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FAMILIES OF CURVES AND ALTERATIONS

by A. Johan de JONG

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1. Introduction.

This paper is a continuation of or sequel to the article [1]. The main point that is in this paper and that was missed by the author while writing [1] is that one can alter *any* family of curves into a semi-stable family. In [1] this was done only in special cases, see [1, Theorem 5.8]. It should be mentioned that this result on families of curves was "known to be true" to mathematiciens working on moduli of varieties.

Thus the main theorem of this paper is Theorem 5.9. The applications are derived by arguments that are similar to the arguments of [1]; the results are slightly stronger than the results of that paper. For example, the theorem implies that any proper dominant morphism $X \longrightarrow S$ of integral excellent schemes may be altered into a composition of semi-stable curve fibrations, see Theorem 5.9, Corollary 5.10 and Remark 5.16. It gives an alternative proof of a stronger version of [1, Theorem 8.2] concerning

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semi-stable families over an excellent base scheme of dimension 1, see Corollary 5.1.

Another point of this paper is to deal systematically with group actions as well; the advantage of this is that one gets resolution of singularities up to quotient singularities in certain cases, see Theorem 5.13 and its corollaries. Finally, the results of [1] are extended to integral schemes of finite type over excellent base schemes of dimension $\leqslant 2$, see Corollary 5.1.

For an indication as to how to get cohomological applications of the geometric results obtained in this paper we refer to the introduction of [1].

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2. Set up and statement of theorem.

Throughout this paper we adopt the notations, conventions, definitions and terminology of [1, Section 2].

Situation 2.1. — Here we have a proper morphism $f : X \to S$ of integral excellent schemes whose generic fibre X_η has dimension 1. We have a finite group G acting on f, i.e., actions of G on X and S such that f is G-equivariant. In addition we have a proper closed subset $Z \subset X$ which is G-stable, i.e., for all $g \in G$ we have $g(Z) \subset Z$. We note that the extreme cases $Z = \varnothing$ or $G = \{1\}$ (or both) are allowed.

In the sequel we will consider the following conditions on $(f : X \to S, G, Z)$.

(2.1.1) The generic fibre X_η is smooth over η and Z_η is étale over η.

(2.1.2) The morphism f is projective.

(2.1.3) All fibres of f are equidimensional of dimension 1.

(2.1.4) The variety X_η is geometrically irreducible over η.

(2.1.5) The quotient X_η/G is geometrically irreducible over $R(S)^G$. We note that (2.1.4) implies this condition.

(2.1.6) (Assume (2.1.3).) For any geometric point $\bar{s}$ of S and any irreducible component C of $X_{\bar{s}}$ the normalization of C has genus at least 2.

2.2. *Alterations of Situation* 2.1. — This refers to a commutative diagram of morphisms of schemes

$$\begin{array}{ccc} X' & \xrightarrow{\psi} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{\phi} & S, \end{array}$$

where $(f' : X' \to S', G', Z')$ and $(f : X \to S, G, Z)$ are as in Situation 2.1. Further we demand that ψ and ϕ are alterations, $\psi^{-1}(Z) = Z'$ and that a surjection $G' \to G$ is given such that ψ and ϕ are G'-equivariant morphisms.

In the sequel we will consider the following conditions on the pair (ψ, ϕ).

(2.2.1) The field extension $R(S)^G \subset R(S')^{G'}$ is purely inseparable and the field $R(X')^{G'}$ is the residue field of the local ring $R(S')^{G'} \otimes_{R(S)^G} R(X)^G$.

(2.2.2) The field $R(X')$ is a quotient of the ring $R(X) \otimes_{R(S)} R(S')$, i.e., the generic fibre $X'_{\eta'}$ maps birationally to an irreducible component of $X_\eta \times_S S'$.

(2.2.3) The field $R(X')$ is separable over the image of the natural map $R(X) \otimes_{R(S)} R(S') \to R(X')$. This implies that the field $R(X')^{G'}$ is separable over the image of the natural map $R(X)^G \otimes_{R(S)^G} R(S')^{G'} \to R(X')^{G'}$.

(2.2.4) Let $N \subset G'$ be the kernel of the map $G' \to G \times \operatorname{Aut} S'$. The field $R(X')^N$ is a quotient of the ring $R(X) \otimes_{R(S)} R(S')$. This means that the curve $X'_{\eta'}/N$ maps birationally onto an irreducible component of $X_\eta \times_S S'$.

(2.2.5) The field extensions $R(S) \subset R(S')$ and $R(X) \subset R(X')$ are separable. This means that the alterations ψ and ϕ are generically étale.

(2.2.6) Let $H = \operatorname{Ker}(G \to \operatorname{Aut} S)$ and $H' = \operatorname{Ker}(G' \to \operatorname{Aut} S')$. Then $H' \to H$ is surjective.

(2.2.7) The morphisms ψ and ϕ are projective.

Remarks 2.3. — We make some obvious observations.

(i) The conditions above are by no means independent. For example the conditions (2.2.1), (2.2.5), (2.2.2) and (2.2.4) all imply (2.2.3).

(ii) Various combinations of the conditions above have interesting consequences. E.g. if (ψ, ϕ) satisfies (2.2.4) and (2.2.6) then $X'_{\eta'}/H'$ is birational to a component of $(X_\eta/H) \otimes \kappa(\eta')$. If (ψ, ϕ) satisfies (2.2.1) and (2.2.5) then we have $R(S)^G = R(S')^{G'}$ and $R(X)^G = R(X')^{G'}$. In particular the field extensions $R(S) \subset R(S')$ and $R(X) \subset R(X')$ are Galois and certain subgroups of G' surject onto their Galois groups. From this we see that (2.2.1) and (2.2.5) imply (2.2.4).

(iii) It is clear how to define compositions $(\psi, \phi) \circ (\psi', \phi')$ of given alterations (ψ, ϕ) and (ψ', ϕ'). If the alterations (ψ, ϕ) and (ψ', ϕ') satisfy one of (2.2.1), (2.2.2), (2.2.5), (2.2.6) or (2.2.7) then so does the composition.

(iv) If, in the situation of (iii), we have that (ψ, ϕ) satisfies (2.2.4) and (ψ', ϕ') satisfies (2.2.6) and (2.2.4), then the composition $(\psi, \phi) \circ (\psi', \phi')$ satisfies (2.2.4).

(v) If X_η is geometrically irreducible, then the condition (2.2.2) implies that the curves X_η and $X'_{\eta'}$ have the same geometric genus. If X_η is also geometrically reduced, then (2.2.2) is equivalent to the birationality of the morphism $X'_{\eta'} \to X_\eta \otimes \kappa(\eta')$.

THEOREM 2.4. — *Let $(f : X \to S, G, Z)$ be as in Situation 2.1. There exists an alteration $(f_1 : X_1 \to S_1, G_1, Z_1)$ as in 2.2*

$$\begin{array}{ccc} X_1 & \xrightarrow{\psi_1} & X \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} \\ S_1 & \xrightarrow{\phi_1} & S, \end{array}$$

with the following properties:

(i) *The morphism f_1 is a projective semi-stable family of curves over S_1 with (irreducible) smooth generic fibre.*

(ii) *There are disjoint sections $\sigma_i : S_1 \to X_1$, $i = 1, \ldots, n$ of f_1 into the smooth locus of X_1 over S_1 such that $Z_{1,\eta_1} = \{\sigma_1(\eta_1), \ldots, \sigma_n(\eta_1)\}$.*

(iii) *The alteration (ψ_1, ϕ_1) satisfies (2.2.7) and (2.2.4) (and hence (2.2.3)).*

(iv) *If $(f : X \to S, G, Z)$ satisfies (2.1.5), then (ψ_1, ϕ_1) satisfies (2.2.1).*

(v) *If $(f : X \to S, G, Z)$ satisfies (2.1.4), then (ψ_1, ϕ_1) satisfies (2.2.6).*

(vi) *If* $(f : X \to S, G, Z)$ *satisfies (2.1.1), then* (ψ_1, ϕ_1) *satisfies (2.2.5).*

In addition we can choose the diagram such that

(vii)(a) *The n-pointed curve* $(X_1, \sigma_1, \dots, \sigma_n)$ *over* S_1 *is stable.*

Or we can choose (ψ_1, ϕ_1) *such that in addition to (i)-(vi) we have*

(vii)(b) *The generic fibre* $(X_1)_{\eta_1}$ *of* f_1 *is the normalization of an irreducible component of* $X_\eta \otimes_{R(S)} R(S_1)$*, i.e., the alteration* (ψ_1, ϕ_1) *satisfies (2.2.2).*

Remark 2.5. — The condition on Z_{1,η_1} implies that

$$Z_1 = \psi_1^{-1}(Z) \subset \sigma_1(S_1) \cup \ldots \cup \sigma_n(S_1) \cup f_1^{-1}(D)$$

for some proper closed subset $D \subset S_1$ invariant under G_1.

