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## ON A VARIANT OF KAZHDAN'S PROPERTY (T) FOR SUBGROUPS OF SEMISIMPLE GROUPS

by M. B. BEKKA and N. LOUVET

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### 1. Introduction and statement of the results.

Lubotzky and Zimmer considered in [LuZ] the following variant of Kazhdan's property (T) for a locally compact group  $G$ . Let  $\mathcal{R}$  be a set of equivalence classes of unitary representations of  $G$  containing the trivial one dimensional representation  $1_G$ . Throughout this paper, all group representations will be assumed to be unitary and strongly continuous representations in non zero Hilbert spaces. The group  $G$  is said to have property (T;  $\mathcal{R}$ ) if  $1_G$  is isolated in  $\mathcal{R}$ . For  $\mathcal{R} = \widehat{G}$ , the unitary dual of  $G$ , this is just the celebrated Kazhdan's property (T) (see [HaV] for an account on this property).

There are interesting examples of (discrete) groups that are not Kazhdan groups and that satisfy property (T;  $\mathcal{R}$ ) for some natural classes of representations  $\mathcal{R}$ . For instance,  $SL(2, \mathbb{Z})$  has property (T;  $\mathcal{R}$ ) for the set  $\mathcal{R}$  of all irreducible representations of  $SL(2, \mathbb{Z})$  that factorize through a quotient by a congruence subgroup of  $SL(2, \mathbb{Z})$ . Another interesting example is the group  $SL(2, \mathbb{Z}[1/p])$  which has the property (T;  $\mathcal{R}$ ),  $\mathcal{R}$  being the set of all finite dimensional representations of  $SL(2, \mathbb{Z}[1/p])$  (see [LuZ]). Such isolation properties allow for instance the construction of expanders graphs (see [Lub]).

In [LuZ], other examples were discovered. They occur as follows. Let  $\Gamma$  be a lattice in the direct product  $G_1 \times G_2$  of two groups. Assume that  $\Gamma$  projects densely on  $G_1$  and that  $G_2$  has property (T). Then  $\Gamma$  has (T;  $\mathcal{R}$ )

for the set  $\mathcal{R}$  of all finite dimensional representations of  $\Gamma$  (see below for some concrete examples of such  $\Gamma$ ).

In this paper, we shall generalize the results in [LuZ]. It is interesting to note that our methods are elementary.

We fix, once and for all, a locally compact group  $G$  and we let  $H$  be a closed (not necessarily discrete) subgroup of  $G$  and  $N$  a closed normal subgroup of  $G$ . Throughout this paper, we shall make the following two assumptions:

- (I)  $HN$  is dense in  $G$ , and
- (II) the homogeneous space  $G/H$  has a finite invariant measure.

Denoting by  $p$  the canonical projection  $G \rightarrow G/N$ , assumption (I) says that  $p(H)$  is dense in  $G/N$ . If  $\pi$  is an irreducible representation of  $G/N$ , it is clear that  $(\pi \circ p)|_H$  is an irreducible representation of  $H$  and that  $\pi$  is determined (up to unitary equivalence) by  $(\pi \circ p)|_H$ . So, we may view  $\widehat{G/N}$  as a subset of  $\widehat{H}$ . Our main result states that if an irreducible representation  $\pi$  of  $H$  is sufficiently close to the trivial representation  $1_H$  and if  $N$  has property (T), then  $\pi$  is in  $\widehat{G/N}$ .

More precisely, the following holds:

**THEOREM A.** — *Let  $G$ ,  $H$ , and  $N$  satisfy the above assumptions (I) and (II). Assume moreover that  $N$  has Kazhdan's property (T). Let  $\mathcal{U}$  be a neighbourhood of  $1_{G/N}$  in  $\widehat{G/N}$ . Then the set of all  $(\pi \circ p)|_H$ ,  $\pi \in \mathcal{U}$ , is a neighbourhood of  $1_H$  in  $\widehat{H}$ .*

In other words,  $H$  has property (T;  $\mathcal{R}$ ) with  $\mathcal{R}$  the set of all irreducible representations of  $H$  that do not factorize through  $G/N$  together with  $1_H$ . Of course, in case  $G$  has Kazhdan's property (T), then taking  $N = G$  shows that Theorem A is a generalization of the fact that property (T) is inherited by subgroups of cofinite volume. (see [HaV, Chapter 3, Theorem 4]).

