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p-ADIC INTERPOLATION OF CONVOLUTIONS OF HILBERT MODULAR FORMS

by Volker DÜNGER

In 1973, Shimura [S1] proved a striking connection between modular forms of half-integral weight and modular forms of integral weight, which was the starting point of a renewed interest in modular forms of half-integral weight. This also led to an investigation of algebraicity properties of certain special values of convolutions of modular forms of half-integral weight by Shimura [S7] respectively convolutions of modular forms of integral weight and modular forms of half-integral weight by Im [I]. This suggests that one might ask for the existence of a p-adic L-function connected to these values, and the purpose of this paper is to show that the answer is affirmative in the following case:

Let F be a totally real number field, $\mathbf{f} \in \mathcal{M}_k(\mathfrak{c}(\mathbf{f}), \psi)$ a primitive Hilbert automorphic form of scalar integral weight $k = k_0 \cdot \mathbf{1}$ and central character ψ and $g \in \mathcal{M}_l(\mathfrak{c}(g), \phi)$ a Hilbert modular form of half-integral scalar weight $l = l_0 \cdot \mathbf{1}$ and character ϕ such that $l_0 < k_0$. The convolution of $\mathbf{f}$ and g is then defined in terms of the Fourier coefficients $c(\mathfrak{m}, \mathbf{f})$ and $\lambda(\xi, \mathfrak{m}; g, \phi)$ as the Dirichlet series

$$D(s; \mathbf{f}, g) = \sum_{(\xi,\mathfrak{m})} c(\xi\mathfrak{m}^2, \mathbf{f})\overline{\lambda(\xi, \mathfrak{m}; g, \phi)}\xi^{-\frac{1}{2}(l-\frac{1}{2}\cdot\mathbf{1})}\mathcal{N}(\xi\mathfrak{m}^2)^{-s}.$$

Here, $(\xi, \mathfrak{m})$ runs over certain pairs of totally positive numbers ξ of F and fractional ideals $\mathfrak{m}$ of F. We fix a rational prime p, an embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ of the algebraic closure of $\mathbb{Q}$ into the Tate field $\mathbb{C}_p$, a finite set S of finite primes of F containing all primes above p, and an integral

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ideal $\mathfrak{q}$. Let Gal_S be the Galois group of the maximal abelian extension of F unramified outside S and the infinite primes, let $X_S = \mathrm{Hom}_c(\mathrm{Gal}_S, \mathbb{C}_p^\times)$ be the p-adic Lie group of continuous $\mathbb{C}_p^\times$-valued characters of the Galois group, and let $\mathcal{N}_p \in X_S$ be that character which associates to a fractional ideal $\mathfrak{m}$ the image under i_p of its norm $\mathcal{N}(\mathfrak{m})$; here we view Gal_S as the projective limit of the quotients of the prime-to-S-part of the ideal group with respect to certain principal ideal subgroups. We will assume that F has class number $h_F = 1$, that the $\mathfrak{p}$-th Hecke polynomial of the Hilbert automorphic form $\mathbf{f}$ is p-ordinary for $\mathfrak{p} \in S$ (i.e. that there exists a root $\alpha(\mathfrak{p})$ of p-adic absolute value 1), that the ideals $\mathfrak{c}(\mathbf{f})$, $4\mathfrak{c}(g)$, $\mathfrak{m}_0$ and $\mathfrak{q}$ are pairwise relatively prime, that the modular form $g\iota$ with a certain inverter ι defined in section 3, is a simultaneous Hecke eigenform for all Hecke operators, and that the Fourier coefficients of g at $\mathbf{i}\infty$ are algebraic and p-adically bounded. Also, let $\theta \in \{0,1\}$ satisfy $\theta \equiv k_0 - l_0 - \frac{1}{2} \bmod 2$. Then the main theorem 6.1 states that for $r \in \mathbb{Z}$ with $0 \leqslant 2r \leqslant k_0 - l_0 - \frac{1}{2} + \theta - 2$ and for $\kappa_r = \theta - 1 - 2r$ there are measures $\mu^{(r)}$ on X_S defined by

$$\mu^{(r)}(\chi) = \int_{\mathrm{Gal}_S} \chi \, d\mu^{(r)} := i_p \left(\gamma(\mathbf{f}, g, \chi) \frac{D(\frac{\kappa_r - 1}{2}; \mathbf{f}_0, g(\overline{\chi}^*_{\mathfrak{mq}}) j_{c,\mathfrak{m}'})}{\langle \, \mathbf{f}, \mathbf{f} \, \rangle_{\mathfrak{c}\mathfrak{m}_0^2}} \right) \quad \text{for } \chi \in X_S^{\mathrm{tor}}$$

with a certain constant $\gamma(\mathbf{f}, g, \chi)$ which is a product of gamma factors, values of Dirichlet L-functions, and elementary factors. $j_{c,\mathfrak{m}'}$ is a certain inverter, and the ideals $\mathfrak{m}$ and $\mathfrak{m}'$, and the modified Hilbert automorphic form $\mathbf{f}_0$ are as stated in the theorem. These measures satisfy $\mu^{(r)} = \mathcal{N}_p^r \mu^{(0)}$ and determine a p-adic L-function as their Mellin-transform:

$$L_\mu^{(r)} : X_S \to \mathbb{C}_p, \quad L_\mu^{(r)}(x) := \mu(x) = \int_{\mathrm{Gal}_S} x \, d\mu.$$

A few remarks regarding the general position of our p-adic L-function $L_p(\mathbf{f}, g) := L_\mu^{(0)}$ are in order at this point. First we note that $L_p(\mathbf{f}, g)$ is different from the L-function of the convolution of $\mathbf{f}$ and the Shimura lift $\mathbf{g}$ of g. We do not have an arithmetic interpretation of $L_p(\mathbf{f}, g)$ in Iwasawa theory yet. Provided such an interpretation existed, could one then extend the Main Conjecture of Greenberg [G] for Panchishkin's p-adic L-function $L_p(\mathbf{f}, \mathbf{g})$ to $L_p(\mathbf{f}, g)$? Another interesting question concerns the so-called Hida families interpolating an integral weight Hilbert modular form $\mathbf{f}$. Is there hope to make this proof work for such a Hida family? Can one show the existence of p-adic L-functions if g varies in a p-adic family of half-integral weight (see [Hi]) ?

Let us now summarize the contents of the sections and the general idea of the proof of the main theorem. In section 1 we fix the general notations, establish some formulas for Gauss sums of imprimitive characters and recall the notions of p-adic distributions and measures. In section 2 and the first half of section 3, we recall the basic facts about Hilbert modular forms of integral and half-integral weight, respectively. We then prove that the twist of a modular form of half-integral weight with a possibly imprimitive character $\chi_{\mathfrak{q}}^*$ is again a modular form of half-integral weight. Next, we investigate how certain operators ι, m and sw which go back to Shimura [S8] commute with each other and the Hecke operators, and we define the inverter $j_{c,\mathfrak{m}'}$ in terms of these operators. The proof of the theorem then tries to follow the lines of the proof of Panchishkin's theorem [P] on convolutions of Hilbert automorphic forms of integral weight. The first step uses the Euler product of partial Dirichlet series of the convolution of $\mathbf{f}_0$ and $g(\overline{\chi}_{\mathfrak{mq}}^*)j_{c,\mathfrak{m}'}$ to show the independence of this definition from both the modulus $\mathfrak{mq}$ and the auxiliary ideal $\mathfrak{m}'$; this is done in section 4. In section 5, we make use of the Rankin-Selberg integral representation derived by Im [I] to prove the algebraicity of the distribution, and we can verify this via the Fourier coefficients of certain Hilbert automorphic forms of integral weight k and fixed level by successively applying a projection operator, a trace operator and a holomorphic projection operator to the product of g with a certain Eisenstein series of half-integral weight. In order to obtain the measure of the main theorem, the distributions of section 5 have to be regularized because of the occurrence of Dirichlet L-factors in the Fourier coefficients of the Eisenstein series. The proof of the boundedness of our measures is given in section 6 and makes use of the p-adic measure of Deligne and Ribet [DR] derived from the values of Hecke L-functions at negative integers.

There are some differences from the integral case of Panchishkin: The Dirichlet series of modular forms of half-integral weight only have Euler products for partial series. This suffices to prove the independence of the definition of $\mu^{(r)}$ from both $\mathfrak{m}$ and $\mathfrak{m}'$, but it prevents a simple expression in terms of $\mathbf{f}$ and $g^{\rho}(\overline{\chi})$. Secondly, the interchange of the twist of a half-integral Hilbert modular form with a character χ and the inverter $\iota\,\mathrm{sw}_{\mathfrak{c}(\chi)}$ defined in section 3, involves a quadratic character as shown in Proposition 3.10. In particular, this requires us to consider twists with non-primitive characters. Thirdly, the auxiliary ideal $\mathfrak{m}'$ used to smooth the expression for the Fourier coefficients when proving the boundedness of the distribution occurs as a square in the level of the half-integral form $g(\overline{\chi}^*_{\mathfrak{m}\mathfrak{q}})j_{c,\mathfrak{m}'}$. The underlying reason is that the operator sw of section 3 commutes with the twist operator, but it requires quadratic levels. Unlike the integral case, the theory of primitive forms has only been established in certain cases. We therefore assume that g has algebraic and p-adically bounded Fourier coefficients, and that g_ι is a simultaneous Hecke eigenform. A theorem of Shimura about the existence of a $\overline{\mathbb{Q}}$-form of $\mathcal{M}_l(\mathfrak{c}, \phi)$ and of "primitive" half-integral forms provides evidence towards the existence of "many" forms g satisfying these conditions. For a discussion we refer to the end of chapter 6 where we also give a concrete example for the case $F = \mathbb{Q}$. Finally, we have imposed the class number condition $h_F = 1$. For Hilbert automorphic forms of integral weight, there are h_F components which are permuted (and acted on) by the standard inverter J of integral forms. However, Hilbert automorphic forms of half-integral weight as defined by Shimura in [S7] only have one component. At some point in section 5 we want to connect the inverter J of the integral case with our inverter j of the half-integral case, and we do not know how to achieve this without our class number assumption. Also, there are distributions defined for certain positive critical points. However, the Fourier expansion of g'^+ of (37) then involves the values of certain Hecke L-functions at positive integers. These can be expressed in terms of the values of the L-functions at negative integers by applying the functional equation, but there occur Euler product factors with the ideal character χ^* evaluated at primes $\mathfrak{p} \in S$. It is for this reason that we can not show the boundedness of the distribution associated to the positive values.

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1. Idele characters and distributions.

Let us introduce some notations first. We will always denote by F a totally real algebraic number field of degree $n = [F : \mathbb{Q}]$ over $\mathbb{Q}$ with maximal order $\mathfrak{o}$, different $\mathfrak{d}$ and discriminant d_F, and we write $\alpha \gg 0$ to indicate that the element α of F is totally positive. The ideal group of fractional ideals of F will be denoted by $J = J_F$. The class group J_F/P_F of order $h = h_F$ is then obtained by factorization of J_F after the subgroup P_F of principal ideals with totally positive generator. $\mathbf{A} = F_{\mathbf{A}}$ denotes the ring of adeles, its unit group $I_F = F_{\mathbf{A}}^\times$ is the group of ideles, and we write $\mathbf{A}_f$ and $\mathbf{A}_\infty$ for the finite and archimedean part of the adeles respectively. If x is an adele, then x_0 and x_∞ will denote the finite and archimedean part of x. The norm map of $F/\mathbb{Q}$ (of elements of F, ideals, adeles) will be denoted by $\mathcal{N}$. For an idele a of I_F or an element a of $F^\times$ the associated fractional ideal will be written as $\widetilde{a}$ or $\mathfrak{o} \cdot a$ or (a). For a finite prime $\mathfrak{p}$, $\nu_\mathfrak{p}$ denotes the discrete normalized valuation associated with $\mathfrak{p}$, $F_\mathfrak{p}$ and $\mathfrak{o}_\mathfrak{p}$ denote the completions of F and $\mathfrak{o}$ with respect to $\nu_\mathfrak{p}$, and $\mathfrak{d}_\mathfrak{p}$ denotes the local different. For every (finite or infinite) prime $\mathfrak{p}$ there are the following exponential maps from $F_\mathfrak{p}$ to $\mathbb{C}$: For infinite primes, this is the map $e_\mathfrak{p}(x) = \exp(2\pi i x)$, and for finite primes, it is the map $e_\mathfrak{p}(x) = \exp(-2\pi i y)$ with any $y \in \mathbb{Q}$ such that $y \in \bigcap_{q \neq p} (\mathbb{Q} \cap \mathbb{Z}_q)$ and $y - \mathrm{Tr}_{F_\mathfrak{p}/\mathbb{Q}_p}(x) \in \mathbb{Z}_p$; here p is the rational prime determined by $\mathfrak{p}|p$. These maps are the local components of the maps $e_{\mathbf{A}} : \mathbf{A} \to \mathbb{C}^\times, x \mapsto \prod_\mathfrak{p} e_\mathfrak{p}(x)$ and $e_\infty = \prod_{\mathfrak{p}|\infty} e_\mathfrak{p}$; the last map is also denoted as e_F by some authors. In particular, e_∞ is the archimedean component of $e_{\mathbf{A}}$, and $e_{\mathbf{A}}$ is trivial on $\mathbb{F}^\times$. Similarly we have absolute values $|\cdot|_{\mathbf{A}} : F_{\mathbf{A}} \to \mathbb{C}$ and $|\cdot|_\infty : \mathbb{R}^n \to \mathbb{C}$ defined by $|x|_{\mathbf{A}} := \prod_\mathfrak{p} |x_\mathfrak{p}|_\mathfrak{p}$ and $|x|_\infty := \prod_{\mathfrak{p}|\infty} |x_\mathfrak{p}|_\mathfrak{p}$ with the local maps $|x|_\mathfrak{p} = \mathcal{N}(\mathfrak{p})^{-\nu_\mathfrak{p}(x)}$ for $\mathfrak{p}$ finite and $|x|_\mathfrak{p} = |x|$ for $\mathfrak{p}$ archimedean. The algebraic closure of $\mathbb{Q}$ will be denoted by $\overline{\mathbb{Q}}$. We fix a rational prime p and an embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ of the algebraic closure of $\mathbb{Q}$ into the Tate field $\mathbb{C}_p$ which is the p-adic closure of the algebraic closure of $\mathbb{Q}_p$. We denote the ring of integers of $\mathbb{C}_p$ by $\mathfrak{O}_p$. For a fractional ideal $\mathfrak{m}$, the "support" of $\mathfrak{m}$ is defined as $S(\mathfrak{m}) := \{\mathfrak{p} \mid \nu_\mathfrak{p}(\mathfrak{m}) \neq 0\}$, and we write $\Phi(\mathfrak{n}) := \#(\mathfrak{o}/\mathfrak{n})^\times$ for the Euler function of integral ideals. Finally, μ will denote the Möbius function (of integral ideals) which is non-zero only on squarefree ideals $\mathfrak{a}$ and takes the value $\mu(\mathfrak{a}) = (-1)^r$ if $\mathfrak{a}$ is the product of r different prime ideals.

By a *Hecke character* of finite order, we understand a continuous character $\chi : I_F \to \mathbb{C}^\times$ which has finite image and is trivial on the principal ideles $F^\times \leqslant I_F$. χ can be written as a product $\chi = \prod_\mathfrak{p} \chi_\mathfrak{p}$ of local characters $\chi_\mathfrak{p} : F_\mathfrak{p}^\times \to \mathbb{C}^\times$, where the product is taken over all primes. In particular, the archimedean part χ_∞ is given as $\chi_\infty(x_\infty) = \prod_{\mathfrak{p}|\infty} x_\mathfrak{p}^{r_\mathfrak{p}} =: x_\infty^r$ for some $r = (r_\mathfrak{p}) \in \mathbb{Z}^n$, which is uniquely determined modulo $(2\mathbb{Z})^n$. We will also write χ_f for the finite part $\prod_{\mathfrak{p} \not| \infty} \chi_\mathfrak{p}$. If $\widetilde{\mathfrak{c}} = \mathfrak{c} \cdot \mathfrak{c}_\infty$ is the conductor of χ with finite part $\mathfrak{c} = \mathfrak{c}(\chi)$ and infinite part $\mathfrak{c}_\infty$, then we can associate a (primitive) ideal character $\chi^* : J_F \to \mathbb{C}$ modulo $\mathfrak{c}$ with χ in the following way: It is non-vanishing only for ideals $\mathfrak{a}$ prime to $\mathfrak{c}$, and in this case it is given as $\chi^*(\mathfrak{a}) = \chi(a)$ for any idele $a \in I_F$ satisfying $\widetilde{a} = \mathfrak{a}$, $a_\mathfrak{p} \equiv 1 \bmod \mathfrak{p}^{\nu_\mathfrak{p}(\mathfrak{c})}$ for $\mathfrak{p}|\mathfrak{c}$, and $a_\mathfrak{p} > 0$ for $\mathfrak{p}|\mathfrak{c}_\infty$. For any ideal $\mathfrak{q}$ with $\mathfrak{c}(\chi)|\mathfrak{q}$ we let $\chi_\mathfrak{q}^* : J \to \mathbb{C}$ be the ideal character which coincides with χ^* on the ideals prime to $\mathfrak{q}$ and takes the value 0 otherwise. We also let $\chi_\mathfrak{q} = \prod_{\mathfrak{p}|\mathfrak{q}} \chi_\mathfrak{p}$ be the $\mathfrak{q}$-part of the idele character χ and write $\chi_0 := \chi_{\mathfrak{c}(\chi)}$. If $\alpha \in F^\times$ then we write ε_α for the quadratic Hecke character corresponding to the quadratic extension $F(\sqrt{\alpha})/F$ by class field theory. In other words, the kernel of ε_α is the norm group $\mathcal{N}_{F(\sqrt{\alpha})/F}(I_{F(\sqrt{\alpha})})$. For the trivial character ε_1, we will also write ε.

For a finite prime $\mathfrak{p}$ and a non-negative integer n we denote the group of higher principal units by $U_\mathfrak{p}^{(n)} = \{x \in \mathfrak{o}_\mathfrak{p}^\times \,|\, x \equiv 1 \ (\mathfrak{p}^n)\}$, and set $U_\mathfrak{p} := U_\mathfrak{p}^{(0)} = \mathfrak{o}_\mathfrak{p}^\times$. If χ is a Hecke character of finite order of conductor $\mathfrak{c}$ and $\mathfrak{p}$ is a finite prime, then the local character $\chi_\mathfrak{p}$ has conductor $\mathfrak{c}_\mathfrak{p} = \mathfrak{p}^{\nu_\mathfrak{p}(\mathfrak{c})}$, and the (local) Gauss sum of $\chi_\mathfrak{p}$ is defined as

$$\tau_\mathfrak{p}(\chi_\mathfrak{p}) := \sum_a \overline{\chi}_\mathfrak{p}\left(\frac{a}{d_\mathfrak{p}}\right) e_\mathfrak{p}\left(-\frac{a}{d_\mathfrak{p}}\right),$$

where $d_\mathfrak{p} \in F_\mathfrak{p}^\times$ is a generator of the ideal $\mathfrak{c}_\mathfrak{p}\mathfrak{d}_\mathfrak{p}$ and a runs through a system of representatives of $U_\mathfrak{p}/U_\mathfrak{p}^{\nu_\mathfrak{p}(\mathfrak{c})}$. If $\mathfrak{q} \subseteq \mathfrak{c}$ is any ideal we define the global Gauss sum of the ideal character $\chi_\mathfrak{q}^*$ as

$$\tau(\chi_\mathfrak{q}^*) := \sum_{a \in (\mathfrak{d}\mathfrak{q})^{-1}/\mathfrak{d}^{-1}} \chi_\infty(a) \chi_\mathfrak{q}^*(a\mathfrak{d}\mathfrak{q}) e_\infty(a), \tag{1}$$

where the summation is understood as a running over a system of representatives of $(\mathfrak{d}\mathfrak{q})^{-1}/\mathfrak{d}^{-1}$ which does not include 0. The Gauss sum $\tau(\chi)$ of χ is then defined as the Gauss sum corresponding to the primitive ideal character χ^*:

$$\tau(\chi) := \tau(\chi^*) = \tau(\chi_\mathfrak{c}^*),$$

and it satisfies

$$\tau(\chi) = \prod_{\mathfrak{p} \nmid \infty} \tau_\mathfrak{p}(\chi_\mathfrak{p}) \quad \text{and} \quad |\tau(\chi)| = \sqrt{\mathcal{N}(\mathfrak{c})}. \tag{2}$$

Notice that for almost all finite primes $\mathfrak{p}$ the local character $\chi_\mathfrak{p}$ is unramified and $\mathfrak{d}_\mathfrak{p} = \mathfrak{o}_\mathfrak{p}$, and hence the above infinite product only has a finite number of factors which are not equal to 1. We give a proof of (2) in a more general situation below.

We will need to deal with character sums of imprimitive characters and therefore introduce the following concept:

DEFINITION 1.1. — *An ideal character* $\chi_{\mathfrak{q}}^*$ *modulo* $\mathfrak{q}$ *of conductor* $\mathfrak{c}$ *is called almost primitive if* $\gcd\left(\mathfrak{c}, \frac{\mathfrak{q}}{\mathfrak{c}}\right) = \mathfrak{o}$ *and* $\frac{\mathfrak{q}}{\mathfrak{c}}$ *is squarefree.*

Remark. — In general, for an imprimitive character $\chi_{\mathfrak{q}}^*$, the Gauss sum $\tau(\chi_{\mathfrak{q}}^*)$ may vanish. However, as shown below, for almost primitive characters this Gauss sum never vanishes. Moreover, almost primitive characters naturally occur when multiplying primitive characters with characters of prime conductor, and the set of almost primitive characters is stable under this operation.

We will now establish an analogue to the character sum (3.11) of [S3] for almost primitive characters. This will be needed later on to define the twist of a half-integral modular function with an almost primitive character.

LEMMA 1.2. — *Let* $\mathfrak{a}$ *be a fractional ideal,* $\chi_{\mathfrak{q}}^*$ *an almost primitive character of conductor* $\mathfrak{c}$, *and* $b \in \mathfrak{a}^{-1}\mathfrak{q}^{-1}\mathfrak{d}^{-1}$.

Then

$$\sum_{x\in\mathfrak{a}/\mathfrak{a}\mathfrak{q}} \chi_\infty(x)\chi_{\mathfrak{q}}^*(x\mathfrak{a}^{-1})e_\infty(bx)$$

$$= \begin{cases} \chi_\infty(b)\overline{\chi}^*(b\mathfrak{a}\mathfrak{c}\mathfrak{d})\mu(\frac{\mathfrak{q}}{\mathfrak{c}})\tau(\chi)\prod_{\mathfrak{p}|\frac{\mathfrak{q}}{\mathfrak{c}}}(1-\mathcal{N}(\mathfrak{p}))^{\min(1,\nu_{\mathfrak{p}}(b\mathfrak{a}\mathfrak{q}\mathfrak{d}))}, & \text{if } b \neq 0,\\ \delta_{\chi,\varepsilon}\mu(\mathfrak{q})\prod_{\mathfrak{p}|\mathfrak{q}}(1-\mathcal{N}(\mathfrak{p})), & \text{if } b = 0 \end{cases}$$

with the Kronecker symbol δ.

Proof. — Choose an idele a such that $\widetilde{a} = \mathfrak{a}$, and for $\mathfrak{p}|\mathfrak{q}$ let $\{\beta_j^{(\mathfrak{p})}\} \subseteq U_{\mathfrak{p}}$ be a system of representatives for $U_{\mathfrak{p}}/U_{\mathfrak{p}}^{(\nu_{\mathfrak{p}}(\mathfrak{q}))}$. If $\gcd(x\mathfrak{a}^{-1},\mathfrak{q}) = \mathfrak{o}$, then write x considered as element of the idele group I_F as

$$x = a \cdot \prod_{\mathfrak{p}|\mathfrak{q}} \iota_{\mathfrak{p}}(\beta_{j(x)}^{(\mathfrak{p})}) \cdot \beta_x;$$

here $\iota_{\mathfrak{p}}$ denotes the embedding $F_{\mathfrak{p}} \hookrightarrow \mathbf{A}_F$ and $\beta_x \in I_F$ is an idele that satisfies $\widetilde{\beta}_x \subseteq \mathfrak{o}$ and $\beta_{x\mathfrak{p}} \equiv 1 \bmod \mathfrak{p}^{\nu_{\mathfrak{p}}(\mathfrak{q})}$ for all $\mathfrak{p}|\mathfrak{q}$. Notice that the map

$$\mathfrak{a}/(\mathfrak{a}\mathfrak{q}) \to \prod_{\mathfrak{p}|\mathfrak{q}} \mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}^{\nu_{\mathfrak{p}}(\mathfrak{q})}, \quad x + \mathfrak{a}\mathfrak{q} \mapsto (xa_{\mathfrak{p}}^{-1} + \mathfrak{p}^{\nu_{\mathfrak{p}}(\mathfrak{q})})_{\mathfrak{p}|\mathfrak{q}}$$

is a bijection, and that the image of $x + \mathfrak{a}\mathfrak{q}$ is a unit if and only if $x\mathfrak{a}^{-1}$ is prime to $\mathfrak{q}$. Therefore, as $x + \mathfrak{a}\mathfrak{q}$ runs over those representatives of

$\mathfrak{a}/\mathfrak{a}\mathfrak{q}$ for which $\chi_{\mathfrak{q}}^*(x\mathfrak{a}^{-1})$ does not vanish, the tupel $(\beta_{j(x)}^{(\mathfrak{p})})_{\mathfrak{p}|\mathfrak{q}}$ runs over $\prod_{\mathfrak{p}|\mathfrak{q}} U_{\mathfrak{p}}/U_{\mathfrak{p}}^{(\nu_{\mathfrak{p}}(\mathfrak{q}))}$. Making use of

$$\begin{aligned}\chi_\infty(x)\chi_{\mathfrak{q}}^*(x\mathfrak{a}^{-1}) &= \chi_\infty(x) \prod_{\mathfrak{p}\nmid\mathfrak{q},\mathfrak{p}\nmid\infty} \chi_{\mathfrak{p}}(xa_{\mathfrak{p}}^{-1}) \\ &= \prod_{\mathfrak{p}\nmid\mathfrak{q},\mathfrak{p}\nmid\infty} \overline{\chi}_{\mathfrak{p}}(a_{\mathfrak{p}}) \cdot \prod_{\mathfrak{p}|\mathfrak{q}} \overline{\chi}_{\mathfrak{p}}(a_{\mathfrak{p}})\overline{\chi}_{\mathfrak{p}}(\beta_{j(x)}^{(\mathfrak{p})})\end{aligned}$$

and

$$e_\infty(bx) = \prod_{\mathfrak{p}|\infty} e_{\mathfrak{p}}(bx) = \prod_{\mathfrak{p}\nmid\infty} e_{\mathfrak{p}}(-bx) = \prod_{\mathfrak{p}|\mathfrak{q}} e_{\mathfrak{p}}(-ba_{\mathfrak{p}}\beta_{j(x)}^{(\mathfrak{p})}),$$

the character sum of the lemma now evaluates to

$$\prod_{\mathfrak{p}\nmid\mathfrak{q},\mathfrak{p}\nmid\infty} \overline{\chi}_{\mathfrak{p}}(a_{\mathfrak{p}}) \cdot \prod_{\mathfrak{p}|\mathfrak{q}} \overline{\chi}_{\mathfrak{p}}(a_{\mathfrak{p}}) \sum_{u\in U_{\mathfrak{p}}/U_{\mathfrak{p}}^{(\nu_{\mathfrak{p}}(\mathfrak{q}))}} \overline{\chi}_{\mathfrak{p}}(u)e_{\mathfrak{p}}(-ba_{\mathfrak{p}}u).$$

For $\mathfrak{p}|\frac{\mathfrak{q}}{\mathfrak{c}}$ the local character sum in this expression takes the value

$$\begin{aligned}\sum_{u\in U_{\mathfrak{p}}/U_{\mathfrak{p}}^{(1)}} \overline{\chi}_{\mathfrak{p}}(u)e_{\mathfrak{p}}(-ba_{\mathfrak{p}}u) &= \sum_{u\in U_{\mathfrak{p}}/U_{\mathfrak{p}}^{(1)}} e_{\mathfrak{p}}(-ba_{\mathfrak{p}}u) \\ &= \begin{cases} \mathcal{N}(\mathfrak{p})-1, & \text{if } b\in \mathfrak{a}^{-1}\mathfrak{q}^{-1}\mathfrak{d}^{-1}\mathfrak{p}, \\ -1, & \text{otherwise.}\end{cases}\end{aligned}$$

For $\mathfrak{p}|\mathfrak{c}$ the local character sum takes the value

$$\sum_{u\in U_{\mathfrak{p}}/U_{\mathfrak{p}}^{(\nu_{\mathfrak{p}}(\mathfrak{c}))}} \overline{\chi}_{\mathfrak{p}}(u)e_{\mathfrak{p}}(-ba_{\mathfrak{p}}u) = \tau_{\mathfrak{p}}(\chi_{\mathfrak{p}}) \cdot \begin{cases} \chi_{\mathfrak{p}}(ba_{\mathfrak{p}}), & \text{if } \nu_{\mathfrak{p}}(b\mathfrak{a}\mathfrak{c}\mathfrak{d}) = 0, \\ 0, & \text{if } b=0 \text{ or } \nu_{\mathfrak{p}}(b\mathfrak{a}\mathfrak{c}\mathfrak{d}) > 0.\end{cases}$$

This follows directly from the definition of the local Gauss sum if $\nu_{\mathfrak{p}}(b\mathfrak{a}\mathfrak{c}\mathfrak{d}) = 0$. In the other case there exists some $u' \in U_{\mathfrak{p}}^{(\nu_{\mathfrak{p}}(\mathfrak{c})-1)} \backslash U_{\mathfrak{p}}^{(\nu_{\mathfrak{p}}(\mathfrak{c}))}$ such that $\chi_{\mathfrak{p}}(u') \neq 1$ because the conductor of $\chi_{\mathfrak{p}}$ is $\mathfrak{p}^{\nu_{\mathfrak{p}}(\mathfrak{c})}$. But $e_{\mathfrak{p}}(-ba_{\mathfrak{p}}uu') = e_{\mathfrak{p}}(-ba_{\mathfrak{p}}u)$ for all $u \in U_{\mathfrak{p}}$, and therefore we obtain

$$\begin{aligned}\sum_{u\in U_{\mathfrak{p}}/U_{\mathfrak{p}}^{(\nu_{\mathfrak{p}}(\mathfrak{q}))}} \overline{\chi}_{\mathfrak{p}}(u)e_{\mathfrak{p}}(-ba_{\mathfrak{p}}u) &= \sum_{u\in U_{\mathfrak{p}}/U_{\mathfrak{p}}^{(\nu_{\mathfrak{p}}(\mathfrak{q}))}} \overline{\chi}_{\mathfrak{p}}(uu')e_{\mathfrak{p}}(-ba_{\mathfrak{p}}uu') \\ &= \overline{\chi}_{\mathfrak{p}}(u') \sum_{u\in U_{\mathfrak{p}}/U_{\mathfrak{p}}^{(\nu_{\mathfrak{p}}(\mathfrak{q}))}} \overline{\chi}_{\mathfrak{p}}(u)e_{\mathfrak{p}}(-ba_{\mathfrak{p}}u)\end{aligned}$$

which implies the vanishing of the local character sum as claimed. Now, for $\mathfrak{p}\nmid\mathfrak{c}$ choose a generator $d_{\mathfrak{p}} \in F_{\mathfrak{p}}^\times$ of the local different $\mathfrak{d}_{\mathfrak{p}}$ and observe that the local Gauss sum is then by definition $\tau_\mathfrak{p}(\chi_\mathfrak{p}) = \chi_\mathfrak{p}(d_\mathfrak{p})$. The lemma is now proved as follows:

First assume $b = 0$. If $\chi \neq \varepsilon$ then the character sum equals 0 and otherwise its value is

$$\prod_{\mathfrak{p}|\mathfrak{q}} \{(-1)(1-\mathcal{N}(\mathfrak{p})\} = \mu(\mathfrak{q}) \prod_{\mathfrak{p}|\mathfrak{q}} (1-\mathcal{N}(\mathfrak{p})).$$

Now assume $b \neq 0$. If $\nu_\mathfrak{p}(b\mathfrak{a}\mathfrak{c}\mathfrak{d}) > 0$ for some $\mathfrak{p}|\mathfrak{c}$ then the character sum of the lemma equals 0, and otherwise we obtain as its value

$$\prod_{\mathfrak{p}\nmid\mathfrak{q},\mathfrak{p}\nmid\infty} \overline{\chi}_\mathfrak{p}(a_\mathfrak{p}) \prod_{\mathfrak{p}|\frac{\mathfrak{q}}{\mathfrak{c}}} \{\overline{\chi}_\mathfrak{p}(a_\mathfrak{p}) \cdot (-1) \cdot (1-\mathcal{N}(\mathfrak{p}))^{\min(1,\nu_\mathfrak{p}(b\mathfrak{a}\mathfrak{q}\mathfrak{d}))}\} \prod_{\mathfrak{p}|\mathfrak{c}} \chi_\mathfrak{p}(b)\tau_\mathfrak{p}(\chi_\mathfrak{p})$$
$$= \prod_{\mathfrak{p}\nmid\mathfrak{c},\mathfrak{p}\nmid\infty} \overline{\chi}_\mathfrak{p}(abd_\mathfrak{p}) \cdot \chi_\infty(b) \cdot \mu\Big(\frac{\mathfrak{q}}{\mathfrak{c}}\Big) \cdot \prod_{\mathfrak{p}\nmid\infty} \tau_\mathfrak{p}(\chi_\mathfrak{p}) \cdot \prod_{\mathfrak{p}|\frac{\mathfrak{q}}{\mathfrak{c}}} (1-\mathcal{N}(p))^{\min(1,\nu_\mathfrak{p}(b\mathfrak{a}\mathfrak{q}\mathfrak{d}))}.$$

In all cases this is is the value given in the lemma. □

The proof of the lemma for $\mathfrak{a} = (\mathfrak{d}\mathfrak{c})^{-1}$, $\mathfrak{q} = \mathfrak{c}$ and $b = 1$ actually proves the product expression (2) for the global Gauss sum given above. If we choose $\mathfrak{a} = (\mathfrak{q}\mathfrak{d})^{-1}$ and $b = 1$, then the lemma shows that

$$\tau(\chi_\mathfrak{q}^*) = \mu\left(\frac{\mathfrak{q}}{\mathfrak{c}}\right) \chi^*\left(\frac{\mathfrak{q}}{\mathfrak{c}}\right) \tau(\chi).$$

In particular, the non-vanishing of $\tau(\chi)$ implies the non-vanishing of $\tau(\chi_\mathfrak{q}^*)$. The following lemma is a simple rephrase of Lemma 1.2 more suited to our intended application.