2.6. Procedure of the proof of Theorem 2.4. We are going to alter (2.2) repeatedly our family $f : X \to S$, finally reaching a situation which satisfies (vii)(a) or (vii)(b). This is allowed, because a composition of alterations as in 2.2 is another such. Of course we have to check that the final alteration has all the properties required of it in Theorem 2.4, e.g. in case (vii)(b) we will have to make sure the alterations we use all have property (2.2.2). However, the first step in 2.9 will be to reduce to the case where X_η is geometrically irreducible over η and (X_η, Z_η) is smooth over η; after that we will have alterations satisfying the properties (2.2.1), (2.2.5), (2.2.6) and (2.2.7) (these imply (2.2.3) and (2.2.4)). In case (vii)(b) we also require (2.2.2). It is then easily verified that the end result is what we want (compare Remarks 2.3).

2.7. Let us produce a useful alteration of $(f : X \to S, G, Z)$ in the case that $S = \operatorname{Spec} k$ where k is a field. Choose a finite purely inseparable extension $k \subset k'$ as in Lemma 2.8 below. In the case that X is smooth and Z étale over k we take $k' = k$. Put k'' equal to the compositum of the fields $g(k')$, $g \in G$ inside the perfect closure of k. (Note that G acts on the perfect closure of k.) Then k'' works as well in Lemma 2.8 and G acts on k''. Let X_1 be the normalization of the reduction of $X \otimes_k k''$ and let Z_1 be the inverse image of Z in X_1. The group G acts on the smooth pair (X_1, Z_1) over k''. Let $X_1 \to \operatorname{Spec} k_1 \to \operatorname{Spec} k''$ be the Stein factorization of the morphism $X_1 \to \operatorname{Spec} k''$. The group G acts on k_1 by uniqueness of Stein factorization. Put $S_1 = \operatorname{Spec} k_1$. The triple $(X_1 \to S_1, G, Z_1)$ has properties (2.1.4) and (2.1.1).

Note that we trivially have (2.2.4) and (2.2.2) for the alteration $(X_1, S_1) \to (X, S)$ so obtained. In case the pair (X, Z) is smooth over k, we have that $k \subset k_1$ is separable, hence (2.2.5).

In case X/G is geometrically irreducible over k^G, we see that the extension $k^G \subset k_1^G$ is purely inseparable as $X_1/G \to X/G$ is purely inseparable. Let $Y \hookrightarrow X/G \otimes_{k^G} (k'')^G$ be the reduction of $X/G \otimes_{k^G} (k'')^G$. The morphism $\alpha : X \otimes_k k'' \to X/G \otimes_{k^G} (k'')^G$ is generically étale and an open part of X_1 is equal to an open part of $\alpha^{-1}(Y)$. Thus

$$\begin{aligned}\deg(X_1 \to Y) &= \deg(X \otimes_k k'' \to X/G \otimes_{k^G} (k'')^G) \\ &= \deg(X \to X/G) \cdot [k'' : (k'')^G]/[k : k^G] \\ &= \deg(X \to X/G).\end{aligned}$$

But $\deg(X_1 \to X_1/G) = \deg(X \to X/G)$ (we did not make the group bigger), hence we conclude that $X_1/G \to Y$ is birational. This proves the last assertion of (2.2.1).

Finally, assume that X is geometrically irreducible over k. In that case we have $k_1 = k''$, and the property (2.2.6) is clear for the alteration $(X_1, S_1) \to (X, S)$.

LEMMA 2.8. — *Let X be a variety of dimension 1 over a field k and let $Z \subset X$ be a proper closed subset. There exists a finite purely inseparable extension $k \subset k'$ such that the normalization X' of the reduction of $X \otimes_k k'$ is smooth over k' and the inverse image $Z' \subset X'$ of Z is étale over k'. Any further extension $k' \subset k''$ has the same property.*

Proof. — Omitted. □

2.9. (Reduction to the case where we have (2.1.1) and (2.1.4).) Let $(f : X \to S, G, Z)$ be as in 2.1. Let $(Y \to \operatorname{Spec} K, G, Z_Y)$ be the alteration of $(X_\eta \to \eta, G, Z_\eta)$ constructed in 2.7. We set S' equal to the normalization of S in K, and X' equal to the normalization of X in $R(Y)$. The universal property of normalization assures that we get a proper morphism $f' : X' \to S'$ and that G acts on f'. In other words, we have an alteration $(\psi, \phi) : (X', S') \to (X, S)$ as in 2.2. The generic fibre of X' is Y and hence $X' \to S'$ satisfies (2.1.1) and (2.1.4).

As was shown in 2.7, the pair (ψ, ϕ) satisfies (2.2.2) and (2.2.4) and if f satisfies one of the conditions mentioned in Theorem 2.4, then (ψ, ϕ) satisfies the corresponding condition. The morphisms ψ and ϕ are finite hence projective. As explained in 2.6, we reduce to the case that we have (2.1.1) and (2.1.4).

2.10. Assume (2.1.1) and (2.1.4). By Chow's lemma there exists a projective modification $X' \to X$ such that X' is projective over S. By averaging over G, as explained in [1, 7.6], we may assume that G acts on X'. Note that $X'_\eta \cong X_\eta$ as X_η is smooth over η. Thus we may replace X by X' and assume that we have (2.1.2).

By [6, page 37-38] there is a canonical minimal modification $S' \to S$ which is projective, such that the strict transform X' of X with respect to $S' \to S$ is flat over S'. Thus G acts on $X' \to S'$ and we have $X'_{\eta'} \cong X_\eta$. The alteration $(X', S') \to (X, S)$ satisfies all properties listed in 2.2. Thus we may replace (X, S) by (X', S') and assume that we have (2.1.1), (2.1.3), (2.1.2) and (2.1.4).

In Section 3 we prove case (vii)(a) of Theorem 2.4 under these assumptions. In Section 4 we deduce case (vii)(b) from case (vii)(a) (under the same assumptions).

3. Genus goes up.

3.1. Throughout this section the triple $(f : X \to S, G, Z)$ satisfies (2.1.1), (2.1.3), (2.1.2) and (2.1.4). In this section we will prove Theorem 2.4 in this case.

3.2. Suppose that $R(S) \subset L$ is a finite separable extension such that $R(S)^G \subset L$ is Galois. Then we can put S' equal to the normalization of S in L and $X' = (X \times_S S')_{\mathrm{red}}$. For the group G' we take the fibre product $G' = G \times_{\mathrm{Gal}(R(S)/R(S)^G)} \mathrm{Gal}(L/R(S)^G)$ (compare with [1, 7.12]). It is easy to see that the resulting alteration $(f' : X' \to S', G', Z')$ of $(f : X \to S, G, Z)$ satisfies all the conditions listed in 2.2 and that $(f' : X' \to S', G', Z')$ satisfies (2.1.1), (2.1.3), (2.1.2) and (2.1.4) as well. In this way we may enlarge $R(S)$.