Theorem A is a rigidity result in the following sense. It says that any irreducible representation of  $H$  that is sufficiently close to the trivial representation  $1_H$  extends to a representation of  $G$ . One may speculate whether this is always true for irreducible lattices in a (nontrivial) product of, say, simple Lie groups. Recall that the only simple Lie groups without property (T) are the ones that are locally isomorphic to  $SO(n, 1)$  or  $SU(n, 1)$ .

Let  $FD$  denote the subset of  $\widehat{H}$  consisting of the finite dimensional representations. Recall that a group  $G$  is called minimally almost periodic if  $1_G$  is the unique finite dimensional unitary irreducible representation of  $G$ . An immediate consequence of Theorem A is the following corollary, also noticed in [LuZ].

**COROLLARY.** — *If  $G/N$  is minimally almost periodic then  $H$  has property  $(T, FD)$ .*

Our Theorem A improves and makes more precise Theorem 2.2 in [LuZ] where the following was shown. Under the additional assumption that  $G$  is a direct product  $M \times N$ ,  $H$  has property  $(T; \mathcal{R})$  where  $\mathcal{R}$  is the set of all  $\pi \in \widehat{H}$  such that, for all  $k \geq 1$ , no nontrivial subrepresentation of the symmetric power  $S^k(\pi)$  factorize through  $p : H \rightarrow M$ . Of course, this is sufficient in order to deduce the corollary above.

Our proof of Theorem A is elementary. It is based on the following extension result which may be of independent interest.

**LEMMA 1.** — *Let  $G$ ,  $H$ , and  $N$  satisfy the above assumptions (I) and (II). Let  $\pi$  be a unitary representation of  $H$ . Then the following are equivalent:*

(i)  $\pi$  contains a subrepresentation that factorizes through the canonical projection  $p : G \rightarrow G/N$  i.e. there exists a representation  $\sigma$  of  $G/N$  so that  $\pi$  contains  $(\sigma \circ p)|_H$ .

(ii) The induced representation  $\text{Ind}_H^G \pi$  contains a non zero  $N$ -invariant vector.

For a representation  $\pi$  of a locally compact group  $H$ , let  $H^1(H, \pi)$  be the first cohomology group of  $H$  with coefficients in  $\pi$  (see [Gu1, Gu2]). It is well known that  $H^1(H, \pi) = 0$  for any  $\pi$  if (and only if) the group  $H$  has Kazhdan's property (T) (see [HaV, Chapter 4, Theorem 7]).

Vershik and Karpushev [VeK, Theorem 2] showed that if  $H^1(H, \pi) \neq 0$  for some irreducible representation  $\pi$ , then  $\pi$  is infinitesimally small, that is, there exists a net  $\pi_n$  in  $\widehat{H}$  such that  $\lim \pi_n = 1_H$  and  $\lim \pi_n = \pi$ . This result has been conjectured by Guichardet [Gu2] and some partial results were previously obtained by Delorme [Del].

Using Theorem A and Vershik and Karpushev's result, we give an elementary proof of the following theorem, proved in [LuZ, Theorem 3.1] in the product case  $G = M \times N$  (see also [BoW]).

**THEOREM B.** — *Let  $G, H$  and  $N$  be as in Theorem A and assume  $G/N$  is minimally almost periodic. Then, for any finite dimensional irreducible unitary representation  $\pi$  of  $H$ , we have*

$$H^1(H, \pi) = 0.$$

Here are some examples of groups  $H$  to which the above results apply. It seems that the only interesting examples occur when  $G$  is locally isomorphic to a product  $M \times N$  and  $H$  is a lattice in  $G$ .