LEMMA 1.3. — *Let $\mathfrak{a}$ be a fractional ideal, $\chi_\mathfrak{q}^*$ an almost primitive character of conductor $\mathfrak{c}$, and $t \in F_\mathbf{A}^\times$ an idele with $\tilde{t} \subseteq \mathfrak{a}^{-1}\mathfrak{q}^{-1}\mathfrak{d}^{-1}$. Then*

$$\sum_{x \in \mathfrak{a}/\mathfrak{a}\mathfrak{q}} \chi_\infty(x)\chi_\mathfrak{q}^*(x\mathfrak{a}^{-1}) e_\mathbf{A}(-tx_0)$$
$$= \tau(\chi)\overline{\chi}^*(t\mathfrak{a}\mathfrak{c}\mathfrak{d})\chi_f(t)\mu\left(\frac{\mathfrak{q}}{\mathfrak{c}}\right) \prod_{\mathfrak{p}|\frac{\mathfrak{q}}{\mathfrak{c}}} (1-\mathcal{N}(\mathfrak{p}))^{\min(1,\nu_\mathfrak{p}(t\mathfrak{a}\mathfrak{q}\mathfrak{d}))}.$$

For a prime ideal $\mathfrak{p} \nmid 2$ the next lemma will show the existence of a certain quadratic character $\chi^\mathfrak{p}$, which is a generalization of the character $n\mathbb{Z} \mapsto \left(\frac{|n|}{p}\right)$ of the ideals of $\mathbb{Z}$.

LEMMA 1.4. — *Let F have class number $h_F = 1$, and let $\mathfrak{p}$ be a prime ideal which does not divide 2. Then there exists a unique quadratic* *extension* $F^{\mathfrak{p}} = F(\sqrt{\tau})$ *of* F *which is only ramified in* $\mathfrak{p}$ *and possibly at infinite primes. The corresponding quadratic idele character of* F

$$\chi^{\mathfrak{p}} : I_F/F^{\times} \to \{\pm 1\} \quad \textit{determined by} \quad \ker(\chi^{\mathfrak{p}}) = \mathcal{N}_{F^{\mathfrak{p}}/F}(I_{F^{\mathfrak{p}}}/F^{\mathfrak{p}\times})$$

has the local components

$$\chi^{\mathfrak{p}}_{\mathfrak{q}} : F^{\times}_{\mathfrak{q}} \to \{\pm 1\}, \quad \chi^{\mathfrak{p}}_{\mathfrak{q}}(x) = \left(\frac{x, \tau}{\mathfrak{q}}\right)$$

with the local Hilbert symbol $\left(\frac{\cdot,\cdot}{\mathfrak{q}}\right) : F^{\times}_{\mathfrak{q}} \times F^{\times}_{\mathfrak{q}} \to \{\pm 1\}$ *(see [N], chapter V §3 for the definition).*

Proof. — Let $I^{S_\infty} := \prod_{\mathfrak{q} \nmid \infty} \mathfrak{o}^{\times}_{\mathfrak{q}} \cdot \prod_{\mathfrak{q} | \infty} \mathbb{R}$. The idele class group $C_F := I_F/F^{\times}$ equals $I^{S_\infty}F^{\times}/F^{\times}$ by our assumption $h_F = 1$ (cf. [N], Satz VI.1.3). With the subgroup $\mathfrak{u}_{\mathfrak{p}} \leqslant \mathfrak{o}^{\times}_{\mathfrak{p}}$ of quadratic residues modulo $\mathfrak{p}$ let $C_{\mathfrak{p}} := \prod_{\mathfrak{q} \neq \mathfrak{p}} \mathfrak{o}^{\times}_{\mathfrak{q}} \cdot \mathfrak{u}_{\mathfrak{p}} \cdot \prod_{\mathfrak{q}|\infty} \mathbb{R}_+$. $C_{\mathfrak{p}}F^{\times}/F^{\times}$ is a norm subgroup of C_F, and the quotient is

$$C_F/(C_{\mathfrak{p}}F^{\times}/F^{\times}) \cong I^{S_\infty}F^{\times}/C_{\mathfrak{p}}F^{\times} \cong I^{S_\infty}/(C_{\mathfrak{p}}F^{\times} \cap I^{S_\infty}) = I^{S_\infty}/C_{\mathfrak{p}}\mathfrak{o}^{\times}.$$

Now $I^{S_\infty}/C_{\mathfrak{p}} \cong \mathfrak{o}^{\times}_{\mathfrak{p}}/\mathfrak{u}_{\mathfrak{p}} \cdot \prod_{\mathfrak{q}|\infty} \mathbb{R}/\mathbb{R}_+ \cong (\mathbb{Z}/2\mathbb{Z})^{n+1}$, ${\mathfrak{o}^{\times}}^2 \leqslant C_{\mathfrak{p}}$, and $\mathfrak{o}^{\times}/{\mathfrak{o}^{\times}}^2 \cong (\mathbb{Z}/2\mathbb{Z})^n$ by Dirichlet's unit theorem. This implies that $C_F/(C_{\mathfrak{p}}F^{\times}/F^{\times})$ is 2-elementary abelian and nontrivial. The existence theorem of class field theory now implies the existence of $F^{\mathfrak{p}}$. Since there are no quadratic extensions of F that are unramified at all finite primes (by our assumption $h_F = 1$), $F^{\mathfrak{p}}$ must be ramified at $\mathfrak{p}$ and possibly some infinite primes. If $F^{\mathfrak{p}'} \neq F^{\mathfrak{p}}$ is another such extension, then the third quadratic subextension of $F^{\mathfrak{p}}F^{\mathfrak{p}'}$ over F is unramified, a contradiction which proves the uniqueness of $F^{\mathfrak{p}}$. Finally, the quadratic character $\prod_{\mathfrak{q}} \chi^{\mathfrak{p}}_{\mathfrak{q}}$ is a Hecke character by the product formula of the Hilbert symbol ([N], Satz VI.8.1), and for finite primes $\mathfrak{q} \neq \mathfrak{p}$ the extension $F_{\mathfrak{q}}(\sqrt{\tau})/F_{\mathfrak{q}}$ and hence the local character $\chi^{\mathfrak{p}}_{\mathfrak{q}}$ are unramified. By the uniqueness of $F^{\mathfrak{p}}$ the local characters $\chi^{\mathfrak{p}}_{\mathfrak{q}}$ must be the components of $\chi^{\mathfrak{p}}$. □

We will now recall the notions of p-adic distributions and p-adic measures. The reader is referred to [P], I§3 and IV§4 or [K], 4.0 for details. Let $Y = \varprojlim Y_i$ be a profinite (i.e., compact and totally disconnected) topological space, and R a ring. Denote by $\mathrm{Step}(Y, R)$ the R-module of all locally constant R-valued functions on Y. An R-valued distribution on Y is an R-linear map

$$\mu : \ \mathrm{Step}(Y, R) \to R.$$

If $R_0 \leqslant R$ is a subring of R and μ is an R-valued distribution, then we say it is *defined over* R_0 if $\mu(\delta_{y\,\mathrm{pr}_i^{-1}(\mathrm{Y}_i)}) \in R_0$ for every Y_i, $y \in Y$ and characteristic function $\delta_{y\,\mathrm{pr}_i^{-1}(\mathrm{Y}_i)}$ of the set $y\,\mathrm{pr}_i^{-1}(\mathrm{Y}_i) \subseteq \mathrm{Y}$. Now, let R be a closed subring of the Tate field $\mathbb{C}_p$. Denote by $\mathcal{C}(Y, R)$ the R-module of all continuous R-valued functions on Y. An R-valued p-adic measure on Y is an R-linear continuous map

$$\mu : \mathcal{C}(Y, R) \to R, \quad \text{written symbolically as} \quad \int_Y f\, d\mu.$$

Given an R-valued measure μ on Y and a function $g \in \mathcal{C}(Y, R)$, the product $g\mu$ is the R-valued measure defined by

$$\int_Y f\, d(g\mu) := \int_Y fg\, d\mu \qquad \text{for } f \in \mathcal{C}(Y, R).$$

Given an R-valued p-adic measure μ on Y and a continuous map $\varphi : Y \to Y$, we also have an R-valued measure $\mu \circ \varphi$ on Y defined by

$$\int_Y f\, d\mu \circ \varphi := \int_Y f \circ \varphi\, d\mu \qquad \text{for } f \in \mathcal{C}(Y, R).$$

The restriction of an R-valued measure μ to the subalgebra $\mathrm{Step}(Y, R) \subseteq \mathcal{C}(Y, R)$ defines an R-valued distribution which we denote by the same letter μ. Moreover, the measure μ is uniquely determined by the corresponding distribution because of the density of $\mathrm{Step}(Y, R)$ in $\mathcal{C}(Y, R)$.

Now, let $R = \mathbb{C}_p$. Then a $\mathbb{C}_p$-valued distribution μ on Y can be extended to a $\mathbb{C}_p$-valued measure on Y if and only if μ is bounded on $\mathrm{Step}(Y, \mathfrak{O}_p)$, i.e. if there exists some constant $C \in \mathbb{R}_+$ such that $|\mu(f)|_p \leqslant C$ for all $f \in \mathrm{Step}(Y, \mathfrak{O}_p)$. This implies that every $\mathbb{C}_p$-valued measure on Y becomes an $\mathfrak{O}_p$-valued measure on Y after multiplication with some non-zero constant of $\mathbb{C}_p^\times$. The following proposition gives an important criterion for the existence of a measure with given properties:

PROPOSITION 1.5 (Abstract Kummer congruences). — *Let $\{f_i\}_{i\in I}$ be a collection of elements of $\mathcal{C}(Y, \mathfrak{O}_p)$ such that the $\mathbb{C}_p$-linear span of $\{f_i\}$ is dense in $\mathcal{C}(Y, \mathbb{C}_p)$, and let $\{a_i\}_{i\in I}$ be any system of elements $a_i \in \mathfrak{O}_p$. Then there exists an $\mathfrak{O}_p$-valued measure μ on Y with the property*

$$\int_Y f_i\, d\mu = a_i \quad \text{for all } i \in I$$

if and only if the a_i satisfy the following "Kummer congruences": For every collection $\{b_i\}_{i\in I}$ of elements of $\mathbb{C}_p$ which are zero for all but finitely many i,

$$\sum b_i f_i(y) \in p^n\mathfrak{O}_p \quad \textit{for all } y \in Y \textit{ implies} \quad \sum b_i a_i \in p^n\mathfrak{O}_p.$$

We will apply this criterion to the following situation: p is a fixed prime number and S a finite set of primes of F containing all primes above p. Let $Y = \mathrm{Gal}_S = \mathrm{Gal}(F(S)/F)$ be the Galois group of the maximal abelian extension $F(S)$ of F which is unramified outside S and ∞. Then by class field theory,

$$\mathrm{Gal}_S = \varprojlim_{\mathfrak{m}} J(\mathfrak{m})/P(\mathfrak{m}).$$

Here $\mathfrak{m}$ runs over all ideals of F with support in S, $J(\mathfrak{m}) = \{\mathfrak{a} \in J \mid S(\mathfrak{a}) \cap S(\mathfrak{m}) = \varnothing\}$, and $P(\mathfrak{m})$ is the subgroup $\{(\alpha) \mid \alpha \in F,\ \alpha \equiv 1 \bmod \mathfrak{m}, \alpha \gg 0\}$. Let

$$X_S = \mathrm{Hom}_c(\mathrm{Gal}_S, \mathbb{C}_p^\times)$$

be the p-adic analytic Lie group of all $\mathbb{C}_p^\times$-valued continuous characters of the Galois group Gal_S. The elements $\chi \in X_S$ of finite order can be identified with those Hecke characters of finite order whose conductors $\mathfrak{c}(\chi)$ are only divisible by primes in S; if $\mathfrak{c}(\chi)$ divides $\mathfrak{m}$, then this identification is induced from

$$\chi_{\mathfrak{m}} : J(\mathfrak{m})/P(\mathfrak{m}) \to \mathbb{C}^\times, \quad \mathfrak{a}\cdot P(\mathfrak{m}) \mapsto \chi^*(\mathfrak{a}).$$

The maximal abelian extension $\mathbb{Q}(p)/\mathbb{Q}$ unramified outside p and ∞ is a subfield of $F(S)$ because S contains all primes dividing p. The restriction of Galois automorphisms to $\mathbb{Q}(p)$ determines a natural homomorphism

$$\mathcal{N} : \mathrm{Gal}_S \to \mathrm{Gal}(\mathbb{Q}(p)/\mathbb{Q}) \cong \mathbb{Z}_p^\times,$$

and we shall denote by $\mathcal{N}_p$ the composition of this homomorphism with the inclusion $\mathbb{Z}_p^\times \hookrightarrow \mathbb{C}_p^\times$. Then $\mathcal{N}_p$ is an element of X_S, and $\mathcal{N}_p$ maps the image in Gal_S of an ideal prime to S to its norm. For a fixed $r \in \mathbb{Z}$ the $\mathbb{C}_p$-linear span of the collection $\{\chi\mathcal{N}_p^r \mid \chi \in X_S^{\mathrm{tor}}\}$ is dense in $\mathcal{C}(\mathrm{Gal}_S, \mathbb{C}_p)$ because $\mathrm{Step}(\mathrm{Gal}_S, \mathbb{C}_p)$ has this property, and the $\mathbb{C}_p$-span of the characters of finite order coincides with $\mathrm{Step}(\mathrm{Gal}_S, \mathbb{C}_p)$ by the character relations. Provided that for suitable a_χ the Kummer congruences are satisfied, we obtain a bounded $\mathbb{C}_p$-measure μ which determines in turn a $\mathbb{C}_p$-analytic function, the "p-adic L-function"

$$L_\mu : X_S \to \mathbb{C}_p, \quad L_\mu(x) := \mu(x) = \int_{\mathrm{Gal}_S} x\, d\mu$$

as its non-archimedean Mellin transform. In particular, L_μ takes the value a_χ at the character $\chi\mathcal{N}_p^r$.

An example for the Kummer congruences is given by the values at negative integers of Dirichlet series attached to Hecke characters of finite order. Recall that for $\chi \in X_S^{\mathrm{tor}}$ and an ideal $\mathfrak{m}$ the Dirichlet series

$$L_{\mathfrak{m}}(s,\chi) := \sum_{\mathfrak{n}} \chi^*(\mathfrak{n})\mathcal{N}(\mathfrak{n})^{-s} \tag{3}$$

defines a holomorphic function for $\mathrm{Re}(s) > 1$; here $\mathfrak{n}$ runs over all integral ideals prime to $\mathfrak{m}$. If $\mathfrak{m} = \mathfrak{o}$ we also write $L(s,\chi)$ for $L_{\mathfrak{o}}(s,\chi)$. $L_{\mathfrak{m}}$ can be continued to a meromorphic function with at most a simple pole at $s = 1$, and its values at non-positive integers are algebraic and lie in $\mathbb{Q}(\chi)$. There is the following well-known functional equation for $L(s,\chi)$, which is for example given in [N, Satz VII.8.6] in a slightly different formulation:

Let r_0 be the number of archimedean places at which χ ramifies, write $\mathfrak{c}(\chi)$ for the finite part of the conductor of χ, define the Artin root number $W(\chi) := \dfrac{\tau(\chi)}{i^{r_0}\sqrt{\mathcal{N}(\mathfrak{c}(\chi))}}$, and put

$$\mathcal{L}(s,\chi) := \pi^{-\frac{1}{2}(r_0+ns)} d_F^{\frac{s}{2}} \mathcal{N}(\mathfrak{c}(\chi))^{\frac{s}{2}} \Gamma\left(\frac{s+1}{2}\right)^{r_0} \Gamma\left(\frac{s}{2}\right)^{n-r_0} L(s,\chi).$$

Then the functional equation for $L(s,\chi)$ formulated in terms of $\mathcal{L}(s,\chi)$ is

$$\mathcal{L}(1-s,\chi) = W(\chi)\mathcal{L}(s,\overline{\chi}). \tag{4}$$

THEOREM 1.6 (Deligne-Ribet [DR]). — *Let p be a prime, $S \supseteq \{\mathfrak{p}|p\}$ a finite set of finite primes of F, $\mathfrak{m}_0 = \prod_{\mathfrak{p}\in S} \mathfrak{p}$, ω a Hecke character of finite order, $\mathfrak{a}$ an integral ideal prime to $\mathfrak{m}_0$ with $\mathfrak{c}(\omega)|\mathfrak{a}$, and $\mathfrak{q}$ an integral ideal prime to $\mathfrak{m}_0\mathfrak{a}$. Then there exists an $\mathfrak{O}_p$-valued measure $\mu = \mu(\mathfrak{q},\omega,S)$ on Gal_S which is uniquely determined by*

$$i_p^{-1}\left(\int_{\mathrm{Gal}_S} \chi\mathcal{N}_p^r \, d\mu\right) = (1 - (\chi\overline{\omega})^*(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{r+1}) L_{\mathfrak{a}\mathfrak{m}_0}(-r,\chi\overline{\omega})$$

for $\chi \in X_S^{\mathrm{tor}}$ and $r = 0$, and it satisfies the above equality for all non-positive $r \in \mathbb{Z}$.

Proof. — For $\omega = 1$ this follows immediately from the main theorem of [DR] in the form of Theorem 0.4 by imitating the proof of Theorem 1.9b).

The measure $\mu(\mathfrak{q},\omega,S)$ is then obtained by applying this to the set of primes $\overline{S} = S \cup S(\mathfrak{a})$ and the character $\chi\overline{\omega}$:

$$\int_{\mathrm{Gal}_S} \chi\mathcal{N}_p^r \, d\mu(\mathfrak{q},\omega,S) := \int_{\mathrm{Gal}_{\overline{S}}} \chi\overline{\omega}\mathcal{N}_p^r \, d\mu(\mathfrak{q},1,\overline{S})$$
$$= i_p\left((1-(\chi\overline{\omega})^*(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{r+1})L_{\mathfrak{m}_0\mathfrak{a}}(-r,\chi\overline{\omega})\right)$$

for $\chi \in X_S^{\mathrm{tor}}$ and $r \in \mathbb{Z}$ non-positive. □

2. Hilbert modular forms of integral weight.

In this section we want to recall the basic facts on Hilbert modular forms of integral weight. We refer to [S4] or to chapter IV of [P] for details and proofs.

Let us view the group $\mathrm{GL}_2(F)$ as the group $\widetilde{G}(\mathbb{Q})$ of $\mathbb{Q}$-rational points of a $\mathbb{Q}$-rational algebraic subgroup $\widetilde{G}$ of $\mathrm{GL}_{2n}(\mathbb{Q})$. Then the adelization $\widetilde{G}_{\mathbf{A}}$ of $\widetilde{G}$ can be identified with $\mathrm{GL}_2(F_{\mathbf{A}})$. Now, if $\tau_1,\ldots,\tau_n$ are all injections of F into $\mathbb{R}$, fix the embedding of F into $\mathbb{R}^n$ given by $a \mapsto (a^{\tau_1},\ldots,a^{\tau_n})$. This identifies $\mathbb{R}^n$ and $\mathrm{GL}_2(\mathbb{R})^n$ with the archimedean part of $F_{\mathbf{A}}$ and $\widetilde{G}$, which we denote by F_∞ and $\widetilde{G}_\infty$ respectively. The finite parts of $F_{\mathbf{A}}$ and $\widetilde{G}$ will be denoted by F_f and $\widetilde{G}_f$. With $\mathrm{GL}_2^+(\mathbb{R}) = \{\alpha \in \mathrm{GL}_2(\mathbb{R}) \mid \det(\alpha) > 0\}$ put

$$\widetilde{G}_\infty^+ = \mathrm{GL}_2^+(\mathbb{R})^n, \quad \widetilde{G}_{\mathbf{A}}^+ = \{x \in \widetilde{G}_{\mathbf{A}} \mid x_\infty \in \widetilde{G}_\infty^+\},$$
$$\text{and } \widetilde{G}^+ = \widetilde{G}^+(\mathbb{Q}) = \widetilde{G}(\mathbb{Q}) \cap \widetilde{G}_{\mathbf{A}}^+.$$

Next, we let $\mathbf{1} = (1,\ldots,1) \in \mathbb{Z}^n$. For $k = (k_1,\ldots,k_n) \in \mathbb{Z}^n$ and $z = (z_1,\ldots,z_n) \in \mathbb{C}^n$ we write

$$z^k := \prod_{\nu=1}^n z_\nu^{k_\nu} \quad \text{and} \quad a^{\{k\}} := (a\cdot\mathbf{1})^k = a^{k_1+\cdots+k_n} \quad \text{for } a \in \mathbb{C}.$$

If $z_1,\ldots,z_n$ are real positive we define z^k in the same manner also for $k \in \mathbb{Q}^n$. The group $\mathrm{GL}_2^+(\mathbb{R})^n$ acts as follows on $\mathbb{H}^n$: For $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathrm{GL}_2^+(\mathbb{R})^n$ with $\alpha_\nu = \begin{pmatrix} a_\nu & b_\nu \\ c_\nu & d_\nu \end{pmatrix}$ set

$$\alpha(z) = (\alpha_1(z_1),\ldots,\alpha_n(z_n)) \quad \text{with} \quad \alpha_\nu(z_\nu) = \frac{a_\nu z_\nu + b_\nu}{c_\nu z_\nu + d_\nu}.$$

Let $j(\alpha,z)=(\det(\alpha_1)^{-\frac12}(c_1z_1+d_1),\dots,\det(\alpha_n)^{-\frac12}(c_nz_n+d_n))$. The group $\mathrm{GL}_2^+(\mathbb{R})^n$ acts with the factor of automorphy $j(\alpha,z)^{-k}$ on complex-valued functions f on $\mathbb{H}^n$ by

$$(f\|_k\alpha)(z):=j(\alpha,z)^{-k}f(\alpha z).$$

For a congruence subgroup $\Gamma\leqslant\widetilde{G}^+(\mathbb{Q})$ we denote the $\mathbb{C}$-vector space of *Hilbert modular forms of weight k with repect to* Γ by $\mathcal{M}_k(\Gamma)$ and its subspace of *cusp forms* by $\mathcal{S}_k(\Gamma)$. If $\psi:\Gamma\to\mathbb{C}^\times$ is a character of finite order, we set

$$\begin{aligned}\mathcal{M}_k(\Gamma,\psi)&:=\{f\in\mathcal{M}_k(\ker(\psi))\mid f\|_k\gamma=\psi(\gamma)f\text{ for all }\gamma\in\Gamma\},\\ \mathcal{S}_k(\Gamma,\psi)&:=\mathcal{M}_k(\Gamma,\psi)\cap\mathcal{S}_k(\ker(\psi)).\end{aligned}$$

Furthermore, let $\mathcal{N}_k(\Gamma)$ be the set of all *nearly holomorphic forms* of weight k, i.e. the space of all functions $f:\mathbb{H}^n\to\mathbb{C}$ which are modular with respect to Γ and which have for some $A\in\mathbb{Z}^n$ a Fourier expansion

$$f(z)=\sum_{\xi\in F}\sum_{0\leqslant a\leqslant A}c(a,\xi)(\pi y)^{-a}e_\infty(\xi z)\quad\text{for }z\in\mathbb{H}^n$$

with $y=\operatorname{Im}z$, $c(a,\xi)\in\mathbb{C}$, $c(a,\xi)=0$ unless ξ is an element of a certain lattice in F and $\xi\gg0$ or $\xi=0$, and we also have to require a similar Fourier expansion at each cusp in the case $F=\mathbb{Q}$. Now, $\mathcal{N}_k(\Gamma,\psi)$ is defined similarly as above, and we let

$$\mathcal{M}_k=\bigcup_\Gamma\mathcal{M}_k(\Gamma),\quad\mathcal{S}_k=\bigcup_\Gamma\mathcal{S}_k(\Gamma),\quad\text{and }\mathcal{N}_k=\bigcup_\Gamma\mathcal{N}_k(\Gamma),$$

with Γ running over all congruence subgroups of $\widetilde{G}^+$. For two fractional ideals $\mathfrak{x}$ and $\mathfrak{y}$ in F such that $\mathfrak{x}\mathfrak{y}\subset\mathfrak{o}$ put

$$\widetilde{D}_\mathfrak{p}[\mathfrak{x},\mathfrak{y}]=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in M_2(F_\mathfrak{p})\mid a\in\mathfrak{o}_\mathfrak{p},b\in\mathfrak{x}_\mathfrak{p},c\in\mathfrak{y}_\mathfrak{p},d\in\mathfrak{o}_\mathfrak{p},ad-bc\in\mathfrak{o}_\mathfrak{p}^\times\right\}\quad\text{for }\mathfrak{p}\nmid\infty,$$

$$\begin{aligned}\widetilde{D}[\mathfrak{x},\mathfrak{y}]&=\widetilde{G}_\infty^+\prod_{\mathfrak{p}\nmid\infty}\widetilde{D}_\mathfrak{p}[\mathfrak{x},\mathfrak{y}]\ \subseteq\ \widetilde{G}_\mathbf{A}^+,\\ \widetilde{\Gamma}[\mathfrak{x},\mathfrak{y}]&=\widetilde{G}(\mathbb{Q})\cap\widetilde{D}[\mathfrak{x},\mathfrak{y}].\end{aligned}$$

Fix $h=h_F$ elements $t_1,\dots,t_h$ of $F_\mathbf{A}^\times$ so that $(t_\lambda)_\infty=1$ and $\widetilde{t}_1,\dots,\widetilde{t}_h$ form a complete set of representatives for the ideal classes, and put

$$x_\lambda=\begin{pmatrix}1&0\\0&t_\lambda\end{pmatrix},\quad\widetilde{D}(\mathfrak{c})=\widetilde{D}[\mathfrak{d}^{-1},\mathfrak{c}\mathfrak{d}],\quad\text{and }\widetilde{\Gamma}_\lambda(\mathfrak{c})=\widetilde{\Gamma}[(t_\lambda\mathfrak{d})^{-1},t_\lambda\mathfrak{d}\mathfrak{c}]$$

for an integral ideal $\mathfrak{c}$. Denote by $\iota : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ the main involution of $M_2(F)$ and its extension to $M_2(F_{\mathbf{A}})$, let ψ be a Hecke character of finite order such that the finite part of its conductor divides $\mathfrak{c}$, and for a subgroup $W \leqslant \widetilde{D}[\mathfrak{x}, \mathfrak{y}\mathfrak{c}]$ let

$$\psi_W : W \to \mathbb{C}^\times, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \psi_0(a).$$

The $\mathbb{C}$-vector space $\mathcal{M}_k(\mathfrak{c}, \psi)$ of *Hilbert automorphic forms* is the set of all complex valued functions $\mathbf{f} : \widetilde{G}_{\mathbf{A}} \to \mathbb{C}$ satisfying the following conditions: $\mathbf{f}(s\alpha x w) = \psi(s)\psi_{\widetilde{D}(\mathfrak{c})}(w^\iota)\mathbf{f}(x)$ if $s \in F_{\mathbf{A}}^\times$, $\alpha \in \widetilde{G}(\mathbb{Q})$, and $w \in \widetilde{D}(\mathfrak{c})$ with $w_\infty = 1$, and for each λ there is an element f_λ of $\mathcal{M}_k(\widetilde{\Gamma}_\lambda(\mathfrak{c}), \psi_{\widetilde{\Gamma}_\lambda(\mathfrak{c})})$ such that $\mathbf{f}(x_\lambda^{-\iota} y) = (f_\lambda \| y)(\mathbf{i})$ for all $y \in \widetilde{G}_\infty^+$ where $\mathbf{i}$ denotes the point $\mathbf{i} = (i, \ldots, i) \in \mathbb{H}^n$. The spaces $\mathcal{S}_k(\mathfrak{c}, \psi)$ and $\mathcal{N}_k(\mathfrak{c}, \psi)$ of cusp forms respectively nearly holomorphic forms are similarly defined by requiring that f_λ be an element of $\mathcal{S}_k(\widetilde{\Gamma}_\lambda(\mathfrak{c}), \psi_{\widetilde{\Gamma}_\lambda(\mathfrak{c})})$ respectively $\mathcal{N}_k(\widetilde{\Gamma}_\lambda(\mathfrak{c}), \psi_{\widetilde{\Gamma}_\lambda(\mathfrak{c})})$. Finally, the group $\widetilde{G}_{\mathbf{A}}$ can be expressed as a disjoint union

$$\widetilde{G}_{\mathbf{A}} = \dot{\bigcup_{\lambda=1}^{h}} \widetilde{G}(\mathbb{Q}) x_\lambda \widetilde{D}(\mathfrak{c}) = \dot{\bigcup_{\lambda=1}^{h}} \widetilde{G}(\mathbb{Q}) x_\lambda^{-\iota} \widetilde{D}(\mathfrak{c}),$$

so that $\mathbf{f} \in \mathcal{M}_k(\mathfrak{c}, \psi)$ is uniquely determined by the $f_\lambda \in \mathcal{M}_k(\widetilde{\Gamma}_\lambda, \psi_{\widetilde{\Gamma}_\lambda})$; we also write $\mathbf{f} = (f_1, \ldots, f_h)$. Now, by (2.6) of [S4], $\mathcal{M}_k(\widetilde{\Gamma}_\lambda, \psi_{\widetilde{\Gamma}_\lambda}) = \{0\}$ and hence $\mathbf{f} = 0$ unless ψ_0 satisfies

$$\psi_0(\varepsilon) = \operatorname{sgn}(\varepsilon)^k = \prod_\nu \left(\frac{\varepsilon^{\tau_\nu}}{|\varepsilon^{\tau_\nu}|} \right)^{k_\nu} \quad \text{for every } \varepsilon \in \mathfrak{o}^\times.$$

We will henceforth impose this condition on ψ_0. Now each f_λ has a Fourier expansion

$$f_\lambda(z) = \sum_\xi a_\lambda(\xi) e_\infty(\xi z) \quad \text{with } \xi = 0 \text{ or } 0 \ll \xi \in \widetilde{t}_\lambda.$$

Any non-zero ideal $\mathfrak{m}$ of F can be written as $\mathfrak{m} = \xi \widetilde{t}_\lambda^{-1}$ with $0 \ll \xi \in F$ and a unique λ. Define the $\mathfrak{m}$-th Fourier coefficient of $\mathbf{f}$ by

$$c(\mathfrak{m}, \mathbf{f}) = \begin{cases} a_\lambda(\xi)\xi^{-\frac{k}{2}}, & \text{if } \mathfrak{m} = \xi \widetilde{t}_\lambda^{-1} \text{ is integral,} \\ 0, & \text{if } \mathfrak{m} \text{ is not integral,} \end{cases}$$

and let $C(\mathfrak{m},\mathbf{f}) = \mathcal{N}(\mathfrak{m})^{\frac{k_0}{2}} c(\mathfrak{m},\mathbf{f})$ with $k_0 := \max\{k_1,\dots,k_n\}$. This modified coefficient only depends on $\mathfrak{m}$ and $\mathbf{f}$, and the form $\mathbf{f}$ is uniquely determined by the values of $C(\mathfrak{m},\mathbf{f})$ for all ideals $\mathfrak{m}$. For every integral ideal $\mathfrak{n}$ of F one can define a Hecke operator $T(\mathfrak{n})$ which is a $\mathbb{C}$-linear endomorphism of $\mathcal{M}_k(\mathfrak{c},\psi)$ (cf. p. 648 and (2.21) of [S4]) such that

$$C(\mathfrak{m},\mathbf{f}|T(\mathfrak{n})) = \sum_{\mathfrak{a}} \psi_{\mathfrak{c}}^*(\mathfrak{a})\mathcal{N}(\mathfrak{a})^{k_0-1}C(\mathfrak{a}^{-2}\mathfrak{m}\mathfrak{n},\mathbf{f}) \quad \text{for all } \mathfrak{m},$$

where $\mathfrak{a}$ runs over all integral ideals dividing $\mathfrak{m}+\mathfrak{n}$. If $\mathbf{f}$ is a simultaneous eigenfunction of the Hecke operators $T(\mathfrak{n})$ with eigenvalue $\omega(\mathfrak{n})$, then the Dirichlet series

$$D(s,\mathbf{f}) := \sum_{\mathfrak{m}\subseteq\mathfrak{o}} C(\mathfrak{m},\mathbf{f})\mathcal{N}(\mathfrak{m})^{-s}$$

associated with $\mathbf{f}$ has a factorization as Euler product over all prime ideals:

$$D(s,\mathbf{f}) = C(\mathfrak{o},\mathbf{f})\prod_{\mathfrak{p}} \left(1-\omega(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-s}+\psi_{\mathfrak{c}}^*(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{k_0-1-2s}\right)^{-1}.$$

There are several important operators on the space of Hilbert automorphic forms. First, if $\mathfrak{q}$ is an integral ideal and $q \in F_{\mathbf{A}}^\times$ is an idele with $\widetilde{q} = \mathfrak{q}$, and $q_\infty = 1$, then by Proposition 2.3 of [4] and IV.1.19 of [P] we can define the operators

$$\mathfrak{q} : \mathcal{M}_k(\mathfrak{c},\psi) \to \mathcal{M}_k(\mathfrak{q}\mathfrak{c},\psi), \quad (\mathbf{f}|\mathfrak{q})(x) := \mathcal{N}(\mathfrak{q})^{-\frac{k_0}{2}}\mathbf{f}\left(x\begin{pmatrix} q & \\ & 1\end{pmatrix}^{-1}\right),$$

$$U(\mathfrak{q}) : \mathcal{M}_k(\mathfrak{c},\psi) \to \mathcal{M}_k(\mathfrak{q}\mathfrak{c},\psi),$$

$$(\mathbf{f}|U(\mathfrak{q}))(x) := \mathcal{N}(\mathfrak{q})^{\frac{k_0}{2}-1}\sum_{v\in\mathfrak{d}^{-1}/\mathfrak{q}\mathfrak{d}^{-1}} \mathbf{f}\left(x\begin{pmatrix} q & v_0 \\ & 1\end{pmatrix}\right).$$