3.3. These remarks prove case (vii)(b) of Theorem 2.4 if $S = \operatorname{Spec} k$ is the spectrum of a field, as we can make the points of $Z \subset X$ rational after enlarging k as above. Case (vii)(a) follows by choosing a suitable covering of X, to make the pair (X, Z) stable, details left to the reader. Therefore, from now on we will assume that S is not the spectrum of a field, i.e., $\dim S > 0$.

3.4. Let $H = \operatorname{Ker}(G \to \operatorname{Aut} S)$. As X is projective over S, the quotient $Y = X/H$ exists and is an integral scheme, projective over S of relative dimension 1.

For any closed point $s \in S$ we will construct an affine open neighborhood $U \subset S$ and a *finite* morphism $\varphi : X_U \to \mathbb{P}^1_U$ with the following properties:

(i) φ is H-invariant,

(ii) φ is generically étale, and

(iii) for some G-invariant proper closed subset $T \subset X_\eta$ containing Z_η and the ramification points of $X_\eta \to \mathbb{P}^1_\eta$, we have $\varphi(T) \cap \{0, 1, \infty\} = \varnothing$.

To do this we may assume S affine. Choose a relatively ample line bundle $\mathcal{L}$ on $Y = X/H$ over S. Take n so large that (a) the map $\Gamma(Y, \mathcal{L}^{\otimes n}) \to \Gamma(Y_s, (\mathcal{L}|_{Y_s})^{\otimes n})$ is surjective, (b) there exist $\bar{s}_1, \bar{s}_2 \in \Gamma(Y_s, (\mathcal{L}|_{Y_s})^{\otimes n})$ which define a finite morphism $Y_s \to \mathbb{P}^1_s$, and (c) $\mathcal{L}_\eta^{\otimes n}$ is very ample on Y_η. If $s_1, s_2 \in \Gamma(Y, \mathcal{L}^{\otimes n})$ are arbitrary lifts of $\bar{s}_1, \bar{s}_2$, then there exists an open neighborhood $U = U(s_1, s_2) \subset S$ of s such that s_1, s_2 define a finite morphism $\xi : Y_U \to \mathbb{P}^1_U$. Note that by (c) there exists a nonempty Zariski open subset $\mathcal{V} \subset \Gamma(Y_\eta, \mathcal{L}_\eta^{\otimes n})^2$ such that $(s_{1,\eta}, s_{2,\eta}) \in \mathcal{V}$ implies that ξ_η is generically étale. However, the set of lifts (s_1, s_2) of $(\bar{s}_1, \bar{s}_2)$ form a Zariski dense subset of $\Gamma(Y_\eta, \mathcal{L}_\eta^{\otimes n})^2$, as is easy to show (use that $s \neq \eta$ as $\dim S > 0$).

Let $\varphi : X_U \to \mathbb{P}^1_U$ be the resulting morphism $X_U \to Y_U \to \mathbb{P}^1_U$. We have (i) and (ii) but not yet (iii); we will get (iii) by changing coordinates on $\mathbb{P}^1_U$. Let $T \subset X_\eta$ be a G-stable closed subset as in (iii). Consider the group $\mathcal{G} = \mathrm{PGL}_2(\Gamma(U, \mathcal{O}_U))$. We are looking for an element $g \in \mathcal{G}$ such that $g(T) \cap \{0, 1, \infty\} = \varnothing$. Again this is a Zariski open condition in $\mathrm{PGL}_{2,\eta}$. The residue field $\kappa(\eta)$ is infinite as $\dim S > 0$, and it follows that $\mathcal{G} \subset \mathrm{PGL}_2(\kappa(\eta))$ is Zariski dense. Thus we can find $g \in \mathcal{G}$ having the desired properties.

3.5. Since S is Noetherian we can choose a finite covering by open affines $S = \bigcup_{i=1}^{m} U_i$ such that there are finite morphisms $\varphi_i : X_{U_i} \to \mathbb{P}^1_{U_i}$ satisfying (i), (ii) and (iii) of Subsection 3.4. Further we may assume that on each U_i a prime number $\ell_i \geqslant 5$ is invertible.

Suppose we do a base change by $S' \to S$ as in 3.2. The covering $S = \bigcup U_i$ pulls back to a covering $S' = \bigcup U_i'$ and φ_i to a morphism $\varphi_i' : X'_{U_i'} \to \mathbb{P}^1_{U_i'}$ having properties (i), (ii) and (iii) stated in 3.4. Thus we may assume that $R(S)$ contains a primitive ℓ_ith root of unity for each i. Also S is normal.

Let $\mathcal{C}_i \to \mathbb{P}^1_{U_i}$ be the finite normal covering given by the function field extension $R(\mathbb{P}^1_{U_i}) \subset L_i$ obtained by adjoining the ℓ_ith roots of x and $x-1$ to $R(\mathbb{P}^1_{U_i})$. By our assumption that $\zeta_i, \ell_i \in \Gamma(U_i, \mathcal{O}^*_{U_i})$ we have that $\mathcal{C}_i \to \mathbb{P}^1_{U_i}$ is a (ramified) Galois covering with group $(\mathbb{Z}/\ell_i\mathbb{Z})^2$. More to the point, the morphism $\mathcal{C}_i \to U_i$ is smooth, projective and all its geometric fibres are nonsingular curves of genus $\geqslant 2$.

Let $N \subset R(X)^*$ be the multiplicative subgroup generated by the following elements:

$$f_{i,g} = g(\varphi_i^*(x)) \quad \text{and} \quad f'_{i,g} = g(\varphi_i^*(x-1)), \quad i = 1, \ldots, m, \ g \in G.$$

By our choice of φ_i the zeros and poles of these functions $f_{i,g}, f'_{i,g}$ on the curve X_η are simple and occur in points $p \in X_\eta \setminus Z_\eta$ with $\kappa(\eta) \subset \kappa(p)$ separable. Set ℓ equal to the product of the distinct prime numbers that occur among the ℓ_i. Let $R(X) \subset L$ be the finite Galois extension obtained by adjoining the ℓth roots of the elements of N, or equivalently the ℓth roots of the $f_{i,g}, f'_{i,g}$. In fact the extension $R(X)^G \subset L$ is Galois as $N \subset R(X)^*$ is G-stable.

Suppose the algebraic closure K of $R(S)$ in L strictly contains $R(S)$. Let us do a base change $S' \to S$ as in 3.2 with $R(S') = K$. This is allowed as $R(S)^G \subset K$ is Galois. We replace φ_i by its pullback φ'_i as above. Repeating our construction of N and L in the new situation, we see that we have reached a situation where $R(S)$ is algebraically closed in L.

At this point we let X' be the normalization of the scheme X in L. We put

$$G' = G \times_{\mathrm{Gal}(R(X)/R(X)^G)} \mathrm{Gal}(L/R(X)^G)$$

and we see that G' surjects onto G and G' acts on X'. The curve X'_η is smooth over η as the morphism $X'_\eta \to X_\eta$ is generically étale and has only tame ramification over points $p \in X_\eta \setminus Z_\eta$ with $\kappa(\eta) \subset \kappa(p)$ separable. It follows that X'_η is geometrically irreducible over η as $\kappa(\eta) = R(S)$ is algebraically closed in $R(X'_\eta) = L$. In addition Z'_η is étale over η as $Z'_\eta \to Z_\eta$ is étale.