- (1) As it is well known,  $H = SL(n, \mathbb{Q})$  is, via diagonal embedding, a lattice in

$$SL(n, \mathbb{A}) = SL(n, \mathbb{R}) \times SL(n, \mathbb{A}_f),$$

where  $\mathbb{A}$  is the ring of adèles of  $\mathbb{Q}$  and  $\mathbb{A}_f$  the subring of finite adèles. By the Strong Approximation Theorem (see, e. g., [Hum], 14.3),  $SL(n, \mathbb{Q})$  is dense in  $SL(n, \mathbb{A}_f)$ . The dual space of  $SL(n, \mathbb{A})$  is, as a topological space, a restricted product of the dual spaces of the factors  $SL(n, \mathbb{Q}_p)$ ,  $p \in P = \{\text{primes in } \mathbb{N}\} \cup \{\infty\}$  (see [Gu2, Corollary 11]). When  $n \geq 3$ ,  $SL(n, \mathbb{R})$  has property (T) and Theorem A gives a neat description of the topology in the neighborhood of  $1_{SL(n, \mathbb{Q})}$ .

The above can be generalized to  $H = \mathbf{G}(\mathbb{Q})$ , the rational points of a connected simple algebraic group  $\mathbf{G}$  defined over  $\mathbb{Q}$ . The group  $\mathbf{G}(\mathbb{Q})$  is a lattice in

$$\mathbf{G}(\mathbb{A}) = \mathbf{G}(\mathbb{R}) \times \mathbf{G}(\mathbb{A}_f),$$

and when  $\mathbf{G}$  is simply connected and  $\mathbf{G}(\mathbb{R})$  is non compact, it projects densely into  $\mathbf{G}(\mathbb{A}_f)$  (see [Bor, 5.6.Theorem] and [Mar, Chap. II, (6.8) Theorem]).

- (2) Let  $\mathbf{G} = SO(q)$  be the subgroup of  $SL(n+1)$  preserving the quadratic form  $q(x) = \sum_{i=1}^n x_i^2 - x_{n+1}^2$ . For  $n \geq 2$ , the group  $\mathbf{G}(\mathbb{R}) \simeq SO(n, 1)$  has real rank one. If  $p \equiv 1 \pmod{4}$ , the equation  $x^2 + 1 = 0$  has a solution in  $\mathbb{Q}_p$ . This implies that the  $\mathbb{Q}_p$ -rank of  $\mathbf{G}(\mathbb{Q}_p)$  is at least two

and  $G(\mathbb{Q}_p)$  has Kazhdan's property.  $H = G(\mathbb{Z}[1/p])$  is an irreducible lattice in  $G(\mathbb{R}) \times G(\mathbb{Q}_p)$ .

Note that in case  $n \geq 4$  the equation  $x^2 + y^2 + 1 = 0$  has a solution in  $\mathbb{Q}_p$  for any prime  $p$ , and the  $\mathbb{Q}_p$ -rank of  $G(\mathbb{Q}_p)$  is at least two.

(3) For  $n \geq 2$ , let  $q$  be the quadratic form

$$q(x) = x_1^2 + \dots + x_n^2 - x_{n+1}^2 + \sqrt{2}x_{n+2}^2.$$

Let  $\mathcal{K}$  be the number field  $\mathbb{Q}(\sqrt{2})$ , and let  $\mathcal{O} = \mathbb{Z}[\sqrt{2}]$  be the ring of integers of  $\mathcal{K}$ . Let  $G$  be the subgroup of  $SL(n+2)$  preserving the form  $q$ . Then  $G$  is defined over  $\mathcal{K}$ , the group  $G(\mathbb{R})$  is isomorphic to  $SO(n+1, 1)$  and the real rank of  $G(\mathbb{R})$  is one.

Now, let  $\sigma$  be the automorphism of  $\mathcal{K}$  with  $\sigma(\sqrt{2}) = -\sqrt{2}$  and denote by  $G^\sigma$  the subgroup of  $SL(n+2)$  preserving the form

$${}^\sigma q(x) = x_1^2 + \dots + x_n^2 - x_{n+1}^2 - \sqrt{2}x_{n+2}^2.$$

Then  $G^\sigma(\mathbb{R}) \simeq SO(n, 2)$  has property (T) as it has real rank 2. Take  $H = G(\mathcal{O})$ . Embedded in  $G^\sigma(\mathbb{R}) \times G(\mathbb{R})$  by means of

$$\begin{aligned} G(\mathcal{O}) &\rightarrow G^\sigma(\mathbb{R}) \times G(\mathbb{R}) \\ g &\rightarrow (g^\sigma, g), \end{aligned}$$

$G(\mathcal{O})$  an irreducible lattice in this product .

