Annales de l'institut Fourier

Dmitri Panyushev

On deformation method in invariant theory

Annales de l'institut Fourier, tome 47, nº 4 (1997), p. 985-1012
<http://www.numdam.org/item?id=AIF_1997__47_4_985_0>

Ann. Inst. Fourier, Grenoble
47, 4 (1997), 985–1012

ON DEFORMATION METHOD IN INVARIANT THEORY

by Dmitri PANYUSHEV (1)

Introduction.

The purpose of this paper is to present in a more general form the deformation method in Invariant Theory and to give some new applications.

First, recall what the deformation method is. Let G be a connected reductive algebraic group defined over an algebraically closed field k of characteristic zero. Suppose we are given an affine variety Z endowed with a regular G-action. Then the algebra of regular functions $k[Z]$ carries a natural G-invariant filtration. More precisely, let U denote a maximal unipotent subgroup of G and T denote a maximal torus such that $T \subset N_G(U)$. Denote by $\mathfrak{X}_+$ the monoid of dominant characters relative to T and U. Define $k[Z]_{(\lambda)}$ (resp. $k[Z]_{(<\lambda)}$) to be the sum of all isotypic components of $k[Z]$ associated to weights $\mu \le \lambda$ (resp. $\mu < \lambda$). The family of subspaces $\{k[Z]_{(\lambda)}\}$, $\lambda \in \mathfrak{X}_+$ constitutes a $\mathfrak{X}_+$-filtration in $k[Z]$, i.e., $k[Z] = \cup k[Z]_{(\lambda)}$ and $k[Z]_{(\lambda)} k[Z]_{(\mu)} \subset k[Z]_{(\lambda+\mu)}$. The point of the Deformation Method is that:

(1) the associated graded algebra $\mathrm{gr}_{\mathfrak{X}_+} k[Z] := \bigoplus_{\lambda \in \mathfrak{X}_+} k[Z]_{(\lambda)}/k[Z]_{(<\lambda)}$ can explicitly be described. Namely, it is isomorphic to $(k[Z]^U \otimes k[G]^{U^{\mathrm{op}}})^T$;

(1) This research was supported in part by Alexander von Humboldt Foundation and RFFI Grant 95-01-00783a.
Key words: Algebra of invariants – Reductive group action – Complete intersection – Spherical variety.
Math. classification: 13A20 – 14L30 – 14M10 – 14M17.

(2) $k[Z]$ is a deformation of $\mathrm{gr}_{\mathfrak{X}_+} k[Z]$ and therefore $\mathrm{gr}_{\mathfrak{X}_+} k[Z]$ and $k[Z]$ have many good properties in common;

(3) it is often easier to study $\mathrm{gr}_{\mathfrak{X}_+} k[Z]$ than $k[Z]$. The idea to make use of a G-invariant filtration in $k[Z]$ is due to D. Luna. (A brief outline of his approach appeared in [4], Lemme 1.5.) Then H. Kraft has applied this method to the property of having rational singularities. A detailed development of this method together with numerous applications has been given by V.L. Popov [26] (see [9] for the results in arbitrary characteristic). Various aspects are also touched on in [6], [28], [14]. The above filtration (and deformation) appeals only to the internal structure of the action in question. There are also several papers where a filtration in $k[Z]$ is determined in terms of certain external data (an embedding of Z in a G-module), see [1], [3].

A more general presentation deals with a "relative" situation. Suppose H is a subgroup of G and we are interested in the subalgebra $k[Z]^H$ of H-invariant functions. The above filtration in $k[Z]$ is inherited by $k[Z]^H$. Consider the associated graded algebra $\mathrm{gr}\, k[Z]^H$. Our result is that, when H is spherical in G, this algebra has a nice description. More precisely, one can naturally attach to H a monoid $\mathfrak{E} \in \mathfrak{X}_+$ and a subgroup $\widehat{H} \subset G$ such that $\widehat{H} \supset U$ and $\dim \widehat{H} = \dim H$. The following is proved in Section 2.

0.1. THEOREM.

(i) *There is an isomorphism of $\mathfrak{E}$-graded algebras* $\mathrm{gr}\, k[Z]^H \cong k[Z]^U_{\mathfrak{E}}$;

(ii) *if G/H is quasi-affine and $\mathfrak{E}$ is saturated, then these algebras are isomorphic to* $k[Z]^{\widehat{H}}$.

Here $k[Z]^U_{\mathfrak{E}}$ stands for the subalgebra of $k[Z]^U$ corresponding to $\mathfrak{E}$. (For precise definitions, see Section 2.) It is not immediate after this outline, why the "relative" case may be treated as a generalization of the "absolute" one, because $H = \{e\}$ is not spherical unless G is a torus. In order to establish the desired relationship, we proceed in two steps. The *first* step is also an application of this "relative" result. Consider the diagonal G-action on the product of two G-varieties X and Y. Actions of such form will be called reducible. In this case, $X \times Y$ is acted upon by $G{\times}G$ and the diagonal G-action is nothing but the action of the diagonal subgroup $G_\Delta \subset G{\times}G$. Since $G{\times}G/G_\Delta$ is spherical, Theorem (0.1) applies. That is, we have the large group $G{\times}G$, the spherical subgroup G_Δ, and wish to describe the associated graded algebra of $k[X{\times}Y]^{G_\Delta}$. In this case one proves that $\widehat{G}_\Delta = (U \times U^{\mathrm{op}})T_\Delta$, where U^{op} is the opposite maximal unipotent subgroup and T_Δ is the diagonal in $T \times T$, and that $\mathfrak{C}$ is saturated. Therefore by invoking Theorem (0.1), we get

0.2. THEOREM. — *There is a natural filtration in* $k[X \times Y]^{G_\Delta}$ *such that* $\operatorname{gr} k[X \times Y]^{G_\Delta} \simeq (k[X]^U \otimes k[Y]^{U^{\mathrm{op}}})^T$.

The *second* step is that one considers a special reducible action. Suppose $Y = G$ and the G-action on G is that by right multiplication. Then $k[X \times G]^{G_\Delta} \cong k[X]$ and we recover in this way the G-invariant filtration in $k[X]$ and the description of $\operatorname{gr}_{\mathfrak{X}_+} k[X]$ given in [26], th. 5 (see Section 4). One may observe that the *second part* of Theorem (0.1) may be deduced from the results in [26]. However, I think that the present approach is more natural and certainly allows to give more short and transparent proofs. Another reason is that there is a partial generalization of Theorem (0.1) for the case where G/H has complexity one (the complexity equals zero in the spherical case).

Our applications of (0.1) and (0.2) concern the property of being complete intersection (= c.i.) for algebras of invariants. In Section 3, we give two sufficient conditions for preserving this property under deformations and show that homological dimension does not increase in this case.

Reducible representations of semisimple groups and quadruple cones constitute the main field of applications of these results. A quadruple cone is a G-variety of the form $\prod_{i=1}^{4} C(\lambda_i)$, where $\lambda_i \in \mathfrak{X}_+$ and $C(\lambda)$ is the closure of the G-orbit of highest weight vectors in the irreducible G-module V_λ. In other words, a quadruple cone is the product of two double cones. (See [17] and [21] on double cones.) Making use of Theorem (0.2) and results of Section 3, we find many series of quadruple cones and reducible representations of simple groups such that their algebras of invariants are c.i. This method allows also to get an upper bound for homological dimension of algebras in question (see (4.3)–(4.6), (5.7)).

In Section 5, we consider a particular case of reducible actions. For any G-variety X, one can define the dual G-variety X^* [20]. The diagonal G-action on $X \times X^*$ is called the doubled. It turns out that the associated graded algebra of $k[X \times X^*]^G$ is the algebra of invariants of a doubled action of T. Therefore one may derive some properties of general doubled actions by examining that for tori. Theorem (5.2) gives a sufficient condition for the algebra of invariants of a doubled action to be a hypersurface. The second topic is the canonical module of the doubled action. For sufficiently good *graded* algebras $k[X]$, we prove that $k[X \times X^*]^G$ is Gorenstein and give a formula for degree of the generator of the canonical module (5.6).

Our basic reference for Invariant Theory is [32]. For algebraic groups and their representations we refer to [30].

Remark. — When the main part of this paper was written, I learned that a special case of Theorem (0.2) has been already proved by R. Howe and T. Umeda [11], 6.6. Namely, they considered the case, where $X = V$ is a G-module and $Y = V^*$. However their main interest is in the structure of G-invariant differential operators on G-modules.

Acknowledgement. — Much of this research was conducted while I was visiting the Ruhr-Universität Bochum and I would like to thank Professor A.T. Huckleberry for the invitation and hospitality. I am grateful to D. Shmel'kin for pointing out some errors in the table.

1. Preliminaries.

1.1. *Main notation.* — Throughout the paper, G denotes a connected reductive group with a fixed Borel subgroup B and a fixed maximal torus $T \subset B$. Denote by U the unipotent radical of B and by B^{op} the opposite Borel subgroup (the latter means that $B \cap B^{\mathrm{op}} = T$). Thereupon one also obtains the corresponding objects: the character group $\mathfrak{X}(T)$ of T, the set of simple roots Π, the monoid of dominant weights $\mathfrak{X}_+ \subset \mathfrak{X}(T)$, the Weyl group $W = N_G(T)/T$, etc. For the operation in $\mathfrak{X}(T)$, the additive notation is used. To avoid a confusion with integer 0, we denote by $\underline{0}$ the zero element of $\mathfrak{X}(T)$. Without loss of generality, we may assume whenever it is convenient hereafter that G is a direct product of its connected center and a simply-connected semisimple group. We identify the set of equivalence classes of irreducible rational representations of G with $\mathfrak{X}_+$. For $\lambda \in \mathfrak{X}_+$, we let V_λ denote a representative of the class λ, i.e. an irreducible G-module with highest weight λ. The standard partial ordering on $\mathfrak{X}(T)$ is defined as follows: $\mu < \lambda \Leftrightarrow \lambda - \mu$ is a non-empty sum of positive roots. I use numeration of the fundamental weights of simple groups as in [30].

We let $k[X]$ denote the algebra of regular functions and $k(X)$ the field of rational functions on an algebraic variety X. Given a set M (e.g.

algebra, field, or vector space) equipped with an action of a group $\widetilde{G}$, then $M^{\widetilde{G}}$ denotes the subset of $\widetilde{G}$-fixed elements in it. Unless otherwise stated, all varieties are assumed to be irreducible, and a G-variety is a variety endowed with a regular left action of G.

1.2. *Some standard facts on representations.* — For any $\lambda \in \mathfrak{X}_+$, the one-dimensional subspace $V_\lambda^U \subset V_\lambda$ has a unique T-invariant complement, which is denoted by V_λ^0. Thus the projection

$$\mathrm{pr}_\lambda : V_\lambda \to V_\lambda^U \tag{1}$$

along V_λ^0 is well-defined. Moreover, if μ is a weight of T in V_λ^0, then $\mu < \lambda$. This shows V_λ^0 is even B^{op}-invariant. Given two G-modules V_λ and V_μ, the following relation holds

$$V_\lambda \otimes V_\mu = V_{\lambda+\mu} + \sum_{\nu<\lambda+\mu} c^\nu_{\lambda\mu} V_\nu . \tag{2}$$

The reason is that the subspace of weight $\lambda+\mu$ in $V_\lambda \otimes V_\mu$ is one-dimensional and $\nu < \lambda + \mu$ for the other weights. It follows that the projection $p_{\lambda\mu} : V_\lambda \otimes V_\mu \to V_{\lambda+\mu}$ is well-defined.

1.3. *Some facts on homogeneous spaces.* — Let G/H be a homogeneous space. Given a G-module V, denote by $\mathrm{Mor}_G(G/H, V)$ the vector space of G-equivariant morphisms of G/H into V. It is immediate (and well known) that the following three vector spaces are in a natural one-to-one correspondence: $\mathrm{Hom}_G(V^*, k[G/H])$, $\mathrm{Mor}_G(G/H, V)$, and V^H. We set $\Gamma(G/H) = \{\lambda \in \mathfrak{X}_+ \mid V_\lambda^H \neq \{0\}\}$. Since $k[G/H]$ is a domain, it follows from the previous bijections that $\Gamma(G/H)$ is a monoid.

DEFINITION. — *A monoid $\Gamma \in \mathfrak{X}_+$ is said to be saturated, if $\lambda - \mu \in \Gamma$ whenever $\lambda, \mu \in \Gamma$ and $\lambda - \mu \in \mathfrak{X}_+$. (This is equivalent to saying that $\Gamma = \mathbb{Z}\Gamma \cap \mathfrak{X}_+$.)*

However, it may happen that $\Gamma(G/H)$ is not saturated even for affine spherical homogeneous spaces. Recall that a subgroup H (or a homogeneous space G/H) is called *spherical*, if B has a dense orbit on G/H. In this case, $\dim V_\lambda^H \leq 1$ for any $\lambda \in \mathfrak{X}_+$. If G/H is quasiaffine, then the converse is also true [29].

Example. — In [16], Tabelle 1, one finds the list of all affine spherical homogeneous spaces G/H such that G is simple and H is connected. The description of $\Gamma(G/H)$ is also given therein. Inspecting the list, I found that

$\Gamma = \Gamma(SO_{10}/\operatorname{Spin}_7 \times SO_2)$ is not saturated. More precisely, let $\varphi_1, \dots, \varphi_5$ be the fundamental weights of SO_{10}. Then $2\varphi_1,\ \varphi_1+2\varphi_4 \in \Gamma$, but $4\varphi_4 \notin \Gamma$. I also checked Krämer's computations in this case.

In view of this example and future applications, it is helpful to know whether $\Gamma(G/H)$ is saturated for a given spherical G/H. The only positive result known to me is that $\Gamma(G/H)$ is saturated whenever H is the fixed-point subgroup of an involutory automorphism of G. Apparently, the first algebraic proof of it is due to Th. Vust [33].

1.4. *Horospherical contraction.* — A subgroup of G is called *horospherical*, if it contains a maximal unipotent subgroup. Obviously, any horospherical subgroup is spherical. To any G/H (actually, to any G-variety), one can attach in a natural way the conjugacy class of a horospherical subgroup $\widehat{H}$ [14], Section 2. (This is also implicit in [20].) In the spherical case, an explicit construction of $G/\widehat{H}$ has been given in [6]. If G/H is spherical, then $\dim G/H = \dim G/\widehat{H}$ and $\widehat{H}$ will be referred to as a *horospherical contraction* of H.

For future applications, we need a $\widehat{H}$ which contains the selected U and is normalized by the selected T. We give two descriptions of such a $\widehat{H}$. The first of them, which may be taken as a definition, applies only to quasiaffine G/H. For simplicity, the second description is already stated for spherical G/H.

1) For any $\lambda \in \mathfrak{X}_+$, pick a non-zero vector $v_\lambda \in V_\lambda^U$. Then

$$\widehat{H} = \{g \in G \mid \ gv_\lambda = v_\lambda \text{ for any } \lambda \in \Gamma(G/H)\} = \bigcap_{\lambda \in \Gamma(G/H)} G_{v_\lambda}. \tag{3}$$

2) Let B^{op} be the opposite Borel subgroup, i.e. $B^{\mathrm{op}} \cap B = T$. After eventually replacing H by a conjugate subgroup, one may assume that $B^{\mathrm{op}}H$ is dense in G. Then $B^{\mathrm{op}} \cap H$ is normalized by a maximal torus of G [25]. Having replaced H once more, one can achieve that this maximal torus is the selected T. Then $\widehat{H} = (B^{\mathrm{op}} \cap H)U$ is the desired horospherical contraction. (Of course, some efforts are needed to prove that this is a group.)

The equivalence of these descriptions follows from [20], Section 1. It should only be mentioned that our present approach differs slightly from that of [20]. That paper contains a description of the stabilizer in general position for B-action on quasiaffine X in terms of the monoid $\Gamma(X) \subset \mathfrak{X}_+$. Here 2 correlated alterations are made: (a) we consider the stabilizer in general position for B^{op}-action on $X = G/H$ and (b) our present $\Gamma(G/H)$ should be $\Gamma(G/H)^*$ in the notation of [20].

1.5. PROPOSITION. — $\Gamma(G/\widehat{H}) = \mathbb{Z}\,\Gamma(G/H) \cap \mathfrak{X}_+$.

Proof. — Since $\widehat{H} \supset U$, we have $\mu \in \Gamma(G/\widehat{H})$ if and only if $\widehat{H}v_\mu = v_\mu$.

1. "$\subset$" Suppose $\widehat{H}v_\mu = v_\mu$. Then $(\widehat{H} \cap T)v_\mu = v_\mu$, i.e. $\widehat{H} \cap T \subset \operatorname{Ker} \mu$. Here μ is considered as a group homomorphism $T \to k^*$ and $\operatorname{Ker} \mu$ is its kernel. It follows from Equation (3) that $\widehat{H} \cap T = \bigcap_{\lambda \in \Gamma(G/H)} \operatorname{Ker} \lambda$. Therefore μ belongs to the subgroup of $\mathfrak{X}(T)$ generated by $\Gamma(G/H)$. That is, $\mu \in \mathbb{Z}\Gamma(G/H) \cap \mathfrak{X}_+$.

2. "$\supset$" By Equation (3), we always have $\Gamma(G/\widehat{H}) \supset \Gamma(G/H)$. Suppose $\mu = \lambda_1 - \lambda_2 \in \mathfrak{X}_+$, $\lambda_i \in \Gamma(G/H)$. It then follows from (1.2) that, up to a scalar multiple, $v_\mu \otimes v_{\lambda_2} = v_{\lambda_1} \in V_{\lambda_1} \subset V_\mu \otimes V_{\lambda_2}$. Thus $\widehat{H}v_\mu = v_\mu$. □

2. A filtration of the algebra of invariants.

2.1. *G-invariant filtration.* — Let Z be an affine G-variety. Then one gets the induced G-action on $k[Z]$: $(g, f) \mapsto g \cdot f$, $g \in G$, $f \in k[Z]$. It is defined by $(g \cdot f)(x) = f(g^{-1}x)$. Then $k[Z]$ becomes a locally-finite G-algebra. This means that the linear span $\langle G \cdot f \rangle$ is finite-dimensional for any $f \in k[Z]$. It follows from complete reducibility that $k[Z]$ is the direct sum of its isotypic components $k[Z]_\lambda$, $\lambda \in \mathfrak{X}_+$. Here $k[Z]_\lambda$ stands for the sum of all irreducible G-submodules of $k[Z]$ which are isomorphic to V_λ. This decomposition is not a grading in general, but a minor modification allows us to obtain a G-invariant filtration (see [26]). We set

$$k[Z]_{(<\lambda)} = \bigoplus_{\mu < \lambda} k[Z]_\mu \quad \text{and} \quad k[Z]_{(\lambda)} = k[Z]_\lambda \oplus k[Z]_{(<\lambda)}, \quad \lambda \in \mathfrak{X}_+.$$

The family $\{k[Z]_{(\lambda)}\}$ is a G-invariant $\mathfrak{X}_+$-filtration of $k[Z]$. More precisely, we have $k[Z]_{(\lambda)} k[Z]_{(\mu)} \subset k[Z]_{(\lambda+\mu)}$ and $k[Z]_{(\lambda)} k[Z]_{(<\mu)} \subset k[Z]_{(<\lambda+\mu)}$. This follows from Equation (2), since the product of two irreducible submodules in the algebra $k[Z]$ is a homomorphic image of their tensor product.

Let H be a subgroup of G. Consider the inherited filtration $\{k[Z]^H \cap k[Z]_{(\lambda)}\}$ in $k[Z]^H$. It is immediate that

$$(4) \qquad k[Z]^H \cap k[Z]_\lambda \neq \{0\} \Rightarrow V_\lambda^H \neq \{0\} \Leftrightarrow \lambda \in \Gamma(G/H).$$

To simplify notation, we set $\mathfrak{E} := \Gamma(G/H)$, $C := k[Z]^H$, $C_\lambda := k[Z]^H \cap k[Z]_\lambda$, etc. It follows from Equation (4) that the inherited filtration can be reduced to a $\mathfrak{E}$-filtration. Namely, since $C = \bigoplus_{\lambda\in\mathfrak{E}} C_\lambda$ (vector space decomposition!), we may (and shall) consider $C_{(<\lambda)} := k[Z]^H \cap k[Z]_{(<\lambda)}$ and $C_{(\lambda)}$ for $\lambda \in \mathfrak{E}$ only. Thus one has a family of subspaces $\{C_{(\lambda)}\}$, $\lambda \in \mathfrak{E}$ and the decompositions $C_{(\lambda)} = C_\lambda \oplus C_{(<\lambda)}$. Let us observe that $C_{(\underline{0})} = k[Z]_{(\underline{0})} = k[Z]^G$. The associated $\mathfrak{E}$-graded algebra $\operatorname{gr}_{\mathfrak{E}} C$ is defined by

$$\operatorname{gr}_{\mathfrak{E}} C = \bigoplus_{\lambda\in\mathfrak{E}} C_{(\lambda)}/C_{(<\lambda)}. \tag{5}$$

It follows that $\operatorname{gr}_{\mathfrak{E}} C$ is isomorphic to C as a vector space. Therefore one may consider $\operatorname{gr}_{\mathfrak{E}} C$ as C but endowed with a new multiplication law. More precisely, suppose $f \in C_\lambda \cong C_{(\lambda)}/C_{(<\lambda)}$ and $h \in C_\mu \cong C_{(\mu)}/C_{(<\mu)}$. Then $fh \in C_{(\lambda+\mu)} = C_{\lambda+\mu} \oplus C_{(<\lambda+\mu)}$ for the obvious multiplication in C. While in the new multiplication coming from $\operatorname{gr}_{\mathfrak{E}} C$, only the component of fh lying in $C_{\lambda+\mu}$ survives.

Let us apply the preceding construction to $H = U$. Then the decomposition $k[Z]^U = \bigoplus_{\lambda\in\mathfrak{X}_+} k[Z]^U_\lambda$, where $k[Z]^U_\lambda = k[Z]^U \cap k[Z]_\lambda$, gives already a $\mathfrak{X}_+$-grading. This follows from (1.2). This grading originates also from the obvious T-action on $k[Z]^U$. (Observe that $\mathfrak{X}_+ = \Gamma(G/U)$.) For any subset $\Gamma \subset \mathfrak{X}_+$, we set $k[Z]^U_\Gamma = \bigoplus_{\lambda\in\Gamma} k[Z]^U_\lambda$. Obviously, this is a (finitely generated) algebra whenever Γ is a (finitely generated) monoid.

2.2. THEOREM. — *Suppose H is a spherical subgroup of G and $\mathfrak{E} = \Gamma(G/H)$. Then, for any affine G-variety Z,*

(i) *there is an isomorphism of $\mathfrak{E}$-graded algebras $\operatorname{gr}_{\mathfrak{E}} k[Z]^H \cong k[Z]^U_{\mathfrak{E}}$;*

(ii) *if G/H is quasi-affine and $\mathfrak{E}$ is saturated, then these algebras are isomorphic to $k[Z]^{\widehat{H}}$, where $\widehat{H}$ is a horospherical contraction of H. In this case, the relation $\mathfrak{E} = \Gamma(G/\widehat{H})$ holds.*

Remark. — The point of part (ii) is that it holds for any Z. For some Z, it certainly may happen that $k[Z]^U_{\mathfrak{E}}$ is the algebra of invariants of a horospherical subgroup, which is not $\widehat{H}$ and regardless of saturatedness of $\mathfrak{E}$.

Proof. — (i) We may assume that H is chosen such that $B^{\mathrm{op}}H$ is dense in G. Then the linear span of $B^{\mathrm{op}}V^H_\lambda$ coincides with V_λ for any

$\lambda \in \mathfrak{C}$. Therefore V_λ^H does not lie in V_λ^0 (see (1.2)) and the restriction on V_λ^H of pr_λ is one-to-one. (Recall that $\dim V_\lambda^H = 1$.) Pick a non-zero $f_\lambda \in V_\lambda^H$. Then the functions $\chi(f_\lambda)$ generates C_λ as a vector space, where χ ranges over $\mathrm{Hom}_G(V_\lambda, k[Z])$. Putting $\tau_\lambda(\chi(f_\lambda)) = \chi(\mathrm{pr}_\lambda(f_\lambda))$, one obtains a homomorphism of vector spaces

$$\tau_\lambda \; : \; C_\lambda \to k[Z]^U_\lambda \, .$$

It is easily seen that τ_λ is one-to-one. Considering each τ_λ as a mapping from $C_{(\lambda)}/C_{(<\lambda)}$ and then putting them together, we get a one-to-one mapping $\tau \; : \; \mathrm{gr}_{\mathfrak{C}}\, C \to k[Z]^U_{\mathfrak{C}}$. It then follows from Equation (2) that τ is an algebra homomorphism.

(ii) We may assume that $\widehat{H}$ is defined by Equation (3). Then $\widehat{H} \supset U$ and $N_G(\widehat{H}) \supset T$. Therefore $k[Z]^{\widehat{H}} = \bigoplus_\mu k[Z]^U_\mu$, where μ runs over all dominant weights such that $\widehat{H} v_\mu = v_\mu$. Thus everything follows from (1.5) and saturatedness of $\mathfrak{C}$. □

2.3. *Remark.* — In order to have a geometric content, one should be sure that all algebras being considered above are finitely generated. There are different ways to deduce it from known results. Here is one of them. Let G/S be a spherical homogeneous space of G. Then $\Gamma(G/S)$ is finitely generated. A short conceptual proof is given in [15] (close results were earlier obtained by Guillemonat and Grosshans). This implies the algebra $k[G/S]$ is finitely generated as well. For a G-variety Z, consider the diagonal action of G on $k[Z] \otimes k[G/S]$. By [4], 1.1 or [8], 1.2, the algebras $(k[Z] \otimes k[G/S])^G$ and $k[Z]^S$ are isomorphic. It follows that $k[Z]^S$ is finitely generated for any affine G-variety Z. Finally, an algebra of the form $k[Z]^U_\Gamma$ is finitely generated whenever Γ is. Thus all algebras arising in (2.2) are finitely generated.

Consider the particular case of (2.2) when $Z = G$ and G acts on itself by right multiplications. Then $k[G]^H$ is the algebra of regular functions on G/H and Theorem (2.2) asserts that

$$\mathrm{gr}_{\mathfrak{C}}\, k[G/H] \cong k[G]^U_{\mathfrak{C}} . \tag{6}$$

In this case, G acts also on itself by left multiplications and clearly the $\mathfrak{C}$-filtration in $k[G]^H$ is G-stable. Therefore the above isomorphism is G-equivariant. Since one has an extra G-action in this case, the construction of $\mathrm{gr}_{\mathfrak{C}}\, k[G/H]$ can also be performed in the framework of the usual approach to the deformation method (see [6], [26]). Assume that G/H is quasiaffine.

Put $X = \operatorname{Spec} k[G/H]$ and $Y = \operatorname{Spec} k[G]^U_{\mathfrak{C}}$. Then the G-variety X contains G/H as the open orbit and the G-variety Y contains $G/\widehat{H}$ as the open orbit. In the terminology of [26], Y is a contraction of X (and, otherwise, X is a deformation of Y). Now, one can see why $\mathfrak{C}$ is of some importance in deformation questions. The monoid $\mathfrak{C}$ is saturated if and only if $\operatorname{gr}_{\mathfrak{C}} k[G/H]$ is the algebra of regular functions on $G/\widehat{H}$. As it was already explained in (1.4), the homogeneous space $G/\widehat{H}$ can be explicitly determined without involving any deformation arguments. From geometric point of view, the saturated and non-saturated cases are distinguished by codimension of the complement of $G/\widehat{H}$ in Y. Namely, $\mathfrak{C}$ is saturated if and only if this complement does not contain divisors (see [31], §3).

2.4. *Examples.* — Let H act on a variety Z. Suppose (we are lucky and) there exists an overgroup $G \supset H$ acting on Z such that G/H is spherical. The overgroup provides, as "deus ex machina", a filtration in $k[Z]^H$ and a description of $\operatorname{gr}_{\mathfrak{C}} k[Z]^H$. Next step is to establish a "good" property for the graded algebra and to carry it on $k[Z]^H$. Our applications of (2.2) are related to the property of being complete intersection and are based on the results of Section 3. Two general schemes of realization of this idea are presented in Section 4. Here we consider two sporadic examples. If representations of both a group H and an overgroup G are considered, then $\tilde{\varphi}_i$'s are the fundamental weights of G, and φ_i's are the ones of H. The trivial 1-dimensional representation is denoted by $\mathbf{I}$. For a G-module V, the restriction of the corresponding representation to H is denoted by $V|_H$.

1. Let H be a group of type $\mathbf{G}_2$ and Z be the sum of the fundamental representations, i.e. $Z = V_{\varphi_1} + V_{\varphi_2}$. It is known that H is a subgroup of a group G of type $\mathbf{B}_3$. In this case G/H is spherical and $\mathfrak{C}$ is generated by $\tilde{\varphi}_3$ (see [16]). Therefore $\widehat{H}$ is the stabilizer of a highest weight vector in the spinor representation of $\mathbf{B}_3$. This is a semi-direct product of a simple group of type $\mathbf{A}_2$ and a 6-dimensional unipotent radical. The representation of $\mathbf{G}_2$ in $V_{\varphi_1} + V_{\varphi_2}$ is the restriction of the adjoint representation of $\mathbf{B}_3$. Thus we get an isomorphism

$$\operatorname{gr}_{\mathfrak{C}} k[V_{\varphi_1} + V_{\varphi_2}]^{\mathbf{G}_2} \cong k[V_{\tilde{\varphi}_2}]^{U(\mathbf{B}_3)}_{\mathfrak{C}} .$$

I can prove the latter algebra is a hypersurface. This implies $k[V_{\varphi_1} + V_{\varphi_2}]^{G_2}$ is a hypersurface as well (see (3.5)).

2. $H = SL_n$ and $Z = V_{\varphi_1} + V_{\varphi_{n-1}} + V_{\varphi_1+\varphi_{n-1}}$. Consider the obvious embedding $SL_n \subset SL_{n+1} = G$. This is a spherical pair and

$\mathfrak{E}$ is generated by $\tilde\varphi_1$ and $\tilde\varphi_n$ [16]. It is easy to see that $V_{\tilde\varphi_1+\tilde\varphi_n}\ |_H=$ $V_{\varphi_1}+V_{\varphi_{n-1}}+V_{\varphi_1+\varphi_{n-1}}+\mathbf{I}$. This means that $k[Z+\mathbf{I}]^H$ admits a filtration such that the associated graded algebra is isomorphic to $k[V_{\tilde\varphi_1+\tilde\varphi_n}]_{\mathfrak{E}}^{U(SL_{n+1})}$. I can prove that this is a hypersurface. This result and (3.5) show that $k[Z]^{SL_n}$ is a hypersurface as well.

3. Deformations and complete intersections.

3.1. *On deformations.* — This subsection is an abridged exposition of [26], §5. Several fragments of that theory are also found in [6]. Given a filtered algebra, it is quite natural to compare properties of the algebra itself and its associated graded algebra. For this purpose, one more condition should be satisfied. Namely, we have to produce a $\mathbb{N}$-filtration in $C=k[Z]^H$ such that $\mathrm{gr}_{\mathfrak{E}}\, C$ and $\mathrm{gr}_{\mathbb{N}}\, C$ would be isomorphic as algebras. This can be done in a standard way (see [6], [26]). Let $(\cdot,\cdot)$ be a W-invariant inner product on $\mathfrak{X}(T)\otimes\mathbb{Q}$ and $\varrho^\vee$ be the sum of all positive coroots. Define $l\ :\ \mathfrak{X}(T)\to\mathbb{Z}$ by $l(\gamma)=(\gamma,\varrho^\vee)$. It is easy to see that $l(\mathfrak{X}_+)\subset\mathbb{N}$ and $l(\alpha)=2$ for any $\alpha\in\Pi$. Then the family of subspaces

$$C^{(n)}:=\bigoplus_{l(\lambda)\le n} C_\lambda \qquad (C^{(-1)}:=0)$$

defines an ascending $\mathbb{N}$-filtration such that

$$\mathrm{gr}_{\mathbb{N}}\, C:=\bigoplus_{n=0}^{\infty} C^{(n)}/C^{(n-1)}\cong \mathrm{gr}_{\mathfrak{E}}\, C$$

as algebras. An easy proof makes use of Equation (2) and the above-mentioned properties of l. Let t be an indeterminate and

$$D:=\bigoplus_{n=0}^{\infty} t^n C^{(n)}\subset C[t]\ .$$

Clearly, D is an algebra and $k[t]$ is a subalgebra of it. One introduces a $\mathbb{N}$-grading on D by letting $\deg t=1$. It is immediate that D is a flat $k[t]$-module, $D/(t)\cong \mathrm{gr}_{\mathbb{N}}\, C$, and $D/(t-\alpha)\cong C$ for any $\alpha\in k\setminus\{0\}$. By considering t as the function (morphism)

$$t\ :\ \operatorname{Spec} D\to\mathbb{A}^1, \tag{7}$$

we get: $t^{-1}(0)\cong\operatorname{Spec}(\mathrm{gr}_{\mathbb{N}}\, C)$ and $t^{-1}(\alpha)\cong\operatorname{Spec} C$, $\alpha\neq 0$. That is, one obtains a flat one-parameter family of affine varieties. For this reason, $\mathrm{gr}_{\mathbb{N}}\, C$ is called a *contraction* of C and, in its turn, C is called a *deformation* of

$\mathrm{gr}_{\mathbb{N}}\, C$. The same terminology applies to the respective varieties. One more property of Equation (7) is important: $t^{-1}(\mathbb{A}^1 \setminus \{0\}) \cong (\mathbb{A}^1 \setminus \{0\}) \times \operatorname{Spec} C$. Making use of it, V.L. Popov has shown that presence of a so-called property of open type for $\operatorname{Spec}(\mathrm{gr}\, C)$ implies that for $\operatorname{Spec} C$. For instance, the following properties are of open type: normality, Cohen-Macaulayness, rationality of singularities.

3.2. *On complete intersections.* — Given an affine variety Y, the minimal m such that there exists a closed embedding $Y \subset \mathbb{A}^m$ is called the *embedding dimension* of Y and is denoted by $\mathrm{ed}\, Y$. We set $\mathrm{hd}\, k[Y] = \mathrm{ed}\, Y - \dim Y$. If $k[Y]$ is a Cohen-Macaulay graded domain, this integer is customary called the homological dimension of Y. Recall that Y (or its coordinate algebra $k[Y]$) is said to be a *complete intersection*, if there exists an exact sequence

$$0 \to I_N \to k[T_1, \dots, T_N] \to k[Y] \to 0$$

such that the ideal I_N is generated by $N - \dim Y$ elements. There are the corresponding concepts in the theory of local rings. But, unlike that theory, it is not known whether for the *given* presentation $k[Y] = k[T_1, \dots, T_n]/I_n$ the ideal I_n is generated by $n - \dim Y$ elements, if Y is a complete intersection. (Some results in this direction are found in [8].) In the sequel, c.i. will be shorthand of "complete intersection".

Suppose $k[Y]$ is a c.i. Then the local ring $k[Y]_{\mathfrak{p}}$ is a c.i. for any maximal ideal $\mathfrak{p} \subset k[Y]$ and $\mathrm{hd}\, k[Y]_{\mathfrak{p}} \le \mathrm{hd}\, k[Y]$. But the converse is not true. In other words, the property of being c.i. is not local. Nevertheless, there is an important particular case, when the global theory is essentially equivalent to the local one.

3.3. PROPOSITION [2]. — *Let* $A = \sum_{i=0}^{\infty} A_i$ *be a* $\mathbb{N}$*-graded affine* k*-algebra such that* $A_0 = k$. *Suppose* $A_{\mathfrak{m}}$, *where* $\mathfrak{m} = \sum_{i=1}^{\infty} A_i$, *is a complete intersection. Then* A *is a c.i. as well. More precisely, in the presentation* $0 \to I \to S^\bullet(\mathfrak{m}/\mathfrak{m}^2) \to A \to 0$, *the ideal* I *is generated by a regular sequence.* □

Let us apply this assertion to the $\mathbb{N}$-graded algebra $\mathrm{gr}\, C$ (we shall omit the subscript $\mathbb{N}$ in the sequel).

3.4. PROPOSITION. — *Suppose* $C^{(0)} = k$ *and* $\mathrm{gr}\, C$ *is a c.i. Then* C *is a c.i. as well and* $\mathrm{hd}\, C \le \mathrm{hd}(\mathrm{gr}\, C)$.

Proof. — Since $C^{(0)} = k$, the ideals $\mathfrak{m} := \sum_{n=1}^{\infty} C^{(n)}/C^{(n-1)} \subset \operatorname{gr} C$ and $\hat{\mathfrak{m}} := \sum_{n=1}^{\infty} t^n C^{(n)} \subset D$ are maximal. It is obvious that $t \in \hat{\mathfrak{m}} D_{\hat{\mathfrak{m}}}$ and $D_{\hat{\mathfrak{m}}}/(t) \cong (\operatorname{gr} C)_{\mathfrak{m}}$. Then, by a standard property of local rings, the c.i. property for $(\operatorname{gr} C)_{\mathfrak{m}}$ is equivalent to that for $D_{\hat{\mathfrak{m}}}$ (see [7], 2.3.4). Moreover, since $t \in \hat{\mathfrak{m}} \setminus \hat{\mathfrak{m}}^2$, we have $\operatorname{hd} D_{\hat{\mathfrak{m}}} = \operatorname{hd}(\operatorname{gr} C)_{\mathfrak{m}}$. Thus D is a c.i. and $\operatorname{hd} D = \operatorname{hd}(\operatorname{gr} C)$ by (3.3). Consider the embedding $\operatorname{Spec} D \subset (\hat{\mathfrak{m}}/\hat{\mathfrak{m}}^2)^*$ given by Nakayama's lemma. Then the equality $D/(t-\alpha) = C$ reads that $\operatorname{Spec} C$ is the intersection (as a scheme!) of $\operatorname{Spec} D$ and an affine hyperplane in $(\hat{\mathfrak{m}}/\hat{\mathfrak{m}}^2)^*$. It follows that C is a c.i. as well and $\operatorname{hd} C \le \operatorname{hd} D$.

For $C = k[Z]^H$, the subalgebra $C^{(0)} = \bigoplus_{(\lambda,\varrho^\vee)=0} C_\lambda$ is nothing but the algebra of invariants of the commutator subgroup G' of G. The condition $k[Z]^{G'} = k$ is obviously satisfied, if G' has an open orbit in Z. However one would like to have something like (3.4) for actions with non-constant invariants. For this purpose, consider the case when C is both $\mathbb{N}$-filtered and $\mathbb{N}$-graded. This means that the grading $C = \bigoplus_{n=0}^{\infty} C_n$ and the filtration $C = \bigcup_{i=0}^{\infty} C^{(i)}$ should be compatible in the sense that $C^{(n)} = \bigoplus_{m=0}^{\infty} (C_m \cap C^{(n)})$ for any $n \in \mathbb{N}$. (One do not confuse the vector space decomposition $C = \bigoplus_{\lambda \in \mathfrak{C}} C_\lambda$ with the $\mathbb{N}$-grading, since we use different indices.) Then $\operatorname{gr} C$ is $\mathbb{N}^2$-graded and one can produce the natural "diagonal" $\mathbb{N}$-grading in it:

$$(\operatorname{gr} C)_n := \bigoplus_{m+l=n} (C^{(m)} \cap C_l / C^{(m-1)} \cap C_l) \,.$$

It is immediate that $(\operatorname{gr} C)_0 = k$ whenever $C_0 = k$. In this case, there is an analogue of (3.4). We retain the notation just introduced.

3.5. PROPOSITION. — *Let C be a $\mathbb{N}$-filtered and $\mathbb{N}$-graded algebra such that $C_0 = k$. Suppose* $\operatorname{gr} C$ *is a c.i. Then C is a c.i. as well and* $\operatorname{hd} C \le \operatorname{hd}(\operatorname{gr} C)$.

Proof. — Consider $\operatorname{gr} C$ with the diagonal grading and provide D with the diagonal grading too, i.e. set

$$D_n := \bigoplus_{m+l=n} t^m (C^{(m)} \cap C_l) \,.$$

Then $D_0 = k$ and t still has degree one. Having these gradings at hand, one can repeat the proof of (3.4). The only noticeable point is that the relation

$D_{\hat{\mathfrak{m}}}/(t) \cong (\operatorname{gr} C)_{\mathfrak{m}}$ being used above survives under the new interpretation of $\mathfrak{m}$ and $\hat{\mathfrak{m}}$. □

A typical situation for applying (3.5) is that of a linear representation $G \subset GL(V)$. Then $k[V]$ and $k[V]^H$ are graded by the total degree of polynomials and it is obvious that the filtration and the grading on $k[V]^H$ are compatible. More generally, it suffices to assume that $k[Z]$ is $\mathbb{N}$-graded such that $k[Z]_0 = k$ and G respects this grading. Examples will be given in Section 4.

3.6. *Remark.* — Along with the diagonal $\mathbb{N}$-grading on $\operatorname{gr} C$, one can consider the $\mathbb{N}$-grading which is induced from C. This one is defined by

$$(\operatorname{gr} C)^{\sim}_n = \bigoplus_{m=0}^{\infty} (C^{(m)} \cap C_n)/(C^{(m-1)} \cap C_n)\ .$$

An important property of this grading is that $\dim(\operatorname{gr} C)^{\sim}_n = \dim C_n$ for any n. Suppose we are in the situation of (2.2) and $k[Z]$ is already $\mathbb{N}$-graded. Then $k[Z]^H$ inherits this grading and the induced grading on $\operatorname{gr} k[Z]^H \cong k[Z]^{\widehat{H}}$ coincides with the grading inherited by the subalgebra $k[Z]^{\widehat{H}}$.

For an affine k-algebra A, if $\operatorname{hd} A \le 1$, then A is already a c.i. (a hypersurface). If $\operatorname{hd} A = 2$, A is $\mathbb{N}$-graded with $A_0 = k$, and A is Gorenstein, then A is a c.i. Therefore it is helpful to have a relation between the embedding dimensions of C and $\operatorname{gr} C$. Next assertion is implicit in [26] (cf. also [1], Lemma 3).

3.7. PROPOSITION. — *One always has* $\operatorname{ed}(\operatorname{gr} C) \ge \operatorname{ed} C$. *In particular, if* $\operatorname{gr} C$ *is a polynomial ring then so is* C*; if* $\operatorname{gr} C$ *is a hypersurface, then* C *is either a hypersurface or a polynomial ring.*

Proof. — Suppose $\{f_1, \dots, f_s\}$ is a homogeneous generating system of $\operatorname{gr} C$, where $f_i \in C^{(n_i)}/C^{(n_i-1)}$. Pick an element $\hat{f}_i \in C^{(n_i)}$ lying over f_i and denote by $\hat{C}$ the subalgebra of C generated by $\{\hat{f}_1, \dots, \hat{f}_s\}$. Since $C^{(0)}$ is contained in both C and $\operatorname{gr} C$, it is immediate that the elements $\hat{f}_i$ such that $n_i = 0$ generate $C^{(0)}$. Therefore $C^{(0)} \subset \hat{C}$. Then an obvious induction step: if $C^{(n)} \subset \hat{C}$, then $C^{(n+1)} \subset \hat{C}$. □

4. On invariants of reducible actions.

4.1. *Reducible actions.* — Given a G-action on an affine variety Z, we shall say it is *reducible* whenever Z is the product of G-varieties X, Y and the action on $X\times Y$ is diagonal, i.e. $g(x,y)=(gx,gy)$, $(x,y)\in X\times Y$. It is always assumed that the G-action on both factors is non-trivial. In this case, $X\times Y$ is acted upon by $G\times G$ and the diagonal G-action is nothing but the action of the diagonal subgroup G_Δ of $G\times G$. Since $G\times G/G_\Delta$ is spherical (see below), the results of Section 2 can be used. That is, we have the large group $G\times G$, the subgroup G_Δ, and wish to describe $k[X\times Y]^{G_\Delta}$.

In order to apply (2.2), we have to select a Borel subgroup and a maximal torus in $G\times G$. For G, we retain the notation of (1.1). We set $\mathcal{B} := B\times B^{\rm op}$ to be the selected Borel and $T\times T$ to be the selected maximal torus. Then $\mathcal{B}^{\rm op}=B^{\rm op}\times B$ and therefore $\mathcal{B}^{\rm op}\cap G_\Delta =: T_\Delta$ is the diagonal subgroup of $T\times T$. By counting dimension, we get convinced that $\mathcal{B}^{\rm op}G_\Delta$ is dense in $G\times G$. Therefore G_Δ is spherical and has the required position relative to $\mathcal{B}^{\rm op}$. By (1.4), this implies that $\widehat{G}_\Delta=(U\times U^{\rm op})T_\Delta$. By Schur's lemma, $(V_\lambda\otimes V_\mu)^{G_\Delta}$ is not zero if and only if $\mu=\lambda^*$. Therefore $\mathfrak{E}=\{(\lambda,\lambda^*)\mid \lambda\in\mathfrak{X}_+\}\cong\mathfrak{X}_+$ is saturated. One should be careful at this point, since we choose a Borel in $G\times G$ such that its second factor is $B^{\rm op}$. But, we still denote highest weights with respect to $B\times B$. Anyhow, having in mind the preceding argument, the following result is a straightforward consequence of (2.2). (In what follows, we omit the subscript Δ.)

4.2. THEOREM. — *The algebra of invariants of a reducible action $(G : X\times Y)$ has a natural $\mathfrak{X}_+$-filtration. There is an isomorphism of $\mathfrak{X}_+$-graded algebras*

$$\mathrm{gr}_{\mathfrak{X}_+} k[X\times Y]^G \cong (k[X]^U\otimes k[Y]^{U^{\rm op}})^T,$$

where T acts diagonally on the tensor product. □

Now, we are in position to show that the description of the associated graded algebra given in [26] can be considered as a particular case of (4.2) and hence of (2.2). Take $Y=G$ and consider the G-action on G by right multiplications. This means the action $\rho_r : G\times G\to G$ is given by $\rho_r(s,g)=gs^{-1}$. (The left-hand copy of G acts on the right-hand one.) For the corresponding diagonal G-action on $X\times G$, we then get the isomorphism $\phi : k[X]\to k[X\times G]^G$ given by $\phi(p)(x,g)=p(gx)$, $p\in k[X]$. Along with the diagonal action, one can introduce the following one: $(s,\ (x,g))\mapsto \rho_l(s,\ (x,g)) := (x,sg)$, $s\in G$, $(x,g)\in X\times G$. It is immediate that both actions under consideration commute and ρ_l induces the original (left) G-action on $k[X\times G]^G \cong k[X]$. Moreover the $\mathfrak{X}_+$-filtration in $k[X\times G]^G$ corresponds under this isomorphism to the G-invariant $\mathfrak{X}_+$-filtration in $k[X]$ considered in [26] and which is described in (2.1). Hence one recover in this way Theorem 5 in loc. cit.

4.3. *Example: reducible representations.* — The most obvious example of a reducible action is a reducible representation. First, I present an explicit computation proving that the algebra of invariants of some reducible representation is a complete intersection. This is an application of (4.2) and results of Section 3.

Consider $G = Sp_6$ and its 2nd and 3rd basic representations, i.e. $X = V_{\varphi_2}, Y = V_{\varphi_3}$. In both cases, the algebras of U-invariants are polynomial, and degrees and the weights of free generators are found in [5]. Namely, $k[V_{\varphi_2}]^U$ is generated by functions $a_1,\dots,a_5$ whose weights are φ_2, φ_2, $\varphi_1+\varphi_3$, $\underline{0}$, $\underline{0}$. The functions $b_1,\dots,b_5$ whose weights are $-\varphi_3$, $-\varphi_3$, $-2\varphi_1$, $-2\varphi_2$, $\underline{0}$ generate $k[V_{\varphi_3}]^{U^{\mathrm{op}}}$. Therefore $(k[V_{\varphi_2}]^U\otimes k[V_{\varphi_3}]^{U^{\mathrm{op}}})^T$ is generated by $a_1^2b_4$, $a_1a_2b_4$, $a_2^2b_4$, $a_3^2b_1^2b_3$, $a_3^2b_1b_2b_3$, $a_3^2b_2^2b_3$, a_4, a_5, b_5 and there are two basic relations connecting these functions. Thus, according to (3.5) and (4.2), $C = k[V_{\varphi_2}+V_{\varphi_3}]^{Sp_6}$ is a c.i. and $\mathrm{hd}\, C \le 2$. Actually, it was shown by D. Shmel'kin that $\mathrm{hd}\, C = 1$ in this case.

This example suggests a *general strategy* for finding new ones. Suppose X and Y are G-varieties such that their algebras of U-invariants are polynomial. Then $k[X]^U\otimes k[Y]^{U^{\mathrm{op}}}$ is polynomial and therefore $(k[X]^U\otimes k[Y]^{U^{\mathrm{op}}})^T$ is the algebra of invariants of a linear representation of T. The problem of describing these algebras is manageable.

Let us apply this strategy to finding new examples of reducible representations whose algebras of invariants are complete intersections. In [5], Table 1, one finds the list of all representations of simple groups with polynomial algebras of U-invariants. It turns out that $(k[V_1]^U\otimes k[V_2]^{U^{\mathrm{op}}})^T$ is a c.i. for *all* pairs V_1, V_2 such that either V_i or its dual is contained in that table. The homological dimension is also easily computed. Indeed, the non-zero weights of generators of algebras of U-invariants are fundamental for almost all representations in Brion's table. Therefore the corresponding representation of T decomposes into the sum of representations of one-dimensional subtori corresponding to simple roots. Since the algebra $k[V_1+V_2]^G$ is a deformation of the preceding one, one obtains only in-equality for its homological dimension. Actually, it often turns out that $k[V_1+V_2]^G$ is polynomial, while $(k[V_1]^U \otimes k[V_2]^{U^{\mathrm{op}}})^T$ is not. This reflects a general feature that algebra becomes better after deformation. These cases are not included in the following table. Since the results for $G = SL_2,\ SL_3$ are easy, these cases are also excluded.

CLAIM. — *Let V be a representation of a simple group $G \neq SL_2, SL_3$. Suppose $k[V]^G$ is not polynomial and there is a decomposition $V = V_1 + V_2$ such that both $k[V_i]^U$ are polynomial. Then* (i) *$k[V]^G$ is a c.i.;* (ii) *either V or its dual is contained in the following table.*

Table

Group	$V = V_1 + V_2$	Conditions	hd
A_n	$aV_{\varphi_1} + bV_{\varphi_2} + (2-b)V_{\varphi_{n-1}} + (4-a)V_{\varphi_n}$	$a \leq 2$; $a \geq b$, if n is even	$\leq n$
A_n	$V_{2\varphi_1} + aV_{\varphi_1} + bV_{\varphi_2} + (1-b)V_{\varphi_{n-1}} + (2-a)V_{\varphi_n}$	$a + 2b \neq 2$, if n is even	$\leq n$
A_n	$V_{2\varphi_1} + 2V_{\varphi_1} + 2V_{\varphi_n}$		≤ 3
A_3	$V_{\varphi_1} + 3V_{\varphi_2} + V_{\varphi_3}$		≤ 2
A_3	$2V_{\varphi_1} + 3V_{\varphi_2}$		≤ 2
A_5	$V_{\varphi_1} + V_{\varphi_2} + V_{\varphi_3} + V_{\varphi_5}$		≤ 3
A_5	$2V_{\varphi_1} + V_{\varphi_2} + V_{\varphi_3}$		≤ 3
A_5	$2V_{\varphi_1} + V_{\varphi_3} + V_{\varphi_4}$		≤ 3
B_2	$2V_{\varphi_1} + 2V_{\varphi_2}$		1
B_2	$3V_{\varphi_1} + V_{\varphi_2}$		≤ 2
C_3	$V_{\varphi_2} + V_{\varphi_3}$		≤ 2
D_4	$kV_{\varphi_1} + lV_{\varphi_3} + mV_{\varphi_4}$	$k+l+m = 6$, $\max\{k,l,m\} \leq 4$	≤ 4
D_5	$2V_{\varphi_4} + 2V_{\varphi_5}$		≤ 5
D_6	$2V_{\varphi_1} + aV_{\varphi_5} + bV_{\varphi_6}$	$a + b = 2$	≤ 6

An upper bound for homological dimension is given in the last column. It is an attractive problem to find the precise value of homological dimension. The lower bound (which is at least 1) is often suggested by the explicit form of the Poincaré series. The representations V_1, V_2 are not indicated explicitly, because they are uniquely determined by V. For instance, if G is of type $\mathbf{A}_5$ then V_1 is always V_{φ_3}. The only exception is the first representation of $\mathbf{B}_2$. There are two possibilities for V_1 in this case: $2V_{\varphi_1}$ or $V_{\varphi_1} + V_{\varphi_2}$.

4.4. *Example: double cones I.* — In this subsection G is simple. For $\lambda \in \mathfrak{X}_+$, let $C(\lambda)$ denote the closure of the G-orbit of highest weight vectors in V_λ. This is a variety with rational singularities. A *double cone* is a G-variety of the form $C(\lambda_1)\times C(\lambda_2)$. Since each $C(\lambda)$ is a cone in the respective vector space, $C(\lambda_1)\times C(\lambda_2)$ is acted upon by the extended group $\widetilde{G} = G\times T_2$, where T_2 is a two-dimensional torus. According to [31], $C(\lambda)$ is factorial if and only if λ is fundamental. Factorial double cones which are spherical with respect to the $\widetilde{G}$-action were classified by Littelmann [17]. The algebra $kC(\lambda_1)\times C(\lambda_2)]^U$ is polynomial in this case and one can apply the above general strategy to *quadruple cones*, i.e. to the product of two double cones. We shall say a factorial quadruple cone Z is *excellent*, if it is the product of two $\widetilde{G}$-spherical double cones. Below I give a summary of my computations of algebras of G-invariants on excellent quadruple cones.

CLAIM. — *Let G be of type $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{E}_6$, or $\mathbf{E}_7$ and $Z = X\times Y$ be an excellent quadruple cone. Then*

(i) $(k[X]^U \otimes k[Y]^{U^{\mathrm{op}}})^T$ *is a c.i. and its homological dimension* $\le \operatorname{rk} G$;

(ii) $k[Z]^G$ *is a c.i. as well and* $\operatorname{hd} k[Z]^G \le \operatorname{rk} G$.

In many cases, it turns out that both these algebras are even polynomial.

Proof. — (i) By [17], Table 1 and straightforward computations.

(ii) Since Z is the product of cones, one can apply (3.5) to $C = k[Z]^G$ and $\operatorname{gr} C = (k[X]^U \otimes k[Y]^{U^{\mathrm{op}}})^T$. □

For G of type $\mathbf{A}_n$, every factorial double cone is $\widetilde{G}$-spherical [17]. Therefore a factorial quadruple cone Z may be presented as the product of $\widetilde{G}$-spherical double cones in (at most) three different ways. I did not succeed in getting a *general* description of $k[Z]^G$ by deformation method. In [10], it was stated that $k[Z]^G$ is always polynomial. However, D. Wehlau observed that $k[Z]^G$ is a hypersurface for $G = SL_2$ and its unique factorial quadruple cone Z. This is the only exception.

For G of type $\mathbf{D}_n$, there are also many excellent quadruple cones $Z = X\times Y$ such that $(k[X]^U \otimes k[Y]^{U^{\mathrm{op}}})^T$ and hence $k[Z]^G$ is a c.i. Nevertheless, $(k[X]^U\times k[Y]^{U^{\mathrm{op}}})^T$ is not a c.i. for some of them. For instance, this is the case for $n = 5$ and $X = Y = C(\varphi_3)\times C(\varphi_4)$. However it may be true that $k[X\times Y]^G$ is still a c.i.

There are no excellent quadruple cones for G of type $\mathbf{E}_8$, $\mathbf{F}_4$, $\mathbf{G}_2$.

Most (but not all) of the previous examples are covered by the following general assertion. We shall say X is a *G-cone*, if $k[X]$ admits a grading $k[X] = \bigoplus_{i=0}^{\infty} k[X]_i$ such that G respects it and $k[X]_0 = k$.

4.5. THEOREM. — *Let G be semisimple and X, Y be factorial G-varieties having only rational singularities. Suppose $k[X]^U, k[Y]^U$ are polynomial and are generated by functions whose weights are fundamental. Then $(k[X]^U \otimes k[Y]^{U^{\mathrm{op}}})^T$ is a c.i. and its homological dimension $\le \operatorname{rk} G$. Moreover, if either $k[X]^G = k[Y]^G = k$ or X, Y are G-cones, then the same holds for $k[X\times Y]^G$.*

Proof. — We have a representation of T on $N := \operatorname{Spec}(k[X]^U \otimes k[Y]^{U^{\mathrm{op}}})$. Then $N = \bigoplus_{i=0}^{l} N_i$, where $l = \operatorname{rk} G$, $N_0 = N^T$, and the weights of T in N_i belong to the set $\{\varphi_i, -\varphi_i\}$. Therefore $k[N]^T = k[N_0] \otimes (\otimes_{i=1}^{l} k[N_i]^T)$. Let N_i^+ (resp. N_i^-) denote the subspace corresponding to weight φ_i (resp. $-\varphi_i$). It follows from [22], 2.4 (i) that $\dim N_i^{\pm} \le 2$. Therefore $k[N_i]^T$ is at worst a hypersurface (if $\dim N_i = 4$). This proves the assertion on $k[N]^T$. If anyone of two extra constraints imposed on $X\times Y$ holds, then one completes the proof by applying either (3.4) or (3.5) to $\operatorname{gr} C = k[N]^T$ and $C = k[X\times Y]^G$. □

Another series of examples is connected with symmetric varieties. Let G be semisimple and $\theta \in \operatorname{Aut} G$ be an involution such that the symmetric variety G/G_θ is of the maximal rank, i.e. $\dim G_\theta = \dim U$. Here G_θ denotes the fixed-point subgroup of θ. One may assume that θ is chosen such that $\theta(B) = B^{\mathrm{op}}$ and $\theta(t) = t^{-1}$ for all $t \in T$ (see e.g. [33]). Then $G_\theta \cap B^{\mathrm{op}} = \{t \in T \mid t = t^{-1}\} \cong (\mathbb{Z}_2)^l$ and $G_\theta B^{\mathrm{op}}$ is dense in G. Therefore by (1.4), $\widehat{G}_\theta = (\mathbb{Z}_2)^l \cdot U$ and $\mathfrak{E} = \Gamma(G/G_\theta)$ is generated by $2\varphi_1, \dots, 2\varphi_l$. It then follows from (2.2) that

$$\operatorname{gr}_{\mathfrak{E}} k[X]^{G_\theta} \cong k[X]^U_{\mathfrak{E}} \cong (k[X]^U)^{\mathbb{Z}_2{}^l} \tag{8}$$

for any affine G-variety X. Suppose $k[X]^U$ is polynomial. Then $\operatorname{Spec}(\operatorname{gr} k[X]^{G_\theta})$ is the quotient of a linear space by $(\mathbb{Z}_2)^l$. In each individual case, it is easy to determine whether this algebra is a c.i. It is also possible to give a sufficient condition, which is similar to Theorem (4.5).

4.6. THEOREM. — *Let X be a factorial G-variety having only rational singularities and either X be a G-cone or $k[X]^G = k$. Suppose $k[X]^U$ is polynomial and is generated by functions whose weights are fundamental. Then $k[X]^{G_\theta}$ is a c.i. and $\operatorname{hd} k[X]^{G_\theta} \le l = \operatorname{rk} G$.*

Proof. — It goes through similar to that of (4.5). We have a linear representation of $(\mathbb{Z}_2)^l$ on $M = \mathrm{Spec}(k[X]^U)$. Then $M = \bigoplus_{i=0}^{l} M_i$, where $M_0 = M^T$ and M_i $(i \geq 1)$ corresponds to φ_i. Since the ith factor of $(\mathbb{Z}_2)^l$ acts only on M_i, we have $k(M]^{(\mathbb{Z}_2)^l} = k[M_0] \otimes (\otimes_{i=1}^{l} k[M_i]^{\mathbb{Z}_2})$. Again, by [22], 2.4, $\dim M_i \leq 2$. Therefore $M_i/\mathbb{Z}_2$ is at worst the quadratic hypersurface in $\mathbb{A}^3$ and $\mathrm{gr}\, k[X]^{G_\theta} \cong k[M]^{(\mathbb{Z}_2)^l}$ is a c.i. of homological dimension $\leq l$. Finally, one can apply either (3.4) or (3.5) to $\mathrm{gr}\, k[X]^{G_\theta}$. □

Consider several examples related to this theorem. We retain the conventions from (2.4).

1. Let $G = SL_n$ and $X = V_{\tilde{\varphi}_1} + V_{\tilde{\varphi}_2} + V^*_{\tilde{\varphi}_1}$ $(n \geq 5)$. Then $G_\theta = SO_n$ and $X\,|_{SO_n} = 2V_{\varphi_1} + V_{\varphi_2}$. Since $k[X]^U$ is polynomial and the weights of generators are fundamental [5], Table 1, one concludes $k[2V_{\varphi_1} + V_{\varphi_2}]^{SO_n}$ is a c.i. and its homological dimension $\leq n-1$.

2. Let $G = \mathrm{Spin}_{12}$ and $X = V_{\tilde{\varphi}_1} + V_{\tilde{\varphi}_6}$. Then $G_\theta = SL_4 \times SL_4$ and $X\,|_{G_\theta} = V_{\varphi_2} \otimes \mathbf{1} + \mathbf{1} \otimes V_{\varphi_2} + V_{\varphi_1} \otimes V_{\varphi_1} + V_{\varphi_3} \otimes V_{\varphi_3}$. By the same reason, the algebra of invariants of the latter representation is a c.i. and its homological dimension ≤ 6.

3. Let $G = \mathrm{Spin}_8$ and $X = V_{\tilde{\varphi}_1} + V_{\tilde{\varphi}_3} + V_{\tilde{\varphi}_4}$. Then $G_\theta = (SL_2)^4$ and $X\,|_{G_\theta}$ is the sum of 6 four-dimensional irreducible representations. Each of them is tensor product of 2 trivial representations and two simplest representations, for all possible pairs of simple factors of G_θ. Again, the algebra of G_θ-invariants is a c.i.

4.7. *Remark.* — I think the examples in this section should be helpful in classifying of reducible representations of simple groups such that $k[V]^G$ is a c.i. (The irreducible ones have been classified by H. Nakajima in [19].) Namely, I hope that if a reducible representation does have this property, then in most cases this might be proved by a deformation-type argument.

5. The dual G-variety and doubled actions.

An interesting class of reducible actions consists of the doubled ones. Informally speaking, in this case, X and Y are isomorphic as abstract varieties but $k[X]$ and $k[Y]$ are dual G-modules. The first example of this kind is the sum of a representation and its contragredient. The precise definition makes use of the selected $B, T \subset G$ and goes as follows. (It makes sense for arbitrary varieties, but we content ourselves with the affine case in this paper.) For any G-variety X, we define another ("dual") G-action on X. In order to distinguish these two actions, the variety equipped with the dual action will be denoted by X^*. By abuse of language, we shall say that X^* itself is the *dual* G*-variety*. Let $i : X \to X^*$ be a (fixed) isomorphism and $x^* := i(x)$. The twisted action is defined by

$$(g, x^*) \mapsto g \circ x^* := (\theta(g)x)^*, \quad x \in X, g \in G,$$

where $\theta \in \mathrm{Aut}\, G$ is an involution such that $\theta(B) = B^{\mathrm{op}}$ and $\theta(t) = t^{-1}$ for any $t \in T$. The last condition is automatically satisfied on $G' \cap T$. Therefore it is essential only for the connected center of G. The using of "$*$" presuppose that it is clear which group acting on a variety is born in mind. For a G-module V, the G-module V^* in the sense of above definition is isomorphic to the dual module in the usual sense. It follows that if X is equivariantly embedded in V, then X^* is equivariantly embedded in V^*.

The diagonal G-action on $X \times X^*$ is said to be *doubled* (with respect to the original G-action on X). Denote by $\breve{G}$ the natural semi-direct product of G and $\langle\theta\rangle \cong \mathbb{Z}_2$. The doubled action of G on $X \times X^*$ is extended to $\breve{G}$ by $\theta(x, y^*) = (y, x^*)$.

In what follows, we use the usual notation for quotients. If an affine variety X is acted upon by an algebraic group A, then $X/\!\!/A := \mathrm{Spec}\, k[X]^A$ whenever the algebra $k[X]^A$ is finitely generated. The inclusion $k[X]^A \hookrightarrow k[X]$ induces the quotient morphism $\pi_A : X \to X/\!\!/A$.

5.1. PROPOSITION. — *The T-varieties $(X/\!\!/U)^*$ and $X^*/\!\!/U^{\mathrm{op}}$ are canonically isomorphic. (Here the first "$*$" denotes the dual T-variety and the second one denotes the dual G-variety.)*

Proof. — Since $\theta(U) = U^{\mathrm{op}}$, this is almost tautological. It follows from the fact that the highest weight of an irreducible representation is opposite to the lowest weight of the contragredient one. □

This proposition and (4.2) show that, given a doubled G-action, the associated graded algebra for G-invariants is the algebra of T-invariants of some doubled torus action. This observation allows us to give another proof of the fact that $k[X \times X^*]^G$ is polynomial whenever $k[X]^U$ is polynomial and X is a spherical G-variety (cf. [23], Prop. 5). Indeed, in this case $X/\!\!/U$ is a T-module, and the weights of T in $X/\!\!/U$ are linearly independent. Therefore $k[X/\!\!/U \times (X/\!\!/U)^*]^T$ is generated by the products of the pairs of corresponding coordinates, i.e. it is polynomial. The following result shows there is a substantial application of (4.2), when the algebras of U-invariants are not necessarily polynomial. It relies on the theory developed in [24]. Recall that the *complexity* of a G-variety is the minimal codimension of B-orbits in it. It then easily follows that the complexity of the affine G-variety X equals the minimum of codimension of T-orbits in $X/\!\!/U$.

5.2. THEOREM. — *Let X be an affine factorial unirational G-variety of complexity one such that $k[X]^G = k$ and $k[X]$ does not contain non-constant invertible elements. Suppose* $\mathrm{hd}\, k[X]^U \le 1$. *Then $k[X \times X^*]^G$ is either a hypersurface or a polynomial algebra.*

Proof. — By (3.7) and (4.2), it suffices to obtain the same conclusion for $(k[X]^U \otimes k[X^*]^{U^{\rm op}})^T$. In this form the assertion concerns only properties of the T-variety $X/\!\!/U$ and its dual. Therefore one may assume T acts effectively on $Y := X/\!\!/U$ (simply by taking the quotient of T by the kernel of the action). Then the assumption on complexity reads $\dim T + 1 = \dim Y$. The case $\mathrm{hd}\, k[Y] = 0$ is easy. For then one has a self-dual T-module $Y + Y^*$ of dimension $2\dim T + 2$, and an explicit computation shows $k[Y + Y^*]^T$ is a hypersurface (see [24], 3.10). Further we assume Y is a hypersurface. The variety Y is again factorial and unirational. Since $k[Y]^T = k[X]^G = k$ and the only invertible functions on Y are constants, the single closed T-orbit in Y is a fixed point. Let N be the tangent space at this point. This is a V-module of dimension $\dim Y + 1$ and by Nakayama's lemma there is a closed equivariant embedding $Y \hookrightarrow N$. Obviously, Y is a cone in N. Put $r = \dim N$. Let $x_1, \dots, x_r$ be the coordinates in a weight basis of N. According to [24], Section 2, the equation defining Y in N is a sum of exactly *three* monomials in x_i's, say $m_1(x)$, $m_2(x)$, $m_3(x)$. Let $y_1, \dots, y_r$ be the coordinates in the dual basis of N^*. By (5.1), we have $Y^* = X^*/\!\!/U^{\rm op} \hookrightarrow N^*$ and the equation defining Y^* is the sum of $m_1(y)$, $m_2(y)$, $m_3(y)$. Of course, $m_i(y)$ denotes the same monomial as above but in y_i's. Since x_i and y_i are weight functions of opposite weights, the same holds for $m_j(x)$ and $m_j(y)$. Therefore the functions $x_i y_i$ $(i = 1, \dots, r)$ and $m_i(x) m_j(y)$ $(i \ne j)$ are contained in $k[N \oplus N^*]^T$. From the description of the weighted structure of of $k[Y]$ and $k[N]$ given in [24], Section 2, it follows that $k[N \oplus N^*]^T$ is generated by these functions. Therefore $\mathrm{ed}\, k[N \oplus N^*]^T = r + 6$. Restricting this set of generators on $Y \times Y^*$, one can remove all $m_i(\bar{x}) m_j(\bar{y})$, except one, out of the generator system. (The bar is used for denoting restricted functions.) Indeed, assume that we have $m_1(\bar{x}) m_2(\bar{y})$. Since $m_1(\bar{x}) m_1(\bar{y})$ belongs to the subalgebra generated by $\bar{x}_i\bar{y}_i$'s, one obtains $m_1(\bar{x})m_3(\bar{y})$ by using the relation $m_1(\bar{y})+m_2(\bar{y})+m_3(\bar{y}) = 0$, etc. Thus $k[Y{\times}Y^*]^T$ is generated by $r{+}1$ functions $\bar{x}_1\bar{y}_1, \ldots, \bar{x}_r\bar{y}_r$, $m_1(\bar{x})m_2(\bar{y})$. Since T-action on $Y{\times}Y^*$ is effective and stable, the Krull dimension of $k[Y{\times}Y^*]^T$ equals $2\dim Y - \dim T = r$, and we are done. □

Remark. — In [24], I proved that $X/\!\!/U$ is a c.i. under the hypothesis of (5.2). Therefore the assumption for $X/\!\!/U$ to be a hypersurface is less restrictive than it might appear at first glance. Actually, I do not know examples of X as in (5.2) and such that $\mathrm{hd}\, k[X]^U \geq 2$.

My last goal is to obtain some results on the canonical module of the algebra of invariants of a doubled action. In view of (5.1), we first consider doubled actions of tori in more details. Let Y be a normal affine T-variety. Without loss of generality, one may assume that the T-action on Y is effective. We let Y^{reg} denote the non-singular locus of Y. Since T is reductive, the quotient mapping $\pi_T : Y{\times}Y^* \to Y{\times}Y^*/\!\!/T$ is well defined. Consider two open subsets of $Y{\times}Y^*$:

$$R = \{p \mid T_p \text{ is finite}\},$$

$$S = \{p \mid p \in (Y{\times}Y^*)^{\mathrm{reg}},\ \pi_T(p) \in (Y{\times}Y^*/\!\!/T)^{\mathrm{reg}} \ \text{ and } \ \pi_T \ \text{ is smooth in } \ p\}.$$

5.3. PROPOSITION.

(i) *The complement of R does not contain divisors.*

(ii) *If Y is factorial, the same is true for the complement of S.*

Proof. — (i) It is easy.

(ii) Assume not. Denote by D the union of all irreducible components of $Y{\times}Y^* \setminus S$ which are of codimension one. Since $Y{\times}Y^*$ is factorial, the defining ideal $\mathcal{V}(D)$ of D is principal. Since D is T-stable, $\mathcal{V}(D)$ is generated by a T-semi-invariant, say f. Recall that $\check{T} = T{\times}\langle\theta\rangle$ also acts on $Y{\times}Y^*$. It is immediate that D is $\check{T}$-stable as well. Therefore $\theta\cdot f = \pm f$. On the other side, by definition of θ, the semi-invariants f and $\theta\cdot f$ have opposite weights. Hence f is a T-invariant. This implies $\pi_T(D)$ is a divisor as well and $D = \pi_T^{-1}(\pi_T(D))$. Now the same argument as in [12], Satz 2, shows that $D\cap S \neq \varnothing$. This contradiction proves the assertion. □

This proposition contains the necessary ingredients for obtaining results on the canonical module of the algebra of invariants of doubled T-actions. Recall that the canonical module (canonical sheaf) ω_Y is defined whenever Y is a Cohen-Macaulay variety, and this is a reflexive module of rank one (see e.g. [7], Ch. 3). In this case, Y is called Gorenstein, if ω_Y is invertible (locally free).

5.4. PROPOSITION. — *Let Y be an affine factorial T-variety having only rational singularities. Then*

(i) $Q := Y\times Y^*/\!\!/T$ *is Gorenstein;*

(ii) *there is an T-equivariant injection* $\omega_{Y\times Y^*} \hookrightarrow \pi_T^*\omega_Q$.

Proof. — (i) By (5.3) and [13], Kor. 2, it suffices to prove that $(\omega_{Y\times Y^*})^T$ is a free $k[Y\times Y^*]^T$-module. Since $\omega_{Y\times Y^*} \cong \omega_Y \otimes \omega_{Y^*}$ and ω_Y is free, it follows from the definition of the dual action that $k[Y\times Y^*]$-module $\omega_{Y\times Y^*}$ is generated by a T-invariant element.

(ii) It follows from the preceding exposition that all assumptions of [12], Satz 1, are fulfilled. □

From now on assume also that $k[Y]$ is $\mathbb{N}$-graded such that $k[Y]_0 = k$ and T preserves this grading. Let $F_{kY}(z) = \sum\limits_{i=0}^{\infty} \dim k[Y]_i z^i$ be the corresponding Poincaré series. Since ω_Y is also graded, one can consider the analogous series $F_{\omega_Y}(z)$. It is well known that both of them are the expansions of rational functions in z and that $F_{\omega_Y}(z) = (-1)^d F_{kY}(z^{-1})$, where $d = \dim Y$. This relation holds whenever Y is normal and Cohen-Macaulay, see 4, 4.1. We set $q_Y := -\deg F_{k[Y]}$. Then q_Y equals the smallest degree of non zero elements in ω_Y. Since $k[Y\times Y^*]$ and $kQ = k[Y\times Y^*]^T$ are $\mathbb{N}^2$-graded, one can consider their Poincaré series and introduce similarly the 2-component vectors $\bar{q}_{Y\times Y^*}$ and $\bar{q}_Q$. Clearly $\bar{q}_{Y\times Y^*} = (q_Y, q_Y)$, because $q_Y = q_{Y^*}$.

5.5. COROLLARY. — $\bar{q}_{Y\times Y^*} = \bar{q}_Q$.

Proof. — Since the complement of R in $Y\times Y^*$ is of codimension ≥ 2, one can apply [12], Kor. 4. For the sake of completeness, I shall give some details of that argument. In the graded case, the injection of modules in (5.4) is degree-preserving. Therefore $\bar{q}_{Y\times Y^*} \geq \bar{q}_Q$. Let r_1, r_2 be the homogeneous generators of $\omega_{Y\times Y^*}$ and ω_Q respectively (in the graded case these modules are free). Then $r_1 = fr_2$ for some $f \in k[Y\times Y^*]^T$. Since f does not vanish on R, it must be constant. Therefore $\bar{q}_{Y\times Y^*} = \deg r_1 = \deg r_2 = \bar{q}_Q$. □

Let us return to the G-variety X. Suppose X is a G-cone. We shall obtain some information on the graded structure of $k[X\times X^*]^G$ by using the results on $Y := X/\!\!/U$. Consider $k[X]^U$ with the inherited $\mathbb{N}$-grading, that is, $k[X]^U_i = k[X]^U \cap k[X]_i$. Then $k[X]^U_0 = k$ and $q_{X/\!\!/U}$ is well defined. For $\mathbb{N}^2$-graded algebra $C = k[X\times X^*]^G$, the 2-component vector $\bar{q}_{X\times X^*/\!\!/G}$ can be defined. More precisely, let

$$F_C(z_1, z_2) = \sum_{n,m\geq 0} \dim(k[X]_n \otimes k[X]_m)^G z_1^n z_2^m$$

be the respective Poincaré series. Then

$$\bar{q}_{X\times X^*/\!\!/G} := (-\deg_{z_1} F_C, -\deg_{z_2} F_C).$$

5.6. THEOREM. — *Let X be an affine factorial G-cone having only rational singularities. Then $X\times X^*/\!\!/G$ is Gorenstein and $\bar{q}_{X\times X^*/\!\!/G} = (q_{X/\!\!/U}, q_{X/\!\!/U})$.*

Proof. — (i) Since C is a Cohen-Macaulay domain, the condition of being Gorenstein can be stated in terms of F_C [27]. We consider $k[X]^U$ and hence $\operatorname{gr} C = (k[X]^U \otimes k[X^*]^{U^{\mathrm{op}}})^T$ with the inherited grading. Therefore $F_C = F_{\operatorname{gr} C}$ (see (3.6)). Thus the Gorenstein property for $\operatorname{gr} C$ (5.4) implies that for C.

(ii) In view of (5.5) and the equality $F_C = F_{gr\, C}$, one has $\bar{q}_{X\times X^*/\!\!/G} = \bar{q}_{Y\times Y^*/\!\!/T} = \bar{q}_{Y\times Y^*} = (q_{X/\!\!/U}, q_{X/\!\!/U})$. □

The equality for degrees of the Poincaré series was conjectured in [22]. This allows to predict in many cases degree of relations connecting generators.

5.7. *Example: Double cones II.* — We retain the notation of (4.4).

1. Let G be of type $\mathbf{E}_7$ and $X = C(\varphi_1)\times C(\varphi_2)$. (Recall that we number fundamental weights as in [30].) The complexity of X as $\widetilde{G}$-variety equals one [21], 2.8, and $I_U := k[C(\varphi_1)\times C(\varphi_2)]^U$ is a hypersurface [22], 3.11. Thus, by (5.2), the algebra $I_{\widetilde{G}} = k[C(\varphi_1)\times C(\varphi_2)\times C(\varphi_1)^*\times C(\varphi_2)^*]^{\widetilde{G}}$ is at worst a hypersurface. Note that $C(\varphi_i) \cong C(\varphi_i)^*$ as $\mathbf{E}_7$-variety but not as a $\widetilde{G}$-variety. For varieties of complexity one, there is a recipe for writing the Poincaré series of the algebra of invariants of the doubled action, if the algebra of U-invariants is known [23], th. 6. Using degrees of the generators and of the relation in I_U given in [22], one can write explicitly down the Poincaré series for $I_{\widetilde{G}}$. I am not going to make it here, but this bulky formula shows $I_{\widetilde{G}}$ is not polynomial. Hence this is a hypersurface.

2. Let G be of type $\mathbf{B}_n$ or $\mathbf{C}_n$ and $X = C(\varphi_2)\times C(\varphi_n)$, $n \geq 3$. By [22], 3.13, one has the same situation as in the previous example.

3. Combining results in [17] and [22], one may obtain a description of G-invariants for some quadruple cones which are not excellent. For instance, take $G = \mathbf{E}_7$, $X = C(\varphi_1)\times C(\varphi_2)$, and $Y = C(\varphi_1)\times C(\varphi_7)$. Here $k[X]^U$ is a hypersurface and $k[Y]^U$ is a polynomial algebra (of dimension 8). Then a straightforward verification shows that $(k[X]^U\times k[Y]^{U^{\mathrm{op}}})^T$ is a hypersurface. Hence $\mathrm{hd}\, k[C(\varphi_1)\times C(\varphi_2)\times C(\varphi_1)\times C(\varphi_7)]^{\mathbf{E}_7} \leq 1$.

BIBLIOGRAPHY

[1] N. Andruskiewitsch, H. Tirao, A restriction theorem for modules having a spherical submodule, Trans. Amer. Math. Soc., 331 (1992), 705–725.

[2] N. Beklemishev, Algebras invariants of forms that are complete intersections, Math. USSR Izv., 23 (1984), 423–429.

[3] W. Borho, H. Kraft, Über Bahnen und deren Deformationen bei linearen Aktionen reduktiver Gruppen, Comment. Math. Helv., 54 (1979), 61–104.

[4] M. Brion, Sur la théorie des invariants, Publ. Math. Univ. Pierre et Marie Curie, 45 (1981), 1–92.

[5] M. Brion, Représentations exceptionnelles des groupes semi-simples. Ann. Sci. Éc. Norm. Sup., IV Sér., 18 (1985), 345–387.

[6] M. Brion, Quelques propriétés des espaces homogènes sphériques, Manuscripta Math., 55 (1986), 191–198.

[7] W. Bruns, J. Herzog, Cohen-Macaulay rings, Cambridge University Press, 1993.

[8] F.D. Grosshans, The invariants of unipotent radicals of parabolic subgroups, Invent. Math., 73 (1983), 1–9.

[9] F.D. Grosshans, Contractions of the actions of reductive groups in arbitrary characteristic, Invent. Math., 107 (1992), 127–133.

[10] R. Howe, R. Huang, Projective invariants of four subspaces, Adv. in Math., 118 (1996), 295–336.

[11] R. HOWE, T. UMEDA, The Capelly identity, the double commutant theorem, and multiplicity-free actions, Math. Ann., 290 (1991), 565–619.

[12] F. KNOP, Über die Glattheit von Quotientenabbildungen, Manuscripta Math., 56 (1986), 419–427.

[13] F. KNOP, Der kanonische Modul eines Invariantenrings, J. Algebra, 127 (1989), 40–54.

[14] F. KNOP, Weylgruppe und Momentabbildung, Invent. Math., 99 (1990), 1–23.

[15] F. KNOP, Über Hilberts vierzehntes Problem für Varietäten mit Kompliziertheit eins, Math. Z., 213 (1993), 33–36.

[16] M. KRÄMER, Sphärische Untergruppen in kompakten zusammenhängender Lie Gruppen, Compositio Math., 38 (1979), 129–153.

[17] P. LITTELMANN, On spherical double cones, J. Algebra, 166 (1994), 142–157.

[18] N. MOHAN KUMAR, Complete intersections, J. Math. Kyoto Univ., 17 (1977), 533–538.

[19] H. NAKAJIMA, Representations of a reductive algebraic group whose algebras of invariants are complete intersections, J. reine angew. Math., 367 (1986), 115–138.

[20] D. PANYUSHEV, Complexity and rank of homogeneous spaces, Geom. Dedicata, 34 (1990), 249–269.

[21] D. PANYUSHEV, Complexity and rank of double cones and tensor product decompositions, Comment. Math. Helv., 68 (1993), 455–468.

[22] D. PANYUSHEV, A restriction theorem and the Poincaré series for U-invariants, Math. Annalen, 301 (1995), 655–675.

[23] D. PANYUSHEV, Reductive group actions on affine varieties and their doubling, Ann. Inst. Fourier, 45-4 (1995), 929–950.

[24] D. PANYUSHEV, Good properties of algebras of invariants and defect of linear representations, Journal of Lie Theory, 5 (1995), 81–99.

[25] F. PAUER, Sur les espaces homogènes de complication nulle, Bull. Soc. Math. France, 112 (1984), 377–385.

[26] V.L. POPOV, Contractions of actions of reductive algebraic groups, Math. USSR Sb., 58 (1987), 311–335.

[27] R. STANLEY, Hilbert functions of graded algebras, Adv. Math., 28 (1978), 57–83.

[28] E.B. VINBERG, Complexity of actions of reductive groups, Funct. Anal. Appl., 20 (1986), 1–11.

[29] E.B. VINBERG, B.N. KIMEL'FEL'D, Homogeneous domains on flag varieties and spherical subgroups of semisimple Lie groups, Funct. Anal. Appl., 12 (1978), 168–174.

[30] A.L. ONISHCHIK, E.B. VINBERG, Lie groups and algebraic groups, Berlin Heidelberg New York, Springer, 1990.

[31] E.B. VINBERG, V.L. POPOV, On a class of quasihomogeneous affine varieties, Math. USSR Izv., 6 (1972), 743–758.

[32] V.L. POPOV, E.B. VINBERG, Invariant theory, In: Parshin, A.N., Shafarevich, I.R. (Eds.): Algebraic Geometry IV (Encyclopaedia Math. Sci., vol. 55, pp.123–284) Berlin Heidelberg New York: Springer, 1994.

[33] Th. VUST, Opération de groupes réductifs dans un type de cônes presque homogènes, Bull. Soc. Math. France, 102 (1974), 317–333.

Manuscrit reçu le 15 janvier 1997,
accepté le 24 mars 1997.

Dmitri PANYUSHEV,
Moscow Radioelectronics & Automation Institute
Math. Department
Moscow 117454 (Russie).
panyush@dpa.msk.ru