Any element $h \in H$ fixes all the elements $f_{i,g}, f'_{i,g}$, hence its action on $R(X)$ can be lifted uniquely to an element of $\mathrm{Gal}(L/R(X)^G)$ fixing all the ℓth roots of the functions $f_{i,g}, f'_{i,g}$. This gives an element $h' \in G'$ which lies in $\mathrm{Ker}(G' \to \mathrm{Aut}(S))$ and which maps to h. (The author remarks that it actually wasn't necessary to have the elements $f_{i,g}, f'_{i,g}$ to be H-invariant.) Finally, it is trivial to see that $R(X')^{G'} = R(X)^G$.

Thus we have shown that $(f' : X' \to S, G', Z')$ satisfies (2.1.1), (2.1.2), (2.1.3) and (2.1.4), and that $(X' \to X, S \to S)$ satisfies (2.2.1), (2.2.5), (2.2.6) and (2.2.7). In other words we have produced an alteration which is allowed.

We claim that $(f' : X' \to S, G', Z')$ satisfies (2.1.6). This we get from the fact that the homomorphism $\varphi_i^* : R(\mathbb{P}^1_{U_i}) \to R(X)$ extends to a homomorphism $L_i \to L$ by construction. By the universal property of normalization we get a morphism $X'_{U_i} \to \mathcal{C}_i$. This morphism is finite as $X_{U_i} \to \mathbb{P}^1_{U_i}$ is finite. We conclude that any irreducible component C of any geometric fibre $X'_{\bar{s}}$ of X' over S has a finite morphism to the curve $\mathcal{C}_{i,\bar{s}}$ if $\bar{s} \in U_i$. Thus $g(C) \geqslant g(\mathcal{C}_{i,\bar{s}}) \geqslant 2$.

3.6. We assume (2.1.1), (2.1.3), (2.1.2), (2.1.4) and (2.1.6). Note that a base change $S' \to S$ as in 3.2 preserves the condition (2.1.6) as well. Hence we may assume, after extending $R(S)$, that $Z_\eta = \{p_1, \ldots, p_n\}$ with $\kappa(p_i) = \kappa(\eta)$. Remark that $(X_\eta, p_1, \ldots, p_n)$ is a smooth stable n-pointed curve over η in view of (2.1.6).

LEMMA 3.7. — *Let S be an excellent scheme with generic point η, let $(X_\eta, p_1, \ldots, p_n)$ be a smooth stable n-pointed curve over η. There exists a generically étale alteration $S' \to S$ such that $(X_\eta, p_1, \ldots, p_n) \otimes \kappa(\eta')$ extends to a stable n-pointed curve over S'.*

Proof. — Compare [1, 4.17] and [1, 5.13]. (In order to get $S' \to S$ generically étale use only level structures prime to the characteristic of $R(S)$.) □

3.8. We apply the lemma to our stable n-pointed curve $(X_\eta, p_1, \ldots, p_n)$. After replacing S by a finite covering as in 3.2 we may assume that $(X_\eta, p_1, \ldots, p_n)$ extends to a stable n-pointed curve on a modification of S. We can dominate this modification by a normal modification S' on which G acts [1, 7.6]. Replace X by $X' = (X \times_S S')_{\mathrm{red}}$ and S by S'; this is an allowed alteration. In this way we reach a situation where in addition to (2.1.1), (2.1.3), (2.1.2), (2.1.4) and (2.1.6) we have that S is normal and the following condition:

(*) There exists a stable n-pointed curve $(\mathcal{C}, \sigma_1, \ldots, \sigma_n)$ over S with $(X_\eta, Z_\eta) \cong (\mathcal{C}_\eta, \sigma_1(\eta), \ldots, \sigma_n(\eta))$.

We remark that the group G acts on $(\mathcal{C}, \sigma_1, \ldots, \sigma_n)$ over S. This is due to the fact that $I = \mathrm{Isom}(g^*(\mathcal{C}, \{\sigma_1, \ldots, \sigma_n\}), (\mathcal{C}, \{\sigma_1, \ldots, \sigma_n\}))$ is finite unramified over S, the scheme S is normal and we have an element of $I(\eta)$

given by the action of g on (X_η, Z_η). Here $(\mathcal{C}, \{\sigma_1, \ldots, \sigma_n\})$ denotes the curve with an non-ordered set of sections.

3.9. Let us look at the scheme

$$M = \mathrm{Mor}_S(\mathcal{C}, X).$$

It is a disjoint union of schemes quasi-projective over S, see [5, page 22]. We have a point m_η of $M(\eta)$ given by our isomorphism in (*). We define S' as the closure of m_η in M. Thus S' is an integral scheme, $S' \to S$ is birational and quasi-projective.

Claim 3.10. — The morphism $S' \to S$ is proper.

3.11. If we prove the claim then we are done with the proof of (vii)(a) of Theorem 2.4. Indeed, the group G acts on $\mathcal{C}$ and X over its action on S, and hence induces an action of G on M. The map in (*) is equivariant for these actions, hence we see that G acts on S'. Since S' is quasi-projective and proper over S, it is projective over S. The morphism $\mathcal{C}_{S'} \to X$ obtained from the map $S' \to M$ is birational, G-equivariant and projective as $\mathcal{C}_{S'}$ is projective over S. Thus the alteration $(\mathcal{C}_{S'} \to X, S' \to S)$ satisfies all the conditions listed in 2.2.

3.12 (Proof of Claim 3.10). We prove this by checking the valuative criterion of properness; we need only consider morphisms $T \to S$, where T is a trait and the generic point ξ of T maps to $\eta \in S$. (So ξ lifts to S' as $S' \to S$ is birational.) Thus we have to prove the lemma below.

LEMMA 3.13. — *Let S be a trait, $\mathcal{C} \to S$ a semi-stable curve with smooth generic fibre and $X \to S$ a morphism as in 2.1 that satisfies (2.1.3) and (2.1.6). Any morphism $\mathcal{C}_\eta \to X_\eta$ over η extends to a morphism $\mathcal{C} \to X$.*

Proof. — The case that $\mathcal{C}_\eta \to X_\eta$ is constant is left to the reader. If not, then $R(X_\eta) \subset R(\mathcal{C}_\eta)$ is finite. We replace X by the normalization of X in $R(\mathcal{C}_\eta)$; the conditions (2.1.3) and (2.1.6) still hold. Thus we may assume that $\mathcal{C}_\eta \to X_\eta$ is an isomorphism.

There exists a blow up $\tilde{\mathcal{C}} \to \mathcal{C}$ of the normal surface $\mathcal{C}$ such that the birational morphism $\mathcal{C}_\eta \to X_\eta$ extends to a morphism $\tilde{\mathcal{C}} \to X$. As the surface $\mathcal{C}$ has only rational singularities (loc. eq. $xy - \pi^n = 0$) the exceptional curves of $\tilde{\mathcal{C}} \to \mathcal{C}$ are all genus zero curves. Thus they are contracted to points in X in view of (2.1.6). It follows that the morphism $\tilde{\mathcal{C}} \to X$ factors as $\tilde{\mathcal{C}} \to \mathcal{C} \to X$. □

4. Equal genus.

4.1. Assume the triple $(f : X \to S, G, Z)$ satisfies (2.1.1), (2.1.3), (2.1.2) and (2.1.4). By Section 3 there exists an alteration $(f' : X' \to S', G', Z')$ of $(f : X \to S, G, Z)$ that solves case (vii)(a) of Theorem 2.4:

$$\begin{array}{ccc} X' & \xrightarrow{\psi} & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ S' & \xrightarrow{\phi} & S. \end{array}$$

Thus (ψ, ϕ) satisfies (2.2.1), (2.2.5), (2.2.4) (2.2.6) and (2.2.7).

Let $N \subset G'$ be the kernel of the mapping $G' \to G \times \operatorname{Aut} S'$. By (2.2.4) we have that the morphism $X'/N \to X \times_S S'$ is birational. The proposition below shows that $X_1 = X'/N$ is a semistable curve over $S' = S_1$. The sections σ_i give a number of disjoint sections (one for each N-orbit on $\{\sigma_1, \ldots, \sigma_n\}$), mapping into the smooth locus by Proposition 4.2. The alteration $(X_1 \to S_1, G'/N, Z/N) \to (f : X \to S, G, Z)$ satisfies (2.2.1), (2.2.5), (2.2.4) and (2.2.7) as follows from the corresponding properties of (ψ, ϕ). The property (2.2.2) follows from the construction. We are done with the proof of Theorem 2.4 once we have shown Proposition 4.2.

PROPOSITION 4.2. — *Let $\mathcal{C} \to S$ be a projective semi-stable curve over an excellent integral normal scheme S. Assume $G \subset \operatorname{Aut}_S(\mathcal{C})$ is a finite group of automorphisms of $\mathcal{C}$ over S. The quotient $\mathcal{C}/G$ exists and is a semi-stable curve over S. The map $\mathcal{C} \to \mathcal{C}/G$ is finite and maps $sm(\mathcal{C}/S)$ into the smooth locus of $\mathcal{C}/G \to S$.*

Proof. — The quotient exists as $\mathcal{C}$ is projective over S: any point of $\mathcal{C}$ is contained in a G-stable open affine U of $\mathcal{C}$. The quotient of U by G is the spectrum of the invariants for G in $\Gamma(U, \mathcal{O}_U)$. The construction of the quotient commutes with flat base change. (For these assertions one can consult [5].) Thus to study the local structure of the quotient, we may assume that $S = \operatorname{Spec} R$, where R is a normal Noetherian complete local domain with algebraically closed residue field. We want to show that $\mathcal{C}/G$ has a local description as in [1, 2.23]. Let $x \in \mathcal{C}_s$ be a closed point of the special fibre mapping to $y \in \mathcal{C}/G$. Let $H \subset G$ be the stabilizer of x. Then $(\mathcal{O}_{\mathcal{C},x})^H = \mathcal{O}_{\mathcal{C}/G,y}$ and $(\widehat{\mathcal{O}}_{\mathcal{C},x})^H = \widehat{\mathcal{O}}_{\mathcal{C}/G,y}$. Thus the local description and the last assertion of the proposition follow from the lemma below. □

LEMMA 4.3. — *Let $(R, \mathfrak{m}, k)$ be a Noetherian complete local ring which is a normal domain. Let A be an R-algebra isomorphic to either* *(a) $R[[u]]$ or (b) $R[[u,v]]/(uv-f)$ for some $f \in \mathfrak{m}$. Assume given some finite group H contained in the automorphism group of A over R. In case (a) we have $A^H \cong R[[x]]$ and in case (b) we have either $A^H \cong R[[x]]$ or $A \cong R[[x,y]]/(xy-g)$ for some $g \in \mathfrak{m}$ (in this case g can be taken to be a power of f).*

Proof. — We fix some notations: We set $d = \#H$ and $\mathrm{Nm} : A \to A^H$ is the map $a \mapsto \prod_{\sigma \in H} \sigma(a)$. The symbol $\Phi(B)$ will denote the fraction field of the domain B.

Note that A^H is a normal complete local domain with maximal ideal $A^H \cap \mathfrak{m}_A$ and residue field k.

Case I: $A \cong R[[u]]$. Put $x = \mathrm{Nm}(u) \in A^H$. It is immediate that $x \equiv u^d \cdot (unit)$ modulo $\mathfrak{m}A$. Hence $R[[x]] \to A$ is a finite injective ring homomorphism of degree $\leqslant d$. It follows that $A^H \to A$ and $R[[x]] \to A^H$ are finite. We have $\Phi(R[[x]]) \subset \Phi(A^H) \subset \Phi(A)$. By Galois theory $[\Phi(A) : \Phi(A^H)] = d$, hence $\Phi(R[[x]]) = \Phi(A)^H = \Phi(A^H)$. But $R[[x]]$ is normal, hence integrally closed in $\Phi(R[[x]])$, hence $R[[x]] = A^H$.

Case II: $A \cong R[[u,v]]/(uv-f)$ for some $f \in \mathfrak{m}$. We get an action of H on the set of components of $\operatorname{Spec} A/\mathfrak{m}A = \operatorname{Spec} k[[u,v]]/(uv)$. Let H_0 be the subgroup preserving the components. The index $[H : H_0] \leqslant 2$. Below we will do the case $H = H_0$; this reduces us to the case that $H_0 = \{1\}$ and $H = \{1, \tau\}$. In this case it is easy to see that $A \cong R[[u,v]]/(uv-f)$ with $v = \tau(u)$ and $A^H = R[[x]]$ with $x = u + \tau(u)$.

Case IIa: Here $f = 0$ and $H = H_0$. This case follows from Case I as A^H is the fibre product

$$(A/uA)^H \times_R (A/vA)^H$$

and the morphisms $(A/uA)^H \to R$ and $(A/vA)^H \to R$ are surjective.

Case IIb: Here $H = H_0$ and $f \in \mathfrak{m}$ is not zero. Again let $x = \mathrm{Nm}(u)$ and $y = \mathrm{Nm}(v)$. As $H = H_0$ we have that $x \equiv u^d \cdot (unit)$ and $y \equiv v^d \cdot (unit)$ modulo $\mathfrak{m}A$. We obviously have $xy = f^d$. Consider the finite homomorphism of R-algebras

$$B = R[[x,y]]/(xy - f^d) \longrightarrow A = R[[u,v]]/(uv-f).$$

As $f \neq 0$ both A and B are normal domains. Thus $B \to A$ is injective in view of the fact that $\dim A = \dim B = \dim R + 1$. We want to compute $[\Phi(A) : \Phi(B)]$. For any prime ideal $\mathfrak{p} \subset B$ we have

$$[\Phi(A) : \Phi(B)] \leqslant \dim_{\kappa(\mathfrak{p})} A \otimes_B \kappa(\mathfrak{p}).$$

We take $\mathfrak{p} = \mathfrak{m}B + yB$, and we have $B/\mathfrak{p} = k[[x]]$ and $\kappa(\mathfrak{p}) = k((x))$. We compute

$$A/\mathfrak{p}A = k[[u,v]]/(uv,y) = k[[u,v]]/(uv,v^d),$$

as $y \equiv v^d \cdot (unit)$ modulo $\mathfrak{m}A$. Thus we get $A \otimes_B \kappa(\mathfrak{p}) \cong k((u))$, as a $\kappa(\mathfrak{p}) = k((x))$ module. We have $[k((u)) : k((x))] = d$ as $x \equiv u^d \cdot (unit)$ modulo $\mathfrak{m}A$. Thus we see that $[\Phi(A) : \Phi(B)] \leqslant d$. Since $B \subset A^H$ we get equality of degrees by Galois theory. We get $B = A^H$ from the fact that B is normal. □

Remark 4.4. — The idea of the proof of the proposition above goes back to [3, pp. 508-510]. In the case that S has dimension 1 a proof of the proposition was given by Bas Edixhoven; our proof above closely follows his.

5. Applications.

Here is an immediate application.

COROLLARY 5.1. — *Let S be an excellent scheme of dimension 1. In this situation [1, Theorem 8.2] holds, i.e., any S-variety X can be altered into a semi-stable variety over a finite extension S_1 of S.*

Proof. — This is trivial from Theorem 2.4 and the results of [1]. Indeed, the only problem in the proof of [1, 8.2] was to present X (after an alteration) as a semi-stable family of curves over an S-variety whose dimension is 1 less. However, this is immediate from Theorem 2.4. □

LEMMA 5.2. — *Let $f : X \to S$ be a proper morphism of excellent integral schemes, and let G be a finite group acting on f. Assume that the generic fibre X_η of f is geometrically reduced and irreducible and has dimension $d \geqslant 2$. There exists a commutative diagram on which G acts*

$$\begin{array}{ccc} X' & \xrightarrow{\psi} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{h} & S, \end{array}$$

where ψ is a projective modification, g and h are dominant projective morphisms of integral schemes having geometrically reduced and irreducible generic fibres and such that g has a generic fibre of dimension 1.

Proof. — The conditions on f imply that the field extension $R(S) \subset R(X)$ is separable. Choose $x_1, \ldots, x_{d-1} \in R(X)^G$ which are algebraically independent over $R(S)$ and such that $R(X)$ is separable over $R(S)(x_1, \ldots, x_{d-1})$. This defines a G-equivariant rational map $X \dashrightarrow \mathbb{P}^{d-1}_S$. Therefore there is a projective modification $X' \to X$ on which G acts such that there is a G-equivariant morphism $X' \to \mathbb{P}^{d-1}_S$ realizing r. We may assume that X' is normal.

Let $X' \to Y \to \mathbb{P}^{d-1}_S$ be the Stein factorization. Note that G acts on Y and that $X' \to Y$ and $Y \to S$ are G-equivariant. By construction the field extension $R(Y) \subset R(X') = R(X)$ is separable and $R(Y)$ is algebraically closed in $R(X')$. Hence the generic fibre of $X' \to Y$ is geometrically reduced and irreducible. In the same way one sees that $Y \to S$ has geometrically reduced and irreducible generic fibre. □

Situation 5.3. — Here S is an integral Noetherian scheme, and G is a finite group acting on S. A *Galois alteration of* (S, G) is a system $(\pi : S' \to S, G' \to G, G' \times S' \to S')$, where (a) $\pi : S' \to S$ is an alteration, (b) $G' \to G$ is a surjection of finite groups, (c) $G' \times S' \to S'$ is an action of G' on S' such that π becomes G' equivariant, and (d) we have that the extension $R(S)^G \subset R(S')^{G'}$ is purely inseparable. Condition (d) is equivalent to the following: if $H = \mathrm{Ker}(G' \to \mathrm{Aut}(S))$ then $R(S) \subset R(S')^H$ is purely inseparable.

5.4. Suppose we have a commutative diagram of integral schemes

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S. \end{array}$$

Suppose that $G' \to G$ is a surjection of finite groups and that these act on the various schemes above. Assume that $(X', G') \to (X, G)$ and $(S', G') \to (S, G)$ are Galois alterations. We consider the following condition on this diagram:

(5.4.1) Let $H = \mathrm{Ker}(G \to \mathrm{Aut}(S))$ and let $H' = \mathrm{Ker}(G' \to \mathrm{Aut}(S'))$. Then $H' \to H$ is surjective.

Suppose that $X \to S$ has a geometrically irreducible generic fibre. In this case (5.4.1) implies that the map $X' \to (X \times_S S')^{\sim}$ is a Galois alteration (both with group G'). The notation $(X \times_S S')^{\sim}$ refers to the irreducible component of $X \times_S S'$ dominating S'.

LEMMA 5.5 — *Let $f : X \to S$ be a proper morphism of integral excellent schemes on which the group G acts. Assume that the generic fibre of f is geometrically irreducible. Suppose that (X', G') is a Galois alteration of X. There exists a diagram on which G' acts:*

$$\begin{array}{ccccc} X_1' & \longrightarrow & X_1 & \longrightarrow & S_1 \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & X & \longrightarrow & S \end{array}$$

such that the generic fibres of $X_1' \to S_1$ and $X_1 \to S_1$ are geometrically irreducible and reduced, such that $(S_1, G') \to (S, G)$ is a Galois alteration, such that X_1 is the strict transform of X with respect to $S_1 \to S$, and such that the field extension $R(X') \subset R(X_1')$ is purely inseparable. Finally, $X_1' \to X_1$ is a Galois alteration as well and the diagram 5.4 composed of X_1', X, S_1 and S satisfies (5.4.1).

Proof. — Let S' be the normalization of S in a purely inseparable field extension of $R(S)$ on which the group G acts such that the generic fibres of $(X \times_S S')^\sim$ and $(X' \times_S S')^\sim$ over S' are geometrically reduced. Here $(X' \times_S S')^\sim$ denotes the irreducible component of $X' \times_S S'$ dominating S'; this will be our X_1'. We may replace S by S' and assume that X_η' and X_η are geometrically reduced. We may also assume that X' is normal. Let $X' \to S' \to S$ be the Stein factorization of $X' \to S$. Note that G' acts on S' and that $(S', G') \to (S, G)$ is a Galois alteration. Indeed, if $H = \mathrm{Ker}(G' \to G)$, then by assumption $R(X) \subset R(X')^H$ is purely inseparable, hence $R(S) \subset R(S')^H$ is purely inseparable. Now $X_{\eta'}'$ is normal, geometrically reduced and geometrically connected, hence it is geometrically irreducible. We take $S_1 := S'$ and $X_1 := (X \times_S S')^\sim$. The verification of the last statement of the lemma is an exercise in Galois theory that is left to the reader. □

LEMMA 5.6. — *Let (S, G) be as in Situation 5.3, with S excellent. Let $\{s_1, \ldots, s_n\} \subset S$ be a finite set of points of S, and suppose we are given a finite separable field extension $\kappa(s_i) \subset k_i$ for each i. There exists an alteration $(\pi : S' \to S, G' \to G, G' \times S' \to S')$ as in 5.3 with π finite and generically étale, and such that $\pi^{-1}(s_i) = \{s_{ij}\} \subset S'$ where all the field extensions $\kappa(s_i) \subset \kappa(s_{ij})$ contain k_i.*

Proof. — Clear. □

We say that a semi-stable curve X over a field k is *quasi-split* if (a) its singular points are rational over k, and (b) in each of its singular points the two tangents are rational over k. This means that $\mathcal{O}^{\wedge}_{X,x} \cong k[[uv]]/(uv)$ for any singular point $x \in X$. Another formulation: the inverse image under the normalization mapping $X^n \to X$ of the set of singular points of X consists of k-rational points of X^n. We say a semi-stable curve $X \to S$ over a scheme S is *quasi-split* if for all $s \in S$ the fibre X_s is a quasi-split semi-stable curve over $\kappa(s)$. Compare with [1, 2.22 and 2.23] where split semi-stable curves are defined; a split semi-stable curve is quasi-split and a quasi-split semi-stable curve is split only if all its fibres have smooth components.

LEMMA 5.7. — *Let $f : X \to S$ be a semi-stable curve over an excellent integral scheme S of finite dimension. Assume the finite group G acts on f. There exists a Galois alteration (S', G') of (S, G) which is finite and generically étale such that the pullback $X' = X \times_S S'$ is a quasi-split semi-stable curve over S'.*

Proof. — Let $T \subset S$ be an irreducible and reduced closed subscheme, with generic point ξ and assume that X_ξ is quasi-split. We claim that there exists a nonempty open subscheme $U \subset T$ such that $X|_U$ is quasi-split over U. By [2, Section 3] the curve $X|_V$ has locally constant topological type over V for some nonempty open $V \subset T$. Over V we can find a morphism $X^n_V \to X|_V$ such that $X^n_v \to X_v$ is the normalization mapping of X_v for all $v \in V$. Let $Y \subset X^n_V$ be the inverse image of $\mathrm{Sing}(f)|_V$. Then $Y \to V$ is finite étale and X_v is quasi-split if and only if all points of Y_v are $\kappa(v)$-rational (see above). But we assumed that X_ξ is quasi-split, so all points of Y_ξ are rational, hence the covering $Y \to V$ is trivial over a nonempty open subscheme $U \subset V$. This proves the claim.

Let $Z \subset S$ denote the Zariski closure of the set of points s where X_s is not quasi-split. By the claim the curve X_ξ is not quasi-split for any maximal point ξ of Z. Let $\{s_i\}$ be the set of all these maximal points and let $\kappa(s_i) \subset k_i$ be finite separable extensions such that $X_{s_i} \otimes k_i$ is quasi-split. Apply Lemma 5.6 to this situation: we get $\pi : S' \to S$. If $Z' \subset S'$ denotes the Zariski closure of the set of points of S' where X' is not quasi-split, then we see that $\pi(Z') \subset Z$ is nowhere dense. Since π is finite we conclude $\dim Z' < \dim Z$. We win by induction on this dimension (which is finite as $\dim(S) < \infty$). □

Situation 5.8. — Let (S, G) be a pair as in 5.3. A *G-pluri nodal fibration of relative dimension d* over S is given by a system:

$$(X_d \xrightarrow{f_d} X_{d-1} \longrightarrow \dots \longrightarrow X_1 \xrightarrow{f_1} X_0 = S, \{\sigma_{ij}\}, Z_0),$$

where

(i) $f_i : X_i \to X_{i-1}$ is a projective and quasi-split semi-stable curve over X_{i-1},

(ii) $Z_0 \subset S = X_0$ is a G-stable proper closed subset of S, and

(iii) $\sigma_{ij} : X_{i-1} \to X_i$, $j = 1, \ldots, n_i$ are disjoint sections of f_i into the smooth locus of X_i over X_{i-1}.

Put

$$Z_i = \bigcup_{j=1}^{n_i} \sigma_{ij}(X_{i-1}) \cup f_i^{-1}(Z_{i-1}).$$

The data are subject to the condition that $X_{i+1} \to X_i$ is smooth over the locus $X_i \setminus Z_i$. Furthermore G acts on all X_i such that all the morphisms f_i are G-equivariant, and $\forall i, j, g$, $\sigma_{ij} \circ g = g \circ \sigma_{ij'}$ for some $j' = j'(g, i)$.

THEOREM 5.9. — *Let $f : X \to S$ be a proper morphism of integral excellent schemes of finite dimension. Assume that a finite group G acts on f and that a G-stable proper closed subset $Z \subset X$ is given. Suppose the generic fibre X_η of f is geometrically irreducible of dimension $d \geqslant 1$. In this situation there exist:*

(i) *A projective Galois alteration (5.3) (S_1, G_1) of (S, G).*

(ii) *A G_1-pluri nodal fibration $(X_d \to \ldots \to X_1 \to S_1, \{\sigma_{ij}\}, Z_0)$ over S_1.*

(iii) *A G_1-equivariant alteration $\psi_1 : X_d \to X$ making the following diagram commutative*

$$\begin{array}{ccc} X_d & \xrightarrow{\psi_1} & X \\ \big\downarrow{\scriptstyle f_1 \circ \ldots \circ f_d} & & \big\downarrow{\scriptstyle f} \\ S_1 & \longrightarrow & S. \end{array}$$

These will satisfy the following conditions:

(a) *The map $(X_d, G_1) \to (X, G)$ is a Galois alteration and the diagram above satisfies (5.4.1). In particular, if $N = \mathrm{Ker}(G_1 \to \mathrm{Aut}(S_1 \times_S X))$, then the morphism $X_d/N \to (X \times_S S_1)^\sim$ induces a purely inseparable extension of functions fields.*

(b) *We have $\psi_1^{-1}(Z) \subset Z_d$. (See Subsection 5.8 for the definition of $Z_d \subset X_d$.)*

Proof. — In the case $\dim X_\eta = 1$ this follows from Theorem 2.4 combined with Lemma 5.7 to "split" the curve. In the general case we argue by induction on the dimension of X_η. First we replace S by the normalization of S in a purely inseparable extension of its function field on which G acts such that X_η becomes geometrically reduced. Choose $X' \to Y \to S$ as in Lemma 5.2. Then by Theorem 2.4 we find an alteration (2.2) with group G':

$$\begin{tikzcd} X'' \arrow[r, "\psi"] \arrow[d, "g'"] & X' \arrow[d, "g"] \\ Y' \arrow[r] & Y, \end{tikzcd}$$

having the properties (2.2.1)-(2.2.7) except possibly (2.2.5); in particular $Y' \to Y$ is a Galois alteration. There are sections σ_j of g' adapted to the inverse image $Z' \subset X'$ of $Z \subset X$, i.e., $\psi^{-1}(Z') \subset \bigcup_j \sigma_j(Y') \cup (g')^{-1}(Z'')$ for some proper closed subset $Z'' \subset Y'$. We enlarge Z'' such that it contains the locus over which $X'' \to Y'$ is not smooth. By Lemma 5.7 we may in addition assume that g' is a quasi-split semi-stable curve (this does not destroy any of the properties). We apply Lemma 5.5 to $Y' \to Y \to S$ and we get $Y_1' \to Y_1 \to S'$ with an action of G'.

By induction applied to the pair $Z_1'' \subset Y_1'$ over S' with group G' we get a diagram

$$\begin{tikzcd} X_{d-1} \arrow[r] \arrow[d] & Y_1' \arrow[d] \\ S_1 \arrow[r] & S'. \end{tikzcd}$$

We leave it to the reader to show that by taking $X_d = X'' \times_{Y'} X_{d-1}$, we get a G_1-pluri nodal fibration with all the properties as stated in the theorem. For example (a) can be seen to be true by noting that we have (a) for all the rectangles in the following diagram

$$\begin{tikzcd} X_d = X_{d-1} \times_{Y'} X'' \arrow[rr] \arrow[d] & & X'' \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & X \arrow[dd] \\ X_{d-1} \arrow[r] \arrow[d] & Y_1' \arrow[r] \arrow[d] & Y' \arrow[r] & Y \arrow[d] & \\ S_1 \arrow[r] & S' \arrow[rr] & & S \arrow[r, equal] & S. \end{tikzcd}$$

□

COROLLARY 5.10. — *Let $f : X \to S$ be a dominant and proper morphism of integral excellent schemes. There exists a diagram*

$$\begin{array}{ccc} X_d \longrightarrow X_{d-1} \longrightarrow \ldots \longrightarrow X_1 \longrightarrow & & S_1 \\ \Big\downarrow \psi & & \Big\downarrow \phi \\ X & \xrightarrow{\hspace{6em}} & S \end{array}$$

where ψ and ϕ are alterations and $f_d : X_d \to S_1$ is a composition of projective semi-stable curve fibrations $X_i \to X_{i-1}$. In addition we may assume a finite group N acts on this diagram, acting trivially on X, S and S_1, such that $R(X_d/N)$ is a purely inseparable extension of the function field of the component of $X \times_S S_1$ it maps onto.

Proof. — First take a finite alteration $S' \to S$ such that some irreducible component X' of $X \times_S S'$ dominating S' is geometrically irreducible over S'. Next, apply Theorem 5.9 to $f' : X' \to S'$. (Note that the condition of finite dimensionality was used in the proof of Lemma 5.7 only. However, since we do not require the semi-stable curves to be quasi-split here, we do not need it. In fact $X_d \to S_1$ will have all the properties of an N-pluri nodal fibration over S_1 except for being "quasi-split".) □

PROPOSITION 5.11. — *Let (S, G) be as in 5.3 with S regular, and let $D \subset S$ be a G-strict normal crossings divisor in S (see [1, 7.1]). Suppose that $f : X \to S$ is a quasi-split semi-stable curve, smooth over $S \setminus D$. Furthermore, assume the action of G on S lifts to an action of G to X. There exists a G-equivariant projective modification $X_1 \to X$ with the following properties:*

(i) *The scheme X_1 is regular and the center of $X_1 \to X$ is contained in Sing(X). The inverse image of D in X_1 is a normal crossings divisor.*

(ii) *Let $\sigma_1, \ldots, \sigma_n$ be disjoint sections into the smooth locus of $X \to S$ which are permuted by the action of G. Then these lift to sections σ_i of $f_1 : X_1 \to S$, and the divisor $D_1 := \bigcup_i \sigma_i(S) \cup f_1^{-1}(D)$ is a G-invariant normal crossings divisor of X_1. There is a canonical blow up $b : X_2 \to X_1$ such that X_2 is regular and $D_2 = b^{-1}(D_1)$ is a G-strict normal crossings divisor in X_2.*

Proof. — Consider the blow ups that occur in the proof of [1, Lemma 3.2]. Let $T \subset X$ be as in [1, 3.4]. Then $T \xrightarrow{\cong} D_1$ as f is quasi-split. As D is G-strict, we have $T' = \bigcup_{g \in G} g(T)$ is a disjoint union of irreducible closed subschemes of the type described in [1, 3.4]. Hence the blow up in the closed subscheme T' can be described locally as in [1, 3.4]. Thus we may assume that $X \to S$ is quasi-split semi-stable and $\operatorname{codim}(\operatorname{Sing}(X), X) \geqslant 3$.

The rest of our argument is similar to the arguments of [1, 4.24-4.28] and [1, 7.16]. The types of complete local rings that we have now are

$$A[[u,v]]/(uv - t_1 \cdot \ldots \cdot t_s),$$

where A is a regular complete local ring with a regular system of parameters $t_1, \ldots, t_{d-1}$, and D is given by $t_1 \cdot \ldots \cdot t_r = 0$, for some $1 \leqslant s \leqslant r \leqslant d-1$.

We give the argument that proves that the blow ups that occur in the proof of [1, 4.26 and 4.27] can be made G-equivariantly. Indeed, by the remarks at the end of [1, 3.5] we have $\operatorname{Sing}(X) = \bigcup E_\alpha$, with E_α regular and mapping finite étale onto an irreducible component of some $D_i \cap D_j$. However, since f is quasi-split and $E_\alpha \subset \operatorname{Sing}(f)$ this morphism induces trivial residue field extensions, hence E_α maps isomorphically onto a component of $D_i \cap D_j$. Furthermore, if E_α and E_β have the same image in S, then $E_\alpha \cap E_\beta = \varnothing$. Thus, as D is G-strict, we have that $\bigcup_{g \in G} g(E_\alpha)$ is a disjoint union of components E_β. Thus the last sentence of [1, 4.25] should be replaced with: Any G-orbit of a component of the singular locus of X is nonsingular.

Thus we blow up in orbits of components of the singular locus of X. The resulting scheme has a local description as above by the computations of [1, 4.27]. Hence we arrive at a regular scheme X' with an action of G and a G-stable normal crossings divisor $D' \subset X'$.

If we are given sections σ_i as in (ii) then these lift to sections σ_i' into $X_1 = X'$ (as we modified only in singular loci) and the divisor $D_1 = D' \bigcup \sigma_i'(S)$ is a normal crossings divisor as well. To get D_1 to be G-strict we apply the canonical blow up of [1, 7.2]. □

5.12. We consider the following condition on a pair (S, G) as in 5.3:

(5.12.1) For every Galois alteration (S', G') of (S, G) and proper closed subset $Z' \subset S'$, there exists a Galois alteration (S_1, G_1) of (S', G'), such that S_1 is regular and such that the inverse image of Z' in S_1 is contained in a G_1-strict normal crossings divisor.

THEOREM 5.13. — *Let S be an integral excellent scheme of finite dimension. Let X be an integral scheme and let $f : X \to S$ be a dominant morphism on which the finite group G acts. Assume (S, G) satisfies (5.12.1).*

If f is of finite type, is separated and has geometrically irreducible generic fibre then the pair (X, G) satisfies (5.12.1).

Proof. — This is just a combination of the results obtained so far. First, we can assume that X is proper over S by the usual arguments: first make X quasi-projective by a G-equivariant Chow's lemma (use Chow's lemma and argue as in [1, 7.6]). To make X projective over S, take an embedding $i : X \to \mathbb{P}^n_S$ and let $\bar{X}$ be the closure of $i' : X \to (\mathbb{P}^n \times \ldots \times \mathbb{P}^n)_S$ where $i'(x) = \prod_{g \in G} i(gx)$; G acts on $\bar{X}$. If the theorem holds for $(\bar{X}, G)$, then it holds for X.

Let (X', G') be a Galois alteration of the pair (X, G) and let $Z' \subset X'$ be a proper closed subset. By Lemma 5.5 we reduce to a situation where $X = X' \to S$ has geometrically irreducible (and reduced) generic fibre.

By Theorem 5.9 we may assume that $X \to S$ is a G-pluri nodal fibration of relative dimension d, i.e., $X = X_d \to X_{d-1} \to \ldots \to X_1 \to S$ and that $Z = Z_d$. Next, by (5.12.1) for S, we may assume that S is regular and that $Z_0 \subset S$ is a G-strict normal crossings divisor. Then by Proposition 5.11 applied to $X_1 \to S$ we may assume that X is a split G-pluri nodal fibration of relative dimension $d - 1$ over a regular scheme X'_1 whose associated closed subset is a G-strict normal crossings divisor $Z'_1 \subset X'_1$. Apply Proposition 5.11 to the split semi-stable family of curves over X'_1, etc. □

COROLLARY 5.14. — *Any integral scheme separated, flat and of finite type over* Spec $\mathbb{Z}$ *has resolution of singularities up to quotient singularities.*

COROLLARY 5.15. — *Any integral scheme X separated and of finite type over an excellent scheme S with* $\dim S \leqslant 2$ *satisfies (5.12.1) with* $G = \{1\}$. *In particular the singularities of X can be resolved up to quotient singularities and a purely inseparable extension of $R(X)$.*

Proof. — We can find a finite morphism $S' \to S$ of an integral scheme S' to S, and a finite alteration $X' \to X$ with purely inseparable function field extension such that X' is a scheme over S' with geometrically irreducible geometric fibre. Since $\dim S' \leqslant 2$, we have canonical resolution of singularities for S' [4] and any alteration of S', and hence we get (5.12.1) for S'. By the theorem we get (5.12.1) for X'. This implies (5.12.1) for X as $R(X) \subset R(X')$ is purely inseparable. □

Remark 5.16. — A word about the hypothesis of excellency that is used throughout. Undoubtedly, in several places this hypothesis is too strong and one could get away with working over Noetherian (universally) Japanese base schemes. However, in some cases one can establish a result similar to the results in the text by the techniques of EGA IV §8. For example let us do this for Theorem 5.9. Suppose $f : X \to S$ is a proper dominant morphism of finite presentation of integral schemes with geometrically irreducible generic fibre, with S affine and with a finite group G acting on f. We have $f = f_0 \times \mathrm{id}_S$ for some G-equivariant morphism $S \to S_0$ and some proper dominant morphism $f_0 : X_0 \to S_0$ of integral schemes of finite type over $\operatorname{Spec}\mathbb{Z}$ with geometrically irreducible generic fibre endowed with an action of G. We apply Theorem 5.9 to f_0. This gives a group G_1 surjecting onto G, and G_1-equivariant maps $X_{d,0} \to S_{1,0} \to S_0$ and $X_{d,0} \to X_0$. Put $S_1 = S_{1,0} \times_{S_0} S$ and $X_d = X_{d,0} \times_{S_0} S$. Then $X_d \to S_1$ is a G_1-pluri nodal fibration over S_1. The morphisms $S_1 \to S$ and $X_d \to X$ are "alterations" in the sense that they are of finite presentation, proper and finite flat over an open dense subscheme. If S is Noetherian, then we can replace S_1 by the reduction of S_1 and we get the result of Theorem 5.9 in this situation.

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