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### 2. Proof of Theorem A.

Recall the following formula, valid for any representations  $\sigma$  of  $G$  and  $\pi$  of a closed subgroup  $H$ ,

$$\text{Ind}_H^G(\sigma|_H \otimes \pi) \simeq \sigma \otimes \text{Ind}_H^G \pi,$$

(see, e.g., [Fe1, Lemma 4.2]).

*Proof of Lemma 1.* — Suppose the representation  $\pi$  of  $H$  contains a subrepresentation of the form

$$(\sigma \circ p)|_H$$

where  $\sigma$  is a representation of  $G/N$ , and  $p$  denote the canonical projection of  $G$  onto  $G/N$ .

Then the induced  $\text{Ind}_H^G \pi$  contains the representation

$$\text{Ind}_H^G \left( (\sigma \circ p)|_H \right) = (\sigma \circ p) \otimes \text{Ind}_H^G 1_H.$$

Since  $G/H$  carries a finite invariant measure,  $\text{Ind}_H^G 1_H$  has invariant vectors. Therefore,  $\text{Ind}_H^G \pi$  contains the representation  $\sigma \circ p$ . Restricting to  $N$  gives condition (ii) of the lemma.

The proof of the converse is much more involved. Let  $\pi$  be a representation of  $H$ , with space  $\mathcal{H}_\pi$ , such that  $\text{Ind}_H^G \pi$  has a  $N$ -invariant vector  $\xi$  of norm one, that is, a measurable map

$$\xi : G \rightarrow \mathcal{H}_\pi$$

such that

- (1) for all  $h \in H$ ,  $\xi(xh) = \pi(h^{-1}) \cdot \xi(x)$  for almost all  $x \in G$ ,
- (2) for all  $n \in N$ ,  $\xi(n^{-1}x) = \xi(x)$  for almost all  $x \in G$ ,
- (3)

$$\int_{G/H} \|\xi(\dot{x})\|^2 d\dot{x} = 1$$

where  $\dot{x} = xH$  and  $d\dot{x}$  denotes the  $G$ -invariant measure on  $G/H$ .

Let  $\rho$  denote the induced representation  $\text{Ind}_H^G \pi$  with space  $\mathcal{H}_\rho$ . Using a well known smoothing procedure, we are going to construct a continuous  $N$ -invariant vector in  $\mathcal{H}_\rho$  as follows. For every compact neighbourhood  $U$  of the identity in  $G$ , fix a continuous non negative function  $\varphi_U$  on  $G$  with support in  $U$  and  $\int_G \varphi_U(g) dg = 1$ .

Consider the map

$$\xi_U : G \rightarrow \mathcal{H}_\pi$$

defined by

$$\xi_U(x) = \int_G \varphi_U(xg) \xi(g^{-1}) dg.$$

Then the following holds:

- (a)  $\xi_U$  is continuous on  $G$ .

To see this, observe first that the function

$$G \rightarrow \mathbb{R}, \quad g \mapsto \|\xi(g)\|$$

is integrable over any compact subset  $K$  of  $G$ . Indeed, let the left Haar measure  $dh$  on  $H$  be so normalized that  $dg = dh d\dot{g}$  holds. Then, denoting by  $\chi_K$  the characteristic function of  $K$ , one has

$$\begin{aligned} \int_K \|\xi(g)\| dg &= \int_G \|\xi(g)\| \chi_K(g) dg \\ &= \int_{G/H} \left( \int_H \|\xi(gh)\| \chi_K(gh) dh \right) d\dot{g} \\ &= \int_{G/H} \|\xi(\dot{g})\| \left( \int_H \chi_K(gh) dh \right) d\dot{g} \\ &= \int_{G/H} \|\xi(\dot{g})\| \mu_H(H \cap g^{-1}K) d\dot{g}, \end{aligned}$$

where  $\mu_H(H \cap g^{-1}K)$  denotes the  $H$ -measure of  $H \cap g^{-1}K$  and depends only on  $\dot{g}$ . Observe that  $H \cap g^{-1}K$  is non empty if and only if  $\dot{g} = \dot{k}$  for some  $k$  in  $K$ . Hence,  $\mu_H(H \cap g^{-1}K) \leq \mu_H(H \cap K^{-1}K)$ . Note that  $\mu_H(H \cap K^{-1}K) < \infty$  since  $H \cap K^{-1}K$  is compact. Thus, by Cauchy-Schwarz inequality,

$$\begin{aligned} \int_K \|\xi(g)\| dg &\leq \mu_H(H \cap K^{-1}K) \int_{G/H} \|\xi(\dot{g})\| d\dot{g} \\ &\leq \mu_H(H \cap K^{-1}K) \sqrt{\text{vol}(G/H)} \left( \int_{G/H} \|\xi(\dot{g})\|^2 d\dot{g} \right)^{1/2} \\ &< \infty. \end{aligned}$$

Now, fix a compact neighbourhood  $V$  of the group unit  $e$  of  $G$ , let  $x$  be in  $G$  and  $y$  in  $Vx$ . Denote by  $K$  the compact set  $x^{-1}(U \cup V^{-1}U)$ . Since the support of  $\varphi_U$  is contained in  $U$ , one has

$$\begin{aligned} \|\xi_U(x) - \xi_U(y)\| &\leq \int_G \|\varphi_U(xg)\xi(g^{-1}) - \varphi_U(yg)\xi(g^{-1})\| dg \\ &= \int_{x^{-1}(U \cup V^{-1}U)} \Delta(g^{-1}) |\varphi_U(xg^{-1}) - \varphi_U(yg^{-1})| \|\xi(g)\| dg \\ &\leq \sup_{g \in K} \Delta(g^{-1}) |\varphi_U(xg^{-1}) - \varphi_U(yg^{-1})| \int_K \|\xi(g)\| dg, \end{aligned}$$



where  $\Delta$  denotes the modular function on  $G$ . Let  $\varepsilon > 0$ . As  $\varphi_U$  is uniformly continuous, there exists some neighbourhood  $W$  of  $e$  contained in  $V$  such that

$$|\varphi_U(g) - \varphi_U(zg)| < \varepsilon$$

for all  $z$  in  $W$  and all  $g$  in  $G$ . Hence, for all  $y$  in  $Wx$

$$\|\xi_U(x) - \xi_U(y)\| \leq C\varepsilon,$$

where  $C = \sup_{g \in K} \Delta(g^{-1}) \int_K \|\xi(g)\| dg$  is a constant depending only on  $x, U$ , and  $V$ . Thus,  $\xi_U$  is continuous.

- (b)  $\xi_U$  belongs to the space  $\mathcal{H}_\rho$  of the induced representation  $\rho = \text{Ind}_H^G \pi$ . Indeed, let  $x$  in  $G$  and  $h$  in  $H$ . According to (1) above, for almost all  $g \in G$ ,

$$\xi(g^{-1}xh) = \pi(h^{-1}) \cdot \xi(g^{-1}x)$$

and hence

$$\begin{aligned} \xi_U(xh) &= \int_G \varphi_U(g) \xi(g^{-1}xh) dg \\ &= \int_G \varphi_U(g) \pi(h^{-1}) \cdot \xi(g^{-1}x) dg \\ &= \pi(h^{-1}) \cdot \int_G \varphi_U(g) \xi(g^{-1}x) dg \\ &= \pi(h^{-1}) \cdot \xi_U(x). \end{aligned}$$

Moreover,

$$\begin{aligned} |\langle \xi_U(x), \xi_U(x) \rangle| &\leq \int_G \int_G \varphi_U(g) \varphi_U(g') |\langle \xi(g^{-1}x), \xi(g'^{-1}x) \rangle| dg dg' \\ &\leq \int_G \int_G \varphi_U(g) \varphi_U(g') \|\xi(g^{-1}x)\| \|\xi(g'^{-1}x)\| dg dg', \end{aligned}$$

and hence, using Cauchy-Schwarz inequality in the space  $L^2(G/H)$ ,

$$\begin{aligned} \|\xi_U\|^2 &\leq \int_G \varphi_U(g) \int_G \varphi_U(g') \int_{G/H} \|\xi(g^{-1}x)\| \|\xi(g'^{-1}x)\| dx dg dg' \\ &\leq \int_G \varphi_U(g) \int_G \varphi_U(g') \|\rho(g)\xi\| \|\rho(g')\xi\| dg dg' \\ &\leq \left( \int_G \varphi_U(g) \|\rho(g)\xi\| dg \right)^2 = \|\xi\|^2. \end{aligned}$$

(c)  $\xi_U$  is  $N$ -invariant in  $\mathcal{H}_\rho$ .

Indeed, for  $n \in N$ ,  $x \in G$  and  $\eta \in \mathcal{H}_\rho$

$$\begin{aligned} & \int_{G/H} \langle \xi_U(n\dot{x}), \eta(\dot{x}) \rangle d\dot{x} \\ &= \int_{G/H} \left\langle \int_G \varphi_U(g) \xi(g^{-1}n\dot{x}) dg, \eta(\dot{x}) \right\rangle d\dot{x} \\ &= \int_G \varphi_U(g) \left( \int_{G/H} \langle \xi((g^{-1}ng)g^{-1}\dot{x}), \eta(\dot{x}) \rangle d\dot{x} \right) dg \\ &= \int_G \varphi_U(g) \left( \int_{G/H} \langle \rho(g)\rho(g^{-1}n^{-1}g)\xi(\dot{x}), \eta(\dot{x}) \rangle d\dot{x} \right) dg \\ &= \int_G \varphi_U(g) \left( \int_{G/H} \langle \rho(g)\xi(\dot{x}), \eta(\dot{x}) \rangle d\dot{x} \right) dg \\ &= \int_{G/H} \int_G \varphi_U(g) \langle \xi(g^{-1}\dot{x}), \eta(\dot{x}) \rangle d\dot{x} dg \\ &= \int_{G/H} \langle \xi_U(\dot{x}), \eta(\dot{x}) \rangle d\dot{x} \end{aligned}$$

as  $N$  is a normal subgroup and  $\xi$  is  $N$ -invariant.

(d)  $\xi_U$  is non zero, for  $U$  sufficiently small. This is clear since  $\|\xi_U - \xi\| \rightarrow 0$  when  $U \rightarrow \{e\}$ .

Now, for any  $c_1, \dots, c_n \in \mathbb{C}$ ,  $g_1, \dots, g_n \in G$ ,  $k \in N$ ,  $h \in H$ , using the continuity of  $\xi_U$ , we have

$$\begin{aligned} \sum_{i=1}^n c_i \xi_U(g_i h k) &= \sum_{i=1}^n c_i \xi_U((g_i h) k (g_i h)^{-1} g_i h) \\ &= \sum_{i=1}^n c_i \rho((g_i h) k (g_i h)^{-1}) \xi_U(g_i h) \\ &= \sum_{i=1}^n c_i \xi_U(g_i h) \end{aligned}$$

and

$$\left\| \sum_{i=1}^n c_i \xi_U(g_i h) \right\| = \left\| \pi(h^{-1}) \sum_{i=1}^n c_i \xi_U(g_i) \right\| = \left\| \sum_{i=1}^n c_i \xi_U(g_i) \right\|.$$

Therefore, by density of  $HN$  in  $G$  and, again, by continuity of  $\xi_U$ ,

$$\left\| \sum_{i=1}^n c_i \xi_U(g_i g) \right\| = \left\| \sum_{i=1}^n c_i \xi_U(g_i) \right\|$$

for all  $g$  in  $G$ .

Let  $\mathcal{W}_\pi$  be the (non zero) closed subspace of  $\mathcal{H}_\pi$  generated by  $\xi_U(G)$ . Then, for any  $g$  in  $G$ ,

$$\begin{aligned} \mathcal{W}_\pi &\longrightarrow \mathcal{W}_\pi \\ \sum_{i=1}^n c_i \xi_U(g_i) &\mapsto \sum_{i=1}^n c_i \xi_U(g_i g^{-1}) \end{aligned}$$

is a unitary operator depending only on the class of  $g$  in  $G/N$ . This defines a unitary representation  $\sigma$  of  $G/N$ . As  $\xi_U$  is continuous,  $\sigma$  is continuous. Since

$$\sigma \circ p(h) \xi_U(g) = \xi_U(gh^{-1}) = \pi(h) \xi_U(g),$$

it is clear that

$$\sigma \circ p(h) = \pi(h)$$

on  $\mathcal{W}_\pi$ , for all  $h$  in  $H$ . □

We shall need the following lemma.

LEMMA 2. — *Let  $G$ ,  $N$  and  $H$  be as in Lemma 1. Let  $\rho$  be the quasi-regular representation of  $G$  on  $L^2(G/H)$ . The  $N$ -invariant functions in  $L^2(G/H)$  are constant.*

*Proof.* — Lemma 2 amounts to saying that the action of  $N$  (by left multiplication) on the homogeneous space  $G/H$  is ergodic. By Moore’s duality theorem, ergodicity of the  $N$ -action on  $G/H$  is equivalent to ergodicity of the action of  $H$  on  $G/N$  by left multiplication (see [Zim, Corollary 2. 2. 3]). Density of  $HN$  in  $G$  implies that the subgroup  $p(H)$  is dense in the group  $G/N$ , and this is equivalent with ergodicity of the action of  $H$  on  $G/N$  (see [Zim, Lemma 2. 2. 13]). □

We shall frequently use Fell’s inner hull-kernel topology on the set  $\text{Rep}(G)$  of all equivalence classes of unitary representations of a locally compact group  $G$ . This topology is defined as follows. For  $\pi$  in  $\text{Rep}(G)$ ,

$\varepsilon > 0$ , a compact subset  $K$  of  $G$ , and positive definite functions  $\varphi_1, \dots, \varphi_n$  associated with  $\pi$ , let  $W(\varphi_1, \dots, \varphi_n; K; \varepsilon; \pi)$  be the set of all  $\rho$  in  $\text{Rep}(G)$  such that there exists  $\psi_1, \dots, \psi_n$ , each of which is a sum of positive definite functions associated with  $\rho$ , for which

$$|\varphi_i(x) - \psi_i(x)| < \varepsilon \quad \forall i = 1, \dots, n \quad \forall x \in K.$$

The subsets  $W(\varphi_1, \dots, \varphi_n; K; \varepsilon; \pi)$  form a basis of neighbourhoods of  $\pi$  (see [Fe1, Section 2]). This topology may also be described in terms of weak containment. Recall that  $\pi$  is weakly contained in a set  $\mathcal{S}$  of representations of  $G$  if every positive definite function associated with  $\pi$  is the limit, uniformly on compact subsets of  $G$ , of sums of positive definite functions associated with representations from  $\mathcal{S}$ . It is clear that a net  $\pi_n$  of unitary representations of  $G$  converges to  $\pi$  if and only if, for every subnet  $\pi_{n'}$  of  $\pi_n$ ,  $\pi$  is weakly contained in the set  $\{\pi_{n'}\}$ . Restricted to  $\widehat{G}$ , this is just the usual Fell-Jacobson topology on  $\widehat{G}$  (see also [Dix, Chap.18]). We are now in position to prove Theorem A.

*Proof of Theorem A.* — Let  $\pi_n$  be a net of irreducible representations of  $H$  converging to  $1_H$  in  $\widehat{H}$ . Then, by continuity of inducing (see [Fe1, Theorem 4.1]),

$$\text{Ind}_H^G \pi_n \rightarrow \text{Ind}_H^G 1_H$$

in  $\text{Rep}(G)$ . Since  $H$  has finite covolume,  $1_G$  is contained in  $\text{Ind}_H^G 1_H$  and this implies

$$\text{Ind}_H^G \pi_n \rightarrow 1_G.$$

As  $N$  has Kazhdan's property, we may assume that  $\text{Ind}_H^G \pi_n$  has  $N$ -invariant vectors for all  $n$ . Hence, by Lemma 1, there are (irreducible) representations  $\sigma_n$  of  $G/N$  such that  $\pi_n = (\sigma_n \circ p)|_H$  where  $p : G \rightarrow G/N$  is the canonical projection. The proof will be finished if we show that

$$\sigma_n \circ p \rightarrow 1_G$$

in  $\widehat{G}$ .

Since  $H$  has finite covolume, one has

$$(*) \text{Ind}_H^G \pi_n = \text{Ind}_H^G (\sigma_n \circ p)|_H = (\sigma_n \circ p) \otimes \rho = (\sigma_n \circ p) \oplus ((\sigma_n \circ p) \otimes \rho^0),$$

where  $\rho = \text{Ind}_H^G 1_H$  and  $\rho^0$  is the restriction of  $\rho$  to the orthogonal of the constants in  $L^2(G/N)$ .

Now, the restriction to  $N$  of  $(\sigma_n \circ p) \otimes \rho^0$  is a multiple of  $\rho^0|_N$ . Since  $N$  has property (T), Lemma 2 above implies that  $\rho^0|_N$  does not weakly contain the trivial representation  $1_N$ . So,  $(\sigma_n \circ p) \otimes \rho^0$  cannot converge to  $1_G$ . As

$$\text{Ind}_H^G \pi_n \rightarrow 1_G,$$

we conclude from (\*) that

$$\sigma_n \circ p \rightarrow 1_G. \quad \square$$

### 3. Proof of Theorem B.

The finite dimensional representation  $\pi$  decomposes as a finite sum

$$\pi = \sum_{i=1}^n \pi_i$$

of irreducible subrepresentations  $\pi_i$ . Since, in an obvious way,

$$H^1(H, \pi) \cong \bigoplus_{i=1}^n H^1(H, \pi_i),$$

we may assume that  $\pi$  is irreducible.

We first deal with the case where  $\pi$  is the trivial representation  $1_H$ . Then,  $H^1(H, \pi)$  is the group of all (continuous) homomorphisms from  $H$  to the additive group of the complex numbers  $\mathbb{C}$ . Let  $[\overline{H}, \overline{H}]$  denote the closure of the commutator subgroup of  $H$ . The corollary to Theorem A implies that the dual group of the abelian group  $H/[\overline{H}, \overline{H}]$  is discrete, since the trivial character is isolated. So, by duality theory,  $H/[\overline{H}, \overline{H}]$  is compact. Hence,  $H^1(H, 1_H) = 0$ .

Suppose now that  $\pi \neq 1_H$  and assume, by contradiction, that

$$H^1(H, \pi) \neq 0.$$

Then, by Vershik-Karpushev theorem (see [VeK, Theorem 2]) there exists a net  $\pi_n$  in  $\widehat{H}$  such that

$$\pi_n \rightarrow \pi \quad \text{and} \quad \pi_n \rightarrow 1_H.$$

By Theorem A, we may assume that  $\pi_n = (\sigma_n \circ p)|_H$  for irreducible representations  $\sigma_n$  of  $G/N$ .

Let  $\bar{\pi}$  denote the conjugate representation of  $\pi$ . Then, by continuity of tensoring (see [Fe2, Theorem 1]),

$$\pi_n \otimes \bar{\pi} \rightarrow \pi \otimes \bar{\pi}.$$

Hence,

$$(\sigma_n \circ p) \otimes \text{Ind}_H^G \bar{\pi} = \text{Ind}_H^G (\pi_n \otimes \bar{\pi}) \rightarrow \text{Ind}_H^G (\pi \otimes \bar{\pi}).$$

Restricting to  $N$ , this implies that  $\text{Ind}_H^G \bar{\pi}|_N$  weakly contains the representation  $\text{Ind}_H^G (\pi \otimes \bar{\pi})|_N$ .

Since  $\pi$  is finite dimensional, it is well known that  $\pi \otimes \bar{\pi}$  has an invariant vector. Therefore, as  $H$  has finite covolume,  $\text{Ind}_H^G \bar{\pi}|_N$  weakly contains  $1_N$ . We conclude that  $\text{Ind}_H^G \bar{\pi}$  has  $N$ -invariant vectors. But then, Lemma 1 implies that  $\pi$  factorizes to a representation  $\sigma$  of  $G/N$ , thus,  $\pi = (\sigma \circ p)|_H$ . As  $G/N$  is minimally almost periodic, this forces  $\sigma = 1_{G/N}$  and hence  $\pi = 1_H$ , a contradiction.  $\square$

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