Actually, the definition for $U(\mathfrak{q})$ given in [P] is erroneous and should be replaced by the above expression. These two operators are characterized by their effects on the Fourier coefficients:

$$C(\mathfrak{m},\mathbf{f}|\mathfrak{q}) = C(\mathfrak{q}^{-1}\mathfrak{m},\mathbf{f}) \quad \text{and} \quad C(\mathfrak{m},\mathbf{f}|U(\mathfrak{q})) = C(\mathfrak{q}\mathfrak{m},\mathbf{f}) \quad \text{for all } \mathfrak{m}.$$

Second, there is an inverter $J_{\mathfrak{c}} : \mathcal{M}_k(\mathfrak{c},\psi) \to \mathcal{M}_k(\mathfrak{c},\overline{\psi})$ (cf. (2.46) of [S4]), which is defined as follows: Let $m \in F_{\mathbf{A}}^\times$ be an idele such that $\widetilde{m} = \mathfrak{c}\mathfrak{d}^2$ and set $b := \begin{pmatrix} & 1 \\ m_0 & \end{pmatrix}\cdot\begin{pmatrix} 1 & \\ & 1\end{pmatrix}_\infty \in \widetilde{G}_{\mathbf{A}}$. Then

$$(\mathbf{f}|J_{\mathfrak{c}})(x) := \mathbf{f}(x^{-\iota}b) \quad \text{for all } x \in \widetilde{G}_{\mathbf{A}}.$$

With respect to the decomposition $\mathbf{f} = (f_1, \ldots, f_h)$, $J_\mathfrak{c}$ has the following description: For every λ determine κ and a totally positive element q_λ of F such that $t_\lambda t_\kappa \mathfrak{c}\mathfrak{d}^2 = (q_\lambda)$ and write $\mathbf{f}|J_\mathfrak{c} = (f_1', \ldots, f_h')$. Then

$$f_\kappa' = f_\lambda \| \beta_\lambda^{-1} = (-1)^{\{k\}} f \| \beta_\lambda \quad \text{with} \quad \beta_\lambda = \begin{pmatrix} & 1 \\ -q_\lambda & \end{pmatrix} \in \widetilde{G}(\mathbb{Q}).$$

For $\mathbf{f} = (f_1, \ldots, f_h)$, $\mathbf{g} = (g_1, \ldots, g_h) \in \mathcal{S}_k(\mathfrak{c}, \psi)$, the Petersson inner product is defined by

$$\langle\, \mathbf{f}, \mathbf{g} \,\rangle = \sum_{\lambda=1}^{h} \frac{1}{\mu(\widetilde{\Gamma}_\lambda(\mathfrak{c}) \backslash \mathbb{H}^n)} \int_{\widetilde{\Gamma}_\lambda(\mathfrak{c}) \backslash \mathbb{H}^n} f_\lambda(z) \overline{g_\lambda(z)} y^k \, d\mu(z)$$

with the $\widetilde{G}_\infty^+$-invariant measure $d\mu(z) = \prod_{\nu=1}^{n} y_\nu^{-2} dx_\nu dy_\nu$ on $\mathbb{H}^n$. The product is also defined if one of the forms is only in $\mathcal{M}_k(\mathfrak{c}, \psi)$ or $\mathcal{N}_k(\mathfrak{c}, \psi)$. We will also use the following inner product which depends on the level $\mathfrak{c}$ of the automorphic forms:

$$\langle\, \mathbf{f}, \mathbf{g} \,\rangle_\mathfrak{c} = \sum_{\lambda=1}^{h} \int_{\widetilde{\Gamma}_\lambda(\mathfrak{c}) \backslash \mathbb{H}^n} f_\lambda(z) \overline{g_\lambda(z)} y^k \, d\mu(z).$$

The orthogonal complement $\mathcal{S}_k^0(\mathfrak{c}, \psi) \subseteq \mathcal{S}_k(\mathfrak{c}, \psi)$ of the space of old forms has a basis of common eigenfunctions for all Hecke operators $T(\mathfrak{n})$. A normalized element $\mathbf{f}$ of this basis (i.e. an element with $C(\mathfrak{o}, \mathbf{f}) = 1$) will be called a *primitive (cusp) form of conductor* $\mathfrak{c}$. Now let $\mathfrak{c}_1 \subseteq \mathfrak{c}_2$ be two integral ideals and ψ a Hecke character of finite order whose conductor divides $\mathfrak{c}_2$. Then in VI.4.4 of [P] the following trace operator is defined:

$$\mathrm{Tr}_{\mathfrak{c}_2}^{\mathfrak{c}_1} : \mathcal{N}_k(\mathfrak{c}_1, \psi) \to \mathcal{N}_k(\mathfrak{c}_2, \psi), \quad (\mathbf{f}|\mathrm{Tr}_{\mathfrak{c}_2}^{\mathfrak{c}_1})(x) := \sum_{h \in \widetilde{D}(\mathfrak{c}_1)/\widetilde{D}(\mathfrak{c}_2)} \psi_{\widetilde{D}(\mathfrak{c}_1)}(h^\iota) \mathbf{f}(xh),$$

and it can also be expressed in the form

$$\mathrm{Tr}_{\mathfrak{c}_2}^{\mathfrak{c}_1} = (-1)^{\{k\}} \mathcal{N}\left(\frac{\mathfrak{c}_1}{\mathfrak{c}_2}\right)^{1-\frac{k_0}{2}} J_{\mathfrak{c}_1} U\left(\frac{\mathfrak{c}_1}{\mathfrak{c}_2}\right) J_{\mathfrak{c}_2}.$$

The important property of the trace operator is the reduction of levels: If $\mathfrak{c}_1 \leqslant \mathfrak{c}_2$, $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c}_2, \psi) \subseteq \mathcal{S}_k(\mathfrak{c}_1, \psi)$ and $g \in \mathcal{N}_k(\mathfrak{c}_1, \psi)$, then by (4.10) of [loc. cit.]

$$\langle\, \mathbf{f}, \mathbf{g} \,\rangle_{\mathfrak{c}_1} = \langle\, \mathbf{f}, \mathbf{g}|\mathrm{Tr}_{\mathfrak{c}_2}^{\mathfrak{c}_1} \,\rangle_{\mathfrak{c}_2}.$$

Actually, under suitable conditions, a nearly holomorphic function can be replaced by a holomorphic function without changing the value of the Petersson scalar product. These conditions involve the following notion: A function $\mathbf{f} = (f_1, \dots, f_h) \in \mathcal{N}_k(\mathfrak{c}, \psi)$ will be called *of moderate growth* if for all $\lambda = 1, \dots, h$, all $z \in \mathbb{H}^n$ and for any $s \in \mathbb{C}$ with sufficiently large real part the integral

$$\int_{\mathbb{H}^n} f_\lambda(w)(\overline{w} - z)^{-k-|2s|}\mathrm{Im}(w)^{k+s}\, d\mu(w)$$

converges absolutely and admits an analytic continuation over s to the point 0. Here $z^{-k-|2s|}$ is understood as $\prod_\nu z_\nu^{-k_\nu}|z_\nu|^{-2s}$.

We end this section by quoting Proposition IV.4.7 of [P]:

PROPOSITION 2.1. — *Let $\mathbf{f} \in \mathcal{N}_k(\mathfrak{c}, \psi)$ be a function of moderate growth such that its Fourier expansions $f_\lambda(x + iy) = \sum_{\xi \in \widetilde{t_\lambda}} a_\lambda(\xi, y)e_\infty(\xi x)$ contain only terms with totally positive $\xi \in \widetilde{t_\lambda}$. For $0 \ll \xi \in \widetilde{t_\lambda}$ set*

$$a_\lambda(\xi) = \frac{(4\pi)^{\{k-1\}}\xi^{k-1}}{\prod_{\nu=1}^n \Gamma(k_\nu - 1)} \int_{\mathbb{R}_+^n} a_\lambda(\xi, y)e_\infty(i\xi y)y^{k-2}\, dy,$$

and suppose that the integral is absolutely convergent. Then there is a function $\mathcal{H}ol\,\mathbf{f} = (\mathcal{H}ol\, f_1, \dots, \mathcal{H}ol\, f_h) \in \mathcal{M}_k(\mathfrak{c}, \psi)$, whose Fourier expansion is given by

$$\mathcal{H}ol\, f_\lambda(z) = \sum_{0 \ll \xi \in \widetilde{t_\lambda}} a_\lambda(\xi)e_\infty(\xi z),$$

and which has the property $\langle\, \mathbf{g}, \mathbf{f}\,\rangle_\mathfrak{c} = \langle\, \mathbf{g}, \mathcal{H}ol\,\mathbf{f}\,\rangle_\mathfrak{c}$ for all $\mathbf{g} \in \mathcal{S}_k(\mathfrak{c}, \psi)$.

COROLLARY 2.2. — *Let $\mathbf{f} \in \mathcal{N}_k(\mathfrak{c}, \psi)$ and $\mathbf{g} \in \mathcal{S}_k(\mathfrak{c}, \overline{\psi})$. If $\mathbf{f}$ suffices the conditions of the proposition, then*

$$\langle\, \mathbf{g}, \mathbf{f}|J_\mathfrak{c}\,\rangle_\mathfrak{c} = \langle\, \mathbf{g}, (\mathcal{H}ol\,\mathbf{f})|J_\mathfrak{c}\,\rangle_\mathfrak{c}.$$

Proof. — This follows immediately by observing that $J_\mathfrak{c}^{-1} = (-1)^{\{k\}}J_\mathfrak{c}$ is the adjoint of $J_\mathfrak{c}$ with respect to the Petersson scalar product. □

3. Hilbert modular forms of half-integral weight.

In order to define modular forms of half-integral weight, one needs to define a factor of automorphy of weight $\frac{1}{2}$. The ambiguity of the square root of a complex number leads to a much more complicated automorphic factor. In fact, the property of automorphy is valid only for certain subgroups. We will recall the fundamental definitions and properties of half-integral forms before defining some operators which we will need in the sequel.

For two fractional ideals $\mathfrak{x}$ and $\mathfrak{y}$ of F with $\mathfrak{x}\mathfrak{y} \subseteq \mathfrak{o}$ let us define the following groups:

$$\begin{aligned} &G = \mathrm{SL}_2(F) \text{ with adelization } G_{\mathbf{A}} = G_f G_\infty, \\ &D_\mathfrak{p}[\mathfrak{x},\mathfrak{y}] = \widetilde{D}_\mathfrak{p}[\mathfrak{x},\mathfrak{y}] \cap \mathrm{SL}_2(F_\mathfrak{p}) \text{ for } \mathfrak{p} \not| \infty, \\ &D[\mathfrak{x},\mathfrak{y}] = \widetilde{D}[\mathfrak{x},\mathfrak{y}] \cap \mathrm{SL}_2(F_{\mathbf{A}}), \quad \Gamma[\mathfrak{x},\mathfrak{y}] = \widetilde{\Gamma}[\mathfrak{x},\mathfrak{y}] \cap G. \end{aligned}$$

For an element $g \in G_{\mathbf{A}}$ we denote by g_0 and g_∞ the finite respectively archimedean part. We now fix once and for all, an element δ of $F_{\mathbf{A}}^\times$ such that $\delta_\infty = \mathbf{1}$ and $\widetilde{\delta} = \mathfrak{d}$ and define $\eta = \prod_\mathfrak{p} \eta_\mathfrak{p} \in G_{\mathbf{A}}$ by

$$\eta_\mathfrak{p} = \begin{pmatrix} & -\delta_\mathfrak{p}^{-1} \\ \delta_\mathfrak{p} & \end{pmatrix} \text{ for } \mathfrak{p} \nmid \infty, \quad \eta_\mathfrak{p} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \text{ for } \mathfrak{p}|\infty. \tag{5}$$

Next, we define

$$\begin{aligned} &C' = \mathrm{SO}_2(\mathbb{R})^n \cdot \prod_{\mathfrak{p} \not| \infty} D_\mathfrak{p}[2\mathfrak{d}^{-1}, 2\mathfrak{d}], \quad C'' = C' \cup C'\eta, \\ &P = \{\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix} \in G \,|\, c_\alpha = 0\} \quad \text{with adelization } P_{\mathbf{A}}, \\ &\Omega = \{\alpha \in G \,|\, c_\alpha \neq 0\} \quad \text{with adelization } \Omega_{\mathbf{A}}, \text{ and} \\ &T = \{z \in \mathbb{C} \,|\, |z| = 1\}. \end{aligned}$$

Further, let $M_{\mathbf{A}} = Mp(F_{\mathbf{A}})$ denote Weil's metaplectic group, which is an extension of $G_{\mathbf{A}}$ with kernel T. Let pr denote the projection map of $M_{\mathbf{A}}$ onto $G_{\mathbf{A}}$, and note that

$$xy = yx \quad \text{if } x, y \in M_{\mathbf{A}} \text{ with } \mathrm{pr}(x) \in G_\infty \quad \text{and } \mathrm{pr}(y) \in G_f. \tag{6}$$

There are splitting homomorphisms

$$r : G \to M_{\mathbf{A}}, \quad r_P : P_{\mathbf{A}} \to M_{\mathbf{A}}, \quad \text{and a map} \quad r_\Omega : \Omega_{\mathbf{A}} \to M_{\mathbf{A}},$$

which are consistent in the sense that

$$
\begin{aligned}
&r_\Omega(\alpha\beta\gamma) = r_P(\alpha)r_\Omega(\beta)r_P(\gamma) \quad \text{for } \alpha,\gamma \in P_{\mathbf{A}}, \beta \in \Omega_{\mathbf{A}}, \\
&r = r_P \text{ on } P, \quad \text{and} \quad r = r_\Omega \text{ on } P\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}P.
\end{aligned} \tag{7}
$$

We refer to section 3 of [S6] for the definition of these maps. Via the embedding r we understand G as a subgroup of $M_{\mathbf{A}}$, and we let $M_{\mathbf{A}}$ act on $\mathbb{H}^n$ via the infinite part of the projection pr. Similarly, if $\xi \in M_{\mathbf{A}}$ and $z \in \mathbb{H}^n$, we write $j(\xi, z)$ for $j(\mathrm{pr}(\xi)_\infty, z)$. Then in [loc. cit.] it is proved, that for every $\xi \in M_{\mathbf{A}}$ such that $\mathrm{pr}(\xi) \in P_{\mathbf{A}}C''$ there is a non-vanishing holomorphic function $h(\xi, z)$ on $\mathbb{H}^n$ with the following properties:

$$h(\xi, z)^2 = t \cdot j(\xi, z)^{\mathbf{1}} \quad \text{for some } t \in T, \tag{8}$$

$$h(\beta\xi\tau, z) = h(\beta, z)h(\xi, \tau z)h(\tau, z) \tag{9}$$

$$\text{if } \mathrm{pr}(\beta) \in P_{\mathbf{A}},\ \mathrm{pr}(\xi) \in P_{\mathbf{A}}C'',\ \mathrm{pr}(\tau) \in C''G_\infty,$$

$$h(t \cdot r_P(x), z) = t^{-1}|d_x|_\infty^{1/2} \quad \text{if } t \in T \text{ and } x \in P_{\mathbf{A}}, \tag{10}$$

and if $\sigma = r_P(\mathrm{diag}(r, r^{-1}))$ with $r \in F_f^\times$, then Lemma 2.5 of [S7] shows that

$$h(\xi, z) = h(\eta, z) \quad \text{if } \mathrm{pr}(\xi), \mathrm{pr}(\eta) \in C' \text{ and } \xi = \sigma\eta\sigma^{-1}. \tag{11}$$

Now let us define certain Gauss sums for finite primes $\mathfrak{p}$, $\mu \in F_\mathfrak{p}$ and $x \in F_{\mathbf{A}}$ by

$$\gamma_\mathfrak{p}(\mu) := \int_{\mathfrak{o}_\mathfrak{p}} e_\mathfrak{p}(\mu t^2/2)\, dt \quad \text{for } \mathfrak{p} \text{ finite}, \tag{12}$$

$$\gamma(x) := \prod_{\mathfrak{p} \nmid \infty} \gamma_\mathfrak{p}(x_\mathfrak{p}). \tag{13}$$

Here, dx is the Haar measure on $F_\mathfrak{p}$ normalized by $\int_{\mathfrak{o}_\mathfrak{p}} dx = 1$. By Lemma 3.4 and Lemma 3.5 of [S6], for certain $\xi \in M_{\mathbf{A}}$ the automorphic factor $h(\xi, z)$ is completely determined by (8) and

$$\lim_{\rho\to\infty} \frac{h(\xi, \rho\mathbf{i})}{|h(\xi, \rho\mathbf{i})|} = \frac{\gamma(-c_\xi^{-1}d_\xi)}{|\gamma(-c_\xi^{-1}d_\xi)|} \quad \text{if } \xi \in r_\Omega(\Omega_{\mathbf{A}} \cap P_{\mathbf{A}}C''), \tag{14}$$

$$\lim_{\rho\to\infty} \frac{h(\xi, \rho\mathbf{i})}{|h(\xi, \rho\mathbf{i})|} = \frac{\gamma(\delta^{-2}d_\xi^{-1}c_\xi)}{|\gamma(\delta^{-2}d_\xi^{-1}c_\xi)|} \tag{15}$$

$$\text{if } \xi \in r_\Omega(\Omega_{\mathbf{A}})r_\Omega\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\right) \cap \mathrm{pr}^{-1}(P_{\mathbf{A}}C'');$$

here c_ξ and d_ξ are given by $\mathrm{pr}(\xi) = \begin{pmatrix} a_\xi & b_\xi \\ c_\xi & d_\xi \end{pmatrix} \in G_{\mathbf{A}}$. With the quadratic Hecke character $\omega = \varepsilon_{-1}$ corresponding to $F(\sqrt{-1})/F$, we have the formula

$$h(\gamma, z)^2 = \omega_\infty(d_\gamma) \cdot \omega^*((d_\gamma)) \cdot j(\gamma, z)^{\mathbf{1}} \quad \text{if } \gamma \in \Gamma[2\mathfrak{d}^{-1}, 2\mathfrak{d}]. \tag{16}$$

We are now in a position to define half-integral automorphic forms. Fix a half-integral weight $k \in \left(\frac{1}{2}\mathbb{Z}\right)^n$, let $k' := k - \frac{1}{2} \cdot \mathbf{1} \in \mathbb{Z}^n$, and define a factor of automorphy

$$J_k(\tau, z) := h(\tau, z) j(\tau, z)^{k'} \quad \text{for } \tau \in M_{\mathbf{A}} \cap \mathrm{pr}^{-1}(P_{\mathbf{A}} C'').$$

For a function f on $\mathbb{H}^n$ we write

$$(f\|_k \tau)(z) := J_k(\tau, z)^{-1} f(\tau z) \quad \text{for } \tau \in \mathrm{pr}^{-1}(P_{\mathbf{A}} C'').$$

We will now tacitly assume that all congruence subgroups Γ of G are contained in $C'' G_\infty$. With the operation $\|_k \tau$, a *Hilbert modular form* $f \in \mathcal{M}_k(\Gamma)$, a *cusp form* $f \in \mathcal{S}_k(\Gamma)$, and a *nearly holomorphic form* $f \in \mathcal{N}_k(\Gamma)$ with respect to some congruence subgroup Γ of G are defined as in the integral case. We also define $\mathcal{M}_k$, $\mathcal{S}_k$ and $\mathcal{N}_k$ as the union over all congruence subgroups Γ of the respective groups $\mathcal{M}_k(\Gamma)$, $\mathcal{S}_k(\Gamma)$, and $\mathcal{N}_k(\Gamma)$ as in the integral case. An automorphic function on $M_{\mathbf{A}}$ with respect to the factor of automorphy J_k, is a function $f_{\mathbf{A}} : M_{\mathbf{A}} \to \mathbb{C}$ such that

$$f_{\mathbf{A}}(\alpha \xi w) = J_k(w, \mathbf{i})^{-1} f_{\mathbf{A}}(\xi) \quad \text{for } \alpha \in G,\ \xi \in M_{\mathbf{A}}, \text{ and } w \in \mathrm{pr}^{-1}(B) \tag{17}$$

for some open subgroup $B \leqslant C''$. There is the following connection between modular and automorphic functions (see [S7], section 1): Given $f_{\mathbf{A}}$, the function $f : \mathbb{H}^n \to \mathbb{C}$ defined by

$$f(\xi(\mathbf{i})) := f_{\mathbf{A}}(\xi) J(\xi, \mathbf{i}) \quad \text{for } \xi \in \mathrm{pr}^{-1}(B G_\infty)$$

is automorphic with respect to $\Gamma := G \cap B G_\infty$, and conversely a complex valued function f on $\mathbb{H}^n$ with the automorphy property with respect to $G \cap B G_\infty$ determines an automorphic function $f_{\mathbf{A}}$ on $M_{\mathbf{A}}$ by

$$f_{\mathbf{A}}(\alpha \xi) := (f\|_k \xi)(\mathbf{i}) \quad \text{for } \alpha \in G \text{ and } \xi \in \mathrm{pr}^{-1}(B G_\infty).$$

An *automorphic form* is an automorphic function on $M_{\mathbf{A}}$ whose corresponding function on $\mathbb{H}^n$ is a modular form. Given two integral ideals $\mathfrak{b}$, $\mathfrak{b}'$ and a Hecke character ψ of F of finite order whose conductor divides $4\mathfrak{b}\mathfrak{b}'$ and whose archimedean component satisfies $\psi_\infty(-\mathbf{1}) = (-\mathbf{1})^{k'}$, we denote by $\mathcal{M}_k(\mathfrak{b}, \mathfrak{b}', \psi)$ the set of all f in $\mathcal{M}_k$ such that

$$f\|_k\gamma = \psi_{4\mathfrak{b}\mathfrak{b}'}(a_\gamma)f \quad \text{for every } \gamma \in \Gamma[2\mathfrak{b}\mathfrak{d}^{-1}, 2\mathfrak{b}'\mathfrak{d}]$$

and let $\mathcal{S}_k(\mathfrak{b}, \mathfrak{b}', \psi) = \mathcal{S}_k \cap \mathcal{M}_k(\mathfrak{b}, \mathfrak{b}', \psi)$. The ideal $4\mathfrak{b}\mathfrak{b}'$ is the level of our group. We remark that for $f \in \mathcal{M}_k(\mathfrak{b}, \mathfrak{b}', \psi)$ the automorphy property (17) for the associated adelic form $f_\mathbf{A}$ takes the form

$$f_\mathbf{A}(\alpha\xi w) = \overline{\psi}_{4\mathfrak{b}\mathfrak{b}'}(a_w) \cdot J_k(w, \mathbf{i})^{-1} f_\mathbf{A}(\xi) \tag{18}$$

for $\alpha \in G$, $\xi \in M_\mathbf{A}$, and $w \in \mathrm{pr}^{-1}(D[2\mathfrak{b}\mathfrak{d}^{-1}, 2\mathfrak{b}'\mathfrak{d}])$ with $w(\mathbf{i}) = \mathbf{i}$. The modular forms of $\mathcal{M}_k(\mathfrak{b}, \mathfrak{b}', \psi)$ have the following Fourier expansion (cf. [S8] Proposition) 1.1):

PROPOSITION 3.1. — *Given $f \in \mathcal{M}_k(\mathfrak{b}, \mathfrak{b}', \psi)$, there is a complex number $\lambda_f(\xi, \mathfrak{m}) = \lambda(\xi, \mathfrak{m}; f, \psi)$ determined for every $\xi \in F$ and every fractional ideal $\mathfrak{m}$ such that*

$$f_\mathbf{A}(r_P\begin{pmatrix} t & s \\ & t^{-1}\end{pmatrix}) = \overline{\psi}_f(t) t_\infty^{k'} |t|_\mathbf{A}^{\frac{1}{2}} \sum_{\xi \in F} \lambda(\xi, \widetilde{t}; f, \psi) e_\infty(\mathbf{i}t^2\xi/2) e_\mathbf{A}(ts\xi/2)$$

for every $t \in F_\mathbf{A}^\times$ and $s \in F_\mathbf{A}$. Moreover, $\lambda(\xi, \mathfrak{m}; f, \psi)$ has the following properties:

$\lambda(\xi, \mathfrak{m}; f, \psi) \neq 0$ *only if $\xi \in \mathfrak{b}^{-1}\mathfrak{m}^{-2}$, and $\xi = 0$ or ξ is totally positive,*
$\lambda(\xi b^2, \mathfrak{m}; f, \psi) = b^{k'}\psi_\infty(b)\lambda(\xi, b\mathfrak{m}; f, \psi)$ *for every $b \in F^\times$.*

Furthermore, if $\beta \in G \cap \mathrm{diag}(r, r^{-1})D[2\mathfrak{b}\mathfrak{d}^{-1}, 2\mathfrak{b}'\mathfrak{d}]$ with $r \in F_\mathbf{A}^\times$, then

$$J(\beta, \beta^{-1}z) f(\beta^{-1}z) = \overline{\psi}_f(r)\psi_0(d_\beta r)|r_f|^{\frac{1}{2}} \sum_{\xi \in F} \lambda(\xi, \widetilde{r}; f, \psi) e_\infty(\xi z/2).$$

The form f is uniquely determined by its Fourier coefficients $\lambda(\xi, \mathfrak{m}; f, \psi)$ for $\xi \in F$ and fractional ideals $\mathfrak{m}$ (actually, $\mathfrak{m} = \mathfrak{o}$ is sufficient). As in the integral case there is an algebra of Hecke operators acting on $\mathcal{M}_k(\mathfrak{b}, \mathfrak{b}', \psi)$; for a definition see section 2 of [S8]. For our purpose the following description of the Hecke operators on the Fourier coefficients is sufficient: For $f \in \mathcal{M}_k(\mathfrak{b}, \mathfrak{b}', \psi)$, a prime ideal $\mathfrak{p}$ and a fractional ideal $\mathfrak{m}$ with $\xi\mathfrak{b}\mathfrak{m}^2 \subseteq \mathfrak{o}$ we have

$$\begin{aligned} \lambda(\xi, \mathfrak{m}; f|T_\mathfrak{p}, \psi) &= \lambda(\xi, \mathfrak{p}\mathfrak{m}; f, \psi) \\ &+ \psi_{4\mathfrak{b}\mathfrak{b}'}^*(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-1}\left(\frac{\xi c^2}{\mathfrak{p}}\right)\lambda(\xi, \mathfrak{m}; f, \psi) + \psi_{4\mathfrak{b}\mathfrak{b}'}^*(\mathfrak{p})^2\mathcal{N}(\mathfrak{p})^{-1}\lambda(\xi, \mathfrak{m}\mathfrak{p}^{-1}; f, \psi). \end{aligned} \tag{19}$$

Here c is an element of $F_\mathfrak{p}$ such that $c\mathfrak{o}_\mathfrak{p} = \mathfrak{m}_\mathfrak{p}$, $\left(\frac{\cdot}{\mathfrak{p}}\right)$ denotes the Legendre symbol, and the last two summands are 0 if $\mathfrak{p}|4\mathfrak{b}\mathfrak{b}'$.

PROPOSITION 3.2. — *Let $f \in \mathcal{M}_k(\mathfrak{b}, \mathfrak{b}', \psi)$. For $0 \ll \tau \in F$ let $\tau\mathfrak{b} = \mathfrak{q}^2\mathfrak{x}$ with a fractional ideal $\mathfrak{q}$ and a squarefree integral ideal $\mathfrak{x}$, and write ε_τ for the quadratic Hecke character of F corresponding to $F(\sqrt{\tau})/F$. Then the following are equivalent:*

(a) *f is a Hecke eigenform of $T_\mathfrak{p}$ for all primes $\mathfrak{p}$: $f|T_\mathfrak{p} = \omega_\mathfrak{p} f$ with $\omega_\mathfrak{p} \in \mathbb{C}$.*

(b) *For all $0 \ll \tau \in F$, the following formal identity holds:*

$$\sum_{\mathfrak{m}} \lambda_f(\tau, \mathfrak{q}^{-1}\mathfrak{m})\mathcal{N}(\mathfrak{m})^{-s} = \lambda_f(\tau, \mathfrak{q}^{-1}) \prod_{\mathfrak{p}} \frac{1 - (\psi\varepsilon_\tau)^*_{4\mathfrak{b}\mathfrak{b}'}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-1-s}}{1 - \omega_\mathfrak{p}\mathcal{N}(\mathfrak{p})^{-s} + \psi^*_{4\mathfrak{b}\mathfrak{b}'}(\mathfrak{p})^2\mathcal{N}(\mathfrak{p})^{-1-2s}}.$$

Here, $\mathfrak{m}$ runs over all integral and $\mathfrak{p}$ over all prime ideals of F.

Proof. — (a) implies (b) is proved in Proposition 2.2 of [S8]. Conversely, (b) implies

$$\lambda_f(\tau, \mathfrak{q}^{-1}\mathfrak{n})\lambda_f(\tau, \mathfrak{q}^{-1}\mathfrak{m}) = \lambda_f(\tau, \mathfrak{q}^{-1})\lambda_f(\tau, \mathfrak{q}^{-1}\mathfrak{m}\mathfrak{n})$$

for relatively prime integral ideals $\mathfrak{m}$ and $\mathfrak{n}$. Now, multiplication of the above partial Dirichlet series with the denominator on the right side and comparison of the coefficients yields

$$\begin{aligned}
&\lambda_f(\tau, \mathfrak{q}^{-1}\mathfrak{p}) - \omega_\mathfrak{p}\lambda_f(\tau, \mathfrak{q}^{-1}) = -(\psi\varepsilon_\tau)^*_{4\mathfrak{b}\mathfrak{b}'}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-1}\lambda_f(\tau, \mathfrak{q}^{-1}),\\
&\psi^*_{4\mathfrak{b}\mathfrak{b}'}(\mathfrak{p})^2\mathcal{N}(\mathfrak{p})^{-1}\lambda_f(\tau, \mathfrak{q}^{-1}\mathfrak{p}^r) - \omega_\mathfrak{p}\lambda_f(\tau, \mathfrak{q}^{-1}\mathfrak{p}^{r+1}) + \lambda_f(\tau, \mathfrak{q}^{-1}\mathfrak{p}^{r+2}) = 0
\end{aligned}$$

for non-negative integers r. Together with (19) this proves

$$\lambda(\tau, \mathfrak{q}^{-1}\mathfrak{n}\mathfrak{p}^r; f|T_\mathfrak{p}, \psi) = \omega_\mathfrak{p}\lambda(\tau, \mathfrak{q}^{-1}\mathfrak{n}\mathfrak{p}^r; f, \psi)$$

for all integral ideals $\mathfrak{n}$ prime to $\mathfrak{p}$ and all $r \geqslant 0$. But this holds trivially for $r < 0$, and hence (b) implies (a). □

We remark that by Proposition 3.1 of [S8], there is the following relation between modular forms of integral and half-integral weight: If $f \in \mathcal{S}_k(\mathfrak{b}, \mathfrak{b}', \psi)$ is a simultaneous Hecke eigenform of half integral weight, then there is an integral automorphic form $\mathbf{f} \in \mathcal{M}_k(2\mathfrak{b}\mathfrak{b}', \psi^2)$, which is called the Shimura lift of f, such that the denominator of the product in

(b) of the above proposition coincides with the following Dirichlet series:
(20)

$$\sum_{\mathfrak{m}} c(\mathfrak{m},\mathbf{f})\mathcal{N}(\mathfrak{m})^{-(s+1)} = \prod_{\mathfrak{p}} \left(1-\omega_{\mathfrak{p}}\mathcal{N}(\mathfrak{p})^{-s}+\psi^{*}_{4\mathfrak{b}\mathfrak{b}'}(\mathfrak{p})^2\mathcal{N}(\mathfrak{p})^{-1-2s}\right)^{-1}.$$

Let us now define certain operators which will play a vital role in the definition and proof of boundedness of a distribution associated with convolutions of Hilbert automorphic forms of integral weight and Hilbert modular forms of half-integral weight. We will start with the twist of a modular form of half-integral weight and an almost primitive character χ. The final form of the twist is then obtained in the next corollary. Apart from the adaptation to almost primitive characters the proof goes along the well known lines (cf. Proposition 4.4 of [S4]), but the automorphic factors require more attention.

PROPOSITION 3.3. — *Let $f \in \mathcal{M}_k(\mathfrak{b},\mathfrak{b}',\psi)$ be a Hilbert modular form of half-integral weight k, $\chi^*_{\mathfrak{q}}$ an almost primitive ideal class character modulo $\mathfrak{q}$ of conductor $\mathfrak{c}$, and define $\widetilde{\mathfrak{b}}$ by requiring that $4\mathfrak{b}\widetilde{\mathfrak{b}}$ be the least common multiple of $4\mathfrak{b}\mathfrak{b}'$, $\mathfrak{q}^2 4\mathfrak{b}$ and $\mathfrak{q}\mathfrak{c}(\psi)$. Then*

$$f(\chi^*_{\mathfrak{q}})^0_{\mathbf{A}}(x) := \frac{\chi^*(\frac{\mathfrak{q}}{\mathfrak{c}})}{\tau(\overline{\chi})}\sum_{u\in R}\chi_\infty(u)\overline{\chi}^*_{\mathfrak{q}}(u\mathfrak{q}2^{-1}\mathfrak{b}^{-1}\mathfrak{d})f_{\mathbf{A}}\left(xr_P\left(\begin{pmatrix}1 & -u\\ & 1\end{pmatrix}_0\right)\right)$$

for $x \in M_{\mathbf{A}}$

with a system of representatives R of $\mathfrak{q}^{-1}2\mathfrak{b}\mathfrak{d}^{-1}/2\mathfrak{b}\mathfrak{d}^{-1}$ defines a Hilbert modular form $f(\chi^*_{\mathfrak{q}})^0 \in \mathcal{M}_k(\mathfrak{b},\widetilde{\mathfrak{b}},\psi\chi^2)$ with Fourier coefficients

$$\lambda(\xi,\mathfrak{m};f(\chi^*_{\mathfrak{q}})^0,\psi\chi^2) = \begin{cases}\chi^*(\xi\mathfrak{b}\mathfrak{m}^2)\lambda(\xi,\mathfrak{m};f,\psi)\mu(\frac{\mathfrak{q}}{\mathfrak{c}})\prod_{\mathfrak{p}\mid\frac{\mathfrak{q}}{\mathfrak{c}}}(1-\mathcal{N}(\mathfrak{p}))^{\min(1,\nu_{\mathfrak{p}}(\xi\mathfrak{b}\mathfrak{m}^2))}, & \text{if } \xi\neq 0,\\ \delta_{\chi,\varepsilon}\lambda(0,\mathfrak{m};f,\psi)\mu(\mathfrak{q})\prod_{\mathfrak{p}\mid\mathfrak{q}}(1-\mathcal{N}(\mathfrak{p})), & \text{if } \xi = 0.\end{cases}$$

Proof. — First notice that $f(\chi^*_{\mathfrak{q}})^0_{\mathbf{A}}$ is well-defined because $\chi_\infty(u)\overline{\chi}^*_{\mathfrak{q}}$ $(u\mathfrak{q}2^{-1}\mathfrak{b}^{-1}\mathfrak{d})$ and $f_{\mathbf{A}}(xr_P(\begin{pmatrix}1 & -u\\ & 1\end{pmatrix}_0))$ only depend on $u \bmod 2\mathfrak{b}\mathfrak{d}^{-1}$. To prove the automorphy property (18) for $f(\chi^*_{\mathfrak{q}})_{\mathbf{A}}$ it suffices to show

$$f(\chi^*_{\mathfrak{q}})^0_{\mathbf{A}}(xw) = (\psi\chi^2)_{4\mathfrak{b}\mathfrak{b}'}(a_w)^{-1}f(\chi^*_{\mathfrak{q}})^0_{\mathbf{A}}(x)$$

for $x \in M_{\mathbf{A}}$ and $w \in \mathrm{pr}^{-1}(D[2\mathfrak{b}\mathfrak{d}^{-1}, 2\widetilde{\mathfrak{b}\mathfrak{d}}])$ with $w_\infty = 1$ and $h(w,z) = 1$. Write $w = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By the approximation theorem find $\widetilde{d} \in \mathfrak{o}$ with

$$\widetilde{d} \equiv d^{-2} \mod \mathfrak{q}, \quad \widetilde{d} \equiv 1 \ \text{ for } \mathfrak{p}|2\mathfrak{b}\mathfrak{d},\ \mathfrak{p}\not|\,\mathfrak{q}, \quad \text{and } \widetilde{d} \gg 0.$$

Then $\{v := \widetilde{d}u\}$ is also a system of representatives for $\mathfrak{q}^{-1}2\mathfrak{b}\mathfrak{d}^{-1}/2\mathfrak{b}\mathfrak{d}^{-1}$. For each $u \in R$ define $w' = w'_u = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{pr}^{-1}(D[2\mathfrak{b}\mathfrak{d}^{-1}, 2\mathfrak{b}', \mathfrak{d}])$ by $\mathrm{pr}(w')_\infty = 1$, $h(w',z) = 1$, and the relation

$$\begin{pmatrix} 1 & \widetilde{d}u \\ & 1 \end{pmatrix}_0 \begin{pmatrix} a & b \\ c & d \end{pmatrix}_0 = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}_0 \begin{pmatrix} 1 & d^2\widetilde{d}u \\ & 1 \end{pmatrix}_0.$$

Now let $q \in F_{\mathbf{A}}^\times$ with $\widetilde{q} = \mathfrak{q}$ and set $\sigma_q = r_P(\begin{pmatrix} q & \\ & q^{-1} \end{pmatrix}_0)$. Then

$$\begin{aligned}
&h\left(r_P\left(\begin{pmatrix} 1 & \widetilde{d}u \\ & 1 \end{pmatrix}_0\right) w r_P\left(\begin{pmatrix} 1 & -d^2\widetilde{d}u \\ & 1 \end{pmatrix}_0\right), z\right) \\
&\underset{(11)}{=} h\left(\sigma_q r_P\left(\begin{pmatrix} 1 & \widetilde{d}u \\ & 1 \end{pmatrix}_0\right) w r_P\left(\begin{pmatrix} 1 & -d^2\widetilde{d}u \\ & 1 \end{pmatrix}_0\right)\sigma_q^{-1}, z\right) \\
&= h\left(r_P\left(\begin{pmatrix} 1 & q^2\widetilde{d}u \\ & 1 \end{pmatrix}_0\right) \sigma_q w \sigma_q^{-1} r_P\left(\begin{pmatrix} 1 & -q^2d^2\widetilde{d}u \\ & 1 \end{pmatrix}_0\right), z)\right) \\
&\underset{(9),(10)}{=} h\left(\sigma_q w \sigma_q^{-1}, z\right) \underset{(11)}{=} h(w,z) = 1,
\end{aligned}$$

which proves the identity $r_P\left(\begin{pmatrix} 1 & \widetilde{d}u \\ & 1 \end{pmatrix}_0\right) \cdot w = w'_u \cdot r_P\left(\begin{pmatrix} 1 & d^2\widetilde{d}u \\ & 1 \end{pmatrix}_0\right)$ on $M_{\mathbf{A}}$. Now the automorphy property follows from the calculation

$$\begin{aligned}
f(\chi_{\mathfrak{q}}^*)_{\mathbf{A}}^0(xw) &= \frac{\chi^*(\frac{\mathfrak{q}}{\mathfrak{c}})}{\tau(\overline{\chi})} \sum_{u \in R} \chi_\infty(u)\overline{\chi}_{\mathfrak{q}}^*(u\mathfrak{q}2^{-1}\mathfrak{b}^{-1}\mathfrak{d}) \\
&\quad \cdot f_{\mathbf{A}}\left(x r_P\left(\begin{pmatrix} 1 & -\widetilde{d}u \\ & 1 \end{pmatrix}_0\right) w'_u r_P\left(\begin{pmatrix} 1 & u(d^2\widetilde{d}-1) \\ & 1 \end{pmatrix}_0\right)\right) \\
&= \psi_{4\mathfrak{b}\mathfrak{b}'}(a')^{-1}\chi_{\mathfrak{q}}(\widetilde{d}) f(\chi_{\mathfrak{q}}^*)_{\mathbf{A}}^0(x)
\end{aligned}$$

and the congruence properties of $\widetilde{d}$. Finally, to calculate the Fourier coefficients of $f(\chi_{\mathfrak{q}}^*)^0$, let $t \in F_{\mathbf{A}}^\times$ and $s \in F_{\mathbf{A}}$. Then

$$f(\chi_{\mathfrak{q}}^*)^0_{\mathbf{A}}\left(r_P\left(\begin{pmatrix} t & s \\ & t^{-1}\end{pmatrix}\right)\right)$$

$$= \frac{\chi^*(\frac{\mathfrak{q}}{\mathfrak{c}})}{\tau(\overline{\chi})}\sum_{u\in R}\chi_\infty(u)\overline{\chi}_{\mathfrak{q}}^*(u\mathfrak{q}2^{-1}\mathfrak{b}^{-1}\mathfrak{d})f_{\mathbf{A}}\left(r_P\left(\begin{pmatrix} t & -tu_0+s \\ & t^{-1}\end{pmatrix}\right)\right)$$

$$= \chi^*(\frac{\mathfrak{q}}{\mathfrak{c}})\overline{\psi}_f(t)t_\infty^{k'}|t|_{\mathbf{A}}^{\frac{1}{2}}\sum_{\xi\in F}\lambda(\xi,\widetilde{t};f,\psi)e_\infty(\mathbf{i}t^2\xi/2)e_{\mathbf{A}}(ts\xi/2)$$

$$\cdot\frac{1}{\tau(\overline{\chi})}\sum_{u\in R}\chi_\infty(u)\overline{\chi}_{\mathfrak{q}}^*(u\mathfrak{q}2^{-1}\mathfrak{b}^{-1}\mathfrak{d})e_{\mathbf{A}}(-t^2u_0\xi/2)$$

by Proposition 3.1. The desired expression for $\lambda(\xi,\widetilde{t};f(\chi_{\mathfrak{q}}^*)^0,\psi\chi^2)$ now follows from Lemma 1.2 for $\xi=0$ and from Lemma 1.3 otherwise. □

COROLLARY 3.4. — *For a Hecke character χ of finite order let $\mathfrak{q}\subseteq\mathfrak{c}$ be an integral ideal contained in the finite part $\mathfrak{c}$ of the conductor of χ and set $\mathfrak{n}_0 := \prod_{\mathfrak{p}|\mathfrak{q},\mathfrak{p}\not|\mathfrak{c}}\mathfrak{p}$. Then for $f\in\mathcal{M}_k(\mathfrak{b},\mathfrak{b}',\psi)$ the twist $f(\chi_{\mathfrak{q}}^*)$ with the ideal character $\chi_{\mathfrak{q}}^*$ mod $\mathfrak{q}$ defined by*

$$f(\chi_{\mathfrak{q}}^*) := \Phi(\mathfrak{n}_0)\mathcal{N}(\mathfrak{n}_0)^{-1}\sum_{\mathfrak{n}|\mathfrak{n}_0}\mu(\mathfrak{n})\Phi(\mathfrak{n})^{-1}f(\chi_{\mathfrak{cn}}^*)^0$$

is an element of $\mathcal{M}_k(\mathfrak{b},\widetilde{\mathfrak{b}},\psi\chi^2)$ with $\widetilde{\mathfrak{b}} = \mathrm{lcm}(4\mathfrak{b}\mathfrak{b}',(\mathfrak{c}\mathfrak{n}_0)^2 4\mathfrak{b},\mathfrak{c}\mathfrak{n}_0\mathfrak{c}(\psi))/4\mathfrak{b}$. Moreover, its Fourier coefficients are the Fourier coefficients of f twisted with $\chi_{\mathfrak{q}}^*$:

$$\lambda(\xi,\mathfrak{m};f(\chi_{\mathfrak{q}}^*),\psi\chi^2) = \begin{cases}\chi_{\mathfrak{q}}^*(\xi\mathfrak{b}\mathfrak{m}^2)\lambda(\xi,\mathfrak{m};f,\psi), & \text{if } \xi\neq 0,\\ \delta_{\chi_{\mathfrak{q}}^*,\varepsilon_{\mathfrak{o}}^*}\lambda(0,\mathfrak{m};f,\psi), & \text{if } \xi=0.\end{cases}$$

In particular, if $\mathfrak{q}\neq\mathfrak{o}$ or $\chi\neq\varepsilon$ then $f(\chi_{\mathfrak{q}}^*)$ is a cusp form. Conversely, we have

$$f(\chi_{\mathfrak{cn}_0}^*)^0 = \Phi(\mathfrak{n}_0)\sum_{\mathfrak{n}|\mathfrak{n}_0}\mu(\mathfrak{n})\Phi(\mathfrak{n})^{-1}\mathcal{N}(\mathfrak{n})f(\chi_{\mathfrak{cn}}^*).$$

Proof. — It suffices to prove the corollary for $\chi=\varepsilon$ and $\mathfrak{q}=\mathfrak{n}_0$. Now, for a prime ideal $\mathfrak{p}$ the Fourier coefficients of $f(\varepsilon_{\mathfrak{p}}^*)$ are given by the last proposition as

$$\begin{aligned}\lambda(\xi,\mathfrak{m};f(\varepsilon_{\mathfrak{p}}^{*}),\psi) &= \frac{1}{\mathcal{N}(\mathfrak{p})}\left(\Phi(\mathfrak{p})\lambda(\xi,\mathfrak{m};f,\psi) - \lambda(\xi,\mathfrak{m};f(\varepsilon_{\mathfrak{p}}^{*})^{0},\psi)\right) \\ &= \frac{1}{\mathcal{N}(\mathfrak{p})}\left(\Phi(\mathfrak{p})\lambda(\xi,\mathfrak{m};f,\psi) + (-\Phi(\mathfrak{p}))^{\min(1,\nu_{\mathfrak{p}}(\xi\mathfrak{b}\mathfrak{m}^{2}))}\lambda(\xi,\mathfrak{m};f,\psi)\right) \\ &= \begin{cases} \varepsilon_{\mathfrak{p}}^{*}(\xi\mathfrak{b}\mathfrak{m}^{2})\lambda(\xi,\mathfrak{m};f,\psi), & \text{if } \xi\neq 0, \\ 0, & \text{if } \xi = 0.\end{cases}\end{aligned}$$

The general formula for the Fourier coefficients now follows by iteration of the different prime divisors of $\mathfrak{n}_0$. Finally, notice that the converse formula is a special case of the Möbius inversion formula. □

Remark. — It is this twist $f(\chi_{\mathfrak{q}}^{*})$ that we will work with. If a Hilbert automorphic form $\mathbf{f}$ of integral weight l is a Hecke eigenform of $T_{\mathfrak{p}}$, then

$$\mathbf{f}(\varepsilon_{\mathfrak{p}}^{*}) = \mathbf{f} - C(\mathfrak{p},\mathbf{f})\mathbf{f}|\mathfrak{p} + \psi_{\mathfrak{c}(\mathbf{f})}^{*}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{l_0-1}\mathbf{f}|\mathfrak{p}^{2}.$$

This can be used as a definition for $\mathbf{f}(\varepsilon_{\mathfrak{p}}^{*})$. In other words, using the operators $|\mathfrak{p}$, one can cancel the $\mathfrak{p}$-th Hecke polynomial in the denominator of the Euler product. However, in the case of half-integral weight, the Euler product has a nontrivial numerator which depends on the quadratic partial series. Therefore, it is better to work with almost primitive characters. Moreover, our construction also has the advantage of obtaining a twist for every half-integral modular form.

In the rest of this section we want to construct an inverter j on $\mathcal{M}_k(\mathfrak{b},\mathfrak{b}',\psi)$ similar to $J_{\mathfrak{c}}$ in the integral case. In particular, we want to have some commutation relation between j and the above twist similar to Proposition 4.5 of [S4]. We also would like j to map (certain) Hecke eigenforms to Hecke eigenforms and to be "compatible" with $J_{\mathfrak{c}}$ in the sense that there is a relation between $f_1f_2|J_{\mathfrak{c}}$ and $f_1|j\cdot f_2|j$ for two half-integral forms whose product is an integral modular form. The problem we encounter is that the definition of $J_{\mathfrak{c}}$ involves matrices of the group GL_2 having determinants different from 1. Our inverter will only be defined for certain ideals $\mathfrak{b}$, $\mathfrak{b}'$ and it will depend on a choice of a generator of an ideal. However, if we assume that the class number h_F equals 1, we can always define an appropriate inverter.

PROPOSITION 3.5. — a) *Let* $\mathfrak{y}$ *be a fractional ideal and choose any* $y \in F_{\mathbf{A}}^{\times}$ *such that* $y_\infty = 1$ *and* $\widetilde{y} = \mathfrak{y}$. *Then there is a "swop" operator*

$$\mathrm{sw}_{\mathfrak{y}} : \mathcal{M}_k(\mathfrak{b}, \mathfrak{b}', \psi) \to \mathcal{M}_k(\mathfrak{y}^{-2}\mathfrak{b} \cap \mathfrak{o}, \mathfrak{y}^2\mathfrak{b}' \cap \mathfrak{o}, \psi),$$
$$(f\,\mathrm{sw}_{\mathfrak{y}})_{\mathbf{A}}(x) := \overline{\psi}_f(y)\mathcal{N}(\mathfrak{y})^{-\frac{1}{2}} f_{\mathbf{A}}\left(xr_P\left(\begin{pmatrix} y^{-1} & \\ & y\end{pmatrix}\right)\right),$$

which is a bijection if both $\mathfrak{y}^{-2}\mathfrak{b}$ *and* $\mathfrak{y}^2\mathfrak{b}'$ *are integral. The definition only depends on* $\widetilde{y} = \mathfrak{y}$, *and the Fourier coefficients satisfy*

$$\lambda(\xi, \mathfrak{m}; f\,\mathrm{sw}_{\mathfrak{y}}, \psi) = \lambda(\xi, \mathfrak{y}^{-1}\mathfrak{m}; f, \psi) \quad \textit{for all } 0 \ll \xi \in F \textit{ and ideals } \mathfrak{m} \textit{ of } F.$$

Moreover, if $\mathfrak{y} = (\beta)$ *for some (not necessarily totally positive)* $\beta \in F$, *then*

$$(f\,\mathrm{sw}_{(\beta)})(z) = \overline{\psi}_f(\beta)\beta^{k'} f(\beta^2 z).$$

(b) *For every totally positive* $\tau \in F$ *there is an operator (multiplication by* τ*)*

$$\mathrm{m}_\tau : \mathcal{M}_k(\mathfrak{b}, \mathfrak{b}', \psi) \to \mathcal{M}_k(\tau^{-1}\mathfrak{b} \cap \mathfrak{o}, \tau\mathfrak{b}' \cap \mathfrak{o}, \psi\varepsilon_\tau), \quad (f\,\mathrm{m}_\tau)(z) := f(\tau z)$$

where ε_τ *denotes the quadratic Hecke character of* F *corresponding to* $F(\sqrt{\tau})/F$. *The effect on the Fourier coefficients is given by*

$$\lambda(\xi, \mathfrak{m}; f\,\mathrm{m}_\tau, \psi\varepsilon_\tau) = \lambda(\tau^{-1}\xi, \mathfrak{m}; f, \psi) \quad \textit{for all } 0 \ll \xi \in F \textit{ and ideals } \mathfrak{m} \textit{ of } F.$$

(c) *There exists a unique element* η' *of* $M_{\mathbf{A}}$ *such that* $h(\eta', z) = J_k(\eta', z) = 1$ *and* $\mathrm{pr}(\eta') = \eta$ *with* η *of* (5). *Then*

$$\iota : \mathcal{M}_k(\mathfrak{b}, \mathfrak{b}', \psi) \to \mathcal{M}_k(\mathfrak{b}', \mathfrak{b}, \overline{\psi}), \quad (f\iota)_{\mathbf{A}}(x) := \psi_f(\delta) f_{\mathbf{A}}(x\eta')$$

defines an inverter independent of the choice of δ *with the property* $f\iota^2 = \psi_\infty(-\mathbf{1})f$ *for* $f \in \mathcal{M}_k(\mathfrak{b}, \mathfrak{b}', \psi)$. *Moreover, if* B *is an open subgroup of* C'' *such that* $f_{\mathbf{A}}(x) = (f\|_k x)(\mathbf{i})$ *for* $x \in \mathrm{pr}^{-1}(BG_\infty)$, *and if* $\hat{\eta}$ *is an element of* G *such that* $\hat{\eta}\eta \in BG_\infty$, *then*

$$f\iota = \psi_f(\delta) f\|_k \hat{\eta}.$$

Proof. — a) and b) are mainly a restatement of Proposition 1.4 of [S8]. c) is given in Lemma 2.3 of [loc. cit.] and the discussion following it, and the same arguments as for the independence of ι from δ also show the independence of $\mathrm{sw}_{\mathfrak{y}}$ from y. Finally, the formula for $f\,\mathrm{sw}_{(\beta)}$ is obtained as follows: Let $z = x + iy \in \mathbb{H}^n$. Then

$$\begin{aligned}(f\,\mathrm{sw}_{(\beta)})_{\mathbf{A}}\left(r_P\left(\begin{pmatrix}\sqrt{y} & \frac{x}{\sqrt{y}}\\ & \frac{1}{\sqrt{y}}\end{pmatrix}_{\infty}\right)\right) &= \sqrt{y}^{k}\sum_{\xi\in F}\lambda(\xi,\mathfrak{o};f\,\mathrm{sw}_{(\beta)},\psi)e_{\infty}(\xi z/2)\\ &= \sqrt{y}^{k}(f\,\mathrm{sw}_{(\beta)})(z)\end{aligned}$$

by applying Proposition 3.1, but we also have

$$\begin{aligned}(f\,\mathrm{sw}_{(\beta)})_{\mathbf{A}}\left(r_P\left(\begin{pmatrix}\sqrt{y} & \frac{x}{\sqrt{y}}\\ & \frac{1}{\sqrt{y}}\end{pmatrix}_{\infty}\right)\right) &= \overline{\psi}_f(\beta)|\mathcal{N}(\beta)|^{-\frac{1}{2}}\\ \cdot f_{\mathbf{A}}\left(r_P\left(\begin{pmatrix}\beta & \\ & \beta^{-1}\end{pmatrix}\right)r_P\left(\begin{pmatrix}\sqrt{y} & \frac{x}{\sqrt{y}}\\ & \frac{1}{\sqrt{y}}\end{pmatrix}_{\infty}\right)\right.&\left.r_P\left(\begin{pmatrix}\beta^{-1} & \\ & \beta\end{pmatrix}_0\right)\right)\\ &= \overline{\psi}_f(\beta)|\mathcal{N}(\beta)|^{-\frac{1}{2}}(\beta\sqrt{y})^{k'}|\beta\sqrt{y}|_{\infty}^{\frac{1}{2}}f(\beta^2 z).\end{aligned}$$

This proves the last formula of a). □

Proposition 3.6. — *The following diagrams are commutative for all* $0 \ll \tau \in \mathfrak{o}$ *and integral ideals* $\mathfrak{y}$ *of* F*:*

a)
$$\begin{array}{ccc}\mathcal{M}_k(\tau\mathfrak{y}^2\mathfrak{b},\mathfrak{b}',\psi) & \xrightarrow{\mathrm{m}_\tau} & \mathcal{M}_k(\mathfrak{y}^2\mathfrak{b},\tau\mathfrak{b}',\psi\varepsilon_\tau)\\ {\scriptstyle \mathrm{sw}_{\mathfrak{y}}}\downarrow & & \downarrow{\scriptstyle \mathrm{sw}_{\mathfrak{y}}}\\ \mathcal{M}_k(\tau\mathfrak{b},\mathfrak{y}^2\mathfrak{b}',\psi) & \xrightarrow{\mathrm{m}_\tau} & \mathcal{M}_k(\mathfrak{b},\tau\mathfrak{y}^2\mathfrak{b}',\psi\varepsilon_\tau)\end{array}$$

b)
$$\begin{array}{ccc}\mathcal{M}_k(\mathfrak{b},\mathfrak{y}^2\mathfrak{b}',\psi) & \xrightarrow{\iota} & \mathcal{M}_k(\mathfrak{y}^2\mathfrak{b}',\mathfrak{b},\overline{\psi})\\ {\scriptstyle \mathcal{N}(\mathfrak{y})^{-1}\mathrm{sw}_{\mathfrak{y}^{-1}}}\downarrow & & \downarrow{\scriptstyle \mathrm{sw}_{\mathfrak{y}}}\\ \mathcal{M}_k(\mathfrak{y}^2\mathfrak{b},\mathfrak{b}',\psi) & \xrightarrow{\iota} & \mathcal{M}_k(\mathfrak{b}',\mathfrak{y}^2\mathfrak{b},\overline{\psi})\end{array}$$

c)
$$\begin{array}{ccc}\mathcal{M}_k(\mathfrak{b},\tau\mathfrak{b}',\psi) & \xrightarrow{\iota} & \mathcal{M}_k(\tau\mathfrak{b}',\mathfrak{b},\overline{\psi})\\ {\scriptstyle v(\tau)\tau^{-k'}\mathcal{N}(\tau)^{-\frac{1}{2}}\mathrm{m}_{\tau^{-1}}}\downarrow & & \downarrow{\scriptstyle \mathrm{m}_\tau}\\ \mathcal{M}_k(\tau\mathfrak{b},\mathfrak{b}',\psi\varepsilon_\tau) & \xrightarrow{\iota} & \mathcal{M}_k(\mathfrak{b}',\tau\mathfrak{b},\overline{\psi\varepsilon_\tau})\end{array}$$

with some homomorphism v *of* $\mathfrak{o}^\times$ *into* $\{\pm 1\}$.

Proof. — a) follows directly by computing the Fourier coefficients of $f\,\mathrm{sw}_{\mathfrak{y}}\,\mathrm{m}_\tau$ and $f\,\mathrm{m}_\tau\,\mathrm{sw}_{\mathfrak{y}}$. For b), let f be an element of $\mathcal{M}_k(\mathfrak{b},\mathfrak{y}^2\mathfrak{b}',\psi)$, y a finite idele with $\widetilde{y} = \mathfrak{y}$, η' as in Proposition 3.5c) and observe that

$r_P\left(\begin{pmatrix} y^{-1} & \\ & y\end{pmatrix}\right) r_\Omega(\eta)\, r_P\left(\begin{pmatrix} y^{-1} & \\ & y\end{pmatrix}\right) = r_\Omega(\eta)$ by (7). This gives

$$\begin{aligned}(f\iota\,\mathrm{sw}_{\mathfrak{y}})_{\mathbf{A}}(x) &= \psi_f(y\delta)\mathcal{N}(\mathfrak{y})^{-\frac{1}{2}} f_{\mathbf{A}}\left(x r_P\left(\begin{pmatrix} y^{-1} & \\ & y\end{pmatrix}\right)\eta'\right)\\ &= \psi_f(y\delta)\mathcal{N}(\mathfrak{y})^{-\frac{1}{2}} f_{\mathbf{A}}\left(x\eta' r_P\left(\begin{pmatrix} y & \\ & y^{-1}\end{pmatrix}\right)\right)\\ &= \mathcal{N}(\mathfrak{y})^{-1}(f\,\mathrm{sw}_{\mathfrak{y}^{-1}}\,\iota)_{\mathbf{A}}(x)\end{aligned}$$

and proves b). For c) let $f \in \mathcal{M}_k(\mathfrak{b}, \tau\mathfrak{b}', \psi)$, let B be an open subgroup of C' such that $f_{\mathbf{A}}(x) = (f\|_k x)(\mathbf{i})$ for $x \in \mathrm{pr}^{-1}(BG_\infty)$, and let $\hat\eta = \hat\eta(\tau)$ be an element of G such that $\hat\eta\eta \in BG_\infty \cap \begin{pmatrix} 1 & \\ & \tau\end{pmatrix} BG_\infty \begin{pmatrix} 1 & \\ & \tau^{-1}\end{pmatrix}$. Then

$$(f\,\mathrm{m}_\tau\,\iota\,\mathrm{m}_\tau)(z) = \psi_f(\delta) J_k(\hat\eta, \tau z)^{-1} f(\tau\hat\eta\tau z).$$

Set $\xi := \begin{pmatrix} 1 & \\ & \tau^{-1}\end{pmatrix}\hat\eta\begin{pmatrix} \tau & \\ & 1\end{pmatrix} \in G$. Then $\xi\eta = \begin{pmatrix} 1 & \\ & \tau^{-1}\end{pmatrix}\hat\eta\eta\begin{pmatrix} 1 & \\ & \tau\end{pmatrix}\begin{pmatrix} \tau & \\ & \tau^{-1}\end{pmatrix}_\infty$ $\in BG_\infty$, and hence $f\iota = \psi_f(\delta) f\|_k\xi$ by Proposition 3.5c). In order to calculate $J_k(\xi, z)/J_k(\hat\eta, \tau z)$ define $\zeta := \begin{pmatrix} \tau^{-1} & \\ & \tau\end{pmatrix}\xi = \begin{pmatrix} \tau^{-1} & \\ & 1\end{pmatrix}\hat\eta\begin{pmatrix} \tau & \\ & 1\end{pmatrix}$ and notice that $\zeta \in \Omega_{\mathbf{A}} \cap P_{\mathbf{A}} C''$: From $\eta \in C''$ and $\xi\eta \in BG_\infty \subseteq C''G_\infty$ it follows that $\xi \in C''G_\infty$ and hence $\zeta \in P_{\mathbf{A}}C''$. If $\xi \notin \Omega_{\mathbf{A}}$, then $\xi = \begin{pmatrix} a^{-1} & b \\ & a\end{pmatrix} \in P$, and thus $(\xi\eta)_0 = \begin{pmatrix} b\delta & -a^{-1}\delta^{-1} \\ a\delta & \end{pmatrix} \in B_0 \subseteq C_0'$, a contradiction which shows that both ξ and ζ are elements of $\Omega_{\mathbf{A}}$. We now understand G embedded in $M_{\mathbf{A}}$ via r and note that

$$J_k(\xi, z) = J_k\left(\begin{pmatrix} \tau & \\ & \tau^{-1}\end{pmatrix}\zeta, z\right) = |\tau^{-1}|_\infty^{\frac{1}{2}}\tau^{-k'} J_k(\zeta, z),$$

$$j(\zeta, z) = j\left(\begin{pmatrix} \tau^{-1} & \\ & 1\end{pmatrix}, \hat\eta\tau z\right) j(\hat\eta, \tau z) j\left(\begin{pmatrix} \tau & \\ & 1\end{pmatrix}, z\right) = j(\hat\eta, \tau z).$$

Therefore, the quotient $J_k(\xi, z)/J_k(\hat\eta, \tau z)$ is $\mathcal{N}(\tau)^{-\frac{1}{2}}\tau^{-k'}v(\tau)$ with

$$v(\tau) := \frac{J_k(\zeta, z)}{J_k(\hat\eta, \tau z)} = \lim_{\rho\to\infty} \frac{h(\zeta, \rho\mathbf{i})}{|h(\zeta, \rho\mathbf{i})|} \Big/ \frac{h(\hat\eta, \rho\mathbf{i})}{|h(\hat\eta, \rho\mathbf{i})|} \in \mathbb{C}.$$

This is a quotient of Gauss sums and absolute values thereof, see (14). Replacing f by $f\,\mathrm{m}_{\tau^{-1}}$ now gives the commutativity of the diagram. v is necessarily a homomorphism, as can be seen by considering $\iota\,\mathrm{m}_{\tau_1}\,\mathrm{m}_{\tau_2} = \iota\,\mathrm{m}_{\tau_1\tau_2}$. Also, $v(\tau)$ does not really depend on f because the subgroup B used in the definition of $\hat\eta = \hat\eta(\tau)$ can be replaced by any sufficiently small subgroup of C'. Finally, we calculate $f\iota\,\mathrm{m}_{\tau^2}$ in two different ways using Proposition 3.5a) and obtain

$$\begin{aligned} f\iota\,\mathrm{m}_{\tau^2} &= (f\iota\,\mathrm{m}_\tau)\,\mathrm{m}_\tau = \mathcal{N}(\tau)^{-1}v(\tau)^2\tau^{-2k'}f\,\mathrm{m}_{\tau^{-2}}\,\iota \\ &= \overline{\psi}_f(\tau)\tau^{-k'}f\iota\,\mathrm{sw}_{(\tau)} = \overline{\psi}_f(\tau)\tau^{-k'}\mathcal{N}(\tau)^{-1}f\,\mathrm{sw}_{(\tau^{-1})}\,\iota \\ &= \mathcal{N}(\tau)^{-1}\tau^{-2k'}f\,\mathrm{m}_{\tau^{-2}}\,\iota. \end{aligned}$$

This shows that $v(\tau) \in \{\pm 1\}$ as claimed. □

PROPOSITION 3.7. — *The following diagrams are commutative for all integral ideals* $\mathfrak{b}$, $\mathfrak{b}'$, *totally positive* $\tau \in \mathfrak{o}$, *integral ideals* $\mathfrak{y}$ *of* F *and ideal characters* $\chi_\mathfrak{q}^*$ *mod* $\mathfrak{q}$:

a)
$$\begin{array}{ccc} \mathcal{M}_k(\mathfrak{b},\mathfrak{b}',\psi) & \xrightarrow{(\chi_\mathfrak{q}^*)} & \mathcal{M}_k(\mathfrak{b},\widetilde{\mathfrak{b}},\psi\chi^2) \\ T_\mathfrak{p}\downarrow & & \downarrow T_\mathfrak{p} \\ \mathcal{M}_k(\mathfrak{b},\mathfrak{b}',\psi) & \xrightarrow{\chi_\mathfrak{q}^*(\mathfrak{p}^2)(\chi_\mathfrak{q}^*)} & \mathcal{M}_k(\mathfrak{b},\widetilde{\mathfrak{b}},\psi\chi^2) \end{array}$$

with $\widetilde{\mathfrak{b}}$ *as in Corollary 3.4.*

b)
$$\begin{array}{ccc} \mathcal{M}_k(\tau\mathfrak{b},\mathfrak{b}',\psi) & \xrightarrow{\mathrm{m}_\tau} & \mathcal{M}_k(\mathfrak{b},\tau\mathfrak{b}',\psi\varepsilon_\tau) \\ T_\mathfrak{p}\downarrow & & \downarrow T_\mathfrak{p} \\ \mathcal{M}_k(\tau\mathfrak{b},\mathfrak{b}',\psi) & \xrightarrow{\mathrm{m}_\tau} & \mathcal{M}_k(\mathfrak{b},\tau\mathfrak{b}',\psi\varepsilon_\tau) \end{array}$$

c)
$$\begin{array}{ccc} \mathcal{M}_k(\mathfrak{y}^2\mathfrak{b},\mathfrak{b}',\psi) & \xrightarrow{\mathrm{sw}_\mathfrak{y}} & \mathcal{M}_k(\mathfrak{b},\mathfrak{y}^2\mathfrak{b}',\psi) \\ T_\mathfrak{p}\downarrow & & \downarrow T_\mathfrak{p} \\ \mathcal{M}_k(\mathfrak{y}^2\mathfrak{b},\mathfrak{b}',\psi) & \xrightarrow{\mathrm{sw}_\mathfrak{y}} & \mathcal{M}_k(\mathfrak{b},\mathfrak{y}^2\mathfrak{b}',\psi). \end{array}$$

In particular, $\chi_\mathfrak{q}^*$, m_τ *and* $\mathrm{sw}_\mathfrak{y}$ *map Hecke eigenforms to Hecke eigenforms.*

Proof. — The effect of all three operators on the Fourier coefficients of a modular form is given in Corollary 3.4 and Proposition 3.5. Thus the diagram relations can be verified via the Fourier coefficients using the description (19) of $T_\mathfrak{p}$. □

LEMMA 3.8. — *For a finite prime* $\mathfrak{p}$ *the local Gauss sum* $\gamma_\mathfrak{p}$ *of* (12) *has the following properties:*

a) $\gamma_\mathfrak{p}(a) = 1$ *if* $a \in 2\mathfrak{d}_\mathfrak{p}^{-1}$.

b) $\gamma_{\mathfrak{p}}(\pi^{-2i}a) = \mathcal{N}(\mathfrak{p})^{-i}\gamma_{\mathfrak{p}}(a)$ *if* $\mathfrak{p} \nmid 2$, π *is a prime element of* $F_{\mathfrak{p}}$, $a \in F_{\mathfrak{p}} \backslash \mathfrak{p}_{\mathfrak{p}}\mathfrak{d}_{\mathfrak{p}}^{-1}$, *and* $i \geqslant 0$.

c) $\gamma_{\mathfrak{p}}(a) = \dfrac{1}{\mathcal{N}(\mathfrak{p})} \sum_{x \bmod \mathfrak{p}} \left(\dfrac{x}{\mathfrak{p}}\right) e_{\mathfrak{p}}(ax/2)$ *if* $\mathfrak{p} \nmid 2$, *and* $a \in \mathfrak{p}_{\mathfrak{p}}^{-1}\mathfrak{d}_{\mathfrak{p}}^{-1} \backslash \mathfrak{d}_{\mathfrak{p}}^{-1}$.

This is - in a different notation - proved in §54 of [H].

LEMMA 3.9. — *Assume* $h_F = 1$, *let* $\mathfrak{q}$ *be an integral ideal prime to* 2 *and fix an idele* $q \in F_{\mathbf{A}}^{\times}$ *with* $\widetilde{q} = \mathfrak{q}$. *If* $u, v \in F_{\mathbf{A}}^{\times}$ *are such that*

$$u \in 2\mathfrak{q}^{-1}\mathfrak{d}^{-1}, \quad v \in 2\mathfrak{q}^{-1}\mathfrak{d}^{-1}, \quad \textit{and} \quad uvq^2\delta^2 \equiv -1 \bmod \mathfrak{q},$$

then the Gauss sum γ *of* (13) *evaluates to*

$$\frac{\gamma(u)}{|\gamma(u)|} = \varepsilon_{\mathfrak{q}} \left(\frac{2}{\mathfrak{q}}\right) \chi^{\mathfrak{q}*}(v\mathfrak{q}\mathfrak{d})\chi_{\infty}^{\mathfrak{q}}(v).$$

Here, the root of unity $\varepsilon_{\mathfrak{q}}$ *depends only on* $\mathfrak{q}$ *modulo square ideals and is explicitly given as*

$$\varepsilon_{\mathfrak{q}} = \prod_{\mathfrak{p}|\mathfrak{q}} \frac{\tau(\chi^{\mathfrak{p}})}{\sqrt{\mathcal{N}(\mathfrak{p})}} \chi^{\mathfrak{p}*}(\mathfrak{q}/\mathfrak{p}^{\nu_{\mathfrak{p}}(\mathfrak{q})})$$

for $\mathfrak{q}$ *squarefree. If* $F = \mathbb{Q}$ *and* $q > 0$, *then* $\varepsilon_{(q)} = \begin{cases} 1, & \textit{if } q \equiv 1 \bmod 4, \\ i, & \textit{if } q \equiv 3 \bmod 4. \end{cases}$

Proof. — Let $\mathfrak{p}|\mathfrak{q}$ be a prime ideal, and $\chi^{\mathfrak{p}}$ respectively $F^{\mathfrak{p}} = F(\sqrt{\tau})$ the quadratic idele character respectively quadratic extension of F of Lemma 1.4 which is only ramified in $\mathfrak{p}$ and possibly at infinite primes. We may assume that $\nu_{\mathfrak{p}}(\tau) = 1$ and hence can take $\pi = \tau$ as prime element of $F_{\mathfrak{p}}$. Then

$$\left(\frac{x}{\mathfrak{p}}\right) = \left(\frac{x, \pi}{\mathfrak{p}}\right) = \chi_{\mathfrak{p}}^{\mathfrak{p}}(x) \quad \text{for } x \in \mathfrak{o}_{\mathfrak{p}}^{\times}.$$

If $\nu_{\mathfrak{p}}(\mathfrak{q})$ is even, then a) and b) of the previous lemma show $\dfrac{\gamma_{\mathfrak{p}}(u)}{|\gamma_{\mathfrak{p}}(u)|} = 1$. If $\nu_{\mathfrak{p}}(\mathfrak{q}) = 1+2m$ we obtain by part c) of the previous lemma and the product formula (2) for the Gauss sum of $\chi^{\mathfrak{p}}$

$$\begin{aligned}\gamma_{\mathfrak{p}}(\pi^{2m}u) &= \frac{1}{\mathcal{N}(\mathfrak{p})}\sum_{x \bmod \mathfrak{p}}\left(\frac{x}{\mathfrak{p}}\right)e_{\mathfrak{p}}\left(\frac{\pi^{2m}u}{2}x\right)\\ &= \frac{1}{\mathcal{N}(\mathfrak{p})}\sum_{x \bmod \mathfrak{p}, x\notin \mathfrak{p}_{\mathfrak{p}}}\chi_{\mathfrak{p}}^{\mathfrak{p}}(x)e_{\mathfrak{p}}\left(\frac{\pi^{2m}u}{2}x\right)\\ &= \frac{1}{\mathcal{N}(\mathfrak{p})}\chi_{\mathfrak{p}}^{\mathfrak{p}}\left(-\frac{\pi^{2m}u}{2}\right)\tau_{\mathfrak{p}}(\chi_{\mathfrak{p}}^{\mathfrak{p}})\\ &= \frac{1}{\mathcal{N}(\mathfrak{p})}\left(\frac{2}{\mathfrak{p}}\right)\chi_{\mathfrak{p}}^{\mathfrak{p}}(-u)\tau(\chi^{\mathfrak{p}})\chi^{\mathfrak{p}*}(\mathfrak{d}/\mathfrak{p}^{\nu_{\mathfrak{p}}(\mathfrak{d})}).\end{aligned}$$

The conditions on u and v imply $\chi_{\mathfrak{p}}^{\mathfrak{p}}(-u) = \chi_{\mathfrak{p}}^{\mathfrak{p}}(v)$, and together with part b) of the previous lemma we obtain

$$\frac{\gamma_{\mathfrak{p}}(u)}{|\gamma_{\mathfrak{p}}(u)|} = \frac{\gamma_{\mathfrak{p}}(\pi^{2m}u)}{|\gamma_{\mathfrak{p}}(\pi^{2m}u)|} = \left(\frac{2}{\mathfrak{p}}\right)\frac{\tau(\chi^{\mathfrak{p}})}{\sqrt{\mathcal{N}(\mathfrak{p})}}\chi^{\mathfrak{p}*}(\mathfrak{q}/\mathfrak{p}^{\nu_{\mathfrak{p}}(\mathfrak{d})})\chi^{\mathfrak{p}*}(\mathfrak{q}\mathfrak{d}/\mathfrak{p}^{\nu_{\mathfrak{p}}(\mathfrak{q}\mathfrak{d})})\chi_{\mathfrak{p}}^{\mathfrak{p}}(v).$$

This proves the formula of the lemma. If $F = \mathbb{Q}$, then it is well-known that

$$\tau(\chi^{p}) = \sqrt{p^*} = \begin{cases}\sqrt{p}, & \text{if } p \equiv 1 \bmod 4,\\ i\sqrt{p}, & \text{if } p \equiv 3 \bmod 4\end{cases}$$

for a positive rational prime number p. Let s be the number of prime factors of q congruent to 3 mod 4. If $q > 0$ and q is squarefree, then

$$\varepsilon_{(q)} = i^s \prod_{p_1|q}\prod_{\substack{p_2|q\\ p_2\neq p_1}}\left(\frac{p_2}{p_1}\right) = i^s(-1)^{\binom{s}{2}}.$$

The proof is finished by observing that the formula for $\varepsilon_{(q)}$ only depends on q modulo squares and coincides with the expression of the last formula for q squarefree. □

Proposition 3.10. — *Let $\chi_{\mathfrak{q}}^*$ be an almost primitive character mod $\mathfrak{q}$ of conductor $\mathfrak{c}$ dividing $\mathfrak{q}$, $\tau \in \mathfrak{o}$ a totally positive integral element of F, and $\mathfrak{y}$ an integral ideal. Set $\widetilde{\mathfrak{b}} = \operatorname{lcm}(4\mathfrak{b}\mathfrak{b}', \mathfrak{q}^2 4\mathfrak{b}, \mathfrak{q}\mathfrak{c}(\psi))/4\mathfrak{b}$. Then the following diagrams are commutative:*

a)

$$\begin{array}{ccc}\mathcal{M}_k(\tau\mathfrak{b},\mathfrak{b}',\psi) & \xrightarrow{(\chi_{\mathfrak{q}}^*)^0} & \mathcal{M}_k(\tau\mathfrak{b},\widetilde{\mathfrak{b}},\psi\chi^2)\\ \Big\downarrow{\scriptstyle \mathrm{m}_\tau} & & \Big\downarrow{\scriptstyle \mathrm{m}_\tau}\\ \mathcal{M}_k(\mathfrak{b},\tau\mathfrak{b}',\psi\varepsilon_\tau) & \xrightarrow{(\chi_{\mathfrak{q}}^*)^0} & \mathcal{M}_k(\mathfrak{b},\tau\widetilde{\mathfrak{b}},\psi\varepsilon_\tau\chi^2)\end{array}$$

b)
$$\begin{array}{ccc} \mathcal{M}_k(\mathfrak{y}^2\mathfrak{b},\mathfrak{b}',\psi) & \xrightarrow{(\chi_{\mathfrak{q}}^*)^0} & \mathcal{M}_k(\mathfrak{y}^2\mathfrak{b},\widetilde{\mathfrak{b}},\psi\chi^2) \\ {\scriptstyle \mathrm{sw}_{\mathfrak{y}}}\downarrow & & \downarrow{\scriptstyle \mathrm{sw}_{\mathfrak{y}}} \\ \mathcal{M}_k(\mathfrak{b},\mathfrak{y}^2\mathfrak{b}',\psi) & \xrightarrow{(\chi_{\mathfrak{q}}^*)^0} & \mathcal{M}_k(\mathfrak{b},\mathfrak{y}^2\widetilde{\mathfrak{b}},\psi\chi^2) \end{array}$$

c)
$$\begin{array}{ccc} \mathcal{M}_k(\mathfrak{b},\mathfrak{b}',\psi) & \xrightarrow{(\chi_{\mathfrak{q}}^*)^0} & \mathcal{M}_k(\mathfrak{b},\mathfrak{q}^2\mathfrak{b}',\psi\chi^2) \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota\,\mathrm{sw}_{\mathfrak{q}}} \\ \mathcal{M}_k(\mathfrak{b}',\mathfrak{b},\overline{\psi}) & \xrightarrow{\Lambda(\mathfrak{q},\chi)(\overline{(\chi\chi^{\mathfrak{q}})_{\mathfrak{q}}}^*)^0} & \mathcal{M}_k(\mathfrak{b}',\mathfrak{q}^2\mathfrak{b},\overline{\psi\chi^2}) \end{array}$$

if $h_F = 1$ and $\mathfrak{q}$, $4\mathfrak{b}\mathfrak{b}'$ are relatively prime. Here $\Lambda(\mathfrak{q},\chi) = \Lambda(\mathfrak{q},\chi;\psi)$ is the constant

$$\Lambda(\mathfrak{q},\chi) = \psi^*(\mathfrak{q})(\chi\chi^{\mathfrak{q}})^*(4\mathfrak{b}\mathfrak{b}'\frac{\mathfrak{q}}{\mathfrak{c}(\chi\chi^{\mathfrak{q}})})\varepsilon_{\mathfrak{q}}^{-1}i^n(\chi^{\mathfrak{q}})^*(\mathfrak{b})\chi^*(\frac{\mathfrak{q}}{\mathfrak{c}(\chi)})\frac{\tau(\chi\chi^{\mathfrak{q}})}{\sqrt{\mathcal{N}(\mathfrak{q})}\cdot\overline{\tau(\chi)}}.$$

Proof. — We will only prove c) since it is only here that the automorphic factors require special attention. Let R be a system of representatives for $\mathfrak{q}^{-1}2\mathfrak{b}\mathfrak{d}^{-1}/2\mathfrak{b}\mathfrak{d}^{-1}$, fix a finite idele $q \in F_{\mathbf{A}}^{\times}$ such that $\widetilde{q} = \mathfrak{q}$, and let η' be as in Proposition 3.5c). If $f \in \mathcal{M}_k(\mathfrak{b},\mathfrak{b}',\psi)$ then

$$(f(\chi_{\mathfrak{q}}^*)^0\iota\,\mathrm{sw}_{\mathfrak{q}})_{\mathbf{A}}(x) = \psi\chi^2(q\delta)\mathcal{N}(\mathfrak{q})^{-\frac{1}{2}}\frac{\chi^*(\frac{\mathfrak{q}}{\mathfrak{c}})}{\tau(\overline{\chi})} \cdot \sum_{u\in R}\chi_\infty(u)\overline{\chi}_{\mathfrak{q}}^*(u\mathfrak{q}2^{-1}\mathfrak{b}^{-1}\mathfrak{d})f_{\mathbf{A}}\left(xr_P\begin{pmatrix}q^{-1}&\\&q\end{pmatrix}\eta' r_P\begin{pmatrix}1&-u\\&1\end{pmatrix}_0\right).$$

For $u \in R$ such that $u\mathfrak{q}2^{-1}\mathfrak{b}^{-1}\mathfrak{d}$ is relatively prime to $\mathfrak{q}$ find a $v \in \mathfrak{q}^{-1}2\mathfrak{b}'\mathfrak{d}^{-1}$ with

$$uvq^2\delta^2 \equiv -1 \mod \mathfrak{q}, \qquad uvq^2\delta^2 \equiv 0 \mod 4\mathfrak{b}\mathfrak{b}'.$$

Now, choose a system of representatives S for $\mathfrak{q}^{-1}2\mathfrak{b}'\mathfrak{d}^{-1}/2\mathfrak{b}'\mathfrak{d}^{-1}$ containing these v, and notice that these v are precisely those representatives whose associated ideals $v\mathfrak{q}2^{-1}\mathfrak{b}'^{-1}\mathfrak{d}$ are relatively prime to $\mathfrak{q}$. Furthermore,

$$\begin{pmatrix}q^{-1}&\\&q\end{pmatrix}\eta\begin{pmatrix}1&-u\\&1\end{pmatrix}_0 = \begin{pmatrix}1&-v\\&1\end{pmatrix}_0 w\eta$$

with an element $w \in D[2\mathfrak{b}'\mathfrak{d}^{-1}, 2\mathfrak{b}\mathfrak{d}]$ which is given by $w_f = \begin{pmatrix} (1+uvq^2\delta^2)/q^{-1} & vq \\ uq\delta^{-1} & q \end{pmatrix}$ and $w_\infty = \mathbf{1}$. Therefore, there is a relation

$$(*) \qquad r_P(\begin{pmatrix} 1 & v \\ & 1 \end{pmatrix}_0) r_P(\begin{pmatrix} q^{-1} & \\ & q \end{pmatrix}) \eta' r_P(\begin{pmatrix} 1 & -u \\ & 1 \end{pmatrix}_0) = w'\eta' \cdot t_u^{-1}$$

on the metaplectic group with some $t_u \in T$ where w' is the unique element of $\mathrm{pr}^{-1}(w)$ such that $h(w', z) = 1$. Calculating the automorphic factors of both sides gives

$$h\left(\eta' r_P \begin{pmatrix} 1 & -u \\ & 1 \end{pmatrix}_0, z\right) = t_u \cdot h(w', \eta' z) h(\eta', z) = t_u.$$

We will now determine the value of t_u. Define $\zeta \in G_{\mathbf{A}}$ by $\zeta_0 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}_0$ and $\zeta_\infty = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}_\infty$. Then there is a unique element $\zeta' \in M_{\mathbf{A}}$ which satisfies $\mathrm{pr}(\zeta') = \zeta$ and $h(\zeta', z) = \prod_{\nu=1}^{n} \sqrt{(-z_\nu)^{-1}}$ with the complex square root $\sqrt{w}$ for $w \in \mathbb{C}$ determined by $-\frac{\pi}{2} < \arg(\sqrt{w}) \leqslant \frac{\pi}{2}$. Now $\zeta\eta \in \Omega_{\mathbf{A}} \cap P_{\mathbf{A}} C''$, and from

$$\lim_{\rho\to\infty} \frac{h(r_\Omega(\zeta\eta), \rho\mathbf{i})}{|h(r_\Omega(\zeta\eta), \rho\mathbf{i})|} \underset{(14)}{=} \frac{\gamma(0)}{|\gamma(0)|} = 1,$$

$$\lim_{\rho\to\infty} \frac{h(\zeta'\eta', \rho\mathbf{i})}{|h(\zeta'\eta', \rho\mathbf{i})|} \underset{(9)}{=} \lim_{\rho\to\infty} \frac{h(\zeta', \eta'(\rho\mathbf{i}))}{|h(\zeta', \eta'(\rho\mathbf{i}))|} \cdot \frac{h(\eta', \rho\mathbf{i})}{|h(\eta', \rho\mathbf{i})|} = \sqrt{i}^{\,n}$$

we conclude that $\zeta'\eta' = \sqrt{i}^{\,-n} r_\Omega(\zeta\eta)$. This gives

$$\eta' r_P\left(\begin{pmatrix} 1 & -u \\ & 1 \end{pmatrix}_0\right) \zeta'\zeta'\eta' r_P\left(\begin{pmatrix} 1 & -u \\ & 1 \end{pmatrix}_0\right) \sqrt{i}^{\,-n} r_\Omega\left(\zeta\eta \begin{pmatrix} 1 & -u \\ & 1 \end{pmatrix}_0\right).$$

Applying Lemma 3.9 we now obtain the following expression for t_u:

$$t_u = \frac{h\left(\eta' r_P\left(\begin{pmatrix} 1 & -u \\ & 1 \end{pmatrix}_0\right), \zeta' z\right)}{\left|h\left(\eta' r_P\left(\begin{pmatrix} 1 & -u \\ & 1 \end{pmatrix}_0\right), \zeta' z\right)\right|}$$

$$\underset{(9)}{=} \lim_{\rho\to\infty} \frac{h\left(\eta' r_P\left(\begin{pmatrix} 1 & -u \\ & 1 \end{pmatrix} \to 0\right)\zeta', \rho\mathbf{i}\right)}{\left|h\left(\eta' r_P\left(\begin{pmatrix} 1 & -u \\ & 1 \end{pmatrix}_0\right)\zeta', \rho\mathbf{i}\right)\right|} \Bigg/ \frac{h(\zeta', \rho\mathbf{i})}{|h(\zeta', \rho\mathbf{i})|}$$

$$= \sqrt{i}^{-2n} \lim_{\rho\to\infty} \frac{h\Big(r_\Omega\Big(\zeta\eta\begin{pmatrix}1 & -u\\ & 1\end{pmatrix}_0\Big),\rho\mathbf{i}\Big)}{\Big|h\Big(r_\Omega\Big(\zeta\eta\begin{pmatrix}1 & -u\\ & 1\end{pmatrix}_0\Big),\rho\mathbf{i}\Big)\Big|} i^{-n} \frac{\gamma(u)}{|\gamma(u)|}$$

$$= \varepsilon_{\mathfrak{q}} \left(\frac{2}{\mathfrak{q}}\right) i^{-n}\chi^{\mathfrak{q}*}(2\mathfrak{b}')\chi^{\mathfrak{q}*}(v\mathfrak{q}(2\mathfrak{b}')^{-1}\mathfrak{d})\chi^{\mathfrak{q}}_\infty(v).$$

Making use of $(*)$, the automorphy property of $f_{\mathbf{A}}$, and of the equality

$$\chi^2(q\delta)\chi_\infty(u)\overline{\chi}^*_{\mathfrak{q}}(uq\mathfrak{d}) = \chi_\infty(-1)\chi_\infty(v)\chi^*_{\mathfrak{q}}(vq\mathfrak{d}),$$

the above expression for $(f(\chi^*_{\mathfrak{q}})^0\iota\,\mathrm{sw}_{\mathfrak{q}})_{\mathbf{A}}(x)$ evaluates to

$$\psi(q\delta)\frac{\chi^*(\frac{\mathfrak{q}}{\mathfrak{c}})\chi_\infty(-1)}{\sqrt{\mathcal{N}(\mathfrak{q})}\tau(\overline{\chi})}\chi^*(4\mathfrak{b}\mathfrak{b}')\sum_{v\in S}\chi_\infty(v)\chi^*_{\mathfrak{q}}(v\mathfrak{q}2^{-1}\mathfrak{b}'^{-1}\mathfrak{d}) \cdot f_{\mathbf{A}}\Big(xr_P\Big(\begin{pmatrix}1 & -v\\ & 1\end{pmatrix}_0\Big)w'\eta' t_u^{-1}\Big)$$

$$= \psi^*(\mathfrak{q})\frac{\chi^*(\frac{\mathfrak{q}}{\mathfrak{c}})\chi_\infty(-1)}{\sqrt{\mathcal{N}(\mathfrak{q})}\tau(\overline{\chi})}\chi^*(4\mathfrak{b}\mathfrak{b}')\varepsilon_{\mathfrak{q}}^{-1}\left(\frac{2}{\mathfrak{q}}\right)i^n\chi^{\mathfrak{q}*}(2\mathfrak{b}')$$
$$\cdot\sum_{v\in S}(\chi\chi^{\mathfrak{q}})_\infty(v)(\chi\chi^{\mathfrak{q}})^*_{\mathfrak{q}}(v\mathfrak{q}2^{-1}\mathfrak{b}'^{-1}\mathfrak{d})(f\iota)_{\mathbf{A}}\left(xr_P\begin{pmatrix}1 & -v\\ & 1\end{pmatrix}_0\right)$$
$$= \psi^*(\mathfrak{q})\chi^*(4\mathfrak{b}\mathfrak{b}')\varepsilon_{\mathfrak{q}}^{-1}\left(\frac{2}{\mathfrak{q}}\right)i^n\chi^{\mathfrak{q}*}(2\mathfrak{b}')\frac{\chi^*(\frac{\mathfrak{q}}{\mathfrak{c}})\chi_\infty(-1)}{\sqrt{\mathcal{N}(\mathfrak{q})}\tau(\overline{\chi})}$$
$$\cdot(\chi\chi^{\mathfrak{q}})^*\Big(\frac{\mathfrak{q}}{\mathfrak{c}(\chi\chi^{\mathfrak{q}})}\Big)\tau(\chi\chi^{\mathfrak{q}})(f\iota(\overline{\chi\chi^{\mathfrak{q}}}^*_{\mathfrak{q}})^0)_{\mathbf{A}}(x).$$

This proves part c). □

Remark. — Part c) of the above proposition as well as its proof are analogous to Proposition 4.5 of [S4]. However, the half-integral case differs from the integral case because of the occurrence of the quadratic character $\chi^{\mathfrak{q}}$.

For an integral ideal $\mathfrak{c}$ write $\mathcal{M}_k(\mathfrak{c},\psi)$ for the space $\mathcal{M}_k(\mathfrak{o},\mathfrak{c},\psi)$. If $\mathfrak{c}$ is of the form $\mathfrak{c} = c_1\mathfrak{c}_2^2$ with a totally positive $c_1 \in \mathfrak{o}$ and an integral ideal $\mathfrak{c}_2$ of F, we define an *inverter*

$$j = j_{c_1,\mathfrak{c}_2} : \mathcal{M}_k(\mathfrak{c},\psi) \to \mathcal{M}_k(\mathfrak{c},\overline{\psi\varepsilon_{c_1}}) \quad \text{by } j_{c_1,\mathfrak{c}_2} := \mathcal{N}(\mathfrak{c})^{\frac{1}{4}}c_1^{\frac{1}{2}k'}\iota\,\mathrm{m}_{c_1}\,\mathrm{sw}_{\mathfrak{c}_2}.$$

Proposition 3.6 shows that $j^2 = v(c_1)\psi_\infty(-\mathbf{1})$, and Proposition 3.6, 3.7 and 3.10 describe the action of the operator j on the space of Hilbert modular forms.

4. Distributions related to convolutions of Hilbert modular forms.

From now on, we assume that the totally real number field F has class number $h_F = 1$. Let $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c}(\mathbf{f}), \psi)$ be a primitive Hilbert automorphic form of integral weight $k = (k_1, \ldots, k_n)$, conductor $\mathfrak{c}(\mathbf{f})$ and character ψ and let $k_0 := \max\{k_1, \ldots, k_n\}$. Also, let $g \in \mathcal{M}_l(\mathfrak{c}(g), \phi)$ be a Hilbert modular form of half-integral weight $l = (l_1, \ldots, l_n)$ and character ϕ, set $l' = l - \frac{1}{2}\mathbf{1}$ and assume that the Hecke character ϕ satisfies $\phi_\infty(x) = \operatorname{sgn}(x)^{l'}$ and that g_ι is a simultaneous Hecke eigenform for all Hecke operators. Fix a prime number p of $\mathbb{Z}$, a finite set S of finite primes containing all primes $\mathfrak{p}$ dividing p, and set $\mathfrak{m}_0 := \prod_{\mathfrak{p} \in S} \mathfrak{p}$. For every $\mathfrak{p} \in S$ let $\alpha(\mathfrak{p}) = \alpha_\mathfrak{p}$ and $\alpha'(\mathfrak{p}) = \alpha'_\mathfrak{p}$ denote the roots of the $\mathfrak{g}$-th Hecke polynomial of $\mathbf{f}$:

$$X^2 - C(\mathfrak{p}, \mathbf{f})X + \psi^*_{\mathfrak{c}(\mathbf{f})}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{k_0-1} = (X - \alpha_\mathfrak{p})(X - \alpha'_\mathfrak{p}).$$

We extend this definition of α and α' to any integral ideal $\mathfrak{m}$ with $S(\mathfrak{m}) \subseteq S$ by multiplicativity. Define also

$$\mathbf{f}_0 := \sum_{\mathfrak{a}|\mathfrak{m}_0} \mu(\mathfrak{a})\alpha'(\mathfrak{a})\mathbf{f}|\mathfrak{a} \in \mathcal{M}_k(\mathfrak{c}(\mathbf{f})\mathfrak{m}_0, \psi);$$

this is the Hilbert automorphic form whose Dirichlet series factors into the following Euler product:

$$\sum_{\mathfrak{m}} C(\mathfrak{m}, \mathbf{f}_0)\mathcal{N}(\mathfrak{m})^{-s} = \prod_{\mathfrak{p} \in S} (1 - \alpha(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-s})^{-1} \cdot \prod_{\mathfrak{p} \notin S} (1 - C(\mathfrak{p}, \mathbf{f})\mathcal{N}(\mathfrak{p})^{-s} + \psi^*_{\mathfrak{c}(\mathbf{f})}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{k_0-1-2s})^{-1}.$$

We will always assume the following:

$l < k$, i.e. $l_1 < k_1, \ldots, l_n < k_n$,
$k_1 \equiv \ldots \equiv k_n \bmod 2$, $l_1 \equiv \ldots \equiv l_n \bmod 2$,
$\mathfrak{c}(\mathbf{f})$, $4\mathfrak{c}(g)$ and $\mathfrak{m}_0$ are pairwise relatively prime, and
the Fourier coefficient $C(\mathfrak{c}(\mathbf{f}), \mathbf{f})$ does not vanish.

Determine $\theta \in \{0, 1\}$ by $k_1 - l_1 - \frac{1}{2} \equiv \ldots \equiv k_n - l_n - \frac{1}{2} \equiv \theta \bmod 2$, set $\mathfrak{c} = \mathfrak{c}(\mathbf{f})4\mathfrak{c}(g)$, fix once and for all totally positive numbers $c(\mathbf{f})$, $c(g) \in F$

with $(c(\mathbf{f})) = \mathfrak{c}(\mathbf{f})$ and $(c(g)) = \mathfrak{c}(g)$, and set $c := c(\mathbf{f})c(g)$. Let $(\xi, \mathfrak{m})$ denote a pair of a totally positive number ξ of F and a fractional ideal $\mathfrak{m}$ of F subject to $\xi\mathfrak{m}^2 \subseteq \mathfrak{o}$. If $(\xi', \mathfrak{m}')$ is another such pair, we say that $(\xi, \mathfrak{m})$ and $(\xi', \mathfrak{m}')$ are equivalent if $\xi = \eta^2\xi'$ and $\mathfrak{m} = \eta^{-1}\mathfrak{m}'$ for some $\eta \in F^\times$. The convolution of $\mathbf{f}$ and g is then defined as

$$D(s; \mathbf{f}, g) = \sum_{(\xi,\mathfrak{m})} c(\xi\mathfrak{m}^2, \mathbf{f})\overline{\lambda(\xi, \mathfrak{m}; g, \phi)}\xi^{-\frac{l'}{2}}\mathcal{N}(\xi\mathfrak{m}^2)^{-s},$$

where $(\xi, \mathfrak{m})$ runs over a set of representatives for the equivalence defined above. With the quadratic Hecke character $\omega = \varepsilon_{-1}$ corresponding to $F(\sqrt{-1})/F$ define the complex functions

$$\mathcal{D}(s) = L_{\mathfrak{c}\mathfrak{m}_0}(4s - 1, (\omega\psi\overline{\phi})^2)D\Big(s - \frac{3}{4}; \mathbf{f}_0, g\Big),$$

$$\Psi(s) = \mathcal{D}(s)\prod_{\nu=1}^{n}\Gamma\left(s - 1 + \frac{k_\nu + l_\nu}{2}\right),$$

$$\Psi_0(s) = \Psi(s)\prod_{\nu=1}^{n}\left\{\Gamma\Big(s + \frac{\theta}{2} - \frac{1}{4}\Big)\Gamma\Big(s + \frac{k_\nu - l_\nu}{2}\Big)\right\} \cdot \begin{cases} s - \dfrac{3}{4}, & \text{if } (\omega\psi\overline{\phi})^2 = 1 \text{ and } \theta = 0, \\ 1, & \text{otherwise.} \end{cases}$$

The additional gamma and L-factors guarantee that Ψ_0 is a holomorphic function in s, cf. the proof of Proposition 5.1 and Proposition 3.1 of [I]. Recall now that a $\mathbb{C}$-valued distribution μ on $\mathrm{Gal}_S = \varprojlim H(\mathfrak{m})$ with $H(\mathfrak{m}) = J(\mathfrak{m})/P(\mathfrak{m})$ can be uniquely defined by giving its values on the characters $\chi \in X_S^{\mathrm{tor}}$. Now the projection $\mathrm{pr}_{\mathfrak{m}} : \mathrm{Gal}_S = \varprojlim H(\mathfrak{n}) \to H(\mathfrak{m})$ induces the map

$$\mu_{\mathfrak{m}} : F(H(\mathfrak{m}), \mathbb{C}) \xrightarrow{\mathrm{pr}_{\mathfrak{m}}^*} \mathrm{Step}(\mathrm{Gal}_S, \mathbb{C}) \xrightarrow{\mu} \mathbb{C},$$

where $F(H(\mathfrak{m}), \mathbb{C})$ denotes the set of all $\mathbb{C}$-valued functions on $H(\mathfrak{m})$. The distribution μ is also uniquely given by the maps $\mu_{\mathfrak{m}}$, and the $\mu_{\mathfrak{m}}$ fulfill a certain compatibility relation. If $\mathfrak{c}(\chi)|\mathfrak{m}$ then the map $\chi : \mathrm{Gal}_S \to \mathbb{C}$ factors through $\chi_{\mathfrak{m}}^* : H(\mathfrak{m}) \to \mathbb{C}$, and by the character relations $\mu_{\mathfrak{m}}$ is determined by $\mu_{\mathfrak{m}}(\chi_{\mathfrak{m}}^*)$ for all χ with $\mathfrak{c}(\chi)|\mathfrak{m}$. For every $s \in \mathbb{C}$ we will now define a $\mathbb{C}$-valued distribution $\widetilde{\mu}_s^0$ related to Ψ_0 by the values of $\widetilde{\mu}_{s,\mathfrak{m}}^0(\chi_{\mathfrak{m}}^*)$. It will turn out that for certain values of s and non-vanishing factors c_s the distribution $\widetilde{\mu}_s := c_s\widetilde{\mu}_s^0$ will take values in the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and thus these $\widetilde{\mu}_s$ will give rise to a p-adic distribution.

PROPOSITION 4.1. — *For* $\mathbf{f}_0$ *and* g *as above and for every* $s \in \mathbb{C}$ *there exists a uniquely determined complex valued distribution* $\widetilde{\mu}_s^0 : \mathrm{Step}(\mathrm{Gal}_S, \mathbb{C}) \to \mathbb{C}$ *such that*

$$\widetilde{\mu}_s^0(\chi) = \widetilde{\mu}_{s,\mathfrak{m}}^0(\chi_\mathfrak{m}^*) = \frac{\mathcal{N}(\mathfrak{m}')^{k_0+2(s-1)}}{\alpha(\mathfrak{m}')^2} \Psi_0(s; \mathbf{f}_0, g(\overline{\chi}_\mathfrak{m}^*) j_{c,\mathfrak{m}'})$$

for any Hecke character $\chi \in X_S^{\mathrm{tor}}$ *and any choice of integral ideals* $\mathfrak{m}, \mathfrak{m}'$ *satisfying* $\mathrm{lcm}(\mathfrak{m}_0, \mathfrak{c}(\chi))|\mathfrak{m}$, $\mathfrak{m}_0\mathfrak{m}|\mathfrak{m}'$, *and* $S(\mathfrak{m}') = S$.

The proof is based on the following basic lemma which extends Lemma 1 of [S3] to Dirichlet series with a more complex Euler product. We will omit the proof which is completely analogous to [loc. cit.].

LEMMA 4.2. — *Assume that we have formally*

$$\sum_\mathfrak{n} A(\mathfrak{n})\mathcal{N}(\mathfrak{n})^{-s} = \prod_\mathfrak{p} \frac{1 - a(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-s}}{(1 - \alpha_\mathfrak{p}\mathcal{N}(\mathfrak{p})^{-s})(1 - \alpha'_\mathfrak{p}\mathcal{N}(\mathfrak{p})^{-s})},$$

$$\sum_\mathfrak{n} B(\mathfrak{n})\mathcal{N}(\mathfrak{n})^{-s} = \prod_\mathfrak{p} \frac{1 - b(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-s}}{(1 - \beta_\mathfrak{p}\mathcal{N}(\mathfrak{p})^{-s})(1 - \beta'_\mathfrak{p}\mathcal{N}(\mathfrak{p})^{-s})},$$

where $\mathfrak{n}$ *runs over all integral and* $\mathfrak{p}$ *over all prime ideals of* F. *Then for each prime ideal* $\mathfrak{p}$ *and for each integer* $t \in \mathbb{Z}$ *there exist polynomials*

$X_{\mathfrak{p},t}(T)$ *of degree less than* 4,
$Y_\mathfrak{p}(T) = (1 - \alpha_\mathfrak{p}\beta_\mathfrak{p}T)(1 - \alpha_\mathfrak{p}\beta'_\mathfrak{p}T)(1 - \alpha'_\mathfrak{p}\beta_\mathfrak{p}T)(1 - \alpha'_\mathfrak{p}\beta'_\mathfrak{p}T)$,

such that for arbitrary integral ideals $\mathfrak{u}$ *and* $\mathfrak{v}$ *the following holds:*

$$\sum_\mathfrak{n} A\Big(\frac{\mathfrak{n}}{\mathfrak{u}}\Big) B\Big(\frac{\mathfrak{n}}{\mathfrak{v}}\Big)\mathcal{N}(\mathfrak{n})^{-s} = \mathcal{N}\left(\frac{\mathfrak{u}\mathfrak{v}}{\mathfrak{g}}\right)^{-s} \prod_\mathfrak{p} \frac{X_{\mathfrak{p},\nu_\mathfrak{p}(\mathfrak{v})-\nu_\mathfrak{p}(\mathfrak{u})}(\mathcal{N}(\mathfrak{p})^{-s})}{Y_\mathfrak{p}(\mathcal{N}(\mathfrak{p})^{-s})}$$

where $\mathfrak{g} = \gcd(\mathfrak{u}, \mathfrak{v})$ *is the greatest common divisor of* $\mathfrak{u}$ *and* $\mathfrak{v}$, *and* $A\Big(\frac{\mathfrak{n}}{\mathfrak{u}}\Big) = 0$ *if* $\mathfrak{u} \nmid \mathfrak{n}$, *and similarly,* $B\Big(\frac{\mathfrak{n}}{\mathfrak{v}}\Big) = 0$ *if* $\mathfrak{v} \nmid \mathfrak{n}$. *Moreover, the polynomials* $X_{\mathfrak{p},t}$ *and* $Y_\mathfrak{p}$ *have coefficients in* $\mathbb{Q}[A(\mathfrak{n}), B(\mathfrak{n}), a(\mathfrak{n}), b(\mathfrak{n})]$, *and*

$$X_{\mathfrak{p},0}(T) = 1 - \alpha_\mathfrak{p}\alpha'_\mathfrak{p}\beta_\mathfrak{p}\beta'_\mathfrak{p}T^2 \qquad \text{if } a(\mathfrak{p}) = b(\mathfrak{p}) = 0,$$
$$X_{\mathfrak{p},t}(T) = \alpha_\mathfrak{p}^t X_{\mathfrak{p},0}(T) \qquad \text{if } \alpha'_\mathfrak{p} = a(\mathfrak{p}) = 0 \text{ and } t \geqslant 0.$$

Proof of Proposition 4.1. — $\widetilde{\mu}_s^0$ will be a distribution if and only if the value $\widetilde{\mu}_{s,\mathfrak{m}}^0(\chi_\mathfrak{m}^*)$ does not depend on either $\mathfrak{m}$ or $\mathfrak{m}'$. We extend the definition of $\chi^{\mathfrak{p}}$ of Lemma 1.4 for primes $\mathfrak{p}$ not dividing 2 to integral ideals $\mathfrak{n}$ prime to 2 by multiplicativity, so that $\chi^{\mathfrak{n}} = \prod_{\mathfrak{p}|\mathfrak{n}}(\chi^{\mathfrak{p}})^{\nu_{\mathfrak{p}}(\mathfrak{n})}$. For the character χ, let $\mathfrak{q} = \mathfrak{c}(\chi)$ be the finite part of its conductor and define

$$\mathfrak{n}_0 := \prod_{\mathfrak{p}\in S\backslash S(\chi)} \mathfrak{p}, \quad \mathfrak{n}_1 := \prod_{\mathfrak{p}\in S(\chi)\backslash S(\chi\chi^{\mathfrak{q}})} \mathfrak{p}, \quad \chi_0 := \chi\chi^{\mathfrak{n}_1} \text{ with conductor } \mathfrak{q}_0 = \frac{\mathfrak{q}}{\mathfrak{n}_1}.$$

Then by the definition of $j_{c,\mathfrak{m}'}$, Corollary 3.4 and Proposition 3.6 and 3.10

$$\begin{aligned}
g(\overline{\chi}^*_{\mathfrak{m}})j_{c,\mathfrak{m}'} &= \mathcal{N}\Big(\frac{\mathfrak{c}}{4}\mathfrak{m}'^2\Big)^{\frac{1}{4}} c^{\frac{1}{2}l'} g(\overline{\chi}^*_{\mathfrak{q}\mathfrak{m}_0})\iota\, \mathrm{m}_c\, \mathrm{sw}_{\mathfrak{m}'} \\
&= \mathcal{N}\Big(\frac{\mathfrak{c}}{4}\mathfrak{m}'^2\Big)^{\frac{1}{4}} c^{\frac{1}{2}l'} \Phi(\mathfrak{n}_0)\mathcal{N}(\mathfrak{n}_0)^{-1} \sum_{\mathfrak{n}|\mathfrak{n}_0} \mu(\mathfrak{n})\Phi(\mathfrak{n})^{-1} g(\overline{\chi}^*_{\mathfrak{q}\mathfrak{n}})^0 \iota\, \mathrm{m}_c\, \mathrm{sw}_{\mathfrak{m}'} \\
&= \mathcal{N}\Big(\frac{\mathfrak{c}}{4}\mathfrak{m}'^2\Big)^{\frac{1}{4}} c^{\frac{1}{2}l'} \Phi(\mathfrak{n}_0)\mathcal{N}(\mathfrak{n}_0)^{-1} \\
&\quad \cdot \sum_{\mathfrak{n}|\mathfrak{n}_0} \mu(\mathfrak{n})\Phi(\mathfrak{n})^{-1} \Lambda(\mathfrak{q}\mathfrak{n}, \chi; \phi) g\iota\, \mathrm{m}_{c(g)} ((\chi_0\chi^{\mathfrak{q}_0}\chi^{\mathfrak{n}})^*_{\mathfrak{q}\mathfrak{n}})^0\, \mathrm{m}_{c(\mathbf{f})}\, \mathrm{sw}_{\mathfrak{m}'/\mathfrak{q}\mathfrak{n}} \\
&= \mathcal{N}\Big(\frac{\mathfrak{c}}{4}\mathfrak{m}'^2\Big)^{\frac{1}{4}} c^{\frac{1}{2}l'} \Phi(\mathfrak{n}_0)\mathcal{N}(\mathfrak{n}_0)^{-1}\Phi(\mathfrak{n}_1) \sum_{\mathfrak{n}|\mathfrak{n}_0} \mu(\mathfrak{n})\Phi(\mathfrak{n})^{-1}\Lambda(\mathfrak{q}\mathfrak{n}, \chi; \phi) \\
&\quad \cdot \sum_{\mathfrak{m}_1|\mathfrak{n}_1} \mu(\mathfrak{m}_1)\Phi(\mathfrak{m}_1)^{-1}\mathcal{N}(\mathfrak{m}_1) g\iota\, \mathrm{m}_{c(g)} (\chi_0\chi^{\mathfrak{q}_0}\chi^{\mathfrak{n}})^*_{\mathfrak{q}_0\mathfrak{n}\mathfrak{m}_1}\, \mathrm{m}_{c(\mathbf{f})}\, \mathrm{sw}_{\mathfrak{m}'/\mathfrak{q}\mathfrak{n}}\,.
\end{aligned}$$

Put $G = G_{\mathfrak{n},\mathfrak{m}_1} := g\iota\, \mathrm{m}_{c(g)}(\chi_0\chi^{\mathfrak{q}_0}\chi^{\mathfrak{n}})^*_{\mathfrak{q}_0\mathfrak{n}\mathfrak{m}_1}$ and evaluate each summand as follows:

$$\begin{aligned}
&D(s; \mathbf{f}_0, G\, \mathrm{m}_{c(\mathbf{f})}\, \mathrm{sw}_{\mathfrak{m}'/\mathfrak{q}\mathfrak{n}}) \\
&\quad = \sum_{(\xi,\mathfrak{m})} c(\xi\mathfrak{m}^2, \mathbf{f}_0)\overline{\lambda(\xi, \mathfrak{m}; G\, \mathrm{m}_{c(\mathbf{f})}\, \mathrm{sw}_{\mathfrak{m}'/\mathfrak{q}\mathfrak{n}})}\xi^{-l'/2}\mathcal{N}(\xi\mathfrak{m}^2)^{-s} \\
&\quad = \sum_{\substack{0\ll\xi\in\mathfrak{o} \\ (\xi)\ \text{squarefree},\ \xi \bmod \mathfrak{o}^{\times 2}}} \xi^{-k_0/2\cdot\mathbf{1}-l'/2-s\cdot\mathbf{1}} \\
&\quad\quad \cdot \sum_{\mathfrak{m}\subseteq\mathfrak{o}} C(\xi\mathfrak{m}^2, \mathbf{f}_0)\overline{\lambda(\xi, \mathfrak{m}; G\, \mathrm{m}_{c(\mathbf{f})}\, \mathrm{sw}_{\mathfrak{m}'/\mathfrak{q}\mathfrak{n}})}\mathcal{N}(\mathfrak{m})^{-(k_0+2s)}.
\end{aligned}$$

Here, we have made use of the assumption $h_F = 1$ to find a representative $(\xi, \mathfrak{m})$ in each equivalence class such that $0 \ll \xi \in \mathfrak{o}$, (ξ) is squarefree, and $\mathfrak{m}$ is integral. Then, $\mathfrak{m}$ is uniquely determined, and ξ is determined modulo $\mathfrak{o}^{\times 2}$. Provided that $C((\xi), \mathbf{f}_0) \neq 0$ and $\lambda(\xi, \mathfrak{o}; G) \neq 0$, the following two Dirichlet series now have an Euler product as in Lemma 4.2:

$$\sum_{\mathfrak{m}\subseteq\mathfrak{o}} \frac{C(\xi\mathfrak{m}^2,\mathbf{f}_0)}{C((\xi),\mathbf{f}_0)}\mathcal{N}(\mathfrak{m})^{-s} = \prod_{\mathfrak{p}}\sum_{n=0}^{\infty} \frac{C(\mathfrak{p}^{\nu_\mathfrak{p}(\xi)+2n},\mathbf{f}_0)}{C(\mathfrak{p}^{\nu_\mathfrak{p}(\xi)},\mathbf{f}_0)}\mathcal{N}(\mathfrak{p})^{-ns}$$
$$= \prod_{\mathfrak{p}}\sum_{n=0}^{\infty} \frac{1}{C(\mathfrak{p}^{\nu_\mathfrak{p}(\xi)},\mathbf{f}_0)} \frac{\hat{\alpha}_\mathfrak{p}{}^{\nu_\mathfrak{p}(\xi)+2n+1} - \hat{\alpha}_\mathfrak{p}'{}^{\nu_\mathfrak{p}(\xi)+2n+1}}{\hat{\alpha}_\mathfrak{p} - \hat{\alpha}_\mathfrak{p}'}\mathcal{N}(\mathfrak{p})^{-ns}$$
$$= \prod_{\mathfrak{p}} \frac{1 - \hat{\alpha}_\mathfrak{p}\hat{\alpha}_\mathfrak{p}' \frac{\hat{\alpha}_\mathfrak{p}{}^{\nu_\mathfrak{p}(\xi)}\hat{\alpha}_\mathfrak{p}' - \hat{\alpha}_\mathfrak{p}\hat{\alpha}_\mathfrak{p}'{}^{\nu_\mathfrak{p}(\xi)}}{\hat{\alpha}_\mathfrak{p} - \hat{\alpha}_\mathfrak{p}'} \frac{1}{C(\mathfrak{p}^{\nu_\mathfrak{p}(\xi)},\mathbf{f}_0)}\mathcal{N}(\mathfrak{p})^{-s}}{(1-\hat{\alpha}_\mathfrak{p}{}^2\mathcal{N}(\mathfrak{p})^{-s})(1-\hat{\alpha}_\mathfrak{p}'{}^2\mathcal{N}(\mathfrak{p})^{-s})}$$

where we have denoted by $\hat{\alpha}_\mathfrak{p}$ and $\hat{\alpha}_\mathfrak{p}'$ the roots of the $\mathfrak{p}$-th Hecke polynomial of $\mathbf{f}_0$. If $\alpha_\mathfrak{p}$ and $\alpha_\mathfrak{p}'$ are the roots of the Hecke polynomial of $\mathbf{f}$ then $\hat{\alpha}_\mathfrak{p} = \alpha_\mathfrak{p}$ and $\hat{\alpha}_\mathfrak{p}' = \alpha_\mathfrak{p}'$ if $\mathfrak{p} \notin S$, and $\hat{\alpha}_\mathfrak{p} = \alpha_\mathfrak{p}$ and $\hat{\alpha}_\mathfrak{p}' = 0$ for $\mathfrak{p} \in S$. In the case $\hat{\alpha}_\mathfrak{p} = \hat{\alpha}_\mathfrak{p}'$ the above formulas have to be interpreted as before.

The second Euler product is given by Proposition 3.2 because G is a simultaneous Hecke eigenform by our assumption on g and Proposition 3.7:

$$\sum_{\mathfrak{m}\subseteq\mathfrak{o}} \frac{\lambda(\xi,\mathfrak{m};G)}{\lambda(\xi,\mathfrak{o};G)}\mathcal{N}(\mathfrak{m})^{-s}$$
$$= \prod_{\mathfrak{p}} \frac{1-(\overline{\phi\varepsilon_{c(g)}}\chi^2)_\mathfrak{a}^*(\mathfrak{p})\varepsilon_\xi^*(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-1}\mathcal{N}(\mathfrak{p})^{-s}}{1-\omega_\mathfrak{p}(G)\mathcal{N}(\mathfrak{p})^{-s} + (\overline{\phi\varepsilon_{c(g)}}\chi^2)_\mathfrak{a}^*(\mathfrak{p})^2\mathcal{N}(\mathfrak{p})^{-1}\mathcal{N}(\mathfrak{p})^{-2s}}$$

with $\mathfrak{a} = 4\mathfrak{c}(g)\mathfrak{q}_0^2\mathfrak{n}^2\mathfrak{m}_1^2$. Applying once more the class number hypothesis $h_F = 1$, we fix totally positive integral numbers $c_1(\mathbf{f})$ and $c_2(\mathbf{f})$ such that $c(\mathbf{f}) = c_1(\mathbf{f})c_2(\mathbf{f})^2$ and $(c_1(\mathbf{f}))$ is squarefree. For $0 \ll \xi \in \mathfrak{o}$ find $c_3(\mathbf{f})$ and η such that

$$c_1(\mathbf{f})\xi = c_3(\mathbf{f})^2\eta \quad \text{with } 0 \ll \eta \in \mathfrak{o},\ 0 \ll c_3(\mathbf{f}) \in \mathfrak{o}, \text{ and } (\eta) \text{ squarefree.}$$

Consider $G \in \mathcal{M}_l(\mathfrak{c}(g)\mathfrak{q}_0^2\mathfrak{n}^2\mathfrak{m}_1^2, \overline{\phi\varepsilon_{c(g)}}\chi^2)$ as a form of $\mathcal{M}_l(c(\mathbf{f})\frac{\mathfrak{m}'}{\mathfrak{q}\mathfrak{n}}, \mathfrak{c}(g)\mathfrak{q}_0^2\mathfrak{n}^2\ \mathfrak{m}_1^2, \overline{\phi\varepsilon_{c(g)}}\chi^2)$. Then Proposition 3.1 and 3.5 show

$$\lambda(\xi,\mathfrak{m}; G\,\mathrm{m}_{c(\mathbf{f})}\,\mathrm{sw}_{\mathfrak{m}'/\mathfrak{q}\mathfrak{n}}) = \lambda\Big(c(\mathbf{f})^{-1}\xi, \mathfrak{m}/\frac{\mathfrak{m}'}{\mathfrak{q}\mathfrak{n}}; G\Big)$$
$$= (c_1(\mathbf{f})^{-1}c_2(\mathbf{f})^{-1}c_3(\mathbf{f}))^{l'}\lambda\Big(\eta, \mathfrak{m}/\frac{c_1(\mathbf{f})c_2(\mathbf{f})\mathfrak{m}'}{\mathfrak{q}\mathfrak{n}c_3(\mathbf{f})}; G\Big).$$

Assuming $C((\xi), \mathfrak{f}_0) \neq 0$ and $\lambda(\eta, \mathfrak{o}; G) \neq 0$ we can apply Lemma 4.2 with $A_\xi(\mathfrak{m}) := \dfrac{C(\xi\mathfrak{m}^2, \mathfrak{f}_0)}{C((\xi), \mathfrak{f}_0)}$ and $B_\eta(\mathfrak{m}) := \dfrac{\lambda(\eta, \mathfrak{m}; G)}{\lambda(\eta, \mathfrak{o}; G)}$ to obtain

$$\begin{aligned}
&\sum_{\mathfrak{m}\subseteq\mathfrak{o}} C(\xi\mathfrak{m}^2, \mathfrak{f}_0)\overline{\lambda(\xi, \mathfrak{m}, G\,\mathrm{m}_{c(\mathfrak{f})}\,\mathrm{sw}_{\mathfrak{m}'/\mathfrak{q}\mathfrak{n}})}\mathcal{N}(\mathfrak{m})^{-(k_0+2s)} \\
&= C((\xi), \mathfrak{f}_0)\lambda(\eta, \mathfrak{o}, G)\left(\frac{c_1(\mathfrak{f})c_2(\mathfrak{f})}{c_3(\mathfrak{f})}\right)^{-l'} \\
&\qquad\qquad \cdot \sum_{\mathfrak{m}\subseteq\mathfrak{o}} A_\xi(\mathfrak{m})\overline{B_\eta\left(\mathfrak{m}/\frac{c_1(\mathfrak{f})c_2(\mathfrak{f})\mathfrak{m}'}{c_3(\mathfrak{f})\mathfrak{q}\mathfrak{n}}\right)}\mathcal{N}(\mathfrak{m})^{-(k_0+2s)} \\
&= C((\xi), \mathfrak{f}_0)\lambda(\eta, \mathfrak{o}, G)\left(\frac{c_1(\mathfrak{f})c_2(\mathfrak{f})}{c_3(\mathfrak{f})}\right)^{-l'} \mathcal{N}\left(\frac{c_1(\mathfrak{f})c_2(\mathfrak{f})\mathfrak{m}'}{c_3(\mathfrak{f})\mathfrak{q}\mathfrak{n}}\right)^{-(k_0+2s)} \\
&\qquad\qquad \cdot \prod_{\mathfrak{p}} \frac{X_{\mathfrak{p},\nu_\mathfrak{p}(\frac{c_1(\mathfrak{f})c_2(\mathfrak{f})\mathfrak{m}'}{c_3(\mathfrak{f})\mathfrak{q}\mathfrak{n}})}\mathcal{N}(\mathfrak{p})^{-(k_0+2s)}}{Y_\mathfrak{p}(\mathcal{N}(\mathfrak{p})^{-(k_0+2s)})} \\
&= \left(\frac{c_1(\mathfrak{f})c_2(\mathfrak{f})}{c_3(\mathfrak{f})}\right)^{-l'} \mathcal{N}\left(\frac{c_1(\mathfrak{f})c_2(\mathfrak{f})\mathfrak{m}'}{c_3(\mathfrak{f})\mathfrak{q}\mathfrak{n}}\right)^{-(k_0+2s)} C(c(\mathfrak{f}), \mathfrak{f})\alpha^2\left(\frac{\mathfrak{m}'}{\mathfrak{q}\mathfrak{n}}\right) \\
&\qquad\qquad \cdot \sum_{\mathfrak{m}\subseteq\mathfrak{o}} C(\eta\mathfrak{m}^2, \mathfrak{f}_0)\lambda(\eta, \mathfrak{m}; G)\mathcal{N}(\mathfrak{m})^{-(k_0+2s)}.
\end{aligned}$$

Now, this equation holds regardless of our assumption $C((\xi), \mathfrak{f}_0)\lambda(\eta, \mathfrak{o}; G) \neq 0$: If $\lambda(\eta, \mathfrak{o}; G) = 0$ then $\lambda(\xi, \mathfrak{m}; G) = \lambda(\eta, \mathfrak{m}; G) = 0$ for all $\mathfrak{m}$. If $C((\xi), \mathfrak{f}_0) = 0$, then there is a $\mathfrak{p}|(\xi)$ with $0 = C(\mathfrak{p}, \mathfrak{f}_0)$ because (ξ) is squarefree. But then, $\nu_\mathfrak{p}((\xi)) = 1$ and the $\mathfrak{p}$-th Euler factor of $\sum_\mathfrak{m} C(\xi\mathfrak{m}^2, \mathfrak{f}_0)\mathcal{N}(\mathfrak{m})^{-s}$ is 0, and hence $C(\xi\mathfrak{m}^2, \mathfrak{f}_0) = C(\eta\mathfrak{m}^2, \mathfrak{f}_0) = 0$ for all $\mathfrak{m}$. Thus, in the excluded case both sides of the above equation equal 0 and we have equality as well. From $c_1(\mathfrak{f})\eta = \left(\dfrac{c_1(\mathfrak{f})}{c_2(\mathfrak{f})}\right)^2 \xi$ it follows that $\xi \bmod \mathfrak{o}^{\times 2} \mapsto \eta \bmod \mathfrak{o}^{\times 2}$ is a bijection. Therefore

$$\begin{aligned}
&D(s; \mathfrak{f}_0, G\,\mathrm{m}_{c(\mathfrak{f})}\,\mathrm{sw}_{\mathfrak{m}'/\mathfrak{q}\mathfrak{n}}) \\
&= (c_1(\mathfrak{f})c_2(\mathfrak{f}))^{-l'}\mathcal{N}\left(\frac{c_1(\mathfrak{f})c_2(\mathfrak{f})\mathfrak{m}'}{\mathfrak{q}\mathfrak{n}}\right)^{-(k_0+2s)} C(c(\mathfrak{f}), \mathfrak{f})\alpha^2\left(\frac{\mathfrak{m}'}{\mathfrak{q}\mathfrak{n}}\right) \\
&\cdot \sum_\xi \xi^{-k_0/2\cdot 1 - l'/2 - s\cdot 1} c_3(\mathfrak{f})^{(k_0+2s)1+l'} \sum_\mathfrak{m} C(\eta\mathfrak{m}^2, \mathfrak{f}_0)\overline{\lambda(\eta, \mathfrak{m}; G)}\mathcal{N}(\mathfrak{m})^{-(k_0+2s)} \\
&= c(\mathfrak{f})^{-l'/2}\mathcal{N}(\mathfrak{c}(\mathfrak{f}))^{-(k_0+2s)/2}C(\mathfrak{c}(\mathfrak{f}), \mathfrak{f})\mathcal{N}\left(\frac{\mathfrak{m}'}{\mathfrak{q}\mathfrak{n}}\right)^{-(k_0+2s)} \alpha^2\left(\frac{\mathfrak{m}'}{\mathfrak{q}\mathfrak{n}}\right) D(s; \mathfrak{f}_0, G).
\end{aligned}$$

This, together with the expression for $g(\overline{\chi}^*_{\mathfrak{m}'})j_{c,\mathfrak{m}'}$, shows the independence of the value of the distribution of both $\mathfrak{m}$ and $\mathfrak{m}'$. □

5. Algebraicity of special values.

Theorem 5.1 of Im [I] shows, that for certain special values s and non-vanishing factors c_s, the values of $c_s\widetilde{\mu}^0_s$ are contained in the field $\mathbb{Q}(\mathbf{f}, g)$, which is obtained by adjoining the Fourier coefficients of $\mathbf{f}$ and g at $i\infty$ to the rational numbers. This was done for primitive forms $\mathbf{f}$, and we could try to prove the algebraicity of our distribution at these special values by making use of this result and calculations similar to those of the previous section. However, we will proceed in the following way: The distribution has a certain integral representation derived by Im. We will then apply a projection operator different from Im's as well as the trace operator and the holomorphic projection operator of section 2. Then, the theory of (integral) primitive forms allows us to verify the algebraicity via the Fourier coefficients of a certain Hilbert automorphic form of integral weight obtained from g and an Eisenstein series. In the next section the formulas that we obtain will then be used to show the boundedness of the p-adic distributions obtained by regularizing the distributions associated with the negative special values.

Let $\mathbf{f}_0$ and g be as in the previous section and determine $\theta \in \{0, 1\}$ by $k_\nu - l_\nu - \frac{1}{2} \equiv \theta \bmod 2$ for all $\nu = 1, \dots, n$ as before. Now, define the meromorphic functions

$$\begin{aligned}\Phi_1(s) &= \Phi_1(s; \mathbf{f}, g) \\ &:= \prod_{\nu=1}^{n} \left\{ \Gamma\left(s - 1 + \frac{k_\nu + l_\nu}{2}\right) \Gamma\left(s + \frac{\theta}{2} - \frac{1}{4}\right) \Gamma\left(s + \frac{k_\nu - l_\nu}{2}\right) \right\}, \\ \Phi_2(s) &= \Phi_2(s; \mathbf{f}, g) := \Phi_1\left(2\left(s - \frac{1}{4}\right)\right),\end{aligned}$$

and notice that Φ_1 is just the product of the gamma factors occurring in the definition of $\Psi_0(s; \mathbf{f}_0, g)$. We do not know if Ψ or Ψ_0 satisfy a functional equation; if we had a functional equation of the form $\Psi'(s) = \varepsilon\Psi'(1-s)$ for some Ψ' related to Ψ_0 we could define the critical points of $\mathcal{D}(s; \mathbf{f}, g)$ as in Deligne [D]. However, in our case the following definition turns out to be adequate: we say that s is a critical point of $\mathcal{D}(s; \mathbf{f}, g)$ if $\kappa := 2\left(s - \frac{3}{4}\right) + 1$ is a rational integer and neither $\Phi_2(s')$ nor $\Phi_2(1 - s')$ have a pole at $s' = \kappa$.

Since the only poles of the Γ-function are the non-positive integers, one can easily verify that the critical points of $\mathcal{D}(s;\mathbf{f},g)$ are given by $s_r = \frac{1}{2}\Big(\kappa_r + \frac{1}{2}\Big)$ with $\kappa_r \in K^+ \cup K^-$ and

$$K^+ = \big\{\kappa_r := 2-\theta+2r \,|\, r\in\mathbb{Z},\quad 0 \leqslant 2r \leqslant k_\nu - l_\nu - \frac{1}{2} + \theta - 2 \text{ for } \nu = 1,\dots,n\big\},$$

$$K^- = \big\{\kappa_r := \theta-1+2r \,|\, r\in\mathbb{Z},\quad 0 \leqslant 2r \leqslant k_\nu - l_\nu - \frac{1}{2} + \theta - 2 \text{ for } \nu = 1,\dots,n\big\}.$$

For these critical points s_r of $\mathcal{D}(s;\mathbf{f},g)$ we define $\mathbb{C}$-valued distributions $\widetilde{\mu}^+_{s_r}$ and $\widetilde{\mu}^-_{s_r}$ according to $\kappa_r \in K^+$ respectively $\kappa_r \in K^-$ by

$$\widetilde{\mu}^\pm_{s_r,\mathfrak{m}}(\chi^*_\mathfrak{m}) = \gamma^\pm(s_r)\frac{\mathcal{N}(\mathfrak{m}')^{k_0+2(s_r-1)}}{\alpha(\mathfrak{m}')^2}\,\frac{\Psi(s_r;\mathbf{f}_0,g(\overline{\chi}^*_\mathfrak{m})j_{\mathfrak{c},\mathfrak{m}'})}{\langle\, \mathbf{f},\mathbf{f}\,\rangle_{\mathfrak{c}\mathfrak{m}_0^2}} \tag{21}$$

with the factors

$$\gamma^-(s_r) = \pi^{-2ns_r+n-\{k\}} d_F^{2s_r} \mathcal{N}\Big(\frac{\mathfrak{c}}{4}\Big)^{s_r} i^{-\{k+l-\frac{3}{2}\cdot\mathbf{1}\}} e^{2\pi i\frac{n}{8}} \prod_{\nu=1}^{n} \Gamma\Big(s_r + \frac{k_\nu - l_\nu}{2}\Big),$$

$$\gamma^+(s_r) = \gamma^-(s_r)\pi^{-n\kappa_r},$$

and arbitrary integral ideals $\mathfrak{m}$, $\mathfrak{m}'$ subject to $\operatorname{lcm}(\mathfrak{m}_0,\mathfrak{c}(\chi))|\mathfrak{m}$ and $\mathfrak{m}_0\mathfrak{m}|\mathfrak{m}'$. Let us denote by F^G the Galois closure of F in $\mathbb{C}$. For any subfield $K \subseteq \mathbb{C}$ and any set $M \subseteq \mathbb{C}$, we will shortly write $K(M)$ and $K(\psi)$ for the field obtained by adjoining the elements of M or the values of the character ψ to the field K. By a scalar weight $m \in \Big(\frac{1}{2}\mathbb{Z}\Big)^n$, we understand that $m = m_0\cdot\mathbf{1}$ for some $m_0 \in \frac{1}{2}\mathbb{Z}^n$. Recall now that we always impose the following conditions:

$$h_F = 1,$$
$$l < k,\quad k_1 \equiv \ldots \equiv k_n \bmod 2,\quad l_1 \equiv \ldots \equiv l_n \bmod 2,$$
$\mathfrak{c}(\mathbf{f}), 4\mathfrak{c}(g)$ and $\mathfrak{m}_0$ are pairwise relatively prime, and
$$C(\mathfrak{c}(\mathbf{f}),\mathbf{f}) \neq 0.$$

THEOREM 5.1. — *Let $\mathbf{f}_0$ and g be as above.*

a) *Let $\kappa_r \in K^+$ and assume $\kappa_r \neq 1$ if $F=\mathbb{Q}$ and $\psi^2 = \phi^2 = 1$. Then the distributions $\widetilde{\mu}^+_{s_r}$ associated with the critical points $s_r := \frac{1}{2}\Big(\kappa_r + \frac{1}{2}\Big)$ of $\mathcal{D}(s;\mathbf{f},g)$ are defined over $\overline{\mathbb{Q}}(g)$.*

b) *There exists a number field K depending only on $\mathbf{f}$ and S such that, for the critical points $s_r = \frac{1}{2}\left(\kappa_r + \frac{1}{2}\right)$ of $\mathcal{D}(s;\mathbf{f},g)$ with $\kappa_r \in K^-$, the associated distributions $\widetilde{\mu}_{s_r}^-$ are defined over $K(F^G, \phi, \psi, g, \sqrt{\mathfrak{o}_+^\times}, \{\alpha^2(\mathfrak{p}) \mid \mathfrak{p} \in S\})$. If the weights k and l are scalar, then the distributions $\widetilde{\mu}_{s_r}^-$ are already defined over $K(\phi, \psi, g, \{\alpha^2(\mathfrak{p}) \mid \mathfrak{p} \in S\})$.*

The proof of the theorem proceeds in several steps throughout the rest of this section.

1. Integral representation of the distribution.

To start the proof, we will always take the ideal $\widetilde{t}_1 = \mathfrak{o}$ in the definition of the components of a Hilbert automorphic form of integral weight. This is the most natural choice in view of the class number hypothesis $h_F = 1$ and simplifies the notation. First, we note that $\widetilde{\mu}_{s_r}^\pm$ is well-defined by Proposition 4.1, so it remains to prove that the distributions are defined over the fields stated in the theorem. Now, from the proof of Theorem 5.1, (3.13) and (1.4b) of [I] we have the following integral representation for Ψ:

$$(22)\quad \Psi(s;\mathbf{f}_0, g(\overline{\chi}_{\mathfrak{m}}^*)j_{c,\mathfrak{m}'}) = 2^{\{k\}/2-n-1} d_F^{1/2} (2\pi)^{n(s-1)+\frac{\{k+l\}}{2}} \frac{1}{[\Gamma_0(\mathfrak{c}\mathfrak{m}'^2) : \{\pm 1\}\Gamma_1(\mathfrak{c}\mathfrak{m}'^2)]} \cdot \int_{\Gamma_1(\mathfrak{c}\mathfrak{m}'^2)\backslash \mathbb{H}^n} f_0\|_k \begin{pmatrix} 2^{-1} & \\ & 1 \end{pmatrix}(z) \overline{g(\overline{\chi}_{\mathfrak{m}}^*)j_{c,\mathfrak{m}'}(z)\mathcal{E}(z,\overline{s})} y^k \, d\mu(z)$$

with the following notation: $\mathfrak{c} = 4\mathfrak{c}(\mathbf{f})\mathfrak{c}(g)$ as before, $\Gamma_0(\mathfrak{n}) = \Gamma[2\mathfrak{d}^{-1}, 2^{-1}\mathfrak{d}\mathfrak{n}]$ and $\Gamma_1(\mathfrak{n}) = \{\gamma \in \Gamma_0(\mathfrak{n}) \mid a_\gamma \equiv 1 \bmod \mathfrak{m}\}$ for an integral ideal $\mathfrak{n}$ with $4|\mathfrak{n}$, $d\mu(z) = y^{-2\cdot\mathbf{1}}dxdy$ is the invariant measure on $\mathbb{H}^n$, $f_0 \in \mathcal{S}_k(\widetilde{\Gamma}[\mathfrak{d}^{-1}, \mathfrak{c}(\mathbf{f})\mathfrak{m}_0\mathfrak{d}], \psi_{\widetilde{\Gamma}[\mathfrak{d}^{-1},\mathfrak{c}(\mathbf{f})\mathfrak{m}_0\mathfrak{d}]})$ is the component of $\mathbf{f}_0 = (f_0)$, and $\mathcal{E}$ is given by

$$\mathcal{E}(z,s) = \mathcal{E}\left(z,s;\frac{k-l}{2},\Omega,\mathfrak{c}\mathfrak{m}'^2\right) = L_{\mathfrak{c}\mathfrak{m}'^2}(4s-1,\Omega^2) y^{-q/2} E\left(z, s-\frac{1}{4}; \frac{1}{2}, q, \Omega, \mathfrak{c}\mathfrak{m}'^2\right)$$

with $\Omega = \overline{\omega\psi\phi\varepsilon_c}\chi^2$, $q = k - l - \frac{1}{2}\cdot\mathbf{1}$, and the Eisenstein series E which equals H of (1.4b) of [I], or equivalently, E of (4.7b) of [S6]. Then

$$f_0\|_k \begin{pmatrix} 2^{-1} & \\ & 1 \end{pmatrix} \in \mathcal{S}_k(\widetilde{\Gamma}[2\mathfrak{d}^{-1}, 2\mathfrak{c}(\mathbf{f})\mathfrak{c}(g)\mathfrak{m}_0\mathfrak{d}], \psi_{\widetilde{\Gamma}[2\mathfrak{d}^{-1},2\mathfrak{c}(\mathbf{f})\mathfrak{c}(g)\mathfrak{m}_0\mathfrak{d}]}) \subseteq \mathcal{S}_k(\Gamma_0(\mathfrak{c}\mathfrak{m}'^2), \psi_{\Gamma_0(\mathfrak{c}\mathfrak{m}'^2)}).$$

Our first claim is that $g(\overline{\chi}_{\mathfrak{m}}^{*})j_{c,\mathfrak{m}'}(z)\mathcal{E}(z,\overline{s}) \in \mathcal{N}_k(\Gamma_0(\mathfrak{c}\mathfrak{m}'^2), \psi_{\Gamma_0(\mathfrak{c}\mathfrak{m}'^2)})$: Let $\gamma \in \Gamma_0(\mathfrak{c}\mathfrak{m}'^2) \leqslant C''$. Then

$$\begin{aligned}
&\left(g(\overline{\chi}_{\mathfrak{m}}^{*})j_{c,\mathfrak{m}'}(z)y^{-q/2}E(z,\overline{s})\right)\|_k\gamma \\
&= g(\overline{\chi}_{\mathfrak{m}}^{*})j_{c,\mathfrak{m}'}(\gamma(z))j(\gamma,z)^{-l'}h(\gamma,z)^{-1} \\
&\quad \cdot \mathrm{Im}(\gamma(z))^{-q/2}E(\gamma(z),\overline{s})j(\gamma,z)^{-q}h(\gamma,z)^{-1}\cdot\ j(\gamma,z)^{l'+q-k}\cdot h(\gamma,z)^2 \\
&= (\overline{\phi\varepsilon_c}\chi^2)_0(a_\gamma)\overline{\Omega}_0(a_\gamma)\omega_0(a_\gamma)\cdot g(\overline{\chi}_{\mathfrak{m}}^{*})j_{c,\mathfrak{m}'}(z)y^{-q/2}E(z,\overline{s}) \\
&= \psi_0(a_\gamma)g(\overline{\chi}_{\mathfrak{m}}^{*})j_{c,\mathfrak{m}'}(z)y^{-q/2}E(z,\overline{s}).
\end{aligned}$$

Here, we have made use of $g(\overline{\chi}_{\mathfrak{m}}^{*})j_{c,\mathfrak{m}'} \in \mathcal{M}_l(\mathfrak{c}\mathfrak{m}'^2, \overline{\phi\varepsilon_c}\chi^2)$, the automorphy property of the Eisenstein series as shown on p. 299 of [S6], and (16). Now, for the critical points s, $\mathcal{E}(z,\overline{s})$ is nearly holomorphic by 1.12 of [I], and our claim follows. We write $G(z) := g(\overline{\chi}_{\mathfrak{m}}^{*})j_{c,\mathfrak{m}'}(2z)\mathcal{E}(2z,\overline{s}) \in \mathcal{N}_k(\Gamma[\mathfrak{d}^{-1}, \mathfrak{c}\mathfrak{m}'^2\mathfrak{d}], \psi_{\Gamma[\mathfrak{d}^{-1},\mathfrak{c}\mathfrak{m}'^2\mathfrak{d}]})$ for a short notation. Then

$$\begin{aligned}
(23)\quad &\frac{1}{[\Gamma_0(\mathfrak{c}\mathfrak{m}'^2):\{\pm1\}\Gamma_1(\mathfrak{c}\mathfrak{m}'^2)]}\int_{\Gamma_1(\mathfrak{c}\mathfrak{m}'^2)\backslash\mathbb{H}^n} f_0\|_k\begin{pmatrix}2^{-1} & \\ & 1\end{pmatrix}\overline{G(\tfrac{1}{2}z)}y^k\,d\mu(z) \\
&= 2^{\{k\}/2}\int_{\Gamma_0(\mathfrak{c}\mathfrak{m}'^2)\backslash\mathbb{H}^n} f_0\|_k\begin{pmatrix}2^{-1} & \\ & 1\end{pmatrix}\overline{G(z)\|_k\begin{pmatrix}2^{-1} & \\ & 1\end{pmatrix}}y^k\,d\mu(z) \\
&= 2^{\{k\}/2}\int_{\Gamma[\mathfrak{d}^{-1},\mathfrak{c}\mathfrak{m}'^2\mathfrak{d}]\backslash\mathbb{H}^n} f_0\overline{G}y^k\,d\mu(z).
\end{aligned}$$

2. A projection operator.

We introduce a certain projection operator, which enables us to write this last integral as the Petersson scalar product of Hilbert automorphic forms of integral weight.

LEMMA 5.2. — *Let R_0 be a system of representatives of $\mathfrak{o}_+^\times$ mod $\mathfrak{o}_+^{\times 2}$ and $\mathfrak{n}$ an integral ideal. Then there is a projection*

$$P_k : \mathcal{N}_k(\Gamma[\mathfrak{d}^{-1},\mathfrak{n}\mathfrak{d}], \psi_{\Gamma[\mathfrak{d}^{-1},\mathfrak{n}\mathfrak{d}]}) \to \mathcal{N}_k(\widetilde{\Gamma}[\mathfrak{d}^{-1},\mathfrak{n}\mathfrak{d}], \psi_{\widetilde{\Gamma}[\mathfrak{d}^{-1},\mathfrak{n}\mathfrak{d}]}),$$
$$f \mapsto 2^{-n+1}\sum_{\gamma\in R_0} f\|_k\begin{pmatrix}1 & \\ & \gamma\end{pmatrix}$$

whose effect on the Fourier expansion of $f(z) = \sum_{\xi=0,\xi\gg0} a(\xi,y)e_\infty(\xi z)$ *is*

$$P_k(f)(z) = \sum_{\xi=0,0\ll\xi}\left(2^{-n+1}\sum_{\gamma\in R_0}\gamma^{-\frac{k}{2}}a(\gamma\xi,\gamma^{-1}y)\right)e_\infty(\xi z).$$

Proof. — For $\gamma, \gamma' \in R_0$ determine γ'' by $\gamma\gamma' = \gamma''\gamma_1^2$ for some $\gamma_1 \in \mathfrak{o}_+^\times$. Then for γ fixed the map $\gamma' \mapsto \gamma''$ is a bijection of R_0. This shows

$$\begin{aligned} P_k(f)\|_k\begin{pmatrix}1 & \\ & \gamma\end{pmatrix} &= 2^{-n+1}\sum_{\gamma'\in R_0} f\|_k\begin{pmatrix}1 & \\ & \gamma\gamma'\end{pmatrix} \\ &= 2^{-n+1}\sum_{\gamma''\in R_0} f\|_k\begin{pmatrix}\gamma_1^{-1} & \\ & \gamma_1\end{pmatrix}\|_k\begin{pmatrix}\gamma_1 & \\ & \gamma_1\end{pmatrix}\|_k\begin{pmatrix}1 & \\ & \gamma''\end{pmatrix} \\ &= 2^{-n+1}\sum_{\gamma''\in R_0} \psi_0(\gamma_1^{-1}) f\|_k\begin{pmatrix}1 & \\ & \gamma''\end{pmatrix} = P_k(f) \end{aligned}$$

because the character ψ satisfies $\psi_0(\varepsilon) = \mathrm{sgn}(\varepsilon)^k$ for every $\varepsilon \in \mathfrak{o}^\times$. Secondly, let $\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma[\mathfrak{d}^{-1}, \mathfrak{n}\mathfrak{d}]$. Then

$$\begin{aligned} P_k(f)\|_k\begin{pmatrix}a & b\\ c & d\end{pmatrix} &= 2^{-n+1}\sum_{\gamma\in R_0} f\|_k\begin{pmatrix}1 & \\ & \gamma\end{pmatrix}\|_k\begin{pmatrix}a & b\\ c & d\end{pmatrix} \\ &= 2^{-n+1}\sum_{\gamma\in R_0} f\|_k\begin{pmatrix}a & \gamma^{-1}b\\ \gamma c & d\end{pmatrix}\|_k\begin{pmatrix}1 & \\ & \gamma\end{pmatrix} \\ &= \psi_0(a)P_k(f). \end{aligned}$$

But $\widetilde{\Gamma}[\mathfrak{d}^{-1}, \mathfrak{n}\mathfrak{d}]\cdot\mathfrak{o}_+^\times$ is generated by $\Gamma[\mathfrak{d}^{-1}, \mathfrak{n}\mathfrak{d}]\cdot\mathfrak{o}_+^\times$ and $\{\begin{pmatrix}1 & \\ & \gamma\end{pmatrix} \mid \gamma \in R_0\}\cdot\mathfrak{o}_+^\times$, hence P_k is a projection of the given spaces as claimed. The Fourier expansion of $P_k(f)$ is then derived from the explicit formula of P_k. □

The Hecke character ψ of finite order is uniquely determined by its $\mathfrak{c}(\psi)$-part ψ_0 because the class number h_F equals 1. Thus $P_k(G) \in \mathcal{N}_k(\widetilde{\Gamma}[\mathfrak{d}^{-1}, \mathfrak{c}\mathfrak{m}'^2\mathfrak{d}], \psi_{\widetilde{\Gamma}[\mathfrak{d}^{-1},\mathfrak{c}\mathfrak{m}'^2\mathfrak{d}]})$ is already the component of a nearly holomorphic Hilbert automorphic form of central character ψ which we will denote by $P_k(G)_{\mathbf{A}}$. We can then apply the trace operator for integral Hilbert automorphic forms to obtain

(24)

$$\begin{aligned} &\int_{\Gamma[\mathfrak{d}^{-1},\mathfrak{c}\mathfrak{m}'^2\mathfrak{d}]\backslash\mathbb{H}^n} f_0\overline{G}y^k\,d\mu(z) \\ &\quad= 2^{n-1}\int_{\widetilde{\Gamma}[\mathfrak{d}^{-1},\mathfrak{c}\mathfrak{m}'^2\mathfrak{d}]\backslash\mathbb{H}^n} f_0\overline{P_k(G)}\,d\mu(z) \\ &\quad= 2^{n-1}\langle\, \mathbf{f}_0, P_k(G)_{\mathbf{A}}\,\rangle_{\mathfrak{c}\mathfrak{m}'^2} \\ &\quad= (-1)^{\{k\}}2^{n-1}\mathcal{N}\Big(\frac{\mathfrak{m}'}{\mathfrak{m}_0}\Big)^{2-k_0}\langle\, \mathbf{f}_0, P_k(G)_{\mathbf{A}}J_{\mathfrak{c}\mathfrak{m}'^2}U\Big(\frac{\mathfrak{m}'^2}{\mathfrak{m}_0^2}\Big)J_{\mathfrak{c}\mathfrak{m}_0^2}\,\rangle_{\mathfrak{c}\mathfrak{m}_0^2}. \end{aligned}$$

3. The Fourier expansion of $P_k(G)_{\mathbf{A}} J_{c\mathfrak{m}'^2} U\left(\frac{\mathfrak{m}'^2}{\mathfrak{m}_0^2}\right)$.

The next step will be to decompose $J_{c\mathfrak{m}'^2}$ into half-integral operators of well-known effect on the factors of G. For this purpose we choose totally positive elements d, $m' \in F^\times$ such that $(d) = \mathfrak{d}$ and $(m') = \mathfrak{m}'$. Set

$$\beta_0 = \begin{pmatrix} & 1 \\ -4cm'^2d^2 & \end{pmatrix} = \begin{pmatrix} 2 & \\ & 2 \end{pmatrix}\begin{pmatrix} 2^{-1} & \\ & 1 \end{pmatrix}\begin{pmatrix} d & \\ & d \end{pmatrix}\begin{pmatrix} & d^{-1} \\ -d & \end{pmatrix}\begin{pmatrix} cm'^2 & \\ & 1 \end{pmatrix}\begin{pmatrix} 2 & \\ & 1 \end{pmatrix},$$

$$\beta = \begin{pmatrix} 2^{-1} & \\ & 1 \end{pmatrix}\begin{pmatrix} & d^{-1} \\ -d & \end{pmatrix}\begin{pmatrix} cm'^2 & \\ & 1 \end{pmatrix}\begin{pmatrix} 2 & \\ & 1 \end{pmatrix}.$$

The component of $P_k(G)_{\mathbf{A}} J_{c\mathfrak{m}'^2}$ is

(25)
$$\begin{aligned}(-1)^{\{k\}} P_k(G)\|_k\beta_0 &= (-1)^{\{k\}} 2^{-n+1} \sum_{\gamma \in R_0} G\|_k \beta_0 \|_k \begin{pmatrix} \gamma & \\ & \gamma \end{pmatrix} \|_k \begin{pmatrix} 1 & \\ & \gamma^{-1} \end{pmatrix} \\ &= (-1)^{\{k\}} P_k(G\|_k\beta).\end{aligned}$$

It is most convenient to have $\delta_f = (\ldots, d, d, d, \ldots)$ for the idele δ used to define η of (5). We then choose an open subgroup $B \leqslant C'$ such that

$$(g(\overline{\chi}^*_{\mathfrak{m}}) j_{c,\mathfrak{m}'})_{\mathbf{A}}(x) = (g(\overline{\chi}^*_{\mathfrak{m}}) j_{c,\mathfrak{m}'}\|_l x)(\mathbf{i}) \qquad \text{for } x \in \mathrm{pr}^{-1}(BG_\infty),$$

$$(y^{-q/2}E(z,\overline{s}))_{\mathbf{A}}(x) = (y^{-q/2}E(z,\overline{s})\|_{k-l}x)(\mathbf{i}) \qquad \text{for } x \in \mathrm{pr}^{-1}(BG_\infty),$$

$$G\|_k \begin{pmatrix} 2^{-1} & \\ & 1 \end{pmatrix} \|_k \gamma = G\|_k \begin{pmatrix} 2^{-1} & \\ & 1 \end{pmatrix} \qquad \text{for } \gamma \in G \cap BG_\infty,$$

and find $\hat\eta \in G$ as in Proposition 3.5c) such that $\hat\eta\eta \in BG_\infty$. Define

$$\gamma := \begin{pmatrix} & d^{-1} \\ -d & \end{pmatrix}\hat\eta^{-1} = \begin{pmatrix} & d^{-1} \\ -d & \end{pmatrix}\eta(\hat\eta\eta)^{-1} \in G \cap BG_\infty.$$

The action of β on G can then be written as

$$\begin{aligned}(G\|_k\beta)(z) &= v_1 2^{-\{k\}/2} L_{c\mathfrak{m}_0}(4\overline{s}-1,\Omega^2) \\ &\cdot \left((g(\overline{\chi}^*_{\mathfrak{m}}) j_{c,\mathfrak{m}'}(z))\|_l\hat\eta \cdot \left(y^{-q/2}E\left(z,\overline{s}-\frac{1}{4}\right)\right)\|_{k-l}\hat\eta\right) \|_k \begin{pmatrix} cm'^2 & \\ & 1 \end{pmatrix}\begin{pmatrix} 2 & \\ & 1 \end{pmatrix}\end{aligned}$$

with $v_1 = \dfrac{h(\hat\eta,z)^2}{j(\hat\eta,z)}$. Next, observe that

$$\begin{aligned}(g(\overline{\chi}^*_{\mathfrak{m}}) j^2_{c,\mathfrak{m}'})(z) &= \mathcal{N}(c\mathfrak{m}'^2)^{\frac14} c^{\frac12 l'} g(\overline{\chi}^*_{\mathfrak{m}}) j_{c,\mathfrak{m}'} \iota\, \mathrm{m}_c\, \mathrm{sw}_{\mathfrak{m}'}(z) \\ &= (cm'^2)^{\frac{l}{2}} g(\overline{\chi}^*_{\mathfrak{m}}) j_{c,\mathfrak{m}'} \iota\, \mathrm{m}_{cm'^2}(z) \\ &= (cm'^2)^{\frac{l}{2}} (g(\overline{\chi}^*_{\mathfrak{m}}) j_{c,\mathfrak{m}'}\|_l\hat\eta)(cm'^2 z)\end{aligned}$$

by Proposition 3.5a) and m' being totally positive. For the other factor we write

$$y^{-q/2}E\left(z,\bar{s}-\frac{1}{4}\right)\|_{k-l}\hat{\eta}=y^{-q/2}E\left(\hat{\eta}(z),\bar{s}-\frac{1}{4}\right)\left(h(\hat{\eta},z)j(\hat{\eta},z)^{q}|j(\hat{\eta},z)|^{-q}\right)^{-1}$$
$$=y^{-q/2}E'\left(z,\bar{s}-\frac{1}{4}\right)$$

by (4.10) of [S6] with an Eisenstein series $E'=E'(z,s)=E'\left(z,s;\frac{1}{2},q,\Omega,\mathfrak{c}\mathfrak{m}'^2\right)$ defined by 4.9a,b) of [loc. cit.]. Summarizing these calculations we obtain

(26)
$$(G\|_k\beta)(z)=v_1v(c)\phi_\infty(-1)(\mathfrak{c}m'^2)^{\frac{k-l}{2}}$$
$$\cdot\, g(\overline{\chi}^*_{\mathfrak{m}})(2z)L_{\mathfrak{c}\mathfrak{m}_0}(4\bar{s}-1,\Omega^2)(2\mathfrak{c}m'^2y)^{-q/2}E'\left(2\mathfrak{c}m'^2z,\bar{s}-\frac{1}{4}\right).$$

Here, $v(c)\in\{\pm1\}$ is determined by Proposition 3.5c). By 3.14b) of [S6] we have $v_1^2=(-1)^n$, so v_1 is a fourth root of unity. v_1 is independent of $\mathfrak{m}'$ because the value of v_1 is independent of the subgroup B used to define $\hat{\eta}$ as can be seen from (26).

Now, the Fourier expansion of E' has been derived by Shimura in [S6]. It involves the confluent hypergeometric function

$$\xi(u,v;\alpha,\beta)=\int_{\mathbb{R}}e^{-2\pi ivx}(x+iu)^{-\alpha}(x-iu)^{-\beta}\,dx\text{ for }u\in\mathbb{R}_+,\ v\in\mathbb{R},\ \alpha,\beta\in\mathbb{C}$$

introduced in (1.25) of [S5]. The integral is defined for $\mathrm{Re}(\alpha+\beta)>1$, but admits analytic continuation in α and β. Let us also define

$$\eta(u,v;\alpha,\beta)=\int_{x>|v|}e^{-ux}(x+v)^{\alpha-1}(x-v)^{\beta-1}\,dx$$

and the classical Whittaker function

$$W(u,\alpha,\beta)=\int_0^\infty e^{-ux}(x+1)^{\alpha-1}x^{\beta-1}\,dx\qquad\text{for }u>0\text{ and }\mathrm{Re}(\beta)>0.$$

W has a meromorphic continuation as shown in [S2]. By (1.29) of [S5] there is the relation

$$\xi(u,v;\alpha,\beta)=i^{\beta-\alpha}(2\pi)\Gamma(\alpha)^{-1}\Gamma(\beta)^{-1}\eta(2u,\pi v;\alpha,\beta).$$

A short calculation shows that ξ can then be expressed in terms of W as follows:

$$(27)\quad \xi(u,v;\alpha,\beta) = i^{\beta-\alpha}2\pi\Gamma(\alpha)^{-1}\Gamma(\beta)^{-1} \begin{cases} e^{-2\pi uv}(2\pi v)^{\alpha+\beta-1}W(4\pi uv,\alpha,\beta) & \text{if } v>0,\\ \Gamma(\alpha+\beta-1)(2u)^{-(\alpha+\beta-1)} & \text{if } v=0,\\ e^{-2\pi u|v|}(2\pi|v|)^{\alpha+\beta-1}W(4\pi u|v|,\beta,\alpha) & \text{if } v<0. \end{cases}$$

Now, let $s_r = \dfrac{1}{2}\left(\kappa_r + \dfrac{1}{2}\right)$ with $\kappa_r \in K^+ \cup K^-$ be a critical point of $\mathcal{D}(s,\mathbf{f},g)$ and define

(28)

$$\alpha_\nu = \alpha_\nu(\kappa_r) := \frac{\kappa_r + 1 + q_\nu}{2}, \quad \beta_\nu = \beta_\nu(\kappa_r) := \frac{\kappa_r - q_\nu}{2} \qquad \text{for } \nu = 1,\dots,n.$$

Then, $\alpha_\nu + \beta_\nu - 1 = \kappa_r - \dfrac{1}{2} = 2s_r - 1$. In particular, we have the following:

$$(29)\quad \begin{array}{ll} \alpha_\nu \equiv \frac{1}{2} \bmod 1,\ \alpha_\nu > 0 \ \text{ and } \ \beta_\nu \in \mathbb{Z},\ \beta_\nu \leqslant 0 & \text{if } \kappa_r \in K^+,\\ \alpha_\nu \in \mathbb{Z},\ \alpha_\nu \geqslant 1 \ \text{ and } \ \beta_\nu \equiv \frac{1}{2} \bmod 1,\ \beta_\nu < 0 & \text{if } \kappa_r \in K^-. \end{array}$$

The Fourier expansion of E' is given by (6.1) and Proposition 6.1 of [S6]:

(30)

$$L_{\mathfrak{cm}_0}(2\kappa_r,\Omega^2)(2cm'^2y)^{-q/2}E'\left(2cm'^2z,\frac{\kappa_r}{2},\frac{1}{2},q,\Omega,\mathfrak{cm}'^2\right) = \sum_{\sigma\in 2\mathfrak{c}^{-1}\mathfrak{m}'^{-2}} c(\sigma,2cm'^2y,\kappa_r)e_\infty(2cm'^2\sigma x),$$

$$c(\sigma,y,\kappa_r) = (-1)^{\{q\}}d_F^{-\kappa_r}\mathcal{N}(2\mathfrak{c}^{-1}\mathfrak{m}'^{-2})e^{2\pi i\frac{n}{8}}c_f(\sigma,\kappa_r)\prod_{\nu=1}^{n} y_\nu^{\beta_\nu}\xi(y_\nu,\sigma_\nu;\alpha_\nu,\beta_\nu),$$

$$c_f(0,\kappa_r) = L_{\mathfrak{cm}_0}(2\kappa_r - 1,\Omega^2),$$

$$c_f(\sigma,\kappa_r) = L_{\mathfrak{cm}_0}(\kappa_r,\Omega_{2\sigma})\sum_{(\mathfrak{a},\mathfrak{b})}\mu(\mathfrak{a})\Omega_{2\sigma}^*(\mathfrak{a})\Omega^*(\mathfrak{b}^2)\mathcal{N}(\mathfrak{a})^{-\kappa_r}\mathcal{N}(\mathfrak{b})^{1-2\kappa_r}.$$

Here, for $\xi \in F$, Ω_ξ denotes the Hecke character Ω multiplied with the Hecke character ε_ξ corresponding to $F(\sqrt{\xi})/F$, $L_{\mathfrak{cm}_0}(\,\cdot\,,\Omega_{2\sigma})$ is the L-function associated with $\Omega_{2\sigma}$ as in (3), and the summation in the last sum is over all ordered pairs $(\mathfrak{a},\mathfrak{b})$ of integral ideals of F prime to $\mathfrak{cm}'^2$ such that

$\mathfrak{a}^2\mathfrak{b}^2 \supseteq \sigma\mathfrak{c}\mathfrak{m}'^2$. Our next claim is that the Fourier coefficients $c(\sigma, y, \kappa_r)$ vanish unless $\sigma = 0$ and $\kappa_r \leqslant 0$ or σ is totally positive.

Let us first consider the case $\kappa_r \in K^+$. If $\sigma = 0$ then the factor $L_{\mathfrak{c}\mathfrak{m}_0}(2\kappa_r - 1, \Omega^2)$ is finite unless $\kappa_r = 1$ and $\Omega^2 = 1$ in which case it has a simple pole. On the other hand, $\xi(y_\nu, 0, \alpha_\nu, \beta_\nu)$ has a simple zero at $\kappa_r = 1$. Hence $c(0, y, \kappa_r) = 0$ except for the excluded case $F = \mathbb{Q}$, $\kappa_r = 1$, and $\phi^2 = \psi^2 = \chi^4 = 1$. Now assume $\sigma_\nu < 0$ for some ν. Then, $\xi(y_\nu, \sigma_\nu; \alpha_\nu, \beta_\nu) = 0$ because Γ^{-1} has a zero at β_ν and $W(4\pi y_\nu|\sigma_\nu|, \beta_\nu, \alpha_\nu)$ is finite. If $\kappa_r > 1$ or $\Omega_{2\sigma} \neq 1$ then $L_{\mathfrak{c}\mathfrak{m}_0}(\kappa_r, \Omega_{2\sigma})$ is finite, and we have $c(\sigma, y, \kappa_r) = 0$. If $\kappa_r = 1$ and $\Omega_{2\sigma} = 1$, then $\theta = 1$ and $L_{\mathfrak{c}\mathfrak{m}_0}(\cdot, \Omega_{2\sigma})$ has a simple pole at 1. By the assumption on the archimedean components of ψ and ϕ we have $\Omega_\infty(x) = \mathrm{sgn}(x)^q = \mathrm{sgn}(x)^{\theta\mathbf{1}}$; this follows from $\omega_\infty(x) = \mathrm{sgn}(x)^{\mathbf{1}}$, $\phi_\infty(x) = \mathrm{sgn}(x)^{l'}$, and $\psi_\infty(x) = \mathrm{sgn}(x)^k$ because $\psi_\infty(\varepsilon) = \mathrm{sgn}(\varepsilon)^k$ for all $\varepsilon \in \mathfrak{o}^\times$ and there exist units ε with any given signature due to the class number assumption $h_F = 1$. Therefore, $\Omega_{2\sigma} = 1$ implies σ is totally negative. Now if $F = \mathbb{Q}$, $\Omega_{2\sigma} = 1$ implies $\Omega^2 = \phi^2 = \psi^2 = \chi^4 = 1$ and thus gives the excluded case again. But for $F > \mathbb{Q}$ the n-fold zero of $\prod_\nu \xi(y_\nu, \sigma_\nu; \alpha_\nu, \cdot)$ at $(\beta_1, \ldots, \beta_n)$ now implies the vanishing of $c(\sigma, y, \kappa_r)$. Let us now treat the case $\kappa_r \in K^-$. If $\kappa_r = 0$ and $\Omega_{2\sigma} = 1$ then $L_{\mathfrak{c}\mathfrak{m}_0}(0, 1) = \prod_{\mathfrak{p}|\mathfrak{c}\mathfrak{m}_0} (1 - \mathcal{N}(\mathfrak{p})^0) L(0, 1) = 0$. In all other cases, $L(1 - \kappa_r, \overline{\Omega}_{2\sigma})$ is finite. This implies again that $L_{\mathfrak{c}\mathfrak{m}_0}(\kappa_r, \Omega_{2\sigma}) = 0$ by the functional equation: we have $\kappa_r \equiv \theta - 1 \bmod 2$ and as above, $\Omega_\infty(x) = \mathrm{sgn}(x)^{\theta\mathbf{1}}$. Let $r_0(\Omega_{2\sigma})$ denote the number of archimedean primes at which $\Omega_{2\sigma}$ ramifies. If $\theta = 0$ and $\sigma \not\gg 0$, then $r_0(\Omega_{2\sigma}) > 0$ and $\Gamma\Big(\frac{\kappa_r + 1}{2}\Big)^{-r_0(\Omega_{2\sigma})} = 0$. If $\theta = 1$ and $\sigma \not\gg 0$ then $r_0(\Omega_{2\sigma}) < n$ and hence $\Gamma\Big(\frac{\kappa_r}{2}\Big)^{-(n - r_0(\Omega_{2\sigma}))} = 0$. In both cases, $L(\kappa_r, \Omega_{2\sigma}) = 0$ by the functional equation (4). This proves the vanishing of $c(\sigma, y, \kappa_r)$ if $\sigma \neq 0$ and σ is not totally positive. Our claim is now established.

The Fourier expansion (30) can now be written such that up to some factor it is independent of m'. Observe that as σ runs over 0 and the totally positive elements of $2\mathfrak{c}^{-1}\mathfrak{m}'^{-2}$, $\sigma' := \sigma 2cm'^2$ runs over 0 and the totally positive elements of $\mathfrak{o}$. Define

$$(31) \qquad B(\sigma', \kappa_r; \mathfrak{c}\mathfrak{m}_0) := \sum_{(\mathfrak{a},\mathfrak{b})} \mu(\mathfrak{a}) \Omega^*_{\sigma' c}(\mathfrak{a}) \Omega^*(\mathfrak{b}^2) \mathcal{N}(\mathfrak{a})^{-\kappa_r} \mathcal{N}(\mathfrak{b})^{1 - 2\kappa_r}$$

where the summation is over all ordered pairs $(\mathfrak{a}, \mathfrak{b})$ of integral ideals of F prime to $\mathfrak{c}\mathfrak{m}_0$ such that $\mathfrak{a}^2\mathfrak{b}^2 \supseteq (\sigma')$. Expressing ξ in terms of W as in (27) gives us the normalized Fourier expansion

$$(32)\qquad \begin{aligned} &L_{\mathfrak{cm}_0}(2\kappa_r,\Omega^2)(2cm'^2y)^{-q/2}E'\Big(2cm'^2z,\frac{\kappa_r}{2},\frac{1}{2},q,\Omega,\mathfrak{cm}'^2\Big)\\ &=\gamma_1(\mathfrak{m}',\kappa_r)\sum_{0\leqq\sigma'\in\mathfrak{o}} c_1(\sigma',y,\kappa_r)e_\infty(\sigma'z),\end{aligned}$$

where the factor γ_1 and the Fourier coefficients c_1 are given by

$$\gamma_1(\mathfrak{m}',\kappa_r)=i^{\{k-l-1\}}d_F^{-\kappa_r}\mathcal{N}(2\mathfrak{c}^{-1}\mathfrak{m}'^{-2})e^{2\pi i\frac{n}{8}}(2\pi)^n\prod_{\nu=1}^{n}\Gamma(\alpha_\nu)^{-1},$$

$$\begin{aligned} c_1(0,y,\kappa_r)&=2^{-n(\kappa_r-\frac{1}{2})}\Gamma(\kappa_r-\frac{1}{2})^nL_{\mathfrak{cm}_0}(2\kappa_r-1,\Omega^2)\\ &\quad\cdot\prod_{\nu=1}^{n}\Big\{(2c_\nu m_\nu'^2y_\nu)^{-\alpha_\nu+1}\Gamma(\beta_\nu)^{-1}\Big\},\\ c_1(\sigma',y,\kappa_r)&=(2\pi)^{n(2s_r-1)}\mathcal{N}(\sigma')^{2s_r-1}L_{\mathfrak{cm}_0}(\kappa_r,\Omega_{\sigma'\mathfrak{c}})B(\sigma',\kappa_r;\mathfrak{cm}_0)\\ &\quad\cdot\prod_{\nu=1}^{n}\Big\{(2c_\nu m_\nu'^2y_\nu)^{-\alpha_\nu+1}\Gamma(\beta_\nu)^{-1}y_\nu^{\beta_\nu}W(4\pi y_\nu\sigma_\nu',\alpha_\nu,\beta_\nu)\Big\}.\end{aligned}$$

By Proposition 3.1 and Corollary 3.4 $g(\overline{\chi}_{\mathfrak{m}}^*)$ has the Fourier expansion

$$g(\overline{\chi}_{\mathfrak{m}}^*)(2z)=\sum_{0\ll\sigma_1\in\mathfrak{o}}\overline{\chi}_{\mathfrak{m}}^*((\sigma_1))\lambda_g(\sigma_1,\mathfrak{o})e_\infty(\sigma_1z).$$

Summarizing (25), (26), (32) and applying Lemma 5.2 to the nearly holomorphic automorphic form $P_k(G)_{\mathbf{A}}J_{\mathfrak{cm}'^2}=(g_1)$ we see that the component g_1 has the Fourier expansion

$$\begin{aligned} g_1(z)&=(-1)^{\{k\}}v_1v(c)\phi_\infty(-1)(cm'^2)^{(k-l)/2}\gamma_1(\mathfrak{m}',\kappa_r)2^{-n+1}\\ &\cdot\sum_{0\ll\sigma\in\mathfrak{o}}\Big(\sum_{\gamma\in\mathfrak{o}_+^\times/\mathfrak{o}_+^{\times2}}\gamma^{-k/2}\sum_{\sigma_1,\sigma_2}\overline{\chi}_{\mathfrak{m}}^*((\sigma_1))\lambda_g(\sigma_1,\mathfrak{o})c_1(\sigma_2,\gamma^{-1}y,\kappa_r)\Big)e_\infty(\sigma z),\end{aligned}$$

where the summation in the third sum is over all σ_1, σ_2 which satisfy $\gamma\sigma=\sigma_1+\sigma_2$, $0\ll\sigma_1\in\mathfrak{o}$ and $\sigma_2=0$ or $0\ll\sigma_2\in\mathfrak{o}$. For the two inner summations, notice that

$$\begin{gathered}\{(\sigma_1,\sigma_2)\mid\mathfrak{o}\ni\sigma_i\gg0,\sigma_1+\sigma_2=\sigma\}\to\{(\sigma_1,\sigma_2)\mid\mathfrak{o}\ni\sigma_i\gg0,\sigma_1+\sigma_2=\gamma\sigma\},\\ (\sigma_1,\sigma_2)\mapsto(\gamma\sigma_1,\gamma\sigma_2)\end{gathered}$$

is a bijection. Also, the effect of the operator U is given as follows: Let $\mathbf{f}_1=(f_1)\in\mathcal{N}_k$ with the Fourier expansion $f_1(z)=\sum_{\xi=0,0\ll\xi\in\mathfrak{o}}a(\xi,\mathbf{f}_1)e_\infty(\xi z)$

with complex coefficients $a(\xi,\mathbf{f}_1) = a(\xi,\mathbf{f}_1,y)$ depending on y. Now, let $\eta \in \mathfrak{o}$ be totally positive. From the definition of $U(\mathfrak{q})$ and (2.18) of [S4] it follows that $a(0,\mathbf{f}_1|U((\eta))) = a(0,\mathbf{f}_1)$ (which can only be different from 0 if $k_1 = \ldots = k_n$), but it will turn out that the 0-th Fourier coefficient $a(0,\mathbf{f}_1|U((\eta)))$ will not occur in the Fourier development of $P_k(G)_{\mathbf{A}}J_{\mathfrak{c}\mathfrak{m}'^2}U\Big(\frac{\mathfrak{m}'^2}{\mathfrak{m}_0^2}\Big)$. Now, let $\xi \in \mathfrak{o}$ be totally positive. Then $C((\xi),\mathbf{f}_1) = \xi^{\frac{1}{2}(k_0\mathbf{1}-k)}a(\xi,\mathbf{f}_1)$ and $C((\xi),\mathbf{f}_1|U((\eta))) = C((\xi\eta),\mathbf{f}_1)$, and hence the Fourier expansion of $\mathbf{f}_1|U((\eta)) = (f_2)$ is given by

$$f_2(z) = \sum_{0 \ll \xi \in \mathfrak{o}} \eta^{\frac{1}{2}(k_0\mathbf{1}-k)} a(\xi\eta,\mathbf{f}_1) e_\infty(\xi z).$$

Fixing a totally positive $m_0 \in F^\times$ with $(m_0) = \mathfrak{m}_0$, we finally obtain the Fourier expansion of $P_k(G)_{\mathbf{A}}J_{\mathfrak{c}\mathfrak{m}'^2}U\Big(\frac{\mathfrak{m}'^2}{\mathfrak{m}_0^2}\Big) = (g_2)$ as follows:

$$g_2(z) = \gamma_2(\mathfrak{m}',\kappa_r) \sum_{0 \ll \sigma \in \mathfrak{o}} b(\sigma,y,\kappa_r) e_\infty(\sigma z) \tag{33}$$

with the factor γ_2 and Fourier coefficient b given by

$$\begin{aligned}\gamma_2(\mathfrak{m}',\kappa_r) &= (-1)^{\{k+l'\}} v_1 v(c)(cm'^2)^{(k-l)/2}\gamma_1(\mathfrak{m}',\kappa_r)2^{-n+1}\Big(\frac{m'}{m_0}\Big)^{k_0\mathbf{1}-k} \\ &\quad \cdot (2\pi)^{n(2s_r-1)}\prod_{\nu=1}^{n}(2c_\nu m_\nu'^2)^{-\alpha_\nu+1}, \\ b(\sigma,y,\kappa_r) &= \sum_{\substack{(\frac{m'}{m_0})^\sigma = \sigma_1+\sigma_2 \\ 0\ll\sigma_i\in\mathfrak{o}}} \overline{\chi}^*_{\mathfrak{m}}((\sigma_1)) \sum_{\gamma \in \mathfrak{o}_+^\times/\mathfrak{o}_+^{\times 2}} \lambda_g(\gamma\sigma_1,\mathfrak{o}) \\ &\quad \cdot L_{\mathfrak{c}\mathfrak{m}_0}(\kappa_r,\Omega_{\gamma\sigma_2 c})B(\gamma\sigma_2,\kappa_r;\mathfrak{c}\mathfrak{m}_0)\gamma^{-k/2}\mathcal{N}(\sigma_2)^{2s_r-1} \\ &\quad \prod_{\nu=1}^{n}\left\{\gamma_\nu^{-\beta_\nu}y_\nu^{\beta_\nu}\Gamma(\beta_\nu)^{-1}W(4\pi y_\nu\sigma_{2,\nu},\alpha_\nu,\beta_\nu)\right\}.\end{aligned}$$

4. Application of the holomorphic projection operator.

We will now apply the holomorphic projection operator of Proposition 2.1 to the function $P_k(G)_{\mathbf{A}}J_{\mathfrak{c}\mathfrak{m}'^2}U\Big(\frac{\mathfrak{m}'^2}{\mathfrak{m}_0^2}\Big)$, which is of moderate growth because it is a cusp form in $\mathcal{N}_k(\mathfrak{c}\mathfrak{m}_0^2,\psi)$. The Fourier expansion of $\mathcal{H}ol\Big(P_k(G)_{\mathbf{A}} J_{\mathfrak{cm}'^2} U\Big(\frac{\mathfrak{m}'^2}{\mathfrak{m}_0^2}\Big)\Big)$ is obtained by replacing $b(\sigma, y, \kappa_r)$ with

$$\frac{4\pi^{\{k-1\}}\sigma^{k-1}}{\prod_{\nu=1}^{n}\Gamma(k_\nu - 1)} \int_{\mathbb{R}_+^n} b(\sigma, y, \kappa_r) e^{-4\pi \sum_\nu \sigma_\nu y_\nu} \prod_{\nu=1}^{n} y_\nu^{k_\nu - 2}\, dy_1 \dots dy_n.$$

This integral can be explicitly evaluated because there exist polynomial expressions in y^{-1} for $y_\nu^{\beta_\nu}\Gamma(\beta_\nu)^{-1}W(y_\nu, \alpha_\nu, \beta_\nu)$ with α_ν and β_ν of (28). This also shows that the integral is absolutely convergent. We will now first treat the case $\kappa_r \in K^+$. From (2.3) of [S2] and (29) it follows that

$$\begin{aligned} y^{\beta_\nu}\Gamma(\beta_\nu)^{-1}W(y, \alpha_\nu, \beta_\nu) &= \frac{e^{-\pi i \beta_\nu}}{2\pi i}\Gamma(1-\beta_\nu)\int_\infty^{(0+)} (1+y^{-1}t)^{\alpha_\nu - 1} t^{\beta_\nu - 1} e^{-t}\, dt \\ &= (-1)^{\beta_\nu}\Gamma(1-\beta_\nu) \operatorname*{Res}_{t=0} \{(1+y^{-1}t)^{\alpha_\nu-1} t^{\beta_\nu - 1} e^{-t}\} \\ &= \sum_{j=0}^{-\beta_\nu} \binom{\alpha_\nu - 1}{j} y^{-j}(-1)^j \frac{\Gamma(1-\beta_\nu)}{\Gamma(1-\beta_\nu - j)}, \end{aligned}$$

with the contour integral as in [loc. cit.]. We then evaluate

(34)
$$\begin{aligned} P_{\kappa_r,\nu}^{+}(\sigma_{2,\nu}, \sigma_\nu) &:= (4\pi)^{\beta_\nu} \frac{4\pi^{k_\nu - 1}\sigma_\nu^{k_\nu - 1}}{\Gamma(k_\nu - 1)} \\ &\cdot \int_0^\infty y^{\beta_\nu}\Gamma(\beta_\nu)^{-1} W(4\pi y \sigma_{2,\nu}, \alpha_\nu, \beta_\nu)(\sigma_{2,\nu})^{2s_r - 1} e^{-4\pi\sigma_\nu y} y^{k_\nu - 2}\, dy \\ &= \sum_{j=0}^{-\beta_\nu} \binom{\alpha_\nu - 1}{j} (-1)^j \frac{\Gamma(1-\beta_\nu)}{\Gamma(1-\beta_\nu - j)} \frac{\Gamma(k_\nu - 1 - j)}{\Gamma(k_\nu - 1)} \sigma_\nu^j \sigma_{2,\nu}^{\alpha_\nu - 1 - j}. \end{aligned}$$

For $\kappa_r \in K^-$, Lemma 2 of [loc. cit.] together with the above calculations gives

$$\begin{aligned} y^{\beta_\nu}\Gamma(\beta_\nu)^{-1}W(y, \alpha_\nu, \beta_\nu) &= y^{1-\alpha_\nu}\Gamma(1-\alpha_\nu)^{-1}W(y, 1-\beta_\nu, 1-\alpha_\nu) \\ &= \sum_{j=0}^{\alpha_\nu - 1} \binom{-\beta_\nu}{j} y^{-j}(-1)^j \frac{\Gamma(\alpha_\nu)}{\Gamma(\alpha_\nu - j)}, \end{aligned}$$

and we can similarly define and evaluate

(35)
$$\begin{aligned} P_{\kappa_r,\nu}^{-}(\sigma_{2,\nu}, \sigma_\nu) &:= (4\pi)^{\beta_\nu} \frac{4\pi^{k_\nu - 1}\sigma_\nu^{k_\nu - 1}}{\Gamma(k_\nu - 1)} \\ &\cdot \int_0^\infty y^{\beta_\nu}\Gamma(\beta_\nu)^{-1} W(4\pi y \sigma_{2,\nu}, \alpha_\nu, \beta_\nu)(\sigma_{2,\nu})^{2s_r - 1} e^{-4\pi\sigma_\nu y} y^{k_\nu - 2}\, dy \end{aligned}$$

$$= \sum_{j=0}^{\alpha_\nu} \binom{-\beta_\nu}{j} (-1)^j \frac{\Gamma(\alpha_\nu)}{\Gamma(\alpha_\nu - j)} \frac{\Gamma(k_\nu - 1 - j)}{\Gamma(k_\nu - 1)} \sigma_\nu^j \sigma_{2,\nu}^{\alpha_\nu - 1 - j}.$$

Summarizing (21), (22), (23), (24) and applying the holomorphic projection operator as in Corollary 2.2, the distribution $\widetilde{\mu}_s^{\pm}$ can be written in the form

$$\widetilde{\mu}_{s_r}^{\pm}(\chi) = \frac{\langle\, \mathbf{f}_0, G_r'^{\pm} | J_{\mathfrak{cm}_0^2} \,\rangle_{\mathfrak{cm}_0^2}}{\langle\, \mathbf{f}, \mathbf{f} \,\rangle_{\mathfrak{cm}_0^2}} \tag{36}$$

with the cusp forms $G_r'^{\pm} = G_r'^{\pm}(\mathbf{f}, g, \chi, \kappa_r) = (g'^{\pm}) \in \mathcal{S}_k(\mathfrak{cm}_0^2, \psi)$ defined by

$$G_r'^{\pm} = \gamma^{\pm}(s_r) \frac{\mathcal{N}(\mathfrak{m}')^{k_0 + 2s_r - 2}}{\alpha(\mathfrak{m}')^2} (-1)^{\{k\}} 2^{\{k\}-2} d_F^{\frac{1}{2}} (2\pi)^{n(s_r - 1) + \frac{1}{2}\{k+l\}}$$
$$\cdot \mathcal{N}\left(\frac{\mathfrak{m}'}{\mathfrak{m}_0}\right)^{2-k_0} \mathcal{H}ol\left(P_k(G)_{\mathbf{A}} J_{\mathfrak{cm}'^2} U\left(\frac{\mathfrak{m}'^2}{\mathfrak{m}_0^2}\right)\right).$$

Moreover, (33) and the above calculations show that the Fourier expansion

$$g'^{\pm}(z) = \sum_{0 \ll \sigma \in \mathfrak{o}} b(\sigma, \kappa_r) e_\infty(\sigma, z) \tag{37}$$

of $g'^{\pm}$ has the following Fourier coefficients:

$$b(\sigma, \kappa_r) = \gamma_3^{\pm}(\mathfrak{m}') \sum_{\substack{(\frac{m'}{m_0})^2 \sigma = \sigma_1 + \sigma_2, \\ \sigma_i \gg 0}} \overline{\chi}_{\mathfrak{m}}^*((\sigma_1)) \sum_{\gamma \in \mathfrak{o}_+^\times / \mathfrak{o}_+^{\times 2}} \gamma^{-k/2} \lambda_g(\gamma\sigma_1, \mathfrak{o})$$
$$\cdot L_{\mathfrak{cm}_0}(\kappa_r, \Omega_{\gamma\sigma_2 c}) B(\gamma\sigma_2, \kappa_r; \mathfrak{cm}_0) \prod_{\nu=1}^{n} \left\{ \gamma_\nu^{-\beta_\nu} P_{\kappa_r,\nu}^{\pm}\left(\sigma_{2,\nu}, \left(\frac{m_\nu'}{m_{0,\nu}}\right)^2 \sigma_\nu\right)\right\},$$
$$\gamma_3^-(\mathfrak{m}') = \alpha(\mathfrak{m}')^{-2} m'^{k_0 1 - k} m_0^{k - 2\cdot 1} v_1 v(c) 2^{2\{k\} - 1 - 2n} d_F,$$
$$\gamma_3^+(\mathfrak{m}') = \pi^{-n\kappa_r} \gamma_3^-(\mathfrak{m}').$$

Now, the Fourier coefficients of $g'^{\pm}$ are elements of $\overline{\mathbb{Q}}(g)$: $L_{\mathfrak{cm}_0}(\kappa_r, \Omega_{\gamma,\sigma_2 c})$ is algebraic for $\kappa_r \in K^-$, and the functional equation (4) shows that $\pi^{-n\kappa_r} L_{\mathfrak{cm}_0}(\kappa_r, \Omega_{\gamma\sigma_2 c})$ is algebraic for $\kappa_r \in K^+$ as well. By Proposition 2.2 of [S4] the coefficients $C(\mathfrak{p}, \mathbf{f})$ are algebraic, and hence so is $\alpha(\mathfrak{m}')$. Apart from $\lambda_g(\gamma\sigma_1, \mathfrak{o})$ all other factors in the Fourier coefficients of $g'^{\pm}$ are clearly algebraic which proves the claim.

5. A linear functional on $\mathcal{S}_k(\mathfrak{c}\mathfrak{m}_0^2, \psi)$.

Let us now show that the linear functional

$$\mathcal{L} : \mathcal{S}_k(\mathfrak{c}\mathfrak{m}_0^2, \psi) \to \mathbb{C}, \qquad \mathbf{\Phi} \mapsto \frac{\langle\, \mathbf{f}_0, \mathbf{\Phi}|J_{\mathfrak{c}\mathfrak{m}_0^2}\,\rangle_{\mathfrak{c}\mathfrak{m}_0^2}}{\langle\, \mathbf{f}, \mathbf{f}\,\rangle_{\mathfrak{c}\mathfrak{m}_0^2}}$$

is defined over some number field K, i.e. that there exists a number field K and a finite number of ideals $\mathfrak{n}_i$ and (algebraic) numbers $l(\mathfrak{n}_i) \in K$ such that

$$\mathcal{L}(\mathbf{\Phi}) = \sum_i C(\mathfrak{n}_i, \mathbf{\Phi}) l(\mathfrak{n}_i). \tag{38}$$

We remark that Panchishkin [P] in chapter IV.5.5 already applied this functional and the property that it is defined over $\overline{\mathbb{Q}}$. By the theory of primitive forms there exists a basis of $\mathcal{S}_k(\mathfrak{c}\mathfrak{m}_0^2, \psi)$, consisting of elements of the form $\mathbf{f}_j|\mathfrak{b}_j$ with primitive forms $\mathbf{f}_j \in \mathcal{S}_k(\mathfrak{a}_j, \psi)$ and integral ideals $\mathfrak{a}_j$ and $\mathfrak{b}_j$ satisfying $\mathfrak{a}_j\mathfrak{b}_j|\mathfrak{c}\mathfrak{m}_0^2$. Now there exist certain integral ideals $\mathfrak{n}_i$ such that $\{\mathcal{L}_{\mathfrak{n}_i} : \mathcal{S}_k(\mathfrak{c}\mathfrak{m}_0^2, \psi) \to \mathbb{C}, \quad \mathbf{\Phi} \mapsto C(\mathfrak{n}_i, \mathbf{\Phi})\}$ is a basis of the dual space of $\mathcal{S}_k(\mathfrak{c}\mathfrak{m}_0^2, \psi)$. If $\mathbf{\Phi} = \sum \alpha_j \mathbf{f}_j|\mathfrak{b}_j$, then $(C(\mathfrak{n}_i, \mathbf{\Phi}))_i = (C(\mathfrak{n}_i, \mathbf{f}_j|\mathfrak{b}_j))_{i,j}(\alpha_j)_j$ with matrix notation, and hence

$$\mathcal{L}(\mathbf{\Phi}) = \big(\mathcal{L}(\mathbf{f}_j|\mathfrak{b}_j)\big)^{\mathrm{t}}(\alpha_j) = \big(\mathcal{L}(\mathbf{f}_j|\mathfrak{b}_j)\big)^{\mathrm{t}}\big(C(\mathfrak{n}_i, \mathbf{f}_j|\mathfrak{b}_j)\big)^{-1}\big(C(\mathfrak{n}_i, \mathbf{\Phi})\big).$$

By Proposition 2.8 of [S4] $C(\mathfrak{n}_i, \mathbf{f}_j|\mathfrak{b}_j) = C(\mathfrak{n}_i\mathfrak{b}_j^{-1}, \mathbf{f}_j)$ is an element of the number field $\mathbb{Q}(\mathbf{f}_j)$, so it remains to show the algebraicity of $\mathcal{L}(\mathbf{f}_j|\mathfrak{b}_j)$.

$$\begin{aligned}\mathcal{L}(\mathbf{f}_j|\mathfrak{b}_j) &= \frac{\langle\, \mathbf{f}_0, \mathbf{f}_j|\mathfrak{b}_j|J_{\mathfrak{c}\mathfrak{m}_0^2}\,\rangle_{\mathfrak{c}\mathfrak{m}_0^2}}{\langle\, \mathbf{f}, \mathbf{f}\,\rangle_{\mathfrak{c}\mathfrak{m}_0^2}} \\ &= \sum_{\mathfrak{a}|\mathfrak{m}_0} \mu(\mathfrak{a})\alpha'(\mathfrak{a})\mathcal{N}(\mathfrak{b}_j)^{-k_0}\mathcal{N}\Big(\frac{\mathfrak{c}\mathfrak{m}_0^2}{\mathfrak{b}_j}\Big)^{k_0/2} \Lambda(\mathbf{f}_j)\frac{\langle\, \mathbf{f}|\mathfrak{a}, \mathbf{f}_j^{\rho}|\frac{\mathfrak{c}\mathfrak{m}_0^2}{\mathfrak{a}_j\mathfrak{b}_j}\,\rangle}{\langle\, \mathbf{f}, \mathbf{f}\,\rangle},\end{aligned}$$

where we have made use of (3.17) of [P] and $\mathbf{f}_j|J_{\mathfrak{a}_j} = \Lambda(\mathbf{f}_j)\mathbf{f}_j^{\rho}$ for some constant $\Lambda(\mathbf{f}_j) \in \mathbb{C}$; the form $\mathbf{f}_j^{\rho} \in \mathcal{M}_k(\mathfrak{a}_j, \overline{\psi})$ is determined by $C(\mathfrak{n}, \mathbf{f}_j^{\rho}) = \overline{C(\mathfrak{n}, \mathbf{f}_j)}$ and is primitive. Moreover, if $\mathbf{f}_j = (f_j)$ and $\mathbf{f}_j|J_{\mathfrak{a}_j} = (f_j')$, then $f_j' = (-1)^{\{k\}} f_j \| \beta$ for some $\beta \in \widetilde{G}^+(\mathbb{Q})$ as described in section 2. Proposition 1.4 of [S4] now asserts that f_j' has algebraic coefficients which implies that $\Lambda(\mathbf{f}_j)$ is algebraic as well. Now write $\widetilde{D}(s; \mathbf{f}, \mathbf{f}_j) := \sum_{\mathfrak{m}} C(\mathfrak{m}, \mathbf{f})C(\mathfrak{m}, \mathbf{f}_j)\mathcal{N}(\mathfrak{m})^{-s}$ for the convolution of Hilbert automorphic forms of integral weight. The

Dirichlet series associated to $\mathbf{f}$ has the Euler product

$$\sum_{\mathfrak{m}} C(\mathfrak{m},\mathbf{f})\mathcal{N}(\mathfrak{m})^{-s} = \prod_{\mathfrak{p}}(1 - C(\mathfrak{p},\mathbf{f})\mathcal{N}(\mathfrak{p})^{-s} + \psi^*_{\mathfrak{c}(\mathbf{f})}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{k_0-1-2s})^{-1},$$

and we have a similar expression for the Dirichlet series of $\mathbf{f}_j$. Applying Proposition 4.13 of [loc. cit.] together with Lemma 4.2 now shows that

$$\begin{aligned}\frac{\langle\, \mathbf{f}|\mathfrak{a}, \mathbf{f}_j^{\rho}|\mathfrak{b}\,\rangle}{\langle\, \mathbf{f}, \mathbf{f}_j{}^{\rho}\,\rangle} &= \Big(\frac{\widetilde{D}(s;\mathbf{f}|\mathfrak{a},\mathbf{f}_j|\mathfrak{b})}{\widetilde{D}(s;\mathbf{f},\mathbf{f}_j)}\Big)\Big|_{s=k_0} \\ &= \mathcal{N}\Big(\frac{\mathfrak{ab}}{\gcd(\mathfrak{a},\mathfrak{b})}\Big)^{-k_0}\prod_{\mathfrak{p}|\mathfrak{ab}}\frac{X_{\mathfrak{p},\nu_{\mathfrak{p}}(\mathfrak{b}/\mathfrak{a})}(\mathcal{N}(\mathfrak{p})^{-k_0})}{X_{\mathfrak{p},0}(\mathcal{N}(\mathfrak{p})^{-k_0})}\end{aligned}$$

with the polynomials $X_{\mathfrak{p},t}$ of Lemma 4.2 with respect to the roots of the Hecke polynomials of $\mathbf{f}$ and $\mathbf{f}_j$. Notice in particular that the $X_{\mathfrak{p},t}$ have coefficients in $\mathbb{Q}(\mathbf{f},\mathbf{f}_j)$, and that $X_{\mathfrak{p},0}(\mathcal{N}(\mathfrak{p})^{-k_0}) = 1-\psi^*_{\mathfrak{c}(\mathbf{f})\mathfrak{c}(\mathbf{f}_j)}(\mathfrak{p})^2\mathcal{N}(\mathfrak{p})^{-2} \neq 0$. The right side now shows that the above quotient of the scalar product is algebraic. The algebraicity of $\mathcal{L}(\mathbf{f}_j|\mathfrak{b}_j)$ now follows because both $\mathbf{f}$ and $\mathbf{f}_j^{\rho}$ are primitive, and hence $\dfrac{\langle\, \mathbf{f}, \mathbf{f}_j^{\rho}\,\rangle}{\langle\, \mathbf{f}, \mathbf{f}\,\rangle}$ equals 0 or 1. This shows that the linear functional $\mathcal{L}$ is defined over a number field K, and we may take $K = \mathbb{Q}(\mathbf{f}_j, \Lambda(\mathbf{f}_j), \{\alpha'(\mathfrak{p}), \sqrt{\mathcal{N}(\mathfrak{p})} \mid \mathfrak{p} \in S\})$ where $\mathbf{f}_j$ runs over all primitive forms contained in $\mathcal{S}_k(\mathfrak{cm}_0^2, \psi)$.

We are now in a position to prove the theorem. If $\mathfrak{n}_i = (n_i)$ with some totally positive $n_i \in \mathfrak{o}$, then

$$C(\mathfrak{n}_i, G_r'^{\,\pm}) = n_i^{\frac{1}{2}(k_0\cdot 1-k)}b(n_i,\kappa_r).$$

This shows immediately that our distributions take values in $\overline{\mathbb{Q}}(g)$. We want to show that $\widetilde{\mu}^-_{s_r}$ is defined over the field $L = K(F^G, \psi, \phi, g, \sqrt{\mathfrak{o}_+^{\times}}, \{\alpha^2(\mathfrak{p}) \mid \mathfrak{p} \in S\})$. We claim that it suffices to show the following: For each integral ideal $\mathfrak{m}$ with $S(\mathfrak{m}) = S$ there exist a finite number of ideals $\mathfrak{a}_i$ prime to $\mathfrak{m}_0$ and elements $\alpha_i \in L$ such that

$$\widetilde{\mu}^-_{s_r}(\chi) = \sum_i \alpha_i\chi^*(\mathfrak{a}_i) \tag{39}$$

for every Hecke character χ of finite order whose conductor divides $\mathfrak{m}$. Denote by $E = E(\mathfrak{m})$ the finite subextension of $F(S)$ which corresponds by class field theory to $J(\mathfrak{m})/P(\mathfrak{m})$. Now, if $\mathfrak{g}$ is any ideal prime to $\mathfrak{m}_0$, and $(\mathfrak{g}, F(S)/F)$ is the image of $\mathfrak{g}$ in Gal$(F(S)/F)$ under the Artin isomorphism, then

$$\delta_{(\mathfrak{g},F(S)/F)\cdot \mathrm{Gal}(F(S)/E)} = \frac{1}{(E:F)} \sum_{\chi \in X_S^{\mathrm{tor}}, \chi|_{\mathrm{Gal}(F(S)/E)}=\mathrm{id}} \overline{\chi}^*(\mathfrak{g})\chi^*$$

by the character relations. Therefore, (39) implies that our distribution is defined over L. Now let $\mathfrak{n} = \mathrm{lcm}(\mathfrak{c}\mathfrak{m}_0, \mathfrak{c}(\Omega_{\gamma\sigma_2 c}))$. Then

$$L_{\mathfrak{c}\mathfrak{m}_0}(\kappa_r, \Omega_{\gamma\sigma_2 c}) = \sum_{\mathcal{K}} \Omega^*_{\gamma\sigma_2 c}(\mathcal{K})\zeta(\mathcal{K}, s)$$

where the summation is over the classes $\mathcal{K}$ of $J(\mathfrak{n})/P(\mathfrak{n})$ and $\zeta(\mathcal{K}, s)$ is the partial zeta function

$$\zeta(\mathcal{K}, s) = \sum_{\mathfrak{b}\subseteq \mathfrak{o}, \mathfrak{b}P(\mathfrak{n})\in\mathcal{K}} \frac{1}{\mathcal{N}(\mathfrak{b})^s}.$$

By a result of Siegel-Klingen, the values of $\zeta(\mathcal{K}, s)$ at non-positive integers are rational (cf. [N, Korollar VII.9.9]). This, together with the explicit form of the Fourier coefficients of g'^{-}, shows that $\widetilde{\mu}_{s_r}^{-}$ can be written as in (39), and hence is defined over L. If the weights k and l are scalar, then $\gamma^{-\frac{k}{2}}$ and the product $\prod_{\nu} \gamma_\nu^{-\beta_\nu} P^{-}_{\kappa_r,\nu}$ become norms of elements of F and are therefore rational. Thus, the same argument as before yields the stronger result in this case as well, and the theorem is proved. □

We finally remark, that it suffices to adjoin the Fourier coefficients $\lambda(\sigma, \mathfrak{o}; g, \phi)$ for (σ) prime to $\mathfrak{m}_0$ because only these coefficients occur in the Fourier expansion of $g'^{\pm}$ in (37).

6. Boundedness of the regularized distributions.

The aim of this section is to show that the distributions $\widetilde{\mu}_{s_r}^{-}$ can be regularized such that they give rise to bounded p-adic measures. Before stating the theorem we describe the general situation with slight changes to the previous section: Let $\mathfrak{q}$ be an integral ideal, $S \supseteq \{\mathfrak{p}|p\}$ a finite set of primes containing all primes above p, $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c}(\mathbf{f}), \psi)$ a primitive Hilbert automorphic form of scalar integral weight $k = k_0 \cdot \mathbf{1}$, $g \in \mathcal{M}_l(\mathfrak{c}(g), \phi)$ a Hilbert modular form of scalar half-integral weight $l = l_0 \cdot \mathbf{1}$ which has p-adically bounded algebraic Fourier coefficients $\lambda(\xi, \mathfrak{o}; g, \phi)$ for (ξ) prime to $\mathfrak{m}_0\mathfrak{q}$, and assume that $g\iota$ is a simultaneous Hecke eigenform. For $\mathfrak{p} \in S \cup S(\mathfrak{q})$

let $\alpha(\mathfrak{p})$ and $\alpha'(\mathfrak{p})$ be the roots of the $\mathfrak{p}$th Hecke polynomial of $\mathbf{f}$. With the definition of α and α' extended by multiplicativity to all integral ideals $\mathfrak{a}$ with $S(\mathfrak{a}) \subseteq S \cup S(\mathfrak{q})$ we set $\mathfrak{m}_0 := \prod_{\mathfrak{p}\in S} \mathfrak{p}$, $\mathfrak{q}_0 := \prod_{\mathfrak{p}|\mathfrak{q}} \mathfrak{p}$, $\mathbf{f}_0 = \sum_{\mathfrak{a}|\mathfrak{q}_0\mathfrak{m}_0} \mu(\mathfrak{a})\alpha'(\mathfrak{a})\mathbf{f}|\mathfrak{a}$, and $\mathfrak{c} = 4\mathfrak{c}(\mathbf{f})\mathfrak{c}(g)$. Also, fix a totally positive number $c \in F$ such that $(c) = \mathfrak{c}(\mathbf{f})\mathfrak{c}(g)$, and let $\theta \in \{0, 1\}$ be determined by $\theta \equiv k_0 - l_0 - \frac{1}{2} \bmod 2$.

THEOREM 6.1. — *Assume that the following conditions hold: F has class number $h_F = 1$, the ideals $\mathfrak{c}(\mathbf{f})$, $4\mathfrak{c}(g)$, $\mathfrak{m}_0$ and $\mathfrak{q}$ are pairwise relatively prime, $l_0 < k_0$, $C(\mathfrak{c}(\mathbf{f}), \mathbf{f}) \neq 0$, and $i_p(\alpha(\mathfrak{p}))$ is a p-adic unit for all $\mathfrak{p} \in S \cup S(\mathfrak{q})$. Then for each $r \in \mathbb{Z}$ with $0 \leqslant 2r \leqslant k_0 - l_0 - \frac{1}{2} + \theta - 2$ there is a p-adic measure $\mu^{(r)}$ on Gal_S associated with the convolution of $\mathbf{f}_0$ and g which is uniquely defined by the following values on the Hecke characters of finite order with $S(\mathfrak{c}(\chi)) \subseteq S$:*

$$\mu^{(r)}(\chi) := \mu^{(r)}_{\mathfrak{m}}(\chi^*_{\mathfrak{m}})$$
$$:= i_p\left(\chi_\infty(-1)(1 - (\overline{\psi\phi}^2\chi^4)^*(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{2(1-\kappa_r)})\gamma(s_r)\frac{\Psi(s_r; \mathbf{f}_0, g(\overline{\chi}^*_{\mathfrak{m}\mathfrak{q}_0})j_{c,\mathfrak{m}'})}{\langle\, \mathbf{f}, \mathbf{f}\,\rangle_{\mathfrak{c}\mathfrak{m}_0^2}}\right),$$

where

$$\kappa_r = \theta - 1 - 2r, \quad s_r = \frac{1}{2}\left(\kappa_r + \frac{1}{2}\right),$$
$$\gamma(s) = \pi^{-2ns-n-nk_0} d_F^{2s} \mathcal{N}\left(\frac{\mathfrak{c}}{4}\right)^s i^{-n(k_0+l_0-2)}\Gamma\left(s + \frac{k_0 - l_0}{2}\right)^n$$
$$\cdot \mathcal{N}(\mathfrak{m}')^{k_0+2(s-1)}\alpha(\mathfrak{m}')^{-2},$$

and $\mathfrak{m}$, $\mathfrak{m}'$ are arbitrary integral ideals satisfying $\mathrm{lcm}(\mathfrak{m}_0, \mathfrak{c}(\chi))|\mathfrak{m}$, $\mathfrak{m}_0\mathfrak{q}_0^2\mathfrak{m}|\mathfrak{m}'$, *$S(\mathfrak{m}) \subseteq S$ and $S(\mathfrak{m}') \subseteq S \cup S(\mathfrak{q})$. Moreover, the measures are normalized such that*

$$\mu^{(r)} = \mathcal{N}_p^r \mu^{(0)}.$$

Proof. — Note first that $i_p^{-1}(\mu^{(r)}(\chi))$ is independent of $\mathfrak{m}$ and $\mathfrak{m}'$ and is algebraic because of Theorem 5.1 applied to $S \cup S(\mathfrak{q})$. Therefore $\mu^{(r)}$ is a p-adic distribution and it remains to prove its boundedness. In view of (36) and (37), this can be verified via the Fourier coefficients of

$$G^*(\mathbf{f}, g, \chi, r) = \chi_\infty(-1)(1 - (\overline{\psi\phi}^2\chi^4)^*(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{2(1-\kappa_r)})G'(\mathbf{f}, g, \chi, \kappa_r). \tag{40}$$

Let us show that the abstract Kummer congruences of Proposition 1.5 are fulfilled. We assume, without loss of generality, that the Fourier coefficients

$\lambda(\xi, \mathfrak{o}; g, \phi)$ of g are p-integral (i.e. $\mathfrak{p}$-integral for all $\mathfrak{p}|p$) if (ξ) is prime to $\mathfrak{q}\mathfrak{m}_0$. Let $C = n\nu_p(\Gamma(k_0 - 1))$ and N an arbitrary positive integer. For a Hecke character $\chi \in X_S^{\mathrm{tor}}$ we choose an ideal $\mathfrak{m}$ such that $\mathrm{lcm}(\mathfrak{m}_0, \mathfrak{c}(\chi))|\mathfrak{m}$ and set $\mathfrak{m}' = \mathfrak{m}_0\mathfrak{q}_0^2\mathfrak{m}^2p^{C+N}$. Now, in the expression (37) for the Fourier coefficient of G' we may restrict the summation to those σ_1, σ_2 whose associated ideals (σ_1) and (σ_2) are prime to $\mathfrak{m}_0\mathfrak{q}_0$. Let $m_0, m' \in F^\times$ be totally positive with $(m_0) = \mathfrak{m}_0\mathfrak{q}_0$ and $(m') = \mathfrak{m}'$. Then

$$\prod_{\nu=1}^{n} P_{\kappa_r,\nu}^{-}\left(\sigma_{2,\nu}, \left(\frac{m'_\nu}{m_{0,\nu}}\right)^2 \sigma_\nu\right) \equiv \mathcal{N}((\sigma_2))^{-r-1+\frac{1}{2}(\theta+k_0+l_0-\frac{1}{2})} \mod p^N.$$

Also notice that from $\left(\dfrac{m'}{m_0}\right)^2\sigma = \sigma_1 + \sigma_2$ and the choice of m' and $\mathfrak{m}'$, we have

$$\chi_\infty(-1)\overline{\chi}^*_{\mathfrak{m}\mathfrak{q}_0}((\sigma_1)) = \overline{\chi}^*_{\mathfrak{m}\mathfrak{q}_0}((\sigma_2)).$$

With the quadratic Hecke character $\varepsilon_{-\gamma\sigma_2}$ associated with $F(\sqrt{-\gamma\sigma_2})/F$, the measure $\mu = \mu(\mathfrak{q}, \varepsilon_{-\gamma\sigma_2}\psi\phi, S)$ on Gal_S of Theorem 1.6 and the measure $\bar{\mu}$ on Gal_S defined by

$$\int_{\mathrm{Gal}_S} f(x)\, d\bar{\mu}(x) := \int_{\mathrm{Gal}_S} f(x^2)\mathcal{N}_p^{1-\theta}(x)\, d\mu(x) \qquad \text{for all } f \in \mathcal{C}(\mathrm{Gal}_S, \mathbb{C}_p)$$

we have the following expression for the "regularized" L-value:

$$\begin{aligned}
&(1 - (\overline{\psi\phi}^2\chi^4)^*(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{2(1-\kappa_r)})L_{\mathfrak{c}\mathfrak{m}_0}(\kappa_r, \overline{\varepsilon_{-\gamma\sigma_2}\psi\phi\chi^2}) \\
&= (1 + (\overline{\varepsilon_{-\gamma\sigma_2}\psi\phi\chi^2})(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{1-\kappa_r})i_p^{-1}\left(\int_{\mathrm{Gal}_S} \chi^2(x)\mathcal{N}_p(x)^{2r+1-\theta}\, d\mu(x)\right) \\
&= (1 + (\overline{\varepsilon_{-\gamma\sigma_2}\psi\phi\chi^2})(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{2-\theta+2r})i_p^{-1}\left(\int_{\mathrm{Gