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SCHUBERT VARIETIES, TORIC VARIETIES, AND LADDER DETERMINANTAL VARIETIES

by N. GONCIULEA and V. LAKSHMIBAI (*)

Introduction.

Let k be the base field which we assume to be algebraically closed of arbitrary characteristic. Let $\mathcal{L}$ be a a finite distributive lattice, $k[\mathcal{L}] = k[x_\tau \mid \tau \in \mathcal{L}]$, $I(\mathcal{L})$ the ideal generated by all binomials of the form $x_\tau x_\phi - x_{\tau\vee\phi}x_{\tau\wedge\phi}$, τ, ϕ being two noncomparable elements of $\mathcal{L}$ (see Section 3 for notations), and $R(\mathcal{L}) = k[\mathcal{L}]/I(\mathcal{L})$. Let $X(\mathcal{L}) = \mathrm{Spec}R(\mathcal{L})$. The main object of study in this paper is the variety $X(\mathcal{L})$ for the case $\mathcal{L} = I_{d,n}$, where $I_{d,n} = \{\underline{i} = (i_1, \ldots, i_d) \mid 1 \leqslant i_1 < \ldots < i_d \leqslant n\}$ (the partial order on $I_{d,n}$ being the natural one, namely $(i_1, \ldots, i_d) \geqslant (j_1, \ldots, j_d) \iff i_t \geqslant j_t$ for all $1 \leqslant t \leqslant d$). Denoting $X(I_{d,n})$ by just $X_{d,n}$, we prove (cf. Theorem 10.4)

THEOREM 1. — *$X_{d,n}$ is a normal toric variety.*

Let $G_{d,n}$ be the Grassmann variety of d-planes in k^n. Using Theorem 1 and the results of [12], we prove (cf. Theorem 10.8).

THEOREM 2. — *The Grassmannian $G_{d,n}$ degenerates to the normal toric variety $X_{d,n}$.*

In the case $d = 2$, the toric variety $X_{2,n}$ may also be identified with a certain ladder determinantal variety. To make it more precise, let L be a one-sided ladder with lower outside corner α_0, and upper outside

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$$
\begin{array}{ccccccccccccccc}
* & * & \dots & * & \alpha_1 & & & & & & & & & & \\
* & * & \dots & * & * & & & & & & & & & & \\
\vdots & \vdots & & \vdots & \vdots & & & & & & & & & & \\
* & * & \dots & * & * & & & & & & & & & & \\
* & * & \dots & * & * & * & \dots & * & \alpha_2 & & & & & & \\
* & * & \dots & * & * & * & \dots & * & * & & & & & & \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & & & & & \\
* & * & \dots & * & * & * & \dots & * & * & & & & & & \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & & & & \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & & & \\
* & * & \cdots & * & * & * & \cdots & * & * & * & * & \cdots & \cdots & * & \alpha_l \\
* & * & \cdots & * & * & * & \cdots & * & * & * & * & \cdots & \cdots & * & * \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots \\
\alpha_0 & * & \cdots & * & * & * & \cdots & * & * & * & * & \cdots & \cdots & * & *
\end{array}
$$

Figure 1. An one-sided ladder.

corners $\alpha_1 = (b_1, a_1), \alpha_2 = (b_2, a_2), \dots, \alpha_l = (b_l, a_l)$. Let $X(\mathcal{L})$ be the ladder determinantal variety determined by the vanishing of all 2-minors in $\mathcal{L}$. (Note that $X_{2,n}$ is simply $X(\mathcal{L})$, where $\mathcal{L}$ is a ladder with $n-3$ outside corners and $a_i = a_{i-1} + 1$, $b_j = b_{j-1} + 1$, $1 \leqslant i, j \leqslant n-3$, $a_0 = 0 = b_0$). For $1 \leqslant i \leqslant l$, let V_i be the subvariety of $X(\mathcal{L})$ defined by $x_\alpha = 0$ for $\alpha \in [\alpha_0 \wedge \alpha_i, \alpha_0 \vee \alpha_i]$ (see Section 2 for notations). We prove (cf. Theorem 12.3)

THEOREM 3. — *The irreducible components of* $\operatorname{Sing} X(L)$ *are precisely* V_i, $1 \leqslant i \leqslant l$, *where* $\operatorname{Sing} X(L)$ *is the singular locus of* $X(L)$.

For arbirary d, we prove some partial results, and state a conjecture(1) on the irreducible components of the singular locus of the variety $X_{d,n}$.

Because of the relationship between certain toric varieties and ladder determinantal varieties as discussed above, we are naturally led to study ladder determinantal varieties (LDVs). LDVs were first considered by Abhyankar (cf. [1]) in his study of the singularities of Schubert varieties. Fixing an one-sided ladder $\mathcal{L}$ as above and a positive integer t, let $X_t(\mathcal{L})$ be the variety defined by the vanishing of all t-minors in $\mathcal{L}$. In [22], Mulay relates $X_t(\mathcal{L})$ to the "opposite cell" in a certain Schubert variety $X(w)$ in

(1) This conjecture has now been proved in [3].

$SL(n)/Q$ for a suitable n and a suitable parabolic subgroup Q. Using this result and Theorem 3, we prove (cf. Theorem 15.15) a refined version of the conjecture of [20] on the components of the singular locus for the class of Schubert varieties arising from LDVs.

We now give a brief outline of the proof of our results. For proving Theorem 1, we first prove (cf. Section 3) that $I(\mathcal{L})$, $\mathcal{L}$ being a finite distributive lattice, is a toric ideal (in the sense of [26]). We then prove (cf. Section 7) the Cohen-Macalayness of $R(\mathcal{L})$. For the case $\mathcal{L} = I_{d,n}$, we futher prove (using the Jacobian criterion for smoothness) that the variety $X(\mathcal{L})$ is nonsingular in codimension 1.

Theorem 2 is proved using Theorem 1 and the result (cf. [12]) that there exists a flat family whose special fiber is $R(I_{d,n})$ and general fiber is $k[G_{d,n}]$, the homegeneous coordinate ring of $G_{d,n}$ for the Plücker embedding. Theorem 3 is again proved using the Jacobian criterion for smoothness.

The sections are organized as follows. In Sections 1 and 2, we recall some generalities on Gröbner bases and distributive lattices. In Section 3, we prove the primality of the ideal $I(\mathcal{L})$, $\mathcal{L}$ being any finite distributive lattice (our proof is very short, and combinatorial in nature). In Section 4, we carry out a short geometric proof of the fact that a binomial prime ideal is toric (see [9] for an algebraic proof). In Section 5, we prove some general results on $X(\mathcal{L})$ (in particular, we compute the dimension of $X(\mathcal{L})$). In Section 6, we construct a "standard monomial basis" for $R(\mathcal{L})$. In Section 7, we prove the Cohen-Macaulayness of $R(\mathcal{L})$. In Section 8, we derive some properties of the distributive lattice $I_{d,n}$. In Section 9, we study the variety $X_{d,n}$, where $X_{d,n} = X(\mathcal{L})$, for $\mathcal{L} = I_{d,n}$. In Section 10, we prove the normality of the variety $X_{d,n}$. In Section 11, we prove some partial results, and state a conjecture, on the irreducible components of $\mathrm{Sing}X_{d,n}$. In Section 12, we verify the conjecture stated in Section 11 for $d = 2$, by determining the irreducible components of $\mathrm{Sing}X(L)$, where $X(L)$ is the determinantal variety defined by the vanishing of all the 2 minors in a general one-sided ladder L. In Section 13, we recall some results on the flag variety $SL(n)/B$, and its Schubert varieties. In Section 14, we relate $X(L)$, as well as the irreducible components of $\mathrm{Sing}X(L)$ to opposite cells in certain Schubert varieties in certain $SL(n)/Q$. In Section 15, we state a refined version of the conjecture in [20] and verify it to be true for the pull-backs in G/B (under $G/B \to G/Q$) of the Schubert varieties obtained in Section 14.

It should be remarked that in [27] a purely combinatorial description of Sing$X(\mathcal{L})$ is obtained. In this paper we have taken a geometric approach to this problem, in the case when $\mathcal{L} = I_{d,n}$, or $\mathcal{L}$ is an one-sided ladder.

In this paper we have considered only those ladder determinantal varieties which are also toric varieties. In a subsequent paper [13], we study a larger class of ladder determinantal varieties (which are not necessarily toric varieties), and prove results similar to those of Sections 12, 14 and 15.

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1. Generalities on Gröbner bases.

Let k be a field, and consider the ring $k[x_1, \ldots, x_n]$ of polynomials in n variables $x_1, \ldots, x_n$. We recall below some generalities on Gröbner bases; for more details one may refer to [7], [8].

DEFINITION 1.1. — *A total order $\preceq$ on the set of monomials in $k[x_1, \ldots, x_n]$ is called a monomial order if for given monomials $\underline{m}$, $\underline{m}_1$, $\underline{m}_2$, with $\underline{m} \neq 1$, $\underline{m}_1 \prec \underline{m}_2$ implies $\underline{m}_1 \prec \underline{m} \cdot \underline{m}_1 \prec \underline{m} \cdot \underline{m}_2$.*

For the rest of this section, a fixed monomial order $\preceq$ is considered.

If f is a nonzero polynomial in $k[x_1, \ldots, x_n]$, then the greatest monomial (with respect to $\preceq$) occuring in f is called the *initial monomial of* f, and we denote it by $\mathrm{in}(f)$; the coefficient of $\mathrm{in}(f)$ is called the initial coefficient of f. For a family of polynomials $\mathcal{F} \subset k[x_1, \ldots, x_n]$, the ideal generated by its elements will be denoted by $\langle \mathcal{F} \rangle$, and the set of the initial monomials of all polynomials in $\mathcal{F}$ will be denoted by $\mathrm{in}(\mathcal{F})$.

DEFINITION 1.2. — *Let $I \subset k[x_1, \ldots, x_n]$ be an ideal. A finite set of polynomials $\mathcal{F} \subset I$ is called a Gröbner basis for I with respect to $\preceq$ if $\langle \mathrm{in}(\mathcal{F}) \rangle = \langle \mathrm{in}(I) \rangle$.*

DEFINITION 1.3. — *A reduced Gröbner basis for I with respect to $\preceq$ is a Gröbner basis $\mathcal{F}$ for I with respect to $\preceq$ such that the initial coefficients of the elements in $\mathcal{F}$ are all 1, and for all $f \in \mathcal{F}$, none of the monomials present in f lies in $\langle \mathrm{in}(\mathcal{F} \setminus \{f\}) \rangle$.*

PROPOSITION 1.4. — *Any Gröbner basis for I generates I as an ideal.*

In the case when I is the defining ideal of an algebraic variety X, a Gröbner basis for I will be also called a *Gröbner basis for* X.

PROPOSITION 1.5. — *A nonzero ideal* $I \subset k[x_1, \ldots, x_n]$ *has a unique reduced Gröbner basis (with respect to a given monomial order).*

1.6. *Reverse lexicographic order.* Assume that the variables $x_1, \ldots, x_n$ are totally ordered as follows: $x_1 < \ldots < x_n$. A monomial $\underline{m}$ of degree r in the polynomial ring $k[x_1, \ldots, x_n]$ will be written in the form $\underline{m} = x_{i_1} \ldots x_{i_r}$, with $1 \leqslant i_1 \leqslant \ldots \leqslant i_r \leqslant n$. The *reverse lexicographic order* on the set of monomials $\underline{m} \in k[x_1, \ldots, x_n]$ is denoted by $\preceq_{rlex}$, and defined as follows: $x_{i_1} \ldots x_{i_r} \prec_{rlex} x_{j_1} \ldots x_{j_s}$ if and only if either $r < s$, or $r = s$ and there exists an $l < r$ such that $i_1 = j_1, \ldots, i_l = j_l, i_{l+1} < j_{l+1}$. It is easy to check that $\preceq_{rlex}$ is a monomial order.

2. Generalities on distributive lattices.

2.1. We recall the following definitions on lattices. A *lattice* is a partially ordered set $(\mathcal{L}, \leqslant)$ such that, for every pair of elements $x, y \in \mathcal{L}$, there exist elements $x \vee y$ and $x \wedge y$, called the *join*, respectively the *meet* of x and y, defined by:

$$x \vee y \geqslant x, \;\; x \vee y \geqslant y, \;\text{ and if } z \geqslant x \text{ and } z \geqslant y, \;\text{ then } z \geqslant x \vee y,$$
$$x \wedge y \leqslant x, \;\; x \wedge y \leqslant y, \;\text{ and if } z \leqslant x \text{ and } z \leqslant y, \;\text{ then } z \leqslant x \wedge y.$$

It is easy to check that the operations $\vee$ and $\wedge$ are commutative and associative.

An element $z \in \mathcal{L}$ is called the *zero* of $\mathcal{L}$, denoted by $\widehat{0}$, if $z \leqslant x$ for all x in $\mathcal{L}$. An element $z \in \mathcal{L}$ is called the *one* of $\mathcal{L}$, denoted by $\widehat{1}$, if $z \geqslant x$ for all x in $\mathcal{L}$.

Given a lattice $\mathcal{L}$, a subset $\mathcal{L}' \subset \mathcal{L}$ is called a *sublattice* of $\mathcal{L}$ if $x, y \in \mathcal{L}'$ implies $x \wedge y \in \mathcal{L}'$, $x \vee y \in \mathcal{L}'$.

Two lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ are *isomorphic* if there exists a bijection $\varphi : \mathcal{L}_1 \to \mathcal{L}_2$ such that, for all $x, y \in \mathcal{L}_1$,

$$\varphi(x \vee y) = \varphi(x) \vee \varphi(y) \text{ and } \varphi(x \wedge y) = \varphi(x) \wedge \varphi(y).$$

A lattice is called *distributive* if the following identities hold:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{1}$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \tag{2}$$

2.2 *An example.* — Given an integer $n \geqslant 1$, $\mathcal{C}(n)$ will denote the chain $\{1 < \ldots < n\}$, and for $n_1, \ldots, n_d > 1$, $\mathcal{C}(n_1, \ldots, n_d)$ will denote the *chain product lattice* $\mathcal{C}(n_1) \times \ldots \times \mathcal{C}(n_d)$ consisting of all d-tuples $(i_1, \ldots, i_d)$, with $1 \leqslant i_1 \leqslant n_1, \ldots, 1 \leqslant i_d \leqslant n_d$. For $(i_1, \ldots, i_d)$, $(j_1, \ldots, j_d)$ in $\mathcal{C}(n_1, \ldots, n_d)$, we define

$$(i_1, \ldots, i_d) \leqslant (j_1, \ldots, j_d) \iff i_1 \leqslant j_1, \ldots, i_d \leqslant j_d \,.$$

We have

$$(i_1, \ldots, i_d) \vee (j_1, \ldots, j_d) = (\max\{i_1, j_1\}, \ldots, \max\{i_d, j_d\})$$
$$(i_1, \ldots, i_d) \wedge (j_1, \ldots, j_d) = (\min\{i_1, j_1\}, \ldots, \min\{i_d, j_d\}).$$

$\mathcal{C}(n_1, \ldots, n_d)$ is a finite distributive lattice, and its zero and one are $(1, \ldots, 1)$, $(n_1, \ldots, n_d)$ respectively.

Note that there is a total order $\lhd$ on $\mathcal{C}(n_1, \ldots, n_d)$ extending $<$, namely the lexicographic order, defined by $(i_1, \ldots, i_d) \lhd (j_1, \ldots, j_d)$ if and only if there exists $l < d$ such that $i_1 = j_1, \ldots, i_l = j_l, i_{l+1} < j_{l+1}$. Also note that two elements $(i_1, \ldots, i_d) \lhd (j_1, \ldots, j_d)$ are non-comparable with respect to $\leqslant$ if and only if there exists $1 < h \leqslant d$ such that $i_h > j_h$.

Sometimes we denote the elements of $\mathcal{C}(n_1, n_2, \ldots, n_d)$ by $x_{i_1 \ldots i_d}$, with $1 \leqslant i_1 \leqslant n_1, \ldots, 1 \leqslant i_d \leqslant n_d$.

2.3. The lattice of all subsets of the set $\{1, 2, \ldots, n\}$ is denoted by $\mathcal{B}(n)$, and called the *Boolean algebra of rank n*. Note that $\mathcal{B}(n)$ is isomorphic to $[\mathcal{C}(2)]^n$.

One has the following (see [2]):

THEOREM 2.4. — *Any finite distributive lattice is isomorphic to a sublattice of a Boolean algebra of finite rank, and hence, in particular, to a sublattice of a finite chain product.*

3. The ideal associated to a distributive lattice.

3.1. Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be a subset of $\mathbb{Z}^m$. Consider the homomorphism

$$\pi : \mathbb{Z}^n \to \mathbb{Z}^m, \quad \mathbf{u} = (u_1, \ldots, u_n) \mapsto u_1 \mathbf{a}_1 + \ldots + u_N \mathbf{a}_N.$$

Let $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{t} = (t_1, \dots, t_m)$, and

$$k[\mathbf{x}] = k[x_1, \dots, x_n], \quad k[\mathbf{t}^{\pm 1}] = k[t_1, \dots, t_m, t_1^{-1}, \dots, t_m^{-1}].$$

The map π induces a homomorphism of semigroup algebras

$$\widehat{\pi} : k[\mathbf{x}] \to k[\mathbf{t}^{\pm 1}], \qquad x_i \mapsto \mathbf{t}^{\mathbf{a}_i}.$$

DEFINITION 3.2. — *The kernel of $\widehat{\pi}$ is denoted by $I_{\mathcal{A}}$ and called the toric ideal associated to $\mathcal{A}$.*

Note that a toric ideal is prime.

Recall the following (see [26]).

PROPOSITION 3.3. — *The toric ideal $I_{\mathcal{A}}$ is spanned as a a k-vector space by the set of binomials*

$$\{\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \mathbf{u}, \mathbf{v} \in \mathbb{Z}_+^n \text{ with } \pi(\mathbf{u}) = \pi(\mathbf{v})\}.$$

3.4. *An important example* (cf. [26]). — Let us fix the integers $n_1, \dots, n_d > 1$, and let $n = \prod_{i=1}^{d} n_i$, $m = \sum_{i=1}^{d} n_i$. Let $\mathbf{e}_1^l, \dots, \mathbf{e}_{n_l}^l$ be the unit vectors in $\mathbb{Z}^{n_l}$, for $1 \leqslant l \leqslant d$. For $1 \leqslant \xi_1 \leqslant n_1, \dots, 1 \leqslant \xi_d \leqslant n_d$, define

$$\mathbf{a}_{\xi_1 \dots \xi_d} = \mathbf{e}_{\xi_1}^1 \oplus \dots \oplus \mathbf{e}_{\xi_d}^d \in \mathbb{Z}^{n_1} \oplus \dots \oplus \mathbb{Z}^{n_d}$$

and

$$\mathcal{A}_{n_1, \dots, n_d} = \{\mathbf{a}_{\xi_1 \dots \xi_d} \mid 1 \leqslant \xi_1 \leqslant n_1, \dots, 1 \leqslant \xi_d \leqslant n_d\}.$$

The corresponding map

$$\pi : \mathbb{Z}^{n_1 \cdot \ldots \cdot n_d} \to \mathbb{Z}^{n_1 + \ldots + n_d}$$

is defined as follows: for $1 \leqslant l \leqslant d$ and $1 \leqslant i_l \leqslant n_l$ fixed, the $(n_1 + \ldots + n_{l-1} + i_l)$-th coordinate of $\pi(\mathbf{u})$ is given by $\sum u_{\xi_1 \dots \xi_{l-1} \xi_l \xi_{l+1} \dots \xi_d}$, the sum being taken over the elements $(\xi_1, \dots, \xi_{l-1}, \xi_l, \xi_{l+1}, \dots, \xi_d)$ of $\mathcal{C}(n_1, \dots, n_d)$ with $\xi_l = i_l$. We call this subset the l-th *slice* of $\mathcal{C}(n_1, \dots, n_d)$ defined by i_l, and denote it by $\{\xi_l = i_l\}$. The components (or *entries*) of an element $\mathbf{u} \in \mathbb{Z}^{n_1 \cdot \ldots \cdot n_d}$ are indexed by the elements $(i_1, \dots, i_d)$ of $\mathcal{C}(n_1, \dots, n_d)$. If $(j_1, \dots, j_d) \in \{\xi_l = i_l\}$, sometimes we also say that $u_{j_1 \dots j_d}$ itself belongs to the slice $\{\xi_l = i_l\}$.

The map π induces the map

$$\widehat{\pi} : k[x_{11\dots 1}, \dots, x_{\xi_1 \xi_2 \dots \xi_d}, \dots, x_{n_1 n_2 \dots n_d}] \to k[t_{11}, \dots, t_{1n_1}, \dots, t_{d1}, \dots, t_{dn_d}]$$

given by

$$x_{\xi_1\ldots\xi_d} \mapsto t_{1\xi_1}\ldots t_{d\xi_d}, \text{ for } 1 \leqslant \xi_1 \leqslant n_1, \ldots, 1 \leqslant \xi_d \leqslant n_d.$$

DEFINITION 3.5. — *Given a finite lattice $\mathcal{L}$, the ideal associated to $\mathcal{L}$, denoted by $I(\mathcal{L})$, is the ideal of the polynomial ring $k[\mathcal{L}] := k[x_\tau \mid \tau \in \mathcal{L}]$ generated by the set of binomials*

$$\mathcal{G}_{\mathcal{L}} = \{xy - (x \wedge y)(x \vee y) \mid x, y \in \mathcal{L} \text{ non-comparable}\}.$$

By Theorem 2.4, a finite distributive lattice $\mathcal{L}$ may be identified with a sublattice of a finite chain product lattice . Hence it inherits a total order extending the given partial order. In turn, this total order induces the reverse lexicographic order on the monomials in $k[\mathcal{L}]$, as in 1.6.

The following theorem shows that the ideal associated to a chain product lattice is toric.

THEOREM 3.6. — 1) *We have $I(\mathcal{C}(n_1, \ldots, n_d)) = I_{\mathcal{A}_{n_1,\ldots,n_d}}$.*

2) *The set of binomials*

$$\mathcal{G} = \{xy - (x \wedge y)(x \vee y) \mid x, y \in \mathcal{C}(n_1, \ldots, n_d) \text{ non-comparable}\}$$

is a Gröbner basis for $I(\mathcal{C}(n_1, \ldots, n_d))$ with respect to the reverse lexicographic order.

Proof. — Let $\mathcal{C} = \mathcal{C}(n_1, \ldots, n_d)$ and $\mathcal{A} = \mathcal{A}_{n_1,\ldots,n_d}$. Let $f \in I_{\mathcal{A}}$; by Proposition 3.3, there exist $\mathbf{u}_i, \mathbf{v}_i \in \mathbb{Z}_+^n$ with $\pi(\mathbf{u}_i) = \pi(\mathbf{v}_i)$, and $c_i \in k^*$, $1 \leqslant i \leqslant s$ such that

$$f = \sum_{i=1}^{s} c_i(\mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{v}_i}) \tag{4}$$

for some $s \geqslant 1$, with the property that s is the smallest integer $\geqslant 1$ such that f can be expressed as a linear combination of s binomials in the set (3). Now we rewrite f as

$$f = \sum_{i=1}^{s} a_i \mathbf{x}^{\mathbf{u}_i} + \sum_{i=1}^{s} b_i \mathbf{x}^{\mathbf{v}_i}, \quad a_i, b_i \in k.$$

Then none of the coefficients $a_1, \ldots, a_s, b_1, \ldots, b_s$ is zero. Indeed, suppose that $a_i = 0$ for some $1 \leqslant i \leqslant s$. This implies that there exists $j \in \{1, \ldots, s\}$,

$j \neq i$ such that either $c_j = c_i$ and $\mathbf{v}_j = \mathbf{u}_i$, or $c_j = -c_i$ and $\mathbf{u}_j = \mathbf{u}_i$. In the first case we have

$$(5) \qquad c_i(\mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{v}_i}) + c_j(\mathbf{x}^{\mathbf{u}_j} - \mathbf{x}^{\mathbf{v}_j}) = c_i(\mathbf{x}^{\mathbf{u}_j} - \mathbf{x}^{\mathbf{v}_i}), \quad \pi(\mathbf{u}_j) = \pi(\mathbf{v}_i).$$

In the second case we have

$$(6) \qquad c_i(\mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{v}_i}) + c_j(\mathbf{x}^{\mathbf{u}_j} - \mathbf{x}^{\mathbf{v}_j}) = c_i(\mathbf{x}^{\mathbf{v}_j} - \mathbf{x}^{\mathbf{v}_i}), \quad \pi(\mathbf{v}_j) = \pi(\mathbf{v}_i).$$

But (4), (5) and (6) imply that f can be written as a linear combination of $s-1$ binomials in the set (3), contradicting the choice of s. Thus $a_i \neq 0$, $1 \leqslant i \leqslant s$, and similarly $b_i \neq 0$, $1 \leqslant i \leqslant s$. This shows that $\operatorname{in}(f) = \operatorname{in}(\mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{v}_i})$ for some $1 \leqslant i \leqslant s$. Let us write

$$\mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{v}_i} = \mathbf{x}^{\mathbf{w}}(\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}),$$

where $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{Z}_+^n$, with $\pi(\mathbf{u}) = \pi(\mathbf{v})$ and $\operatorname{supp}(\mathbf{u}) \cap \operatorname{supp}(\mathbf{v}) = \varnothing$. Let us suppose that $\mathbf{x}^{\mathbf{u}} \succeq_{rlex} \mathbf{x}^{\mathbf{v}}$, i.e. $\operatorname{in}(\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}) = \mathbf{x}^{\mathbf{u}}$ and $\operatorname{in}(f) = \operatorname{in}(\mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{v}_i}) = \mathbf{x}^{\mathbf{u}_i}$. Let $x_{i_1 \dots i_d}$ be the smallest variable appearing in $\mathbf{x}^{\mathbf{u}}$, i.e. $(i_1, \dots, i_d)$ is the smallest element of $\operatorname{supp}(\mathbf{u})$ with respect to $\lhd$. Then $\mathbf{x}^{\mathbf{v}}$ contains a variable $x_{k_1 \dots k_d}$, with $(k_1, \dots, k_d) \lhd (i_1, \dots, i_d)$. Since $\pi(\mathbf{u}) = \pi(\mathbf{v})$, the sum of the entries in every slice is the same for both $\mathbf{u}$ and $\mathbf{v}$. In particular, since all the entries of $\mathbf{u}$ in the slices $\{\xi_1 = i\}$, with $1 \leqslant i < i_1$, are 0 (by the choice of $(i_1, \dots, i_d)$), all the entries of $\mathbf{v}$ in these slices must also be 0. This implies that $(k_1, \dots, k_d) \in \{\xi_1 = i_1\}$. Let $1 < h \leqslant n_1$ such that $k_1 = i_1, \dots, k_{h-1} = i_{h-1}, k_h < i_h$. Then the sum of the elements of $\mathbf{v}$ in the slice $\{\xi_h = k_h\}$ is nonzero, which implies that $\{\xi_h = k_h\} \cap \operatorname{supp}(\mathbf{u}) \neq \varnothing$. Let $(j_1, \dots, j_d)$ be an element in this intersection. We have $(i_1, \dots i_d) \lhd (j_1, \dots, j_d)$ (by the definition of $(i_1, \dots i_d)$), and since $i_h > k_h = j_h$, we conclude that $(i_1, i_2, \dots, i_d)$ and $(j_1, \dots, j_d)$ are non-comparable. Thus we obtain that $\mathbf{x}^{\mathbf{u}}$ is divisible by $x_{i_1 \dots i_d} x_{j_1 \dots j_d}$. Hence $\mathbf{x}^{\mathbf{u}_i}$ is also divisible by $x_{i_1 \dots i_d} x_{j_1 \dots j_d}$. Therefore $\operatorname{in}(f)$ is divisible by the initial term of an element of the set

$$\mathcal{G} = \{xy - (x \wedge y)(x \vee y) \mid x, y \in \mathcal{C} \text{ non-comparable}\}$$

of generators of the ideal $I(\mathcal{C})$. Since $\mathcal{G} \subset I_{\mathcal{A}}$, it follows that $\mathcal{G}$ is a Gröbner basis for $I_{\mathcal{A}}$. In particular it is a set of generators for this ideal. Thus $\mathcal{G}$ generates both $I(\mathcal{C})$ and $I_{\mathcal{A}}$, which implies the equality of the two ideals.

$\square$

THEOREM 3.7. — *Let $\mathcal{L}$ be a finite distributive lattice . Then*

1) *The ideal $I(\mathcal{L})$ is toric (and hence prime).*

2) *The set of binomials*

$$\mathcal{G}_{\mathcal{L}} = \{xy - (x \wedge y)(x \vee y) \mid x, y \in \mathcal{L} \text{ non-comparable}\}$$

is a Gröbner basis for $I(\mathcal{L})$ with respect to the reverse lexicographic order.

Proof. — By Theorem 2.4, we may assume that $\mathcal{L}$ is a sublattice of $\mathcal{C}(n_1, \ldots, n_d)$, for some $n_1, \ldots, n_d \geqslant 1$. Let us denote $\mathcal{C} = \mathcal{C}(n_1, \ldots, n_d)$, $\mathcal{A} = \mathcal{A}_{n_1,\ldots,n_d}$ and $\mathcal{G} = \mathcal{G}_{\mathcal{C}}$. Note that $\mathcal{G}_{\mathcal{L}}$ is the subset of $\mathcal{G}$ consisting of all binomials in $\mathcal{G}$ involving only the variables from $\mathcal{L}$. Let us denote

$$\mathcal{G}_{\mathcal{L}} = \{f_1, \ldots, f_r\}, \quad \mathcal{G} \setminus \mathcal{G}_{\mathcal{L}} = \{g_1, \ldots, g_s\}.$$

Let $g_i = xy - (x \wedge y)(x \vee y)$, with $x, y \in \mathcal{C}$ non-comparable, $1 \leqslant i \leqslant s$; then at least one of x and y does not belong to $\mathcal{L}$ ($\mathcal{L}$ being a sublattice of $\mathcal{C}$, $x, y \in \mathcal{L}$ would imply $x \wedge y, x \vee y \in \mathcal{L}$, so g_i would involve only variables from $\mathcal{L}$).

Let $\mathcal{A}_{\mathcal{L}} \subset \mathcal{A}$ be given by the elements in $\mathcal{A}$ indexed by the elements of $\mathcal{L}$, and let f be an element of

$$I_{\mathcal{A}_{\mathcal{L}}} = \ker\left(\widehat{\pi}\big|_{k[\mathcal{L}]}\right) = (\ker \widehat{\pi}) \cap k[\mathcal{L}] = I_{\mathcal{A}} \cap k[\mathcal{L}].$$

In the course of the proof of Theorem 3.6, we saw that $\mathrm{in}(f)$ is divisible by the initial term of a binomial in $\mathcal{G}$, and since $f \in k[\mathcal{L}]$, this binomial must be one of the f_i's, i.e. an element of $\mathcal{G}_{\mathcal{L}}$. Since $\mathcal{G}_{\mathcal{L}} \subset I_{\mathcal{A}_{\mathcal{L}}}$, it follows that $\mathcal{G}_{\mathcal{L}}$ is a Gröbner basis for $I_{\mathcal{A}_{\mathcal{L}}}$, hence $\mathcal{G}_{\mathcal{L}}$ generates $I_{\mathcal{A}_{\mathcal{L}}}$. Therefore $I(\mathcal{L}) = I_{\mathcal{A}_{\mathcal{L}}}$, and the stated assertions follow now. □

4. Varieties defined by binomials.

Let k be an algebraically closed field of arbitrary characteristic. Consider an integer $m \geqslant 1$, and the m-dimensional torus $T_m = (k^*)^m$.

DEFINITION 4.1 (cf. [17]). — *An equivariant affine embedding of T_m is an affine variety X containing T_m as an open subset, equipped with an action of T_m on X*

$$T_m \times X \to X$$

extending the action $T_m \times T_m \to T_m$ given by the translations in T_m.

DEFINITION 4.2 (cf. [10], [17]). — *An equivariant affine embedding X of a torus is called an affine toric variety if it is normal.*

4.3. Let $N \geqslant 1$. For a multi-index $a = (a_1, \ldots, a_N)$, let $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \ldots x_N^{a_N}$. Let X be an affine variety in $\mathbb{A}^N$, not contained in any of the coordinate hyperplanes $\{x_i = 0\}$. Further, let X be irreducible, and let its defining prime ideal $I(X)$ be generated by l binomials

$$(*) \qquad \mathbf{x}^{\mathbf{a}_i} = \lambda_i \mathbf{x}^{\mathbf{b}_i}, \qquad 1 \leqslant i \leqslant l,$$

where $\mathbf{a}_i = (a_{i1}, \ldots, a_{iN})$, $\mathbf{b}_i = (b_{i1}, \ldots, b_{iN})$, and $\lambda_i \in k$. Consider the natural action of the torus $T_N = (k^*)^N$ on $\mathbb{A}^N$,

$$(t_1, \ldots, t_N) \cdot (a_1, \ldots, a_N) = (t_1 a_1, \ldots, t_N a_N).$$

Let $T = \{\mathbf{t} = (t_1, \ldots, t_N) \in T_N \mid \mathbf{t}^{\mathbf{a}_i} = \mathbf{t}^{\mathbf{b}_i},\ 1 \leqslant i \leqslant l\}$, and $X^\circ = \{(x_1, \ldots, x_N) \in X \mid x_i \neq 0 \text{ for all } 1 \leqslant i \leqslant N\}$.

PROPOSITION 4.4. — *Let notations be as above.*

(1) *There is a canonical action of T on X.*

(2) *X° is T-stable. Further, the action of T on X° is simple and transitive.*

(3) *T is a subtorus of T_N, and X is an equivariant affine embedding of T.*

Proof. — (1) We consider the (obvious) action of T on $\mathbb{A}^N$. Let $(x_1, \ldots, x_N) \in X$, $\mathbf{t} = (t_1, \ldots, t_N) \in T$, and $(y_1, \ldots, y_N) = \mathbf{t} \cdot (x_1, \ldots, x_N) = (t_1 x_1, \ldots, t_N x_N)$. Using the fact that $(x_1, \ldots, x_N)$ satisfies $(*)$, we obtain

$$\begin{aligned} y_1^{a_{i1}} \ldots y_N^{a_{iN}} = t_1^{a_{i1}} \ldots t_N^{a_{iN}} x_1^{a_{i1}} \ldots x_N^{a_{iN}} &= \lambda_i t_1^{b_{i1}} \ldots t_N^{b_{iN}} x_1^{b_{i1}} \ldots x_N^{b_{iN}} \\ &= \lambda_i y_1^{b_{i1}} \ldots y_N^{b_{iN}}, \end{aligned}$$

for all $1 \leqslant i \leqslant l$, i.e. $(y_1, \ldots, y_N) \in X$. Hence $\mathbf{t} \cdot (a_1, \ldots, a_N) \in X$ for all $\mathbf{t} \in X$.

(2) Let $x = (x_1, \ldots, x_N) \in X^\circ$, and $\mathbf{t} = (t_1, \ldots, t_N) \in T$. Then, clearly $\mathbf{t} \cdot (x_1, \ldots, x_N) \in X^\circ$. Considering x as a point in $\mathbb{A}^N$, the isotropy subgroup in T_N at x is $\{\mathrm{id}\}$. Hence the isotropy subgroup in T at x is also $\{\mathrm{id}\}$. Thus the action of T on X° is simple.

Let $(x_1,\dots,x_N)$, $(x'_1,\dots,x'_N) \in X^\circ$. Set $\mathbf{t} = (t_1,\dots,t_N)$, where $t_i = x_i/x'_i$. Then, clearly $\mathbf{t} \in T$. Thus $(x_1,\dots,x_N) = \mathbf{t}\cdot(x'_1,\dots,x'_N)$. Hence the action of T on X° is simple and transitive.

(3) Now, fixing a point $x \in X^\circ$, we obtain from (2) that the orbit map $\mathbf{t} \mapsto \mathbf{t}\cdot x$ is in fact an isomorphism of T onto X°. Also, since X is not contained in any of the coordinate hyperplanes, the open set $X_i = \{(x_1,\dots,x_N) \in X \mid x_i \neq 0\}$ is nonempty for all $1 \leqslant i \leqslant N$. The irreducibility of X implies that the sets X_i, $1 \leqslant i \leqslant N$, are open dense in X, and hence their intersection

$$X^\circ = \bigcap_{i=1}^{N} X_i = \{(x_1,\dots,x_N) \in X \mid x_i \neq 0 \text{ for any } i\}$$

is an open dense set in X, and thus X° is irreducible. This implies that T is irreducible (and hence connected). Thus T is a subtorus of T_N. The assertion that X is an equivariant affine embedding of T follows from (1) and (2). □

Remark 4.5. — With notations as in Section 3, note that the variety $V(I_{\mathcal{A}})$ is an equivariant affine embedding of $T_m = (k^*)^m$.

5. Some general results on the variety associated to a finite distributive lattice.

We first recall some basic definitions on finite partially ordered sets. A partially ordered set is also called a poset.

A finite poset P is called *bounded* if it has a unique maximal, and a unique minimal element, denoted $\widehat{1}$ and $\widehat{0}$ respectively.

A totally ordered subset C of a finite poset P is called a *chain*, and the number $\#C - 1$ is called the *length* of the chain.

A bounded poset P is said to be *graded* if all maximal chains have the same length (note that $\widehat{1}$ and $\widehat{0}$ belong to any maximal chain).

Let P be a graded poset. The length of a maximal chain in P is called the *rank* of P.

Let P be a graded poset. For $\lambda,\mu \in P$ with $\lambda \geqslant \mu$, the graded poset $\{\tau \in P \mid \mu \leqslant \tau \leqslant \lambda\}$ is called the *interval from μ to λ*, and denoted by

$[\mu, \lambda]$. The rank of $[\mu, \lambda]$ is denoted by $l_\mu(\lambda)$; if $\mu = \widehat{0}$, then we denote $l_\mu(\lambda)$ by just $l(\lambda)$.

Let P be a graded poset, and $\lambda, \mu \in P$, with $\lambda \geqslant \mu$. The ordered pair (λ, μ) is called a *cover* (and we also say that λ *covers* μ) if $l_\mu(\lambda) = 1$.

DEFINITION 5.1. — *An element z of a lattice $\mathcal{L}$ is called join-irreducible (respectively meet-irreducible) if $z = x \vee y$ (respectively $z = x \wedge y$) implies $z = x$ or $z = y$. The set of join-irreducible (respectively meet-irreducible) elements of $\mathcal{L}$ is denoted by $J_{\mathcal{L}}$ (respectively $M_{\mathcal{L}}$), or just by J (respectively M) if no confusion is possible.*

DEFINITION 5.2. — *The set $J_{\mathcal{L}} \cap M_{\mathcal{L}}$ of join and meet-irreducible elements is denoted by $JM_{\mathcal{L}}$, or just JM if no confusion is possible.*

DEFINITION 5.3. — *A subset I of a poset P is called an ideal of P if for all x, $y \in P$,*

$$x \in I \text{ and } y \leqslant x \text{ imply } y \in I.$$

THEOREM 5.4 (Birkhoff). — *Let $\mathcal{L}$ be a distributive lattice with* 0, *and P the poset of its nonzero join-irreducible elements. Then $\mathcal{L}$ is isomorphic to the lattice of finite ideals of P, by means of the lattice isomorphism*

$$\alpha \mapsto I_\alpha = \{\tau \in P \mid \tau \leqslant \alpha\}, \qquad \alpha \in \mathcal{L}.$$

DEFINITION 5.5. — *A quadruple of the form $(\tau, \phi, \tau \vee \phi, \tau \wedge \phi)$, with $\tau, \phi \in \mathcal{L}$ non-comparable is called a diamond, and is denoted by $D(\tau, \phi, \tau \vee \phi, \tau \wedge \phi)$.*

LEMMA 5.6. — *With the notations as above, we have*

(a) $J = \{\tau \in \mathcal{L} \mid \text{there exists at most one cover of the form } (\tau, \lambda)\}$.

(b) $M = \{\tau \in \mathcal{L} \mid \text{there exists at most one cover of the form } (\lambda, \tau)\}$.

Proof. — In order to prove part (a), let us denote

$$Z = \{\lambda \in \mathcal{L} \mid \text{there exists at most one cover of the form } (\lambda, \rho)\}.$$

Clearly, $J \subset Z$. Let $\lambda \in Z$, and assume $\lambda \notin J$. Assumption implies that there exists a diamond $D(\tau, \phi, \lambda, \mu)$. In particular, this implies $\tau \leqslant \lambda$, $\phi \leqslant \lambda$. Now $\mathcal{L}$ being graded, there exist α, β such that $\tau \leqslant \alpha$, $\phi \leqslant \beta$, and (λ, α), (λ, β) are covers. Now, $\tau \leqslant \alpha$, $\phi \leqslant \beta$, and $\lambda = \alpha \vee \beta$ imply $\alpha \neq \beta$. But this would imply that λ covers two distinct elements, contradicting the hypothesis that $\lambda \in Z$. Hence the assumption that $\lambda \notin J$ is wrong, which shows (a).

The proof of (b) is similar. □

5.7. Let $X(\mathcal{L})$ be the affine variety of the zeroes in k^N of $I(\mathcal{L})$ (note that $X(\mathcal{L})$ is irreducible, in view of Theorem 3.7). Then $X(\mathcal{L})$ is a variety defined by binomials, and we follow the notations in Section 4. We have $N = \#\mathcal{L}$, $I(X(\mathcal{L})) = I(\mathcal{L})$. Let $\mathcal{I} = \{(\tau, \phi, \tau \vee \phi, \tau \wedge \phi) \mid (\tau, \phi) \in Q\}$, where

$$Q = \{(\tau, \phi) \mid \tau, \phi \in \mathcal{L} \text{ non-comparable}\}.$$

In view of Proposition 4.4, $\dim X(\mathcal{L}) = \dim T$, and we now compute the dimension of T.

Let $\pi : X(T_N) \to X(T)$ be the canonical map, given by restriction, and for $\chi \in X(T_N)$, denote $\pi(\chi)$ by $\overline{\chi}$. Let us fix a $\mathbb{Z}$-basis $\{\chi_\tau \mid \tau \in \mathcal{L}\}$ for $X(T_N)$. For a diamond $D = (\tau, \phi, \tau \vee \phi, \tau \wedge \phi) \in \mathcal{I}$, let $\chi_D = \chi_{\tau\vee\phi} + \chi_{\tau\wedge\phi} - \chi_\tau - \chi_\phi$.

LEMMA 5.8. — *We have*

(1) $X(T) \simeq X(T_N)/\ker\pi$.

(2) $\ker\pi$ *is generated by* $\{\chi_D \mid D \in \mathcal{I}\}$.

Proof. — The canonical map π is, in fact, surjective, since T is a subtorus of T_N. Now (1) follows from this. The assertion (2) follows from the definition of T. □

5.9. For $\alpha \in \mathcal{L}$, let I_α be the ideal corresponding to α under the isomorphism in Theorem 5.10. Let

$$\psi_\alpha = \sum_{\theta \in I_\alpha} \chi_\theta.$$

LEMMA 5.10. — *The set* $\{\overline{\psi}_a \mid a \in \mathcal{L}\}$ *generates* $X(T)$ *as a* $\mathbb{Z}$*-module.*

Proof. — Consider the homomorphism

$$\theta : X(T_N) \to X(T), \quad \chi_a \mapsto \overline{\psi}_a,$$

(note that $\{\chi_a \mid a \in \mathcal{L}\}$ is a $\mathbb{Z}$-basis for $X(T_N)$). For a diamond $D = (\tau, \phi, \mu, \lambda) \in \mathcal{I}$, we have

$$(*) \qquad \overline{\psi}_\tau + \overline{\psi}_\phi = \overline{\psi}_\mu + \overline{\psi}_\lambda,$$

and hence $\chi_D \in \ker\theta$. Conversely, it is clear that any relation among $\overline{\psi}_a$'s is of the form $(*)$. Hence $\ker\theta = \ker\pi$, and $X(T) \simeq X(T_N)/\ker\theta$, (cf. Lemma 5.8). In particular, this implies that θ is onto, and the result follows, since $\{\chi_a \mid a \in \mathcal{L}\}$ generates $X(T_N)$ as a $\mathbb{Z}$-module. □

PROPOSITION 5.11. — *The set $\{\overline{\chi}_\tau \mid \tau \in J\}$ is a $\mathbb{Z}$-basis for $X(T)$.*

Proof. — By Lemma 5.10, $\{\overline{\psi}_a \mid a \in \mathcal{L}\}$ generates the $\mathbb{Z}$-module $X(T)$. Now

$$(\dagger) \qquad \overline{\psi}_a = \sum_{\theta\in I_a} \overline{\chi}_\theta = \sum_{\{\theta\in J_{\mathcal{L}},\theta\leqslant a\}} \overline{\chi}_\theta$$

(cf. 5.9). Hence $\{\overline{\chi}_\tau \mid \tau \in J\}$ generates the $\mathbb{Z}$-module $X(T)$. Also it is clear that no proper subset of $\{\overline{\chi}_\tau \mid \tau \in J\}$ generates $X(T)$. The result now follows. □

Now, since $\dim X(\mathcal{L}) = \dim T$, we obtain

THEOREM 5.12. — *The dimension of $X(\mathcal{L})$ is equal to $\#J_{\mathcal{L}}$.*

DEFINITION 5.13. — *Let $\mathcal{L}$ be a finite distributive lattice. We call the cardinality of $J_{\mathcal{L}}$ the dimension of $\mathcal{L}$, and we denote it by* $\dim\mathcal{L}$. *If $\mathcal{L}'$ is a sublattice of $\mathcal{L}$, then the codimension of $\mathcal{L}'$ in $\mathcal{L}$ is defined as* $\dim\mathcal{L} - \dim\mathcal{L}'$.

LEMMA 5.14. — *Let $P = (P_\theta)_{\theta\in\mathcal{L}} \in X(\mathcal{L})$ be such that $P_\tau \neq 0$ for any $\tau \in J$. Then $P_\theta \neq 0$ for any $\theta \in \mathcal{L}$.*

Proof (by induction). — Let $\theta \in \mathcal{L}$. If $\theta \in J_{\mathcal{L}}$, there is nothing to check. Let then $\theta \in \mathcal{L} \setminus J_{\mathcal{L}}$.

Let θ be a minimal element of $\mathcal{L} \setminus J_{\mathcal{L}}$. This implies that every $\tau \in \mathcal{L}$ such that $\tau < \theta$ belongs to $J_{\mathcal{L}}$. The fact that $\theta \in \mathcal{L} \setminus J_{\mathcal{L}}$ implies that there are at least two elements θ_1, θ_2 of $\mathcal{L}$ which are covered by θ. Note that θ_1, θ_2 are not comparable. We have $\theta_1 \vee \theta_2 = \theta$. Let $\mu = \theta_1 \wedge \theta_2$. We have $P_\theta P_\mu = P_{\theta_1} P_{\theta_2}$. Now $P_{\theta_1} \neq 0$, $P_{\theta_2} \neq 0$, since $\theta_1, \theta_2 \in J_{\mathcal{L}}$. Hence we obtain that $P_\theta \neq 0$.

Let now ϕ be any element of $\mathcal{L} \setminus J_{\mathcal{L}}$. Assume, by induction, that $P_\tau \neq 0$ for any $\tau < \phi$. Since $\phi \notin J_{\mathcal{L}}$, there are at least two elements ϕ_1, ϕ_2 of $\mathcal{L}$ which are covered by ϕ. We have $\phi_1 \vee \phi_2 = \phi$, $P_\phi P_\delta = P_{\phi_1} P_{\phi_2}$, where $\delta = \phi_1 \wedge \phi_2$. Also $P_{\phi_1} \neq 0$, $P_{\phi_2} \neq 0$ (since ϕ_1, ϕ_2 are both $< \phi$). Hence we obtain $P_\phi \neq 0$. □

DEFINITION 5.15 (cf. [27]). — A sublattice $\mathcal{L}'$ of $\mathcal{L}$ is called an *embedded sublattice of* $\mathcal{L}$ if

$$\tau,\, \phi \in \mathcal{L}, \quad \tau \vee \phi,\, \tau \wedge \phi \in \mathcal{L}' \quad \Rightarrow \quad \tau,\, \phi \in \mathcal{L}'.$$

Given a sublattice $\mathcal{L}'$ of $\mathcal{L}$, let us consider the variety $X(\mathcal{L}')$, and consider the canonical embedding $X(\mathcal{L}') \hookrightarrow \mathbb{A}(\mathcal{L}') \hookrightarrow \mathbb{A}(\mathcal{L})$ (here $\mathbb{A}(\mathcal{L}') = \mathbb{A}^{\#\mathcal{L}'}$, $\mathbb{A}(\mathcal{L}) = \mathbb{A}^{\#\mathcal{L}}$).

PROPOSITION 5.16. — *$X(\mathcal{L}')$ is a subvariety of $X(\mathcal{L})$ if and only if $\mathcal{L}'$ is an embedded sublattice of $\mathcal{L}$.*

Proof. — Under the embedding $X(\mathcal{L}') \hookrightarrow \mathbb{A}(\mathcal{L})$, $X(\mathcal{L}')$ can be identified with

$$\{(x_\tau)_{\tau\in\mathcal{L}} \in \mathbb{A}(\mathcal{L}) \mid x_\tau = 0 \text{ if } \tau \notin \mathcal{L}',\ x_\tau x_\phi = x_{\tau\vee\phi}x_{\tau\wedge\phi} \text{ for } \tau,\phi \in \mathcal{L}' \text{ non-comparable}\}.$$

Let η' be the generic point of $X(\mathcal{L}')$. We have $X(\mathcal{L}') \subset X(\mathcal{L})$ if and only if $\eta' \in X(\mathcal{L})$.

Assume that $\eta' \in X(\mathcal{L})$. Let τ, ϕ be two noncomparable elements of $\mathcal{L}$ such that $\tau \vee \phi$, $\tau \vee \phi$ are both in $\mathcal{L}'$. We have to show that τ, $\phi \in \mathcal{L}'$. If possible, let $\tau \notin \mathcal{L}'$. This implies $\eta'_\tau = 0$. Hence either $\eta'_{\tau\vee\phi} = 0$, or $\eta'_{\tau\wedge\phi} = 0$, since $\eta' \in X(\mathcal{L})$. But this is not possible (note that $\tau \vee \phi$, $\tau \wedge \phi$ are in $\mathcal{L}'$, and hence $\eta'_{\tau\vee\phi}$ and $\eta'_{\tau\wedge\phi}$ are both nonzero).

Assume now that $\mathcal{L}'$ is an embedded sublattice. We have to show that $\eta' \in X(\mathcal{L})$. Let τ, ϕ be two noncomparable elements of $\mathcal{L}$. The fact that $\mathcal{L}'$ is a sublattice implies that if $\eta'_{\tau\vee\phi}$ or $\eta'_{\tau\wedge\phi}$ is zero, then either η'_τ, or η'_ϕ is zero. Also, the fact that $\mathcal{L}'$ is an embedded sublattice implies that if η'_τ or η'_ϕ is zero, then either $\eta'_{\tau\vee\phi}$ or $\eta'_{\tau\wedge\phi}$ is zero. Further, when τ, ϕ, $\tau \vee \phi$, $\tau \wedge \phi \in \mathcal{L}'$,

$$\eta'_\tau\eta'_\phi = \eta'_{\tau\vee\phi}\eta'_{\tau\wedge\phi}.$$

Thus η' satisfies the defining equations of $X(\mathcal{L})$, and hence $\eta' \in X(\mathcal{L})$. □

6. A standard monomial basis for $R(\mathcal{L})$.

6.1. Let $R(\mathcal{L}) = k[\mathcal{L}]/I(\mathcal{L})$, $k[\mathcal{L}]$ being as in Section 3. Note that $R(\mathcal{L})$ is a domain (cf. Theorem 3.7). Let $X(\mathcal{L}) = \mathrm{Spec}R(\mathcal{L})$. For $\theta \in \mathcal{L}$, let us consider the sublattice $\mathcal{L}_\theta = \{\lambda \in \mathcal{L} \mid \lambda \leqslant \theta\}$ and the ring $R_\theta = R_{\mathcal{L}}/\langle x_\lambda \mid \lambda \in \mathcal{L}, \lambda \not\leqslant \theta\rangle \cong R(\mathcal{L}_\theta)$ and denote $\mathrm{Spec}R_\theta$ by $X(\mathcal{L}_\theta)$, or just X_θ. Note that $\mathcal{L}_\theta$ is an embedded sublattice (cf. Definition 5.15). In the sequel, for $\tau \in \mathcal{L}$, we shall denote $\bar{x}_\tau$ (in $R(\mathcal{L})$) by just x_τ; similarly, the restriction $x_\tau\big|_{X_\theta}$ will be also denoted by just x_τ.

6.2. We take a total order $\succ$ on $\mathcal{L}$ extending the partial order $>$, and a monomial of degree r in $R(\mathcal{L})$ will be written as $x_{\tau_1} \ldots x_{\tau_r}$, with $\tau_1 \succeq \ldots \succeq \tau_r$.

DEFINITION 6.3. — *A monomial $x_{\tau_1} \ldots x_{\tau_r}$ of degree r is said to be standard on $X(\mathcal{L})$, if $\tau_1 \geqslant \ldots \geqslant \tau_r$. Such a monomial is said to be standard on X_θ if $\theta \geqslant \tau_1 \geqslant \ldots \geqslant \tau_r$.*

PROPOSITION 6.4. — *Standard monomials on X_θ are linearly independent in R_θ.*

Proof. — R_θ being graded, it suffices to prove the linear independence of standard monomials of a given degree, say r. We shall prove this by induction on $l(\theta)$, and r. If $l(\theta) = 0$, or $r = 1$, the result is clear. Let then $l(\theta) \geqslant 1$, $r > 1$, and let

$$(*) \qquad \sum_{i \in I} a_i F_i = 0, \qquad a_i \in k^*$$

be a linear relation, where $F_i = \tau_{i1} \ldots \tau_{ir}$, $\tau_{i1} \geqslant \ldots \geqslant \tau_{ir}$. If $\tau_{i1} < \theta$ for some i, then restricting $(*)$ to $X_{\tau_{i1}}$, we obtain (by induction hypothesis) that $a_j = 0$ for all j with $\tau_{j1} \leqslant \tau_{i1}$ (note that $F_j\big|_{X_{\tau_{i1}}}$ is either identically zero, or it remains standard on $X_{\tau_{i1}}$). Thus we may suppose that $\tau_{i_1} = \theta$ for all $i \in I$. Now x_θ can be cancelled out (since R_θ is a domain), and the result follows by induction on r. □

PROPOSITION 6.5. — *Any monomial in R_θ is standard.*

Proof. — Let $F = x_{\tau_1} \ldots x_{\tau_r}$. If there is an i such that $\tau_i \not\geqslant \tau_{i+1}$, this implies τ_i, τ_{i+1} are non-comparable. Then, denoting $\tau_i \vee \tau_{i+1}$, $\tau_i \wedge \tau_{i+1}$ by λ, μ respectively, we have $F = x_{\tau_1} \ldots x_\lambda x_\mu \ldots x_{\tau_r}$, and the new expression of F is lexicographically greater than the previous one, since $\lambda > \tau_i$. Continuing this process, after a finite number of steps we arrive at a standard monomial expression of F (since $\mathcal{L}$ is finite). □

Combining Propositions 6.4 and 6.5, we obtain

THEOREM 6.6. — *Standard monomials on X_θ form a basis of R_θ.*

7. Cohen-Macaulayness of $R(\mathcal{L})$.

In this section we prove the Cohen-Macaulayness of $R(\mathcal{L})$ using deformation techniques. Let $S(\mathcal{L})$ denote the Stanley-Reisner algebra of $\mathcal{L}$, namely $k[\mathcal{L}]/\langle x_\alpha x_\beta \mid (\alpha,\beta) \in Q\rangle$ (recall that $Q = \{(\alpha,\beta) \mid \alpha, \beta \in \mathcal{L}$ noncomparable $\}$).

Recall the following (cf. [4]):

THEOREM 7.1. — *The ring $S(\mathcal{L})$ is Cohen-Macaulay.*

We now construct a flat family over $k[t]$ whose general fiber is $R(\mathcal{L})$ and special fiber is $S(\mathcal{L})$. This construction is done in the same spirit as in [12].

7.2. We first assign positive integers n_τ, $\tau \in \mathcal{L}$, in such a way that if $\tau > \tau'$, then $n_\tau > n_{\tau'}$ (for example we may take $n_\tau = l(\tau)$). We choose an integer N, and set $N_\tau = N^{l(\tau)}$. Then, since $\mathcal{L}$ is finite, we can choose N sufficiently large so that for any diamond $D(\tau, \phi, \tau\vee\phi, \tau\wedge\phi)$,

$$N_{\tau\vee\phi} + N_{\tau\wedge\phi} > N_\tau + N_\phi.$$

THEOREM 7.3. — *There exists a flat family over $k[t]$ whose special fiber is $S(\mathcal{L})$ and general fiber is $R(\mathcal{L})$.*

Proof. — For $\tau, \phi \in Q$, let $f_{\tau,\phi}$ be the element in $k[\mathcal{L}]$ given by

$$f_{\tau,\phi} = x_\tau x_\phi - x_{\tau\vee\phi} x_{\tau\wedge\phi}.$$

Then $I(\mathcal{L}) = \langle f_{\tau,\phi} \mid (\tau,\phi) \in Q\rangle$. Let us denote $R(\mathcal{L})$, $S(\mathcal{L})$, $k[\mathcal{L}]$, by R, S, P respectively. Let $A = k[t]$, and $P_A = A[x_\alpha, \alpha \in \mathcal{L}]$. For $(\tau,\phi) \in Q$, we define the element $f_{\tau,\phi,t}$ in P_A as

$$f_{\tau,\phi,t} = x_\tau x_\phi - x_{\tau\vee\phi} x_{\tau\wedge\phi} t^{N_{\tau\vee\phi}+N_{\tau\wedge\phi}-N_\tau-N_\phi}.$$

(note that $f_{\tau,\phi,t}$ is well defined, in view of the choice of N_α's).

Let $\mathcal{I}$ be the ideal in P_A generated by $\{f_{\tau,\phi,t} \mid (\tau,\phi) \in Q\}$, and $\mathcal{R} = P_A/\mathcal{I}$.

CLAIM. — (a) *$\mathcal{R}$ is $k[t]$-free.*

(b) $\mathcal{R} \bigotimes\limits_{k[t]} k[t,t^{-1}] \simeq R[t,t^{-1}]$.

(c) $\mathcal{R} \bigotimes\limits_{k[t]} k[t]/(t) \simeq S$.

Proof. — We have

$$\mathcal{R} \bigotimes_{k[t]} k[t]/(t) = P_A/\langle \mathcal{I} + (t)\rangle = S.$$

This proves (c). Let $B = k[t,t^{-1}]$, and $P_B = B[x_\alpha, \alpha \in \mathcal{L}]$. Let $\tilde{I}$(resp. $\tilde{\mathcal{I}}$) be the ideal in P_B generated by $\{f_{\tau,\phi} \mid (\tau,\phi) \in Q\}$(resp. $\{f_{\tau,\phi,t} \mid (\tau,\phi) \in Q\}$. We have

$$P_B/\tilde{I} \simeq R[t,t^{-1}] \tag{7}$$

$$P_B/\tilde{\mathcal{I}} \simeq \mathcal{R} \bigotimes_{k[t]} k[t,t^{-1}]. \tag{8}$$

The automorphism

$$P_B \simeq P_B, \qquad x_\alpha \mapsto t^{N_\alpha} x_\alpha$$

induces an isomorphism

$$P_B/\tilde{I} \simeq P_B/\tilde{\mathcal{I}}. \tag{9}$$

From (7), (8), (9) we obtain (b). Finally, it remains to show (a). Let $X_\alpha = \bar{x}_\alpha$ (in $\mathcal{R} = P_A/\mathcal{I}$), $f_\alpha = t^{N_\alpha} X_\alpha$ and

$$\mathcal{M} = \{f_{\alpha_1} \cdots f_{\alpha_r} \mid \alpha_1 \geqslant \ldots \geqslant \alpha_r,\ r \in \mathbb{Z}_+\}.$$

We shall now show that $\mathcal{M}$ is a $k[t]$-basis for $\mathcal{R}$. We first observe that $\mathcal{M}$ is a $k[t]$-basis for $\mathcal{R}$. We first observe that any monomial $F = f_{\tau_1} \ldots f_{\tau_r}$ is in fact standard. In order to see this, let i be such that $\tau_i \not\geqslant \tau_{i+1}$. Then using the relation

$$X_{\tau_i} X_{\tau_{i+1}} = X_{\tau_i \vee \tau_{i+1}} X_{\tau_i \wedge \tau_{i+1}} t^{N_{\tau_i \vee \tau_{i+1}} + N_{\tau_i \wedge \tau_{i+1}} - N_{\tau_i} - N_{\tau_{i+1}}},$$

we obtain $F = f_{\tau_1} \ldots f_{\tau_{i-1}} f_{\tau_i \vee \tau_{i+1}} f_{\tau_i \wedge \tau_{i+1}} \cdots f_{\tau_r}$. Continuing thus, we find (as in the proof of Proposition 6.5) that at each step the expression for F

is lexicographically greater than the expression at the previous step (here we suppose $\tau_1 \succeq \ldots \succeq \tau_r$), and thus we arrive at an expression for F as a standard monomial (since $\mathcal{L}$ is finite).

It remains to prove the linear independence of $\mathcal{M}$. Since standard monomials form abasis for R (cf. Theorem 6.6), we obtain (by base change), that $\mathcal{M}$ is a $k[t,t^{-1}]$-basis for $R[t,t^{-1}]$. Denoting the isomorphism $P_B/\tilde{\mathcal{I}} \simeq R[t,t^{-1}]$ by φ, we have $\{\varphi^{-1}(f_{\alpha_1}\cdots f_{\alpha_r}) \mid \alpha_1 \geqslant \ldots \geqslant \alpha_r, r \in \mathbb{Z}_+\}$ is a $k[t,t^{-1}]$-basis for $\mathcal{R}[t^{-1}]$. For a monomial $\underline{m} = x_{\tau_1}\ldots x_{\tau_r}$ (in $R[t,t^{-1}]$), we have $\varphi^{-1}(\underline{m}) = t^{-N_{\underline{m}}}X_{\tau_1}\ldots X_{\tau_r}$, where $N_{\underline{m}} = \sum_{i=1}^{r} N_{\tau_i}$. Hence we obtain $\{f_{\alpha_1}\cdots f_{\alpha_r} \mid \alpha_1 \geqslant \ldots \geqslant \alpha_r, r \in \mathbb{Z}_+\}$ is a $k[t,t^{-1}]$-basis for $\mathcal{R}[t^{-1}]$ (since $t^{-N_{\underline{m}}}$ is a unit in $k[t,t^{-1}]$). In particular, we obtain that $\{f_{\alpha_1}\cdots f_{\alpha_r} \mid \alpha_1 \geqslant \ldots \geqslant \alpha_r, r \in \mathbb{Z}_+\}$ is linearly independent over $k[t,t^{-1}]$, and hence over $k[t]$. □

Combining Theorems 7.1 and 7.3, we obtain

THEOREM 7.4. — *The ring $R(\mathcal{L})$ is Cohen-Macaulay.*

8. The distributive lattice $I_{d,n}$ and the variety $X_{d,n}$.

Let

$$I_{d,n} = \{\tau = (i_1,\ldots,i_d) \mid 1 \leqslant i_1 < \ldots < i_d \leqslant n\}.$$

We consider the partial order $\geqslant$ on $I_{d,n}$ given by

$$(i_1,\ldots,i_d) \geqslant (j_1,\ldots,j_d) \iff i_1 \geqslant j_1,\ldots,i_d \geqslant j_d.$$

For $\tau \in I_{d,n}$, we denote the j-th entry in τ by $\tau(j)$, $1 \leqslant j \leqslant d$.

PROPOSITION 8.1. — *$(I_{d,n},\geqslant)$ is a distributive lattice.*

Proof. — Let $\tau,\phi \in I_{d,n}$, say $\tau = (i_1,\ldots,i_d)$, $\phi = (j_1,\ldots,j_d)$. Let $k_t = \max\{i_t,j_t\}$, $l_t = \min\{i_t,j_t\}$, $1 \leqslant t \leqslant d$. Then it is easily checked that $\tau \vee \phi = (k_1,\ldots,k_d)$, $\tau \wedge \phi = (l_1,\ldots,l_d)$, and that $(I_{d,n},\geqslant)$ is a distributive lattice. □

For the rest of this section, the lattice $I_{d,n}$ will be denoted by simply $\mathcal{L}$, and we use the notations introduced in Section 5.

In the discussion below, by a *segment* we shall mean a set consisting of consecutive integers.

LEMMA 8.2. — *We have*

(a) *The element $\tau = (i_1, \dots, i_d)$ is join-irreducible if and only if either τ is a segment, or τ consists of two disjoint segments (μ, ν), with μ starting with* 1.

(b) *The element $\tau = (i_1, \dots, i_d)$ is meet-irreducible if and only if either τ is a segment, or τ consists of two disjoint segments (μ, ν), with ν ending with n.*

(c) *The element $\tau = (i_1, \dots, i_d)$ is join-irreducible and meet-irreducible if and only if either τ is a segment, or τ consists of two disjoint segments (μ, ν), with μ starting with* 1 *and ν ending with n.*

Proof. — We first observe that $(i_1, \dots, i_d)$ is join-irreducible if and only if $(n+1-i_d, \dots, n+1-i_1)$ is meet-irreducible. Thus it suffice to prove part (a). It is easily seen that $\tau = (i_1, \dots, i_d)$ is a cover for $(j_1, \dots, j_d) \in \mathcal{L}$ if and only if $\{j_1, \dots, j_d\}$ is obtained from $(i_1, \dots, i_d)$ by replacing i_t by $i_t - 1$ for precisely one t, and this proves (a). Part (c) follows from (a) and (b). □

8.3. For a join-irreducible or meet-irreducible element $\tau \in \mathcal{L}$ we say that τ is of Type I (resp. Type II) if τ consists of just one segment (resp. two disjoint segments), as in the description given by Lemma 8.2. We denote by $J_{\mathcal{L}}^{(I)}$, $JM_{\mathcal{L}}^{(I)}$ (resp. $J_{\mathcal{L}}^{(II)}$, $JM_{\mathcal{L}}^{(II)}$) respectively the set of elements of $J_{\mathcal{L}}$, $JM_{\mathcal{L}}$ of Type I (resp. Type II). Note that

$$JM_{\mathcal{L}}^{(I)} = \{(i+1, \dots, i+d) \mid 0 \leqslant i \leqslant n-d\},$$

and

$$JM_{\mathcal{L}}^{(II)} = \{(1, \dots, j, n+j+1-d, \dots, n) \mid 1 \leqslant j \leqslant d-1\}.$$

For $\tau_i = (i+1, \dots, i+d) \in JM_{\mathcal{L}}^{(I)}$ and $\phi_j = (1, \dots, j, n+j+1-d, \dots, n) \in JM_{\mathcal{L}}^{(II)}$, let $\lambda_{ij} = \tau_i \vee \phi_j$, $\mu_{ij} = \tau_i \wedge \phi_j$. Note that $\tau_0 = \widehat{0}$ and $\tau_{n-d} = \widehat{1}$. Clearly,

$$\lambda_{ij} = (i+1, \dots, i+j, n+1+j-d, \dots, n), \quad \mu_{ij} = (1, \dots, j, i+j+1, \dots, i+d).$$

In the sequel, $X_{d,n}$ will denote the variety $X(\mathcal{L})$, for $\mathcal{L} = I_{d,n}$.

9. The irreducible components of $X_{d,n} \setminus X^{\circ}_{d,n}$.

We preserve the notations of Section 8. As in Section 4, let $X^{\circ}_{d,n} = \{(x_1, \ldots, x_N) \in X_{d,n} \mid x_i \neq 0 \text{ for all } i\}$.

LEMMA 9.1. — *Let $\theta \in J_{\mathcal{L}} \setminus JM_{\mathcal{L}}$, say $\theta = (1, \ldots, j, t+1, \ldots, t+d-j)$, where $j < t < n + j - d$. Then $\theta = \mu_{ij}$, where $i = t - j$.*

The result follows from the definition of μ_{ij} (note that $0 < i < n-d$).

9.2. Let

$$\mathcal{L}_i^{(r)} = \{\tau \in \mathcal{L} \mid \tau(r) \neq i + r\}$$

where $0 \leqslant i \leqslant n - d$, and $1 \leqslant r \leqslant d$.

PROPOSITION 9.3. — *With notations as above, $\mathcal{L}_i^{(r)}$ is an embedded sublattice of $\mathcal{L}$.*

Proof. — Let γ, $\delta \in \mathcal{L}_i^{(r)}$. Then clearly $\gamma \vee \delta$, $\gamma \wedge \delta \in \mathcal{L}_i^{(r)}$. Similarly, if γ, δ in $\mathcal{L}$ are such $\gamma \vee \delta$, $\gamma \wedge \delta \in \mathcal{L}_i^{(r)}$, again it is clear that γ, $\delta \in \mathcal{L}_i^{(r)}$. □

PROPOSITION 9.4. — *Let notations be as above. For $0 \leqslant i \leqslant n - d$, let*

$$Y_i = X_{d,n} \cap \{x_{\tau_i} = 0\}.$$

Then

(1) *Y_0 and Y_{n-d} are irreducible,*

(2) *the irreducible components of Y_i, $1 \leqslant i \leqslant n - d - 1$, are precisely $X(\mathcal{L}_i^{(r)})$, $1 \leqslant r \leqslant d$.*

Proof. — In order to prove (1), it is enough to observe that $Y_0 = X(\mathcal{L} \setminus \{\widehat{0}\})$, $Y_{n-d} = X(\mathcal{L} \setminus \{\widehat{1}\})$. Now we prove (2). We have, in view of Propositions 5.16 and 9.3, that $X(\mathcal{L}_i^{(r)})$, $1 \leqslant r \leqslant d$ is an irreducible subvariety of $X(\mathcal{L})$. Also $X(\mathcal{L}_i^{(r)}) \subset Y_i$ (since $\tau_i \notin \mathcal{L}_i^{(r)}$).

Let now $Z \subset Y_i$, Z irreducible. Let η be the generic point of Z. Let $\tau \in \mathcal{L}$, $\tau \neq \tau_i$ such that $\eta_\tau = 0$ (note that, since $\eta_{\tau_i} = 0$, for at least one $\tau \in \mathcal{L}$, $\tau \neq \tau_i$, $\eta_\tau = 0$).

CLAIM. — *$\tau \in \mathcal{L}_i^{(r)}$, for some r, $1 \leqslant r \leqslant d$.*

Proof. — Assume that $\tau \notin \mathcal{L}_i^{(r)}$, for all $1 \leqslant r \leqslant d$. This implies $\tau(r) = i + r$ for all $1 \leqslant r \leqslant d$. But this implies that $\tau = \tau_i$, contradicting the choice of τ. Hence our assumption is wrong, and the claim follows. □

Now the claim implies that $\eta \in X(\mathcal{L}_i^{(r)})$, for some r, $1 \leqslant r \leqslant d$, and hence $Z \subset X(\mathcal{L}_i^{(r)})$. □

COROLLARY 9.5. — *The subvariety $X(\mathcal{L}_i^{(r)})$, $1 \leqslant i \leqslant n - d - 1$, $1 \leqslant r \leqslant d$, of $X_{d,n}$, has codimension* 1.

9.6. We next determine the irreducible components of $X_{d,n} \cap \{x_\phi = 0\}$, for $\phi \in JM_{\mathcal{L}}$ of Type II. Let

$$\phi = \phi_j = (1, \ldots, j, n + 1 + j - d, \ldots, n),$$

where $1 \leqslant j \leqslant d - 1$. Fix r, with $j + 1 \leqslant r \leqslant n + j - d$, and let

$$\mathcal{L}_{j,r} = \{\tau \in \mathcal{L} \mid \text{ either } \tau(j + 1) \leqslant r \text{ or } \tau(j) \geqslant r\}.$$

PROPOSITION 9.7. — *With notations as above, $\mathcal{L}_{j,r}$ is an embedded sublattice of $\mathcal{L}$.*

Proof. — Let γ, $\delta \in \mathcal{L}_{j,r}$. Then either $\gamma(j) \geqslant r$, or $\gamma(j + 1) \leqslant r$, and $\delta(j) \geqslant r$, or $\delta(j + 1) \leqslant r$.

CLAIM. — $\gamma \vee \delta$, $\gamma \wedge \delta \in \mathcal{L}_{j,r}$.

Proof. — We distinguish four cases.

Case 1: $\gamma(j) \geqslant r$, $\delta(j) \geqslant r$

This implies that $(\gamma \vee \delta)(j) \geqslant r$, $(\gamma \wedge \delta)(j) \geqslant r$, and hence $\gamma \vee \delta$, $\gamma \wedge \delta \in \mathcal{L}_{j,r}$.

Case 2: $\gamma(j) \geqslant r$, $\delta(j + 1) \leqslant r$

This implies that $(\gamma \vee \delta)(j) \geqslant r$, $(\gamma \wedge \delta)(j + 1) \leqslant r$, and hence $\gamma \vee \delta$, $\gamma \wedge \delta \in \mathcal{L}_{j,r}$.

Case 3: $\gamma(j + 1) \leqslant r$, $\delta(j) \geqslant r$

This is similar to the previous case.

Case 4: $\gamma(j + 1) \leqslant r$, $\delta(j + 1) \leqslant r$

In this case, $(\gamma \vee \delta)(j + 1) \leqslant r$, $(\gamma \wedge \delta)(j + 1) \leqslant r$, and hence $\gamma \vee \delta$, $\gamma \wedge \delta \in \mathcal{L}_{j,r}$. □

CLAIM. — *Let γ, δ be two noncomparable elements of $\mathcal{L}$ such that $\gamma \vee \delta$, $\gamma \wedge \delta \in \mathcal{L}_{j,r}$. Then γ, $\delta \in \mathcal{L}_{j,r}$.*

Proof. — Here, we distinguish three cases.

Case 1: $(\gamma \wedge \delta)(j) \geqslant r$

This implies $\gamma(j) \geqslant r$, $\delta(j) \geqslant r$. Hence γ, $\delta \in \mathcal{L}_{j,r}$.

Case 2: $(\gamma \vee \delta)(j+1) \leqslant r$

This implies $\gamma(j+1) \leqslant r$, $\delta(j+1) \leqslant r$. Hence γ, $\delta \in \mathcal{L}_{j,r}$.

Case 3: $(\gamma \vee \delta)(j) \geqslant r$, $(\gamma \wedge \delta)(j+1) \leqslant r$.

Now $(\gamma \vee \delta)(j) \geqslant r$ implies that at least one of $\{\gamma(j), \delta(j)\}$ is $\geqslant r$. Assume that $\gamma(j) \geqslant r$. This implies $\gamma(j+1) \geqslant r+1$, and hence $\delta(j+1) \leqslant r$ (since $(\gamma \wedge \delta)(j+1) \leqslant r$). Thus, $\gamma(j) \geqslant r$, and $\delta(j+1) \leqslant r$. Hence γ, $\delta \in \mathcal{L}_{j,r}$. □

The above two claims show that $\mathcal{L}_{j,r}$ is an embedded sublattice. □

PROPOSITION 9.8. — *Let notations be as above. For $1 \leqslant j \leqslant d-1$, let*

$$Y_j = X_{d,n} \cap \{x_{\phi_j} = 0\}.$$

Then the irreducible components of Y_j are precisely $X(\mathcal{L}_{j,r})$, $j+1 \leqslant r \leqslant n+j-d$.

Proof. — We have, in view of Propositions 5.16 and 9.7, that $X(\mathcal{L}_{j,r})$ is an irreducible subvariety of $X(\mathcal{L})$. Also $X(\mathcal{L}_{j,r}) \subset Y_j$ (since $\phi_j \notin \mathcal{L}_{j,r}$). Let now $Z \subset Y_j$, Z irreducible. Let η be the generic point of Z. Consider an element $\tau \in \mathcal{L}$, $\tau \neq \phi_j$ such that $\eta_\tau = 0$ (note that, since $\eta_{\phi_j} = 0$, for at least one $\tau \in \mathcal{L}$, $\tau \neq \phi_j$, $\eta_\tau = 0$).

CLAIM. — *$\tau \in \mathcal{L}_{j,r}$, for some $j+1 \leqslant r \leqslant n+j-d$.*

Proof. — Assume that $\tau \notin \mathcal{L}_{j,r}$, for all $j+1 \leqslant r \leqslant r+j-d$. Then for $r = j+1$, we have $\tau(j) < j+1$. Hence we obtain $\tau(j) = j$. Similarly, for $r = n+j-d$, we have $\tau(j+1) > n+j-d$. Hence $\tau(j+1) = n+1+j-d$ (note that $\tau(j+1) \leqslant n+1+j-d$ for any $\tau \in \mathcal{L}$). Thus we obtain that $\tau = (1, \ldots, j, n+1+j-d, \ldots, n)$, i.e. $\tau = \phi_j$, which is not true, by our choice of τ. Hence our assumption is wrong, and the claim follows. □

Now, the claim implies that $\eta \in X(\mathcal{L}_{j,r})$, for some r, with $j+1 \leqslant r \leqslant n+j-d$, and hence $Z \subset X(\mathcal{L}_{j,r})$, for some r, with $j+1 \leqslant r \leqslant n+j-d$.

□

Corollary 9.9. — *The subvariety* $X(\mathcal{L}_{j,r})$, $1 \leqslant j \leqslant d-1$, $j+1 \leqslant r \leqslant n+j-d$, *of* $X_{d,n}$, *has codimension* 1.

Proposition 9.10. — *The irreducible components of* $X_{d,n} \setminus X^{\circ}_{d,n}$ *are precisely the subvarieties* $X(\mathcal{L}_i^{(r)})$, $1 \leqslant i \leqslant n-d-1$, $1 \leqslant r \leqslant d$, $X(\mathcal{L}_{j,r})$, $1 \leqslant j \leqslant d-1$, $j+1 \leqslant r \leqslant n+j-d$, *and* $X(\mathcal{L} \setminus \{\widehat{0}\})$, $X(\mathcal{L} \setminus \{\widehat{1}\})$.

Proof. — Let $Z \subset X_{d,n} \setminus X^{\circ}_{d,n}$, Z irreducible, and let η be the generic point of Z. In view of Propositions 9.4 and 9.8, it is enough to show that there exists a $\tau \in JM_{\mathcal{L}}$ such that $\eta_\tau = 0$.

Assume that $\eta_\tau \neq 0$, for any $\tau \in JM_{\mathcal{L}}$. Let $\theta \in J_{\mathcal{L}} \setminus JM_{\mathcal{L}}$. Then θ is a Type I join-irreducible element of $\mathcal{L}$. Let

$$\theta = (1, \ldots, j, t+1, \ldots, t+d-j).$$

We have $\theta = \mu_{ij}$, where $i = t-j$ (cf. Lemma 9.1). Thus $\theta = \tau_i \wedge \phi_j$, and hence our assumption implies that $\eta_\theta \neq 0$. This, together with our assumption, implies that $\eta_\delta \neq 0$, for any $\delta \in \mathcal{J}_{\mathcal{L}}$. This implies $\eta \in X^{\circ}_{d,n}$ (cf. Lemma 5.14), which is not true. Hence our assumption is wrong. □

Lemma 9.11. — *We have*

(a) *for* $1 \leqslant i \leqslant n-d-1$, $JM_{\mathcal{L}} \setminus \{\tau_i\} \subset \mathcal{L}_i^{(r)}$ *for all* r, $1 \leqslant r \leqslant d$, *where* $\tau_i = (i+1, \ldots, i+d)$,

(b) *for* $1 \leqslant j \leqslant d-1$, $JM_{\mathcal{L}} \setminus \{\phi_j\} \subset \mathcal{L}_{j,r}$, *for all* r, $j+1 \leqslant r \leqslant n+j-d$, *where* $\phi_j = (1, \ldots, j, n+j+1-d, \ldots, i+d)$.

Proof. — (a) Fix r, $1 \leqslant r \leqslant d$. Let $\theta \in JM_{\mathcal{L}}$. First, let θ be of Type I, say $\theta = (j+1, \ldots, j+d)$, where $j \neq i$. Then, clearly, $\theta \in \mathcal{L}_i^{(r)}$. Now let θ be of Type II, say $\theta = (1, \ldots, j, n+1+j-d, \ldots, n)$. Then $\theta(r) = r$ if $r \leqslant j$, and $\theta(r) = n-d+r$ if $r > j$. In either case, it is clear that $\theta \in \mathcal{L}_i^{(r)}$ for all r.

(b) Fix r, $j+1 \leqslant r \leqslant n+j-d$. Let $\theta \in JM_{\mathcal{L}}$. First, let θ be of Type I, say $\theta = (i+1, i+2, \ldots, i+d)$. We have $\theta(j) = i+j$, $\theta(j+1) = i+j+1$. Hence $\theta(j) \geqslant r$, if $i+j \geqslant r$, and $\theta(j+1) \leqslant r$, if $i+j+1 \leqslant r$. Thus $\theta \in \mathcal{L}_{j,r}$.

Now let θ be of Type II, say $\theta = (1, \ldots, s, n+1+s-d, \ldots, n)$. If $j < s$, then $\theta(j+1) = j+1 \leqslant r$, and hence $\theta \in \mathcal{L}_{j,r}$. If $j \geqslant s$, then $\theta(j) = n-d+j$, and $\theta(j+1) = n-d+j+1$. Hence either $n-d+j+1 \leqslant r$, or $n-d+j \geqslant r$. Thus $\theta \in \mathcal{L}_{j,r}$. □

Proposition 9.12. — *Let Y be an irreducible component of $X_{d,n} \setminus X^{\circ}_{d,n}$. Let P be the generic point of Y. Then $P_\tau = 0$, for precisely one element $\tau \in JM_{\mathcal{L}}$.*

Proof. — If Y is equal to $X(\mathcal{L} \setminus \{\widehat{0}\})$ or $X(\mathcal{L} \setminus \{\widehat{1}\})$ (cf. Proposition 9.10), the assertion is obvious. For the other components the result follows from Proposition 9.10, and Lemma 9.11. □

10. Normality of $X_{d,n}$.

In this section we prove that the variety $X_{d,n}$ is normal. We already know that $X_{d,n}$ is Cohen-Macaulay (cf. Section 7), and in view of Serre Criterion, it suffices to show that $X_{d,n}$ is nonsingular in codimension 1. In view of Proposition 9.10, it is enough to prove that generic points of $X(\mathcal{L} \setminus \{\widehat{0}\})$, $X(\mathcal{L} \setminus \{\widehat{1}\})$, $X(\mathcal{L}_i^{(r)})$, $1 \leqslant i \leqslant n-d-1$, $1 \leqslant r \leqslant d$, $X(\mathcal{L}_{j,r})$, $1 \leqslant j \leqslant d-1$, $j+1 \leqslant r \leqslant n+j-d$ are smooth points. We prove this using the Jacobian Criterion.

Let $\mathcal{J}$ be the Jacobian matrix of $X_{d,n}$, where the rows of $\mathcal{J}$ are indexed by

$$\{f_{\tau,\phi} = x_\tau x_\phi - x_{\tau\vee\phi} x_{\tau\wedge\phi} \mid \tau, \phi \in \mathcal{L} \text{ noncomparable }\},$$

and the columns are indexed by $\{x_\theta \mid \theta \in \mathcal{L}\}$. For simplicity, sometimes we consider the rows being indexed by the diamonds in $\mathcal{L}$, and the columns being indexed by just the elements of $\mathcal{L}$.

Recall that for any $P \in X_{d,n}$, if $\mathcal{J}_P$ is the Jacobian matrix evaluated at P, then $\operatorname{rank} \mathcal{J}_P \leqslant \operatorname{codim}_{\mathbb{A}(\mathcal{L})} X_{d,n}$, with equality if and only if P is a smooth point.

Proposition 10.1. — *Let $P \in X_{d,n}$ be such that $P_\tau \neq 0$ for all $\tau \in JM_{\mathcal{L}}^{(I)} \setminus \{\tau_0, \tau_{n-d}\}$. Then P is a smooth point.*

Proof. — Let $\mathcal{J}_P$ be the Jacobian matrix evaluated at P. We shall now exhibit a submatrix of $\mathcal{J}_P$ of maximal rank, equal to

$$\operatorname{codim}_{\mathbb{A}(\mathcal{L})} X_{d,n} = \binom{n}{d} - d(n-d) - 1.$$

For each i, $2 \leqslant i \leqslant n-d$, let

$$Z_i = \{\theta \in \mathcal{L} \mid \theta(d) = i+d\} \setminus \{\tau_{ij},\ 0 \leqslant j \leqslant d-1\},$$

where for $0 \leqslant j \leqslant d-1$,

$$\tau_{ij} = (i, i+1, \ldots, i+j-1, i+j+1, i+j+2, \ldots, i+d)$$

(note that $\tau_{i-1} < \tau_{id-1} < \ldots < \tau_{i0} = \tau_i$, where $\tau_{i-1} = (i, i+1, \ldots, i+d-1)$ and $\tau_i = (i+1, i+2, \ldots, i+d)$, as defined in Section 8).

Now, τ_{ij}, $0 \leqslant j \leqslant d-1$ are precisely the elements with the properties $\tau_{ij}(d) = i+d$, and $\tau_{ij} > \tau_{i-1}$. Hence τ_{i-1} is not comparable with any θ in Z_i. Now, for each $\theta \in Z_i$, we associate the row indexed by $f_{\tau_{i-1},\theta}$. We consider the submatrix $\mathcal{J}'$ of $\mathcal{J}_P$ with columns indexed by $\{\theta \in Z_i \mid 2 \leqslant i \leqslant n-d\}$, and rows indexed by $\{f_{\tau_{i-1},\theta} \mid \theta \in Z_i,\ 2 \leqslant i \leqslant n-d\}$. Then $\mathcal{J}'$ is a square matrix, of size equal to $\sum\limits_{i=2}^{n-d} \#Z_i$ (note that the Z_i's are disjoint). Now the set $\{\theta \in \mathcal{L} \mid \theta(d) = i+d\}$ is in bijection with $I_{d-1,\{1,\ldots,i+d-1\}}$ (here, for $r < s$, $I_{r,\{n_1,\ldots,n_s\}}$ denotes the set of all r-tuples $\underline{i} = (i_1, \ldots, i_r)$, where $i_1 < \ldots < i_r$, and $\{i_1, \ldots, i_r\} \subset \{n_1, \ldots, n_s\}$). Hence

$$\#Z_i = \binom{i+d-1}{d-1} - d,$$

and

$$\begin{aligned}\#\bigcup_{i=2}^{n-d} Z_i &= \sum_{i=2}^{n-d}\left[\binom{i+d-1}{d-1} - d\right] = \binom{n}{d} - \binom{d}{d-1} - 1 - d(n-d-1)\\ &= \binom{n}{d} - d(n-d) - 1\end{aligned}$$

(note that $\sum\limits_{j=1}^{m-r+1}\binom{m-j}{r-1} = \binom{m}{r}$).

Now, a typical row in $\mathcal{J}'$ is indexed by

$$f_{\tau_{i-1},\theta} = x_{\tau_{i-1}} x_\theta - x_{\tau_{i-1}\vee\theta} x_{\tau_{i-1}\wedge\theta},$$

where $\theta \in Z_i$. The only variable appearing in $f_{\tau_{i-1},\theta}$, which is an index for a column of $\mathcal{J}'$, is x_θ. Further, the entry in $\mathcal{J}'$ in the row indexed by $f_{\tau_{i-1},\theta}$ and column indexed by x_θ is nonzero (by hypothesis). Thus, $\mathcal{J}'$ is a diagonal matrix, with nonzero diagonal entries. Henc $\mathcal{J}'$ has rank equal to its size, which is

$$\binom{n}{d} - d(n-d) - 1 = \operatorname{codim}_{\mathbb{A}(\mathcal{L})} X_{d,n}.$$

Thus we obtain that P is a smooth point of $X_{d,n}$. □

PROPOSITION 10.2. — *Let $P \in X_{d,n}$ be such that $P_\tau \neq 0$, for all $\tau \in JM_{\mathcal{L}}^{(II)}$. Then P is a smooth point of $X_{d,n}$.*

Proof. — Let $\mathcal{J}_P$ be the Jacobian matrix evaluated at P. We shall now exhibit a submatrix of $\mathcal{J}_P$ of maximal rank. Fix r and i, $1 \leqslant r \leqslant d-1$, $r+1 \leqslant i \leqslant n-d+r-1$. Set

$$Z_r^{(i)} = \{\theta \in \mathcal{L} \mid \theta(r) = i, \text{ and } \theta(t) = t, \text{ for } t < r\} \setminus \{\theta_i\},$$

where $\theta_i = (1, \dots, r-1, i, n+r-d+1, \dots, n)$. Let

$$Z_r = \bigcup_{i=r+1}^{n-d+r-1} Z_r^{(i)}.$$

Then

$$\bigcup_{s \leqslant r} Z_s = \{\tau \in \mathcal{L} \mid \tau \text{ and } \phi_r \text{ are noncomparable }\},$$

where $\phi_r = (1, \dots, r, n+r-d+1, \dots, n)$, $1 \leqslant r \leqslant d-1$, as defined in Section 8. Let $\mathcal{J}'$ be the submatrix of $\mathcal{J}_P$ with columns indexed by $\{\theta \in Z_r \mid 1 \leqslant r \leqslant d-1\}$, and rows indexed by $\{f_{\phi_r,\theta} \mid \theta \in Z_r, 1 \leqslant r \leqslant d-1\}$. Then $\mathcal{J}'$ is a square matrix of size equal to $\sum_{r=1}^{d-1} \# Z_r$ (note that the Z_r's are disjoint). Now, $\{\theta \in \mathcal{L} \mid \theta(r) = i, \text{ and } \theta(t) = t, \text{ for } t < r\}$ is in bijection with $I_{d-r,\{i+1,\dots,n\}}$. Hence

$$\# Z_r^{(i)} = \binom{n-i}{d-r} - 1,$$

and

$$\begin{aligned} \# Z_r = \sum_{i=r+1}^{n-d+r-1} \left[\binom{n-i}{d-r} - 1 \right] &= \binom{n-r}{d-r+1} - 1 - (n-d-1) \\ &= \binom{n-r}{d-r+1} - (n-d) = \binom{n-r}{n-d-1} - (n-d). \end{aligned}$$

Hence the size of $\mathcal{J}'$ is equal to

$$\begin{aligned}\sum_{r=1}^{d-1}\left[\binom{n-r}{n-d-1}-(n-d)\right] &= \binom{n}{n-d}-\binom{n-d}{n-d-1}-1-(d-1)(n-d)\\ &= \binom{n}{d}-(n-d)-1-(d-1)(n-d)\\ &= \binom{n}{d}-d(n-d)-1.\end{aligned}$$

As in the proof of Proposition 10.1, we find that $\mathcal{J}'$ is a diagonal matrix, with nonzero diagonal entries. Hence the rank of $\mathcal{J}'$ is equal to its size, which is

$$\binom{n}{d}-d(n-d)-1=\operatorname{codim}_{\mathbb{A}(\mathcal{L})}X_{d,n}.$$

Thus P is a smooth point of $X_{d,n}$. □

PROPOSITION 10.3. — *The variety $X_{d,n}$ is nonsingular in codimension 1.*

Proof. — Let Y be an irreducible component of $X_{d,n}\setminus X^{\circ}_{d,n}$. Let P be the generic point of Y. Then, by Proposition 9.10 and Lemma 9.11, $P_\tau=0$ for precisely one element $\tau\in JM_{\mathcal{L}}$. Hence the result follows from Propositions 10.1 and 10.2. □

Proposition 10.3 and Theorem 7.4 yield the following

THEOREM 10.4. — *The variety $X_{d,n}$ is a normal toric variety.*

10.5. Let $A_{d,n}$ denote the homogeneous coordinate ring of $G_{d,n}$ (the Grassmannian of d-planes in k^n) for the Plücker embedding. We recall (cf. [12]):

THEOREM 10.6. — *There exists a flat family whose general fiber is $A_{d,n}$, and whose special fiber is $R(I_{d,n})$.*

Remark 10.7. — The above result is also proved in [26] using SAGBI (Subalgebra Analog to Gröbner Bases for Ideals) theory.

Combining Theorems 10.4 and 10.6, we obtain

THEOREM 10.8. — *The Grassmannian $G_{d,n}$ degenerates to the normal toric variety $X_{d,n}$.*

11. A conjecture on the singular locus of $X_{d,n}$.

In this section we prove a partial result towards the determination of the singular locus of $X_{d,n}$. We also state a conjecture on the singular locus of $X_{d,n}$.

We assume that $n \geqslant 4$ (note that for $n \leqslant 3$, $X_{d,n}$ is smooth). Let notations be as in Section 8. Let $\mathcal{L}_{ij} = \mathcal{L} \setminus [\mu_{ij}, \lambda_{ij}]$, $0 \leqslant i \leqslant n-d$, $1 \leqslant j \leqslant d-1$.

LEMMA 11.1. — *Let $i = 0$, and $1 \leqslant j \leqslant d-1$. Then $\mathcal{L}_{0j}$ is an embedded sublattice. Further, $JM^{(I)} \setminus \{\tau_0\} \subset \mathcal{L}_{0j}$.*

Proof. — We have $\tau_0 = (1, \dots, d)$, $\phi_j = (1, \dots, j, n+1+j-d, \dots, n)$, and hence $\mu_{0j} = \tau_0$, and $\lambda_{ij} = \phi_j$. Hence $\theta \in [\tau_0, \phi_j]$ if and only if $\theta(t) = t$, for $1 \leqslant t \leqslant j$. Therefore $\theta \in \mathcal{L}_{0j}$ if and only if $\theta(t) \neq t$, for some t, $1 \leqslant t \leqslant j$. Let γ, δ be two noncomparable elements in $\mathcal{L}_{0j}$. Let t_1, $t_2 \leqslant j$ be such that $\gamma(t_1) \neq t_1$, $\delta(t_2) \neq t_2$. Then, letting $t = \max(t_1, t_2)$, we have $\gamma \vee \delta(t) \neq t$, $\gamma \wedge \delta(t) \neq t$. Thus $\gamma \vee \delta$, $\gamma \wedge \delta \in \mathcal{L}_{0j}$.

Let now γ, δ be two noncomparable elements of $\mathcal{L}$ such that $\gamma \vee \delta$, $\gamma \wedge \delta \in \mathcal{L}_{0j}$. Let $t \leqslant j$ be such that $\gamma \wedge \delta(t) \neq t$. This implies $\gamma \vee \delta(t) \neq t$. Then, clearly $\gamma(t) \neq t$, $\delta(t) \neq t$. Thus γ, $\delta \in \mathcal{L}_{0j}$. Thus we obtain that $\mathcal{L}_{0j}$ is an embedded sublattice. Let $\tau \in JM^{(I)} \setminus \{\tau_0\}$, say $\tau = (i+1, \dots, i+d)$, where $i \geqslant 1$. Then clearly $\tau \in \mathcal{L}_{0j}$ (note for example that $\tau(1) \neq 1$). □

COROLLARY 11.2. — *Let P be the generic point of $X(\mathcal{L}_{0j})$. Then P is a smooth point of $\mathcal{L}$.*

Proof. — We have $P_\tau \neq 0$, for $\tau \in JM^{(I)} \setminus \{\tau_0\}$. Hence the result follows from Proposition 10.1. □

LEMMA 11.3. — *Let $i = n-d$, and $1 \leqslant j \leqslant d-1$. Then $\mathcal{L}_{n-d\,j}$ is an embedded sublattice. Further, $JM^{(I)} \setminus \{\tau_{n-d}\} \subset \mathcal{L}_{n-d\,j}$.*

COROLLARY 11.4. — *Let P be the generic point of $X(\mathcal{L}_{n-d\,j})$. Then P is a smooth point of $\mathcal{L}$.*

The proofs of Lemma 11.3, and Corollary 11.4 are similar to those of Lemma 11.1, and Corollary 11.2 respectively.

LEMMA 11.5. – *Let $1 \leqslant i \leqslant n-d-1$, $1 \leqslant j \leqslant d-1$. Then $\mathcal{L}_{ij}$ is an embedded sublattice, and*

$$\operatorname{codim}_{\mathbb{A}(\mathcal{L}_{ij})} X(\mathcal{L}_{ij}) < \operatorname{codim}_{\mathbb{A}(\mathcal{L})} X(\mathcal{L}).$$

Proof. — For $\theta \in \mathcal{L}$ we have

$$\theta \in [\mu_{ij}, \lambda_{ij}] \iff \theta(j) \leqslant i+j, \text{ and } \theta(j+1) \geqslant i+j+1$$

(here, by $\theta(j)$, we mean the j-th entry in the d-tuple θ). Let γ, δ two noncomparable elements in $\mathcal{L}_{ij}$. We have either $\gamma(j) \geqslant i+j+1$, or $\gamma(j+1) \leqslant i+j$. Similarly, we have either $\delta(j) \geqslant i+j+1$, or $\delta(j+1) \leqslant i+j$.

CLAIM. — $\gamma \vee \delta$, $\gamma \wedge \delta \in \mathcal{L}_{ij}$.

The proof is the same as in the proof of the first claim in Proposition 9.7.

CLAIM. — *Let γ, δ be two noncomparable elements of $\mathcal{L}$ such that $\gamma \vee \delta$, $\gamma \wedge \delta \in \mathcal{L}_{ij}$. Then γ, $\delta \in \mathcal{L}_{ij}$.*

The proof is the same as in the proof of the second claim in Proposition 9.7.

The above two claims show that $\mathcal{L}_{ij}$ is an embedded sublattice. It remains to prove the inequality

$$\operatorname{codim}_{\mathbb{A}(\mathcal{L}_{ij})} X(\mathcal{L}_{ij}) < \operatorname{codim}_{\mathbb{A}(\mathcal{L})} X(\mathcal{L}).$$

If $\theta = (k+1, \ldots, k+d) \in J_{\mathcal{L}}^{(I)}$, $k \neq i$, is a join-irreducible element of Type I distinct from τ_i, then $\theta(j) = k+j \geqslant i+j+1$, if $k \geqslant i+1$, and $\theta(j+1) = k+j+1 \leqslant i+j$, if $k \leqslant i-1$. Further, the unique element θ' of $\mathcal{L}$ such that (θ, θ') is a cover in $\mathcal{L}$ is given by $\theta' = (k, k+2, \ldots, k+d)$. It is easily seen that if either $k \neq i+1$, or $j \neq 1$ then $\theta' \in \mathcal{L}_{ij}$, and is the unique element of $\mathcal{L}_{ij}$ such that (θ, θ') is a cover in $\mathcal{L}_{ij}$. Let then $k = i+1$, and $j = 1$. In this case, we observe that if $\delta = (i, i+1, i+4, \ldots, i+d+1)$, then $\delta \in \mathcal{L}_{ij}$, and δ is the unique element of $\mathcal{L}_{ij}$ such that (θ, δ) is a cover in $\mathcal{L}_{ij}$ (note that $[\delta, \theta] \cap \mathcal{L}_{ij} = \{\delta, \theta\}$). Thus $\theta \in J_{\mathcal{L}_{ij}}$.

Let now $\theta = (1, \ldots, s, t+1, \ldots, t+d-s) \in J_{\mathcal{L}}^{(II)}$, where $t \neq n+j-d$, if $s = j$, be a join-irreducible element of Type II distinct from ϕ_j. We distinguish three cases.

Case 1: $s < j$

We have $\theta(j) = t+j-s$, $\theta(j+1) = t+j+1-s$. Hence, if $t-s \geqslant i+1$, then $\theta(j) \geqslant i+j+1$, and $\theta \in \mathcal{L}_{ij}$. If $t-s \leqslant i-1$, then $\theta(j+1) \leqslant i+j$, and $\theta \in \mathcal{L}_{ij}$. Further, the unique element θ' of $\mathcal{L}$ such that (θ, θ') is a cover in $\mathcal{L}$ is given by $\theta' = (1, \ldots, s, t, t+2, \ldots, t+d)$. It is easily seen that $\theta' \in \mathcal{L}_{ij}$, and is the unique element of $\mathcal{L}_{ij}$ such that (θ, θ') is a cover, except when $s+1 = j$, and $t-s = i+1$. Thus $\theta \in J_{\mathcal{L}_{ij}}$, except when $s+1 = j$, and $t-s = i+1$. In this case, we observe that if $\delta = (1, \ldots, s, i+s, i+s+1, t+3, \ldots, t+d-j)$, then $\delta \in \mathcal{L}_{ij}$, and is the unique element of $\mathcal{L}_{ij}$ such that (θ, δ) is a cover in $\mathcal{L}_{ij}$. Thus $\theta \in J_{\mathcal{L}_{ij}}$.

Case 2: $s = j$

We have $\theta(j) = j$, $\theta(j+1) = t+1$. Hence, if $\theta(j+1) \leqslant i+j$, i.e. $t \leqslant i+j-1$, then $\theta \in \mathcal{L}_{ij}$. Further, the unique element θ' of $\mathcal{L}$ such that (θ, θ') is a cover in $\mathcal{L}$ (namely $\theta' = (1, \ldots, j, t, t+2. \ldots, t+d-j))$ also belongs to $\mathcal{L}_{ij}$, and is the unique element of $\mathcal{L}_{ij}$ such that (θ, θ') is a cover in $\mathcal{L}_{ij}$. Hence $\theta \in J_{\mathcal{L}_{ij}}$.

Case 3: $s > j$

We have $s \geqslant j+1$, and hence $\theta(j+1) = j+1 \leqslant i+j$. Hence $\theta \in \mathcal{L}_{ij}$. As in *Case 2*, if θ' is the unique element of $\mathcal{L}$ such that (θ, θ') is a cover in $\mathcal{L}$, then $\theta' \in \mathcal{L}_{ij}$, and is the unique element of $\mathcal{L}_{ij}$ such that (θ, θ') is a cover in $\mathcal{L}_{ij}$. Hence $\theta \in J_{\mathcal{L}_{ij}}$.

From above, we have

$$\# J_{\mathcal{L}_{ij}} \geqslant \# J_{\mathcal{L}} - (n-d-i+j+1).$$

On the other hand, we have

$$\# \mathcal{L} - \# \mathcal{L}_{ij} = \# [\mu_{ij}, \lambda_{ij}] = \binom{i+j}{j} \binom{n-i-j}{d-j}.$$

It can be seen easily (by assuming $d \leqslant [n/2]$, since $I_{d,n}$ is isomorphic to $I_{n-d,n}$) that

$$\binom{i+j}{j} \binom{n-i-j}{d-j} > n-d-i+j+1,$$

i.e.

$$\# \mathcal{L} - \# J_{\mathcal{L}} > \# \mathcal{L}_{ij} - \# J_{\mathcal{L}_{ij}}.$$

The result now follows from this. □

PROPOSITION 11.6. — *For* $1 \leqslant i \leqslant n-d-1$, $1 \leqslant j \leqslant d-1$, *we have*

$$X(\mathcal{L}_{ij}) \subset \mathrm{Sing}(X_{d,n}).$$

Proof. — Let P be the generic point of $\mathcal{L}_{ij}$. We have $P_\tau = 0$ for all $\tau \in [\mu_{ij}, \lambda_{ij}]$. Let $\mathcal{J}_P$ be the Jacobian matrix evaluated at P. Then the submatrix of $\mathcal{J}_P$ with rows given by $\{f_{\tau,\phi} \mid \tau, \phi \in [\mu_{ij}, \lambda_{ij}], \tau,\ \phi \text{ noncomparable }\}$, and columns given by $\{x_\tau \mid \tau \in [\mu_{ij}, \lambda_{ij}]\}$ is the zero matrix. Let $\mathcal{J}'$ be the matrix obtained from $\mathcal{J}_P$ by deleting these rows and columns. Then we have

$$\text{(1)} \qquad \operatorname{rank} \mathcal{J}' = \operatorname{rank} \mathcal{J}_P.$$

Further, $\mathcal{J}'$ is precisely the Jacobian matrix of the variety $X(\mathcal{L}_{ij}) \subset \mathbb{A}_{\mathcal{L}_{ij}}$, evaluated at $P' = (P_\tau)_{\tau \in \mathcal{L}_{ij}}$. Hence

$$\text{(2)} \qquad \operatorname{rank} \mathcal{J}' \leqslant \operatorname{codim}_{\mathbb{A}(\mathcal{L}_{ij})} X(\mathcal{L}_{ij}) < \operatorname{codim}_{\mathbb{A}(\mathcal{L})} X_{d,n}$$

(cf. Lemma 11.5). The result now follows from (1) and (2). □

We have the following conjecture[(1)] on the singular locus of $X_{d,n}$.

CONJECTURE. — *The irreducible components of* $\mathrm{Sing}\, X_{d,n}$ *are precisely* $X(\mathcal{L}_{ij})$, $1 \leqslant i \leqslant n-d-1$, $1 \leqslant j \leqslant d-1$.

12. Singular loci of certain ladder determinantal varieties.

In this section we determine the singular loci of certain ladder determinantal varieties. Viewing $X_{2,n}$ as a ladder determinantal variety, we prove the conjecture of Section 11, for the case $d = 2$.

We assume $d = 2$, and view $\mathcal{L}(= I_{2,n}) = \{(i,j) \mid 1 \leqslant i < j \leqslant n\}$ as being contained in $Y = \{(i,j) \mid 1 \leqslant i, j \leqslant n\}$. The equations defining the variety $X(\mathcal{L})$ are precisely the 2 minors of Y which are contained in $\mathcal{L}$. Now we look at a more general type of varieties.

(1) This conjecture has now been proved in [3].

12.1. Let $Y = \{(b,a) \mid 1 \leqslant b \leqslant n,\ 1 \leqslant a \leqslant n\}$. Given $1 \leqslant b_1 < \ldots < b_l < n$, $1 < a_1 < \ldots < a_1 \leqslant n$, we consider the subset L of Y, defined by

$$L = \{(b,a) \mid \text{ there exists } 1 \leqslant i \leqslant l \text{ such that } b_i \leqslant b \leqslant n,\ 1 \leqslant a \leqslant a_i\}.$$

We call L an *one-sided ladder* in Y, defined by the *outer corners* $\alpha_i = (b_i, a_i)$, $1 \leqslant i \leqslant l$ (see Figure 1). For $1 \leqslant i \leqslant l$, let L_i be the subset of Y defined by

$$L_i = \{(b,a) \mid b_i \leqslant b \leqslant n, 1 \leqslant a \leqslant a_i\}.$$

Clearly, $L = \bigcup\limits_{i=1}^{l} L_i$.

We also view Y as the generic $n \times n$ matrix $(x_{ba})_{1 \leqslant b,a \leqslant n}$, and we say that $x_{ba} \in L$ if $(b,a) \in L$. Let $k[L]$ denote the polynomial ring $k[x_{ba} \mid x_{ba} \in L]$, and let $\mathbb{A}(L) = \mathbb{A}^{\#L}$ be the associated affine space. Let $I(L)$ be the ideal in $k[L]$ generated by the 2 minors of Y which are contained in L, and $X(L) \subset \mathbb{A}(L)$ the variety defined by the ideal $I(L)$. A 2 minor in Y formed with two consecutive rows and two consecutive columns will be called a *solid minor*. We have (cf. [11]):

LEMMA 12.2. — *The codimension of $X(L)$ in $\mathbb{A}(L)$ is equal to the number of solid 2 minors in L.*

THEOREM 12.3. — *Let L be an one-sided ladder in Y defined by the outer corners $\alpha_i = (b_i, a_i)$, with $a_i \geqslant 2$ and $b_i \leqslant n-1$, $1 \leqslant i \leqslant l$. For each $1 \leqslant i \leqslant l$, let $V_i = \{P = (P_\alpha)_{\alpha \in L} \in X(L) \mid P_\alpha = 0$ for all $\alpha \in L_i\}$. Then the irreducible components of $\operatorname{Sing} X(L)$ are precisely V_i, $1 \leqslant i \leqslant l$.*

Proof. — First, we prove that $V_i \subset \operatorname{Sing} X(L)$, for all $1 \leqslant i \leqslant l$. Fix $1 \leqslant i \leqslant l$, and let $P = (P_\alpha)_{\alpha \in L} \in V_i$. Let $\mathcal{J}$ be the Jacobian matrix of $X(L)$. Then the rows of $\mathcal{J}$ are indexed by $\{M \mid M \text{ is a 2 minor of } Y \text{ contained in } L\ \}$, and the columns are indexed by $\{x_\alpha \mid \alpha \in L\}$. Let $\mathcal{J}_P$ be the Jacobian matrix evaluated at P. Then the (M, x_α)-th entry in $\mathcal{J}_P$ is equal to $\pm P_{\alpha'}$, where α' is the element in M which is neither in the row, nor in the column containing α, if x_α appears in M, and 0 otherwise. Since $P \in V_i$, the row indexed by a minor involving $x_{\alpha_i} = x_{(b_i,a_i)}$ is 0. Also, the column indexed by x_{α_i} is 0.

Let $\mathcal{J}'$ be the matrix obtained from $\mathcal{J}_P$ by deleting the column indexed by x_{α_i}, and the rows indexed by minors involving x_{α_i}. Then

$$\operatorname{rank} \mathcal{J}_P = \operatorname{rank} \mathcal{J}', \tag{1}$$

since $\mathcal{J}'$ is obtained from $\mathcal{J}_P$ by deleting zero rows and columns.

Now consider the one-sided ladder L' obtained from L by deleting the element α_i. Let $P' = (P_\alpha)_{\alpha \in L'}$. Then $P' \in X(L')$, and $\mathcal{J}'$ is the Jacabian matrix of $X(L')$ evaluated at P'. Thus

$$\text{(2)} \qquad \operatorname{rank} \mathcal{J}' \leqslant \operatorname{codim}_{\mathbb{A}(L')} X(L').$$

By hypothesis, $a_i \geqslant 2$ and $b_i \leqslant n-1$, and hence there exist at least one solid minor in L not contained in L', while every solid minor in L' is also contained in L. Thus, using Lemma 12.2, we deduce that

$$\text{(3)} \qquad \operatorname{codim}_{\mathbb{A}(L')} X(L') < \operatorname{codim}_{\mathbb{A}(L)} X(L).$$

Using (1), (2) and (3), we deduce that

$$\operatorname{rank} \mathcal{J}_P < \operatorname{codim}_{\mathbb{A}(L)} X(L),$$

i.e. $P \in \operatorname{Sing} X(L)$.

Next we prove that $\operatorname{Sing} X(L) \subset \bigcup_{i=1}^{l} V_i$. Let $P \in X(L) \setminus \bigcup_{i=1}^{l} V_i$. For each $1 \leqslant i \leqslant l$, we fix an element $\beta_i \in L_i$ such that $P_{\beta_i} \neq 0$. Let $\mathcal{C}$ be the set obtained from L by deleting the elements appearing either in the first column, or in the last row of L. Then $\#\mathcal{C}$ is equal to the number of solid 2 minors in L, and by Lemma 12.2, we have

$$\#\mathcal{C} = \operatorname{codim}_{\mathbb{A}(L)} X(L).$$

We have $\mathcal{C} = \dot{\bigcup}_{i=1}^{l} \mathcal{C}_i$, where $\mathcal{C}_i = \mathcal{C} \cap (L_i \setminus L_{i+1})$, for $1 \leqslant i \leqslant l$, $L_{l+1} = \varnothing$. Let $\mathcal{T}_l$ be the set of elements in L_l not in the row or the column of β_l. Clearly, $\#\mathcal{T}_l = \#\mathcal{C}_l$. By (decreasing) induction on i, suppose that, for some i, $1 < i \leqslant l$, the sets $\mathcal{T}_i, \ldots, \mathcal{T}_l$ have been constructed, such that

$(1)_i$ $\mathcal{T}_j \subset L_j$, $i \leqslant j \leqslant l$,

$(2)_i$ the sets $\mathcal{T}_i, \ldots, \mathcal{T}_l$ are pairwise disjoint,

$(3)_i$ $\#\mathcal{T}_j = \#\mathcal{C}_j$, $i \leqslant j \leqslant l$,

$(4)_i$ $\mathcal{T}_j$ contains no elements appearing in the column or in the row of β_j, $i \leqslant j \leqslant l$,

$(5)_i$ there exists a row in L_i not containing any element from $\mathcal{T}_i \cup \ldots \cup \mathcal{T}_l$.

We define the set $\mathcal{T}_{i-1}$ as follows. If $\beta_{i-1} \notin L_{i-1} \setminus L_i$, then $\mathcal{T}_{i-1}$ is obtained from $L_{i-1} \setminus L_i$ by deleting the elements in the column of β_{i-1}. If $\beta_{i-1} \in L_{i-1} \setminus L_i$, then $\mathcal{T}_{i-1}$ is obtained as follows. Choose a row R_i in L_i, as given by $(5)_i$. We set $\mathcal{T}_{i-1} = (L_{i-1} \setminus L_i) \cup R_i \setminus A_{\beta_{i-1}}$, where $A_{\beta_{i-1}}$ is the set of elements of L in the row and column of β_{i-1}. Clearly, $\mathcal{T}_{i-1} \subset L_{i-1}$, the sets $\mathcal{T}_{i-1}, \mathcal{T}_i, \ldots, \mathcal{T}_l$ are pairwise disjoint, $\#\mathcal{T}_{i-1} = \#\mathcal{C}_{i-1}$, and $\mathcal{T}_{i-1}$ does not contain any element in the row or the column of β_{i-1}. Also, there exists a row in L_{i-1} which does not contain any element from $\mathcal{T}_{i-1} \cup \mathcal{T}_i \cup \ldots \cup \mathcal{T}_l$. Hence the sets $\mathcal{T}_{i-1}, \mathcal{T}_i, \ldots, \mathcal{T}_l$ satisfy $(1)_{i-1}-(5)_{i-1}$. Thus we obtain pairwise disjoint sets $\mathcal{T}_j \subset L_j$, $1 \leqslant j \leqslant l$, such that $\#\mathcal{T}_j = \#\mathcal{C}_j$, and $\mathcal{T}_j$ does not contain any element in the row or column of β_j. Let $\mathcal{T} = \bigcup_{i=1}^{l} \mathcal{T}_i$. Then $\#\mathcal{T} = \#\mathcal{C}$.

For $\tau \in \mathcal{T}_i$, $1 \leqslant i \leqslant l$, let M^τ be the 2 minor determined by τ and β_i. Clearly, $M^\tau \neq M^{\tau'}$ for τ, $\tau' \in \mathcal{T}$, $\tau \neq \tau'$. Let $\mathcal{J}'$ be the submatrix of $\mathcal{J}_P$ given by rows indexed by M^τ's and the columns indexed by x_τ's, with $\tau \in \mathcal{T}$. We index the rows and columns of $\mathcal{J}'$ by the elements in $\mathcal{T}$, and we arrange them increasingly, with respect to the lexicographic order in Y (namely, $(b, a) > (b', a')$ if $b > b'$, or $b = b'$, $a > a'$).

Let us fix $\tau \in \mathcal{T}$, say $\tau \in \mathcal{T}_i$, $1 \leqslant i \leqslant l$. Since τ is the only entry in M^τ which belongs to $\mathcal{T}$, we deduce that in the τ-th row of $\mathcal{J}'$ all the entries are zero, except the one in the τ-th column, which is equal to $\pm P_{\beta_i}$, and hence it is nonzero. Thus the matrix $\mathcal{J}'$ is diagonal, with nonzero diagonal entries. Therefore its rank is equal to its size, which is $\#\mathcal{T}$. Since $\#\mathcal{T}$ is equal to $\#\mathcal{C}$, and hence equal to the codimension of $X(L)$, we deduce that

$$\operatorname{rank} \mathcal{J}' = \operatorname{codim}_{\mathbb{A}(L)} X(L). \tag{4}$$

Since $\operatorname{rank} \mathcal{J}' \leqslant \operatorname{rank} \mathcal{J}_P \leqslant \operatorname{codim}_{\mathbb{A}(L)} X(L)$, (4) implies that

$$\operatorname{rank} \mathcal{J}_P = \operatorname{codim}_{\mathbb{A}(L)} X(L),$$

i.e. $P \notin \operatorname{Sing} X(L)$. Therefore we conclude that

$$\operatorname{Sing} X(L) = \bigcup_{i=1}^{l} V_i.$$

Let $\mathcal{L}_i = L \setminus L_i$, $1 \leqslant i \leqslant l$. Then $\mathcal{L}_i$ is a distributive lattice, and V_i is identified with $X(\mathcal{L}_i)$. Using §5.7, we deduce that V_i is irreducible, $1 \leqslant i \leqslant l$. The fact that $V_i \not\subset V_j$ for $i \neq j$ is obvious. This completes the proof of the theorem. □

The following theorem shows the validity of the Conjecture stated in Section 11, for the case $d = 2$.

THEOREM 12.4. — *Let $\mathcal{L}_{i1} \subset I_{2,n}$, $1 \leqslant i \leqslant n-3$, be as in Section 11. Then the irreducible components of* $\operatorname{Sing} X_{2,n}$ *are precisely $X(\mathcal{L}_{i1})$, $1 \leqslant i \leqslant n-3$.*

Proof. — First observe that $\phi_1 = (1,n)$ is the only join and meet-irreducible element of Type II in $I_{2,n}$, and the join and meet-irreducible elements of Type I are $\tau_i = (i+1, i+2)$, $0 \leqslant i \leqslant n-2$ (note that $\tau_0 = \widehat{0}$, $\tau_{n-2} = \widehat{1}$). We have $X_{2,n} = X(L) \times \mathbb{A}^2$, where $L = I_{2,n} \setminus \{\widehat{0}, \widehat{1}\}$. Using Theorem 12.3 for the ladder L, we obtain that the irreducible components of $\operatorname{Sing} X(L)$ are V_i , $1 \leqslant i \leqslant n-3$ (V_i being as defined in the statement of the Theorem). Thus the irreducible components of $\operatorname{Sing} X_{2,n}$ are precisely $V_i \times \mathbb{A}^2$, $1 \leqslant i \leqslant n-3$. It is easily seen that $X(\mathcal{L}_{i1}) = V_i \times \mathbb{A}^2$, $1 \leqslant i \leqslant n-3$, and the result follows from this. □

Remark. — We have

$$\operatorname{codim}_{X(L)} V_i = b_{i+1} - b_i + a_i - a_{i-1} + 1, \quad 1 \leqslant i \leqslant l$$

(here, $a_0 = 0$, $b_{l+1} = n$). In particular, taking $L = I_{2,n} \setminus \{\widehat{0}, \widehat{1}\}$, we deduce that $\operatorname{Sing} X_{2,n}$ is of pure codimension three in $X_{d,n}$.

13. Generalities on $SL(n)/B$.

Let $G = SL(n)$, the special linear group of rank $n-1$. Let T be the maximal torus consisting of all the diagonal matrices in G, and B the Borel subgroup consisting of all the upper triangular matrices in G. It is well-known that W can be identified with $\mathcal{S}_n$, the symmetric group on n letters. For $w \in W$ and Q a parabolic subgroup, let $X_Q(w) (= \overline{BwQ} \pmod{Q})$ be the Schubert variety in G/Q associated to w. When $Q = B$, we shall denote $X_Q(w)$ by just $X(w)$.

Following [6], we denote the set S of simple roots by $\{\varepsilon_i - \varepsilon_{i+1},\ 1 \leqslant i \leqslant n-1\}$ (note that $\varepsilon_i - \varepsilon_{i+1}$ is the character sending $\operatorname{diag}(t_1, \ldots, t_n)$ to $t_i t_{i+1}^{-1}$). The reflection $s_{\varepsilon_i - \varepsilon_{i+1}}$ may be identified with the transposition (i,j) in $\mathcal{S}_n$. For $w = (a_1 \ldots a_n) \in \mathcal{S}_n$, it is easily seen that $w(\varepsilon_i - \varepsilon_j) = \varepsilon_{a_i} - \varepsilon_{a_j}$.

13.1. *The Chevalley-Bruhat order on $\mathcal{S}_n$.* For $w_1, w_2 \in W$, we have

$$X(w_1) \subset X(w_2) \iff \pi_d(X(w_1)) \subset \pi_d(X(w_2)) \text{ for all } 1 \leqslant d \leqslant n-1,$$

where π_d is the canonical projection $G/B \to G/P_{\widehat{\alpha_d}}$ (here, for a simple root α, $P_{\widehat{\alpha}}$ is the maximal parabolic subgroup P with $S_P = S \setminus \{\alpha\}$) . Hence we obtain that for $(a_1 \dots a_n)$, $(b_1 \dots b_n) \in \mathcal{S}_n$,

$$(a_1 \dots a_n) \geqslant (b_1 \dots b_n) \iff (a_1 \dots a_d) \uparrow \geqslant (b_1 \dots b_d) \uparrow \text{ for all } 1 \leqslant d \leqslant n-1$$

(here, for an ordered d-tuple $(t_1 \dots t_d)$ of distinct integers, $(t_1 \dots t_d) \uparrow$ denotes the ordered d-tuple obtained from $\{t_1, \dots, t_d\}$ by arranging its elements in ascending order).

13.2. *The partially ordered set* $I_{a_1,\dots,a_k}$. Let Q be a parabolic subgroup in $SL(n)$, with S_Q as the associated set of simple roots. Let $1 \leqslant a_1 < \dots < a_k \leqslant n$, such that $S_Q = S \setminus \{\alpha_{a_1}, \dots, \alpha_{a_k}\}$. Then $Q = P_{\widehat{\alpha_{a_1}}} \cap \dots \cap P_{\widehat{\alpha_{a_k}}}$, and $W_Q = \mathcal{S}_{a_1} \times \mathcal{S}_{a_2 - a_1} \times \dots \times \mathcal{S}_{n-a_k}$. Let

$$I_{a_1,\dots,a_k} = \{(\underline{i}_1, \dots, \underline{i}_k) \in I_{a_1,n} \times \dots \times I_{a_k,n} \mid \underline{i}_t \subset \underline{i}_{t+1} \text{ for all } 1 \leqslant t \leqslant k-1\}.$$

Then it is easily seen that $W_Q^{\min}(=\{w \in W \mid w(\alpha) > 0 \text{ for all } \alpha \in S_Q\})$ may be identified with $I_{a_1,\dots,a_k}$.

The partial order on the set of Schubert varieties in G/Q (given by inclusion) induces a partial order $\geqslant$ on $I_{a_1,\dots,a_k}$, namely, for $\mathbf{i} = (\underline{i}_1, \dots, \underline{i}_k)$, $\mathbf{j} = (\underline{j}_1, \dots, \underline{j}_k) \in I_{a_1,\dots,a_k}$, $\mathbf{i} \geqslant \mathbf{j} \iff \underline{i}_t \geqslant \underline{j}_t$ for all $1 \leqslant t \leqslant k$.

13.3. *The minimal (maximal) representatives.* Let $w \in W$, and let $\mathbf{i} = (\underline{i}_1, \dots, \underline{i}_k)$ be the element in $I_{a_1,\dots,a_k}$ which corresponds to $w_Q^{\min}$. Let $w_Q^{\min}$ (resp. $w_Q^{\max}$) be the minimal (resp. maximal) representative of wW_Q in W. Let w correspond to $\mathbf{i} = (\underline{i}_1, \dots, \underline{i}_k)$ under the identification in §p13.2. Then, as a permutation, the element $w_Q^{\min}$ is given by $\underline{i}_1$, followed by $\underline{i}_2 \setminus \underline{i}_1$ arranged in ascending order, and so on, ending with $\{1, \dots, n\} \setminus \underline{i}_k$ arranged in ascending order. Similarly, as a permutation, the element $w_Q^{\max}$ is given by $\underline{i}_1$ arranged in descending order, followed by $\underline{i}_2 \setminus \underline{i}_1$ arranged in descending order, etc..

13.4. *The opposite big cell in* G/Q. Let $Q = \bigcap_{t=1}^{k} P_{\widehat{\alpha_{a_t}}}$. Let $a = n - a_k$, and Q be the parabolic subgroup consisting of all the elements of G of the form

$$\begin{pmatrix} A_1 & * & * & \cdots & * & * \\ 0 & A_2 & * & \cdots & * & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_k & * \\ 0 & 0 & 0 & \cdots & 0 & A \end{pmatrix},$$

where A_t is a matrix of size $c_t \times c_t$, $c_t = a_t - a_{t-1}$, $a_0 = 0$, $1 \leqslant t \leqslant k$, and A is a matrix of size $a \times a$, and $x_{ml} = 0$, $m > a_t$, $l \leqslant a_t$, $1 \leqslant t \leqslant k$. Denote by O^- the subgroup of G generated by $\{U_\alpha \mid \alpha \in R^- \setminus R_Q^-\}$ (here R^- is the set of negative roots and $R_Q^- = \{\alpha \in R^- \mid \alpha = \sum_{\beta \in S_Q} \alpha_\beta \beta\}$). Then O^- consists of the elements of G of the form

$$\begin{pmatrix} I_1 & 0 & 0 & \cdots & 0 & 0 \\ * & I_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ * & * & * & \cdots & I_k & 0 \\ * & * & * & \cdots & * & I_a \end{pmatrix},$$

where I_t (resp. I_a) is the $c_t \times c_t$ (resp. $a \times a$) identity matrix, and if $x_{ml} \neq 0$, with $m \neq l$, then $m > a_t$, $l \leqslant a_t$ for some t, $1 \leqslant t \leqslant k$. Further, the restriction of the canonical morphism $f : G \to G/Q$ to O^- is an open immersion, and $f(O^-) \simeq B^- e_{\mathrm{id},Q}$. One refers to $B^- e_{\mathrm{id},Q}$ as the *opposite big cell in* G/Q. Thus we obtain an identification of O^- with the opposite big cell in G/Q.

13.5. *Plücker coordinates on the Grassmannian.* Let $G_{d,n}$ be the Grassmannian variety, consisting of d-dimensional subspaces of an n-dimensional vector space V. Let us identify V with k^n, and denote the standard basis of k^n by $\{e_i \mid 1 \leqslant i \leqslant n\}$. Consider the Plücker embedding $f_d : G_{d,n} \hookrightarrow \mathbb{P}(\wedge^d V)$, where $\wedge^d V$ is the d-th exterior power of V. For $\underline{i} = (i_1, \ldots, i_d) \in I_{d,n}$, let $e_{\underline{i}} = e_{i_1} \wedge \ldots \wedge e_{i_d}$. Then the set $\{e_{\underline{i}} \mid \underline{i} \in I_{d,n}\}$ is a basis for $\wedge^d V$. Let us denote the basis of $(\wedge^d V)^*$ (the linear dual of $\wedge^d V$) dual to $\{e_{\underline{i}} \mid \underline{i} \in I_{d,n}\}$ by $\{p_{\underline{j}} \mid \underline{j} \in I_{d,n}\}$. Then $\{p_{\underline{j}} \mid \underline{j} \in I_{d,n}\}$ gives a system of coordinates for $\mathbb{P}(\wedge^d V)$. These are the so-called *Plücker coordinates.*

13.6. *Schubert varieties in the Grassmannian.* Let $Q = P_{\widehat{\alpha_d}}$. For simplicity of notation, let us denote $P_{\widehat{\alpha_d}}$ by just P_d. We have

$$G_{d,n} \simeq G/P_d.$$

Let $\underline{i} = (i_1, \ldots, i_d) \in I_{d,n}$. Then the T-fixed point $e_{\underline{i},P_d}$ is simply the d-dimensional span of $\{e_{i_1}, \ldots, e_{i_d}\}$. Thus $X_{P_d}(\underline{i})$ is simply the Zariski closure of $B \cdot [e_{i_1} \wedge \ldots \wedge e_{i_d}]$ in $\mathbb{P}(\wedge^d V)$.

In view of the decomposition $X_{P_d}(\underline{i}) = \bigcup_{\underline{j} \leqslant \underline{i}} X_{P_d}(\underline{j})$, we have

$$p_{\underline{j}}\big|_{X_{P_d}(\underline{i})} \neq 0 \iff \underline{i} \geqslant \underline{j}.$$

Remark 13.7. — Given $\tau \in I_{d,n}$, $\tau = (i_1, \dots, i_d)$, let Λ_τ denote the Young diagram $(i_d - d, \dots, i_1 - 1)$. Then, using Lemma 8.2, we see that the element $\tau \in I_{d,n}$ is join-irreducible if and only if the associated Young diagram Λ_τ is a rectangle, i.e. all rows have the same number of boxes. We also observe that τ is join-irreducible if and only if $X(\tau)$ is nonsingular (this is a consequence of the fact that $X(\tau)$ is nonsingular if and only if Λ_τ is a rectangle, cf. [21]).

13.8. *Evaluation of Plücker coordinates on the opposite big cell in* G/P_d. Consider the morphim $\phi_d : G \to \mathbb{P}(\wedge^d V)$, where $\phi_d = f_d \circ \theta_d$, θ_d being the natural projection $G \to G/P_d$. Then $p_{\underline{j}}(\phi_d(g))$ is simply the minor of g consisting of the first d columns and the rows with indices $j_1, \dots, j_d$. Now, denote by Z_d the unipotent subgroup of G generated by $\{U_\alpha \mid \alpha \in R^- \setminus R^-_{P_d}\}$. We have, as in §13.4

$$Z_d = \left\{ \begin{pmatrix} I_{d\times d} & 0_{d\times(n-d)} \\ A_{(n-d)\times d} & I_{(n-d)\times(n-d)} \end{pmatrix} \in G \right\}.$$

As in §13.4, we identify Z_d with the opposite big cell in G/P_d. Then, given $z \in Z_d$, the Plücker coordinate $p_{\underline{j}}$ evaluated at z is simply a certain minor of A, which may be explicitly described as follows. Let $\underline{j} = (j_1, \dots, j_d)$, and let j_r be the largest entry $\leqslant d$. Let $\{k_1, \dots, k_{d-r}\}$ be the complement of $\{j_1, \dots, j_r\}$ in $\{1, \dots, d\}$. Then this minor of A is given by column indices $k_1, \dots k_{d-r}$, and row indices $j_{r+1}, \dots, j_d$ (here the rows of A are indexed as $d+1, \dots, n$). Conversely, given a minor of A, say, with column indices $b_1, \dots, b_s$, and row indices $i_{d-s+1}, \dots, i_d$, it is the evaluation of the Plücker coordinate $p_{\underline{i}}$ at z, where $\underline{i} = (i_1, \dots, i_d)$ may be described as follows: $\{i_1, \dots, i_{d-s}\}$ is the complement of $\{b_1, \dots, b_s\}$ in $\{1, \dots, d\}$, and $i_{d-s+1}, \dots, i_d$ are simply the row indices (again, the rows of A are indexed as $d+1, \dots, n$).

13.9. *Evaluation of the Plücker coordinates on the opposite big cell in* G/Q. Consider

$$f : G \to G/Q \hookrightarrow G/P_{a_1} \times \dots \times G/P_{a_k} \hookrightarrow \mathbf{P}_1 \times \dots \times \mathbf{P}_k,$$

where $\mathbf{P}_t = \mathbb{P}(\wedge^{a_t} V)$. Denoting the restriction of f to O^- (O^- being as in §13.4) also by just f, we obtain an embedding $f : O^- \hookrightarrow \mathbf{P}_1 \times \dots \times \mathbf{P}_k$, O^- having been identified with the opposite big cell in G/Q. For $z \in O^-$, the multi-Plücker coordinates of $f(z)$ are simply all the $a_t \times a_t$ minors of z with column indices $\{1, \dots, a_t\}$, $1 \leqslant t \leqslant k$.

13.10. *Equations defining multicones over Schubert varieties in G/Q.* Let $X_Q(w) \subset G/Q$. Let

$$R = \bigoplus_{\underline{a}} H^0\Big(G/Q, \bigotimes_i L_i^{a_i}\Big)$$

$$R_w = \bigoplus_{\underline{a}} H^0\Big(X_Q(w), \bigotimes_i L_i^{a_i}\Big),$$

where $\underline{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_+^k$. The kernel of the restriction map $R \to R_w$ is generated by the kernel of $R_1 \to (R(w))_1$ (cf. [18]); but now, in view of 13.6, this kernel is the span of

$$\{p_{\underline{i}} \mid \underline{i} \in I_{d,n},\ d = a_1, \ldots, a_k,\ w^{(d)} \not\geqslant \underline{i}\},$$

where $w^{(d)}$ is the d-tuple corresponding to the Schubert variety which is the image of $X_Q(w)$ under the projection $G/Q \to G/P_{a_t}$, $1 \leqslant t \leqslant k$.

13.11. *Ideal of the opposite cell in $X(w)$.* Let us denote $B^- e_{\mathrm{id},Q} \cap X_Q(w)$ by just A_w. Then as in §13.4, we identify $B^- e_{\mathrm{id},Q}$ with the unipotent subgroup O^- generated by $\{U_\alpha \mid \alpha \in R^- \setminus R_Q^-\}$, and consider A_w as a closed subvariety of O^- (one refers to A_w as the opposite cell in $X(w)$). In view of §13.10, we obtain

PROPOSITION 13.12. — *The ideal defining A_w in O^- is generated by*

$$\{p_{\underline{i}} \mid \underline{i} \in I_{d,n},\ d = a_1, \ldots, a_k,\ w^{(d)} \not\geqslant \underline{i}\}$$

(here, for $z \in O^-$, $p_{\underline{i}}(z)$ is as in §13.9).

14. Relationship between ladder determinantal varieties and Schubert varieties.

In this section, we relate $X(L)$ as well as the irreducible components of $\operatorname{Sing} X(L)$ to certain Schubert varieties in a certain $SL(n)/Q$. Eventhough the relationship between $X(L)$'s and Schubert varieties is proved in [22], we give an independent proof to suit to our purpose.

Let $Y = (x_{ba})$, $1 \leqslant b, a \leqslant n$ be a matrix of variables, and $L \subset Y$ an one-sided ladder defined by the outer corners $(b_1, a_1), \ldots, (b_l, a_l)$, with $1 \leqslant b_1 < \ldots < b_l < n$, $1 < a_1 < \ldots < a_l \leqslant n$. We also assume that L lies below the main diagonal of Y, i.e. $b_i > a_i$ for all $1 \leqslant i \leqslant l$ (this can be achieved by adding extra rows and columns to Y, if necessary).

Let $G = SL(n)$, and consider the parabolic subgroup $Q = P_{a_1} \cap \ldots \cap P_{a_l}$ in G. Let O^- be the opposite big cell in G/Q. Let $X(L) \subset \mathbb{A}(L)$ be the the variety defined by the vanishing of the 2 minors in L, as in Section 12. Let H be the one-sided ladder defined by the outer corners $(a_i + 1, a_i)$, $1 \leqslant i \leqslant l$, and let Z be the variety in $\mathbb{A}(H) \simeq O^-$ defined by the vanishing of the 2 minors in L. Note that $Z \simeq X(L) \times \mathbb{A}(H \setminus L)$.

Let $Y^- = (y_{ba})$, $1 \leqslant b, a \leqslant n$, where

$$y_{ba} = \begin{cases} x_{ba}, & \text{if } (b,a) \in H \\ 1, & \text{if } b = a \\ 0, & \text{otherwise.} \end{cases}$$

Note that, given $\tau \in W^{a_i}$, $1 \leqslant i \leqslant l$, the function $p_\tau|_{O^-}$ represents the determinant of the submatrix T of Y^- whose row indices are $\{\tau(1), \ldots, \tau(a_i)\}$, and column indices are $\{1, \ldots, a_i\}$.

We shall now define an element $w_L \in W^Q$ such that Z gets identified with the opposite cell in $X(w_L)$ (see also [22]). We define $w_L \in W^Q$ by specifying $w_L^{(a_i)} \in W^{a_i}$, where $\pi_i(X(w_L)) = X(w_L^{(a_i)})$ under the projection $\pi_i : G/Q \to G/P_{a_i}$, $1 \leqslant i \leqslant l$.

Define $w_L^{(a_i)}$, $1 \leqslant i \leqslant l$, inductively, as the maximal element W^{a_i} with the following properties:

(1) $w_L^{(a_i)}(a_i - 1) = b_i - 1$,

(2) if $i > 1$, then $w_L^{(a_i - 1)} \subset w_L^{(a_i)}$

(here, for a d-tuple $\theta = (\theta_1, \ldots, \theta_d)$, by $\theta(t)$ we mean the t-th entry, $1 \leqslant t \leqslant d$).

Remark. — Note that w_L is well defined as an element of W^Q, and it is the unique maximal element τ of W^Q with the property

$$\tau^{(a_i)}(a_i - 1) < b_i,$$

for all $1 \leqslant i \leqslant l$.

In the sequel, we shall denote w_L by just w.

THEOREM 14.2. — *The variety Z identifies with the opposite cell in $X(w)$, i.e. $Z = X(w) \cap O^-$ (scheme theoretically).*

Proof. — Let f be a generator of $I(Z)$, i.e. $f = \det M$, for some 2×2 matrix M contained in L. Let $r_1 < r_2$ (resp. $c_1 < c_2$) be the row (resp. column) indices of M. Let i be the smallest integer such that M is contained in L_i. Thus $r_1, r_2 \geqslant b_i > a_i$, $c_2 > a_{i-1}$ (here $a_{i-1} = 0$ if $i = 1$). Let $\tau = \{1, \ldots, a_i\} \setminus \{c_1, c_2\} \cup \{r_1, r_2\}$. Then $\tau \in W^{a_i}$, and $p_\tau|_{O^-} = \det T$, where T is the $a_i \times a_i$ submatrix of Y^- whose row indices are $\{\tau(1), \ldots, \tau(a_i)\}$, and column indices are $\{1, \ldots, a_i\}$. Using Laplace expansion with respect to the last two rows of T, we obtain

$$(*) \qquad \det T = \sum \pm \det N_{c',c''} \det M_{c',c''},$$

the sum being taken over all subsets $\{c', c''\} \subset \{1, \ldots, a_i\}$, $c' \neq c''$, where $N_{c',c''}$ is the $(a_i - 2) \times (a_i - 2)$ submatrix of Y^- whose row indices are $\{1, \ldots, a_i\} \setminus \{c_1, c_2\}$ and column indices are $\{1, \ldots, a_i\} \setminus \{c', c''\}$, and $M_{c',c''}$ is the 2×2 submatrix of Y^- whose row indices are $\{r_1, r_2\}$ and column indices are $\{c', c''\}$. Note that $M_{c_1,c_2} = M$, and N_{c_1,c_2} is a lower triangular matrix, with all diagonal entries equal to 1, and hence $\det M$ appears in $(*)$, and its coefficient is ± 1. Also note that $N_{c',c''}$ is obtained from N_{c_1,c_2} by replacing the columns with indices c', c'' by the columns with indices c_1, c_2.

By decreasing induction on the index c_1 of the first column of M, we prove that f $(= \det M)$ can be written in the form $f = \sum g_\phi p_\phi|_{O^-}$, with $\phi \in W^{a_i}$, $\{a_i + 1, \ldots, n\} \cap \{\phi(1), \ldots, \phi(a_i)\} = \{r_1, r_2\}$, and $g_\phi \in k[H]$.

If $c_1 > a_{i-1}$, then for $\{c', c''\} \neq \{c_1, c_2\}$ we have $\det N_{c',c''} = 0$, since at least one of c_1, c_2 is an index for a column in $N_{c',c''}$, and all entries of this column are 0. Thus, in this case $(*)$ reduces to $\det T = \pm \det M$, i.e. $\det M = \pm p_\tau|_{O^-}$, with $\tau \in W^{a_i}$ such that $\{a_i + 1, \ldots, n\} \cap \{\tau(1), \ldots, \tau(a_i)\} = \{r_1, r_2\}$.

Let $c_1 \leqslant a_{i-1}$ be such that the above statement is true for $c_1 + 1$. First we observe that if $c_2 \notin \{c', c''\}$, then $\det N_{c',c''} = 0$, since in this case c_2 is an index for a column in $N_{c',c''}$, and all entries of this column are 0. Let then $c_2 \in \{c', c''\}$, and let $\{c', c''\} = \{c, c_2\}$. Then N_{c,c_2} is obtained from N_{c_1,c_2} by replacing the column with index c by the column with index c_1. If $c < c_1$, then N_{c,c_2} is still lower triangular, but the diagonal entry in the column with index c_1 is 0, and hence $\det N_{c,c_2} = 0$. Therefore we obtain

$$\det T = \pm \det M + \sum_{c \in \{c_1+1, \ldots, a_i\} \setminus \{c_2\}} \pm \det N_{c,c_2} \det M_{c,c_2},$$

and hence

$$f = \det M = \pm p_\tau|_{O^-} + \sum_{c \in \{c_1+1,\dots,a_i\}\setminus\{c_2\}} \pm \det N_{c,c_2} \det M_{c,c_2}.$$

Using induction hypothesis for M_{c,c_2}, we obtain $f = \sum g_\phi p_\phi|_{O^-}$, with $\phi \in W^{a_i}$ such that $\{a_i+1,\dots,n\} \cap \{\phi(1),\dots,\phi(a_i)\} = \{r_1, r_2\}$, and $g_\phi \in k[H]$. In particular, we obtain $\phi(a_i - 1) = r_1$. Since $r_1 \geqslant b_i$, we deduce that $\phi(a_i - 1) \geqslant b_i$. We have $w^{(a_i)}(a_i - 1) = b_i - 1$, and hence $\phi(a_i - 1) > w^{(a_i)}(a_i - 1)$. This shows that $\phi \not\leqslant w^{(a_i)}$, and hence $p_\phi|_{O^-} \in I(X(w) \cap O^-)$. Therefore $f \in I(X(w) \cap O^-)$.

Let now g be a generator of the ideal $I(X(w) \cap O^-)$, i.e. $g = p_\tau|_{O^-}$, with $\tau \in W^{a_i}$ for some $1 \leqslant i \leqslant l$, such that $\tau \not\leqslant w^{a_i}$ (cf. §13.6). Since $w^{(a_i)}$ consists of several blocks of consecutive integers ending with $b_t - 1$ at the $(a_t - 1)$-th place, for some t's in $\{1,\dots,i\}$, and a last block ending with n at the a_i-th place, it follows that there exists a $t \in \{1,\dots,i\}$ such that $\tau(a_t - 1) \geqslant b_t$. As above, the function $p_\tau|_{O^-}$ represents the determinant of the submatrix T of Y^- whose row indices are $\{\tau(1),\dots,\tau(a_i)\}$, and column indices are $\{1,\dots,a_i\}$. Using Laplace expansion with respect to the first a_t columns,we have $\det T = \sum_p \det A_p \det B_p$, where A_p (resp. B_p) is an $a_t \times a_t$ (resp. $(a_i - a_t) \times (a_i - a_t)$) matrix. Clearly, all the column indices of A_p are $\leqslant a_t$, and since $\tau(a_t - 1) \geqslant b_t$, at least 2 row indices of A_p are $\geqslant b_t$. Using Laplace expansion for A_p with respect to 2 rows whose indices are $\geqslant b_t$, we obtain $\det A_p = \sum_q \det C_q \det D_q$, where C_q (resp. D_q) is a 2 (resp. $a_t - 2$) minor, with C_q contained in $L_t \subset L$. This shows that $p_\tau|_{O^-} \in I(Z)$. This completes the proof. □

COROLLARY. 14.3. — *The variety $X(L)$ is normal, Cohen-Macaulay, and has rational singularities.*

This follows from Theorem 14.2, and the fact that Schubert varieties are normal, Cohen-Maculay, and have rational singularities (cf. [23], [24], and [18]).

14.4. Let us fix $j \in \{1,\dots,l\}$. We shall now define $\theta_j \in W^Q$ such that $Z_j = V_j \times \mathbb{A}(H \setminus L)$ gets identified with the opposite cell in $X(\theta_j)$.

For $i < j$, let $\theta_j^{(a_i)} = w^{(a_i)} \setminus \{n\} \cup \{b_j - 1\}$.

For $i = j$, let $\theta_j^{(a_j)} = w^{(a_j)} \setminus \{n\} \cup \{x_j\}$, where x_j is the maximal element in $\{1,\dots,b_j - 1\} \setminus w^{(a_j)}$.

For $i > j$, let

$$\theta_j^{(a_i)} = \begin{cases} w^{(a_i)}, & \text{if } x_j \in w^{(a_i)} \\ w^{(a_i)} \setminus \{y_i\} \cup \{x_j\}, & \text{if } x_j \notin w^{(a_i)}, \end{cases}$$

where y_i is the minimal element in $w^{(a_i)} \setminus \theta_j^{(a_{i-1})}$.

LEMMA 14.5. — *With notations as above, we have $\theta_j \leqslant w$. Further, $\theta_j^{\max}$ is the (unique) maximal element $\tau \in W$, $\tau \leqslant w^{\max}$, such that $\tau^{(a_j)}(a_j) < b_j$.*

The assertion is clear from the definition of θ_j.

THEOREM 14.6. — *The subvariety $Z_j \subset Z$ gets identified with the opposite cell in $X(\theta_j)$, i.e. $Z_j = X(\theta_j) \cap O^-$ (scheme theoretically).*

Proof. — Let f be a generator of $I(Z_j)$. If $f \in I(Z)$, then in view of Theorem 14.2 we have $f \in I(X(w) \cap O^-) \subset I(X(\theta_j) \cap O^-)$ (since $w \geqslant \theta_j$), and there is nothing to prove. Assume that $f \notin I(Z)$; then $f = x_\alpha$, for some $\alpha = (b, a) \in L_j$. Then f can be written as $p_\phi|_{O^-}$, with $\phi \in W^{a_j}$, such that $\{b\} = \{a_j + 1, \dots, n\} \cap \{\phi(1), \dots, \phi(a_j)\}$, and $\{a\} = \{1, \dots, a_j\} \setminus \{\phi(1), \dots, \phi(a_i)\}$. Thus $\phi(a_j) = b$, and since $\alpha \in L_j$, we have $b \geqslant b_j$. Therefore $\phi(a_j) \geqslant b_j$. But $\theta_j^{(a_j)}(a_j) = b_j - 1$, and hence $\phi(a_j) > \theta_j^{(a_j)}(a_j)$. This shows that $\phi \nleqslant \theta_j^{(a_j)}$, and therefore $f \in I(X(\theta_j) \cap O^-)$.

Let now g be a generator of the ideal $I(X(\theta_j) \cap O^-)$, i.e. $g = p_\tau|_{O^-}$, with $\tau \in W^{(a_i)}$, for some $1 \leqslant i \leqslant l$, such that $\tau \nleqslant \theta_j^{(a_i)}$.

First assume that $i \leqslant j$. Then $\theta_j^{(a_i)}$ consists of several blocks of consecutive integers ending with $b_t - 1$ at the $(a_t - 1)$-th place, for some t's in $\{1, \dots, i-1\}$, and a last block ending with $b_j - 1$ at the a_i-th place. The condition $\tau \nleqslant \theta_j^{(a_i)}$ implies that either there exists $t \in \{1, \dots, i-1\}$ such that $\tau(a_t - 1) \geqslant b_t$, or $\tau(a_i) \geqslant b_j$. In the first case we have $\tau \nleqslant w^{(a_i)}$, and hence $p_\tau|_{O^-} \in I(X(w) \cap O^-) = I(Z) \subset I(Z_j)$. Suppose now that $\tau(a_i) \geqslant b_j$. The function $p_\tau|_{O^-}$ represents the determinant of the submatrix T of Y^- whose row indices are $\{\tau(1), \dots, \tau(a_i)\}$, and column indices are $\{1, \dots, a_i\}$. Obviously, all column indices are $\leqslant a_j$. On the other hand, since $\tau(a_i) \geqslant b_j$, the last row of T is contained in L_j, and expanding T along this row, we deduce $p_\tau|_{O^-} \in I(Z_j)$.

Assume now that $i > j$. If $\theta_j^{(a_i)} = w^{(a_i)}$, then $\tau \nleqslant w^{(a_i)}$, and hence $p_\tau|_{O^-} \in I(X(w) \cap O^-) = I(Z) \subset I(Z_j)$, and there is nothing to prove. Suppose that $\theta_j^{(a_i)} \neq w^{(a_i)}$. Then $\theta_j^{(a_i)}$ consists of several blocks of consecutive integers ending with b_t-1 at the (a_t-1)-th place, for some t's in $\{1,\ldots,i\}\backslash\{j\}$, a block ending with b_j-1 at the a_j-th place, and a last block ending with n at the a_i-th place. The condition $\tau \not\leqslant \theta_j^{(a_i)}$ implies that either there exists $t \in \{1,\ldots,i\}\backslash\{j\}$ such that $\tau(a_t-1) \geqslant b_t$, or $\tau(a_j) \geqslant b_j$. In the first case we have $\tau \not\leqslant w^{(a_i)}$, and hence $p_\tau|_{O^-} \in I(X(w)\cap O^-) = I(Z) \subset I(Z_j)$. In the second case, the function $p_\tau|_{O^-}$ represents the determinant of the submatrix T of Y^- whose row indices are $\{\tau(1),\ldots,\tau(a_i)\}$, and column indices are $\{1,\ldots,a_i\}$. Using Laplace expansion with respect to the first a_j columns, we have $\det T = \sum\limits_p \det A_p \det B_p$, where A_p (resp. B_p) is an $a_j \times a_j$ (resp. $(a_i-a_j)\times(a_i-a_j)$) matrix. Clearly, all the column indices of A_p are $\leqslant a_j$, and since $\tau(a_j) \geqslant b_j$, at least one row index of A_p is $\geqslant b_j$. Using Laplace expansion for A_p with respect to a row with index $\geqslant b_j$, we obtain $\det A_p = \sum\limits_q C_q \det D_q$, where C_q's are entries of a row of A_p, contained in $L_j \subset L$. This shows that $p_\tau|_{O^-} \in I(Z_j)$. □

15. A conjecture on the irreducible components of a Schubert variety in $SL(n)/B$.

Let $G = SL(n)$. In this section we state a conjecture which is a refinement of the conjecture in [20] on the irreducible components of the singular locus of a Schubert variety, and prove the conjecture for a certain class of Schubert varieties, namely the pull-backs $\pi^{-1}(X_Q(w))$ under $\pi : G/B \to G/Q$, where w and Q are as in Section 14.

For $\tau \in W$, let P_τ (resp. Q_τ) be the maximal element of the set of parabolic subgroups which leave $\overline{B\tau B}$ (in G) stable under multiplication on the left (resp. right).

We recall the following two well-known results (for a proof, see [19] for example).

LEMMA 15.1. — *Let α be a simple root, and let P_α be the rank 1 parabolic subgroup with $S_{P_\alpha} = \{\alpha\}$. Let $\tau \in W$. Then $\overline{B\tau B}$ is stable under multiplication on the right (resp. left) by P_α if and only if $\tau(\alpha) \in R^-$ (resp. $\tau^{-1}(\alpha) \in R^-$).*

COROLLARY. — *Let $\tau \in W$. With notations as in Section 13, we have*

$$S_{P_\tau} = \{\alpha \in S \mid \tau^{-1}(\alpha) \in R^-\},$$
$$S_{Q_\tau} = \{\alpha \in S \mid \tau(\alpha) \in R^-\}.$$

DEFINITION 15.3. — *Given parabolic subgroups P, Q, we say that $\overline{B\tau B}$ is P-Q stable if $P \subset P_\tau$ and $Q \subset Q_\tau$.*

LEMMA 15.4. — *Let $G = SL(n)$. Let $\tau \in \mathcal{S}_n$, say $\tau = (a_1, \dots, a_n)$. Let $\alpha = \varepsilon_i - \varepsilon_{i+1}$. Then*

(1) *$\tau(\alpha) \in R^-$ if and only if $a_i > a_{i+1}$.*

(2) *$\tau^{-1}(\alpha) \in R^-$ if and only if $i+1$ occurs before i in τ.*

Proof. — We have $\tau(\alpha) = \varepsilon_{a_i} - \varepsilon_{a_{i+1}}$ and $\tau^{-1}(\alpha) = \varepsilon_j - \varepsilon_k$, where $a_j = i$ and $a_k = i+1$. The results follow from this.

Let $\eta \in W$. We shall denote $X_B(\eta)$ by just $X(\eta)$. We first recall the criterion given in [20] for $X(\eta)$ to be singular.

THEOREM 15.5. — *Let $\eta = (a_1 \dots a_n) \in \mathcal{S}_n$. Then $X(\eta)$ is singular if and only if there exist i, j, k, m, $1 \leqslant i < j < k < m \leqslant n$ such that*

$$\text{either } a_k < a_m < a_i < a_j \quad \text{or} \quad a_m < a_j < a_k < a_i\,.$$

15.6. *The set F_η.* Let $\eta = (a_1 \dots a_n) \in \mathcal{S}_n$. Let E_η be the set of all $\tau' \leqslant \eta$ such that either 1) or 2) below holds.

1) There exist i, j, k, m, $1 \leqslant i < j < k < m \leqslant n$, such that

(*a*) $a_k < a_m < a_i < a_j$,

(*b*) if $\tau' = (b_1 \dots b_n)$, then there exist i', j', k', m', $1 \leqslant i' < j' < k' < m' \leqslant n$ such that $b_{i'} = a_k$, $b_{j'} = a_i$, $b_{k'} = a_m$, $b_{m'} = a_j$,

(*c*) if τ (resp. η') is the element obtained from η (resp. τ') by replacing a_i, a_j, a_k, a_m respectively by a_k, a_i, a_m, a_j (resp. $b_{i'}, b_{j'}, b_{k'}, b_{m'}$ respectively by $b_{j'}, b_{m'}, b_{i'}, b_{k'}$), then $\tau' \geqslant \tau$ and $\eta' \leqslant \eta$.

2) There exist i, j, k, m, $1 \leqslant i < j < k < m \leqslant n$, such that

(*a*) $a_m < a_j < a_k < a_i$,

(*b*) if $\tau' = (b_1 \dots b_n)$, then there exist i', j', k', m', $1 \leqslant i' < j' < k' < m' \leqslant n$ such that $b_{i'} = a_j$, $b_{j'} = a_m$, $b_{k'} = a_i$, $b_{m'} = a_k$, (*c*) if τ (resp. η') is the element obtained from η (resp. τ') by replacing a_i, a_j, a_k, a_m respectively by a_j, a_m, a_i, a_k (resp. $b_{i'}, b_{j'}, b_{k'}, b_{m'}$ respectively by $b_{k'}, b_{i'}, b_{m'}, b_{j'}$), then $\tau' \geqslant \tau$ and $\eta' \leqslant \eta$.

Let $F_\eta = \{\tau \in E_\eta \mid \overline{B\tau B}$ is P_η-$Q_\eta$stable$\}$.

CONJECTURE. — *The singular locus of $X(\eta)$ is equal to $\cup_\lambda X(\lambda)$, where λ runs over the maximal (under the Bruhat order) elements of F_η.*

15.7. Let $\eta = (a_1 \dots a_n) \in S_n$. Let $\operatorname{Sing} X(\eta) \neq \varnothing$. Let (a, b, c, d) be four distinct entries in $\{1, \dots, n\}$ such that $a < b < c < d$. An occurence in η of the form (d, b, c, a), where $d = a_i$, $b = a_j$, $c = a_k$, $a = a_m$, $i < j < k < m$, will be referred to as a *Type I bad occurance in* η. An occurance in η of the form (c, d, a, b), where $c = a_i$, $d = a_j$, $a = a_k$, $b = a_m$, $i < j < k < m$, will be referred to as a *Type II bad occurance in* η. Let (d, b, c, a) (resp. (c', d', a', b')) be a bad occurance of Type I (resp. Type II), where $a < b < c < d$ (resp. $a' < b' < c' < d'$). Let θ, θ' be both $\leqslant w$. Further, let b, a, d, c (resp. a', c', b', d') appear in that order in θ (resp. θ'). By abuse of language, we shall refer to (b, a, d, c) (resp. (a', c', b', d')) as a bad occurance in θ (resp. θ') corresponding to the bad occurance (d, b, c, a) (resp. (c', d', a', b')) in η.

Let Q be as in Section 15, and let $\pi : G/B \to G/Q$ be the canonical projection. Let $\tau \in W_Q^{\min}$. We have $\pi^{-1}(X_Q(\tau)) = X_B(\tau^{\max})$, where $\tau^{\max}$, as a permutation, is given by $\tau^{(a_1)}$ arranged in descending order, followed by $\tau^{(a_2)} \setminus \tau^{(a_1)}$ arranged in descending order, etc.. We shall refer to the set $\tau^{(a_i)} \setminus \tau^{(a_{i-1})}$, $1 \leqslant i \leqslant l+1$, arranged in descending order, as the i-th block in $\tau^{\max}$ (here, $\tau^{(a_0)} = \varnothing$, and $\tau^{(a_{l+1})}$ is the set $\{1, \dots, n\} \setminus \tau^{(a_l)}$ arranged in descending order).

For the rest of this section, w and Q will be as in Section 14.

Remark 15.8. — All of the entries in the i-th block in $w^{\max}$ are $\leqslant b_i - 1$, $2 \leqslant i \leqslant l$.

LEMMA 15.9. — *We have*

(1) $Q_{w^{\max}} = Q$.

(2) *Let* $I_{w^{\max}} = \{\varepsilon_i - \varepsilon_{i+1} \mid i = b_j - 1, 1 \leqslant j \leqslant l\}$. *Then* $S_{P_{w^{\max}}} = S \setminus I_{w^{\max}}$.

The assertions are clear from the description of $w^{\max}$ (in view of Lemma 15.4).

LEMMA 15.10. — *Let $P = P_{w^{\max}}$, $Q = Q_{w^{\max}}$. Then $\overline{B\theta_j^{\max}B}$ is P-Q stable.*

Proof. — The Q-stability of $\overline{B\theta_j^{\max}B}$ on the right is obvious. Regarding the P-stability of $\overline{B\theta_j^{\max}B}$ on the left, first let x be an entry $x \neq y_i$, for any $i > j$ such that $x_j \notin w^{(a_i)}$ (notations being as in §14.4). It is clear that if $x-1$ occurs after x in $w^{\max}$, then it does so in $\theta_j^{\max}$ also. Let now $x = y_i$ for some $i > j$ such that $x_j \notin w^{(a_i)}$. Further, let $\varepsilon_{x-1} - \varepsilon_x \in S_P$. This implies that $x \neq b_k$ for any k, $1 \leqslant k \leqslant l$ (cf. Lemma 15.9, (2)), and that $x-1$ occurs after x in $w^{\max}$. In particular we have $x \neq b_j$, and hence $x > b_j$. From this it is clear that $x-1$ occurs after x in $\theta_j^{\max}$ also. The result now follows from this. □

LEMMA 15.11. — *Any bad occurance in $w^{\max}$ is of Type I.*

Proof. — Let $w^{\max} = (a_1 \dots a_n)$. Assume that (c, d, a, b) is a bad occurance of Type II in $w^{\max}$, where $a < b < c < d$. Clearly, c and d (resp. a and b) cannot both appear in the same block , in view of the description of $w^{\max}$. Let then c, d, a, b appear in the h-th, i-th, j-th, k-th blocks respectively, where $h < i \leqslant j < k$. This implies that $a < b < c < d \leqslant b_i - 1$ (cf. Remark 15.8). But now, a and b are both $< b_i - 1$, and they both appear after $b_i - 1$; further, a appears before b in $w^{\max}$, which is not possible by the construction of $w^{\max}$ (note that $a < b$). The required result follows from this. □

Remark 15.12. — Of course, there are several bad occurances in $w^{\max}$ of Type I. For example, fix some j, $1 \leqslant j \leqslant l$. Take $d = n$, $b = b_j - 1$, $c = b_j$, $a = x_j$, notations being as in §14.4. Then d, b, c, a occur in the 1-st, j-th, k-th, m-th blocks respectively, where $j < k \leqslant m$. This provides an example of a Type I bad occurance in $w^{\max}$.

LEMMA 15.13. — *Let d, b, c, a be a bad occurance in $w^{\max}$, where $a < b < c < d$. Assume that b belongs to the i-th block, for some i (note that $i \leqslant l$, since $b < c$). Then*

(1) $b \leqslant b_i - 1$,

(2) $d = n$.

Proof. — Let d, b, c, a occur in the h-th, i-th, j-th, m-th blocks respectively in $w^{\max}$, where $h \leqslant i < j \leqslant m$. First observe that $b < n$, since $b < c < d \leqslant n$. Hence, if $i = 1$, then $b \leqslant b_1 - 1$. If $i \geqslant 2$, again $b \leqslant b_i - 1$ (cf. Remark 15.8). Assertion (1) follows from this.

CLAIM. — $d > b_i - 1$.

Proof. — Assume that $d \leqslant b_i - 1$. Then assumption implies $c < b_i - 1$ (since $c < d$). Now both c and b are $< b_i - 1$, and b belongs to the i-th block in $w^{\max}$. This implies that c should occur before b, which is not possible. Hence our assumption is wrong, and the claim follows. □

Note that the claim in fact implies that $d = n$ (and $h = 1$). □

PROPOSITION 15.14. — *The maximal elements in $F_{w^{\max}}$ are precisely $\theta_j^{\max}$, $1 \leqslant i \leqslant h$ (here $F_{w^{\max}}$ is as in §15.6).*

Proof. — We first observe that $\theta_j^{\max} \in F_{w^{\max}}$; for, corresponding to the bad occurance $d = n$, $b = b_j - 1$, $c = b_j$, $a = x_j$ (cf. Remark 15.12), we have the bad occurance (b, a, d, c) (note that b, a, d, c occur in that order in $\theta_j^{\max}$). Let us denote $\theta_j^{\max}$ by τ'. Let w' (resp. τ) be the element of $\mathcal{S}_n$ obtained from τ' (resp. w) by replacing b, a, d, c (resp. d, b, c, a) respectively by d, b, c, a (resp. b, a, d, c). We have $\tau \leqslant w$ (clearly), and $\tau^{(a_j)}(a_j) \leqslant b_j - 1$. From this we conclude $\tau \leqslant \tau'$ (cf. Lemma 14.5). Also, we have

$$\text{(1)} \qquad \text{for } i \leqslant j,\ w'^{(a_i)}(a_i - 1) = w^{(a_i)}(a_i - 1) < b_i$$

(note that, in fact, for $i \leqslant j$, $w'^{(a_i)} = w^{(a_i)}$), and for $i \geqslant j + 1$,

$$w'^{(a_i)} = \begin{cases} \theta_j^{(a_i)}, & \text{if } b_j \in \theta_j^{(a_i)} \\ \theta_j^{(a_i)} \setminus \{x_j\} \cup \{b_j\}, & \text{if } b_j \notin \theta_j^{(a_i)}. \end{cases}$$

From this it follows that

$$\text{(2) for} \quad i \geqslant j+1,\ \omega'^{(a_i)}(a_i - 1) = \theta_j^{(a_i)}(a_i - 1) \leqslant \omega^{(a_i)}(a_i - 1) = b_i - 1 < b_i.$$

From (1) and (2) we conclude $\omega(\leqslant \omega^{\max}$ (*cf* Remark 14.1).

Thus we obtain $\theta_j^{\max} \in F_{w^{\max}}$.

Let now $\tau' \in F_{w^{\max}}$. In particular, we have $\tau' \in W_Q^{\max}$.

We have a bad occurrance in τ' which has to be of the form (b, a, d, c), $a < b < c < d$, corresponding to the occurrance (d, b, c, a) in $w^{\max}$ (cf. Lemma 15.11). Let b, a, d, c occur in the p-th, q-th, r-th, s-th blocks respectively in τ', where $p \leqslant q < r \leqslant s$ (note that $\tau' \in W_Q^{\max}$). Let w'

be obtained from τ' by replacing b, a, d, c by d, b, c, a respectively. We have $w' \leqslant w^{\max}$ (cf. §15.6). Hence

$$w'^{(a_q)}(a_q - 1) \leqslant w^{(a_q)}(a_q - 1) = b_q - 1.$$

Further, $\tau'^{(a_q)}$ is obtained from $w'^{(a_q)}$ by replacing d by a (note that $d = n$, cf. Lemma 15.13). Hence we obtain

$$\tau'^{(a_q)}(a_q) = w'^{(a_q)}(a_q - 1) \leqslant b_q - 1.$$

This, together with the fact that $\tau' \leqslant w^{\max}$, implies $\tau' \leqslant \theta_q^{\max}$ (cf. Lemma 14.5) □

THEOREM 15.15. — *The conjecture 15.6 holds for $X(w^{\max})$.*

Proof. — In view of Theorems 12.3, 14.2 and 14.6, $X(\theta_j^{\max})$, $1 \leqslant j \leqslant l$ are precisely the irreducible components of $\operatorname{Sing} X(w^{\max})$. On the other hand, we have (cf. Proposition 15.14) that the maximal elements in $F_{w^{\max}}$ are precisely $\theta_j^{\max}$, $1 \leqslant j \leqslant l$. Hence the irreducible components of $\operatorname{Sing} X(w^{\max})$ are precisely $\{X(\theta) \mid \theta \text{ a maximal element of } F_{w^{\max}}\}$. Thus the conjecture holds for $X(w^{\max})$. □

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