\mathrm{Hom}(P, F)$ is a finite set (its number is not greater than the number of all maps from $\{x_i : 1 \leq i \leq n\}$ to $\{y_j : 1 \leq j \leq m\}$, the latter number is m^n).

Considering the map $\mathrm{Hom}(P/N, F) \to \mathrm{Hom}(P, F)$ given by $f \mapsto f \circ \pi$, where $\pi : P \to P/N$ natural projection, it is easy to see that this map is injective, so $\mathrm{Hom}(P/N, F)$ is a finite set, and then $k_F(M)$ is a finite set. □

2.8. — Now we are ready to prove Theorem 1.

Proof of Theorem 1. — By a result of P. Heinzner (*cf.* 4.1 in [11]), the K-complexification $D^{\mathbb{C}}$ of D exists and is Stein $K^{\mathbb{C}}$-homogeneous, the corresponding map $\ell : D \to D^{\mathbb{C}}$ is K-equivariant open embedding such that $eL_1^{\mathbb{C}} \in \ell(D)$ and there is a commutative diagram

$$\begin{tikzcd} D \arrow[r, "\ell"] \arrow[dr, hook, "i"'] & D^{\mathbb{C}} \arrow[d, "\pi"] \\ & K^{\mathbb{C}}/L^{\mathbb{C}} \end{tikzcd}$$

where i is the inclusion map, π is a $K^{\mathbb{C}}$-equivariant holomorphic covering. In particular, $D^{\mathbb{C}} = K^{\mathbb{C}}/L_1^{\mathbb{C}}$ for some group L_1 with $L_0 \subset L_1 \subset L$.

Since $(D^{\mathbb{C}}, \ell(D))$ is a Runge pair (*cf.* Theorem 3.4 in [11], also [36]), the envelope of holomorphy $E(\ell(D))$ is univalent in $D^{\mathbb{C}}$, and $E(\ell(D))$ is orbit convex (*cf. ibid.*).

By Proposition 2.6, one can deduce that $\mathrm{Aut}(E(\ell(D)))$ is compact. Since $\mathrm{Aut}(D) = \mathrm{Aut}(\ell(D)) \subset \mathrm{Aut}(E(\ell(D)))$ is a closed subgroup of $\mathrm{Aut}(E(\ell(D)))$, so $\mathrm{Aut}(D)$ is compact.

Remark. — As argued in Lemma 2.5, using our theorem and the Heinzner's result (*cf.* 1.7), one can prove the following:

THEOREM. — *Let $D \subset\subset K^{\mathbb{C}}/L^{\mathbb{C}}$ be an orbit convex and Stein K-domain, then D contains a minimal K-orbit and* $\mathrm{Aut}(D)$ *fixes this minimal K-orbit.*

3. Proof of Theorem 2.

3.1. LEMMA. — *Let Y be a connected countable CW-complex, X a topological space, $f_v, f \in C^0(X,Y)$ (the space of all continuous mappings from X to Y with the compact-open topology), if $f_v \to f$ in the C-O topology, then given $[\gamma] \in \pi_i(X)$, $i \geq 1$, one has $(f_v)_*[\gamma] = f_*[\gamma]$ for v sufficiently large (maybe dependent on $[\gamma]$). If $\pi_i(X)$ is finitely generated, then $(f_v)_* = f_*$ on $\pi_i(X)$ for v sufficiently large. If X has a homotopy type of a compact metric space, then f_v, f are homotopic equivalent for v large enough.*

Proof. — This lemma is essentially a consequence of the following two facts:

1) Milnor's theorem (see [24]): If Y is a countable CW-complex, C is a compact metric space, then $C^0(C, Y)$ with respect to C-O topology is also a countable CW-complex.

2) A CW-complex is always locally contractible and then locally arcwise connected.

Considering $f_v \circ \gamma, f \circ \gamma \in C^0(S^i, Y)$, $f_v \circ \gamma \to f \circ \gamma$ in C-O topology. When v is sufficiently large, $f_v \circ \gamma$ is contained in a contractible neighborhood of $f \circ \gamma$, and therefore $f_v \circ \gamma$ and $f \circ \gamma$ are homotopic equivalent, so $f_*[\gamma] = (f_v)_*[\gamma]$.

If $\pi_i(X)$ is finitely generated, one can do the same for these generators, then $(f_v)_* = f_*$

Let g denote by the homotopic equivalence between X and a compact metric space C, one has $f_v \circ g \to f \circ g \in C^0(C, Y)$, by the above two facts again, one can deduce that $f_v \circ g$ and $f \circ g$ are homotopic equivalent (for v large enough), and therefore f_v and f are homotopic equivalent when v is large enough. □

3.2. *Remark.* — Let X be a Stein manifold, $D \subset\subset X$ be a domain such that $\pi_1(D)$ is finitely generated and $i : D \to \widetilde{D}$ (where $D \subset\subset \widetilde{D} \subset\subset X$) induces an isomorphism $i_* : \pi_1(D) \to \pi_1(\widetilde{D})$, if $f : D \to D$ is an unbranched proper holomorphic mapping, then f is biholomorphic.

Proof. — Denote by $f^{(v)} = f \circ \cdots \circ f$ the v times iteration of f.

Since $\widetilde{D}$ is hyperbolic, by Montel's theorem, one can choose a convergent subsequence $f^{(v_k)} \to g : D \to \overline{D} \subset \widetilde{D}$.

By the above lemma, $(i \circ f^{(v_k)})_* = g_*$ on $\pi_1(D)$.

Since f is an unbranched covering, f_* is injective.

On the other hand, by assumption, i_* is isomorphic, so $(f^{(v_k)})_* = (f^{(v_{k+1})})_*$, therefore $(f^{(v_{k+1}-v_k)})_* = \mathrm{id}$.

We then deduce that f_* is surjective, and therefore f is bijective. □

3.3. Lemma. — *Let X be a Stein manifold, $D \subset\subset X$ be a domain in X, $f : D \to D$ be a holomorphic surjective mapping. Suppose that $f^{(v_i)} \to F : D \to \overline{D} \subset \widetilde{D} \subset\subset X$ then either F is degenerate everywhere as* *a holomorphic mapping from D to $\widetilde{D}$; or F is biholomorphic and f is also biholomorphic.*

Proof. — Suppose that F is not degenerate at some point $x \in D$, F is then open at $x \in D$.

Choose a neighborhood U_x of x such that $U_x \subset\subset D$ and $F(U_x) \subset\subset D$.

If necessary, passing to a subsequence, we can assume that $f^{(v_{i+1}-v_i)} \to G$ in C-O topology, where G is a holomorphic mapping from D to $\overline{D}$. Passing to limit, one has $G \circ F(x) = F(x)$ on U_x.

Since $F(U_x)$ is open, by $G_{|F(U_x)} = \mathrm{id}$, and D is connected, one has $G = \mathrm{id}$ on D.

We claim that f is bijective by $G = \mathrm{id}$ on D.

On the one hand, let $x_1 \neq x_2 \in D, f(x_1) = f(x_2)$, then $f^{(v_{i+1}-v_i)}(x_1) = f^{(v_{i+1}-v_i)}(x_2)$, passing to the limit, $G(x_1) = G(x_2)$, *i.e.*, $x_1 = x_2$. So f is injective.

Now f is bijective holomorphic, it is known that, f is then biholomorphic, and by Cartan's theorem (*cf.*2.1), F is biholomorphic.

□

Remark. — This is a slight modification of a proposition in Mok [25].

3.4. PROPOSITION. — *Let X be a Stein manifold with $\dim_{\mathbb{C}} X = n$, $D \subset\subset X$ be a Stein domain in X. If there exists a compact oriented n-manifold which is of the same homotopy type of D, and there exists a domain $\widetilde{D}$ with $D \subset\subset \widetilde{D} \subset\subset X$ such that the inclusion $i : D \to \widetilde{D}$ induces an isomorphism $i^* : H^n(\widetilde{D};\mathbb{C}) \to H^n(D;\mathbb{C})$, and if $f : D \to D$ is a proper holomorphic mapping, then f is biholomorphic.*

Proof. — Let $f^{(v_k)} \to F$, where F is a holomorphic mapping from D to $\overline{D} \subset \widetilde{D}$. If F is degenerate everywhere, then for every holomorphic n-form ω on $\widetilde{D}$, $F^*\omega = 0$. By holomorphic version of de Rham's theorem over a Stein manifold, one has $F^* = 0$, where $F^* : H^n(\widetilde{D};\mathbb{C}) \to H^n(D;\mathbb{C})$. By Lemma 3.1, $(i \circ f^{(v_k)})^* = F^*$. By assumption, one then deduces that

$$(*) \qquad \left(f^{(v_k)}\right)^* = 0.$$

On the other hand, let $[\omega]$ be the generator of $H^n(D;\mathbb{C})$ which is a nontrivial finite dimensional vector space, by Poincaré's duality theorem, one can find an element $[\eta]$ in $H_c^n(D;\mathbb{C})$, where η is a differential n-form with compact support of D, so that $[\omega \wedge \eta]$ is a generator of $H_c^{2n}(D;\mathbb{C})$.

By definition of degree of a proper mapping,

$$f^*[\omega \wedge \eta] = \deg f[\omega \wedge \eta].$$

Combining with (*), one has $\deg f = 0$. But $\deg f$ is always greater than 0 for a proper holomorphic mapping from a Stein manifold to another Stein manifold. This contradiction shows that, by Lemma 3.3, f is biholomorphic. □

3.5. COROLLARY. — *Let $D \subset\subset K^{\mathbb{C}}/L^{\mathbb{C}}$ be a Stein K-domain, L be connected. If $f : D \to D$ is a proper holomorphic mapping, then f is biholomorphic.*

Proof. — As argued in Proposition 2.4, we can assume that $eL^{\mathbb{C}} \in D$. Since L is connected, K/L is oriented and D is orbit convex. By Lemma 2.3 and Proposition 3.4, we get the corollary. □

3.6. PROPOSITION. — *Let X be a Stein manifold with $\dim_{\mathbb{C}} X = n$, $D \subset\subset X$ be a Stein domain. If D has the same homotopy type of a compact manifold such that $H^p(D;\mathbb{R})$ is nontrivial, and there exists a domain $\widetilde{D}$ with $D \subset\subset \widetilde{D} \subset\subset X$ such that $i : D \to \widetilde{D}$ induces an isomorphism $H^p(\widetilde{D};\mathbb{R}) \to H^p(D;\mathbb{R})$, and if $f : D \to D$ is a proper holomorphic mapping, then $f^* : H^p(D;\mathbb{R}) \to H^p(D;\mathbb{R})$ is isomorphic, in fact, there exists $\ell > 0$ such that $(f^\ell)^* = \mathrm{id}$.*

3.7. PROPOSITION. — *Let $D \subset\subset K^{\mathbb{C}}/L^{\mathbb{C}}$ be an orbit convex and Stein K-domain, $eL^{\mathbb{C}} \in D$, K be semisimple, if $f : D \to D$ proper holomorphic mapping, then f is biholomorphic.*

Proof. — Since K is semisimple and compact, then the universal covering group $\widetilde{K}$ is still semisimple and compact.

It is known that $(\widetilde{K})^{\mathbb{C}}$ is the universal covering group of $K^{\mathbb{C}}$ and transitively act on $K^{\mathbb{C}}/L^{\mathbb{C}}$. Then we can let $K^{\mathbb{C}}/L^{\mathbb{C}} = \widetilde{K}^{\mathbb{C}}/\widetilde{L}^{\mathbb{C}}$ and regard D as an orbit convex and Stein $\widetilde{K}$-domain with $e\widetilde{L}^{\mathbb{C}} \in D$.

Denote $\widetilde{L}_0^{\mathbb{C}}$ by the identity component of $\widetilde{L}^{\mathbb{C}}$.

Let $p : \widetilde{K}^{\mathbb{C}}/\widetilde{L}_0^{\mathbb{C}} \to \widetilde{K}^{\mathbb{C}}/\widetilde{L}^{\mathbb{C}}$ be the natural projection which is a principal bundle. It is known that $\widetilde{K}^{\mathbb{C}}/\widetilde{L}_0^{\mathbb{C}}$ is simply connected, p is a universal covering.

By the lifting existence theorem, there exists a holomorphic mapping $\tilde{f}$ from $p^{-1}(D)$ to $p^{-1}(D)$ such that the diagram

$$\begin{tikzcd} p^{-1}(D) \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & p^{-1}(D) \arrow[d, "p"] \\ D \arrow[r, "f"] & D \end{tikzcd}$$

is commutative.

One can check that $\tilde{f}$ is proper, in fact, given a compact set C in $p^{-1}(D)$, we have $(\tilde{f})^{-1}(C) \subset \tilde{f}^{-1} \circ p^{-1} \circ p(C) = p^{-1} \circ f^{-1} \circ p(C)$ is compact because p is finite and f is proper.

It is known that $p^{-1}(D)$ is Stein and connected since D is Stein and orbit convex.

Using Corollary 3.5, we get that $\tilde{f}$ is biholomorphic onto. Then $\tilde{f}$ maps $p^{-1}(b)$ one-one onto $p^{-1}(f(b))$ for each $b \in D$, any other point not in $p^{-1}(b)$ is not mapped into $p^{-1}(f(b))$ under $\tilde{f}$, we deduce that f is injective. f is surjective since f is proper holomorphic. So f is biholomorphic. □

3.8. — Now we are ready to prove Theorem 2.

Proof of Theorem 2. — Given a K-domain $D \subset\subset K^{\mathbb{C}}/L^{\mathbb{C}}$, by a result of Heinzner (see [11]), one has a Stein $K^{\mathbb{C}}$-manifold $K^{\mathbb{C}}/L_1^{\mathbb{C}}$ with $L_0 \subset L_1 \subset L$ (where L_0 is the identity component of L), which together with an open embedding $\imath : D \to K^{\mathbb{C}}/L_1^{\mathbb{C}}$ is a K-complexification of D, and $eL_1^{\mathbb{C}} \in \imath(D)$.

Since $(K^{\mathbb{C}}/L_1^{\mathbb{C}}, \imath(D))$ is a Runge pair (*cf.* Thm 3.4 in [11]), the envelope of holomorphy $E(\imath(D))$ is univalent in $K^{\mathbb{C}}/L_1^{\mathbb{C}}$, and $E(\imath(D))$ is then orbit convex.

It is known that a proper holomorphic mapping $\imath \circ f \circ \imath^{-1} : \imath(D) \to \imath(D)$ can be extended to a proper holomorphic mapping $\tilde{f} : E(\imath(D)) \to E(\imath(D))$.

By Proposition 3.7, one has $\tilde{f}$ is biholomorphic and then f is also biholomorphic. Theorem 2 is proved.

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Xiang–Yu ZHOU,
Academia Sinica
Institute of Mathematics
Beijing (P. R. China).
xyzhou@math08.math.oc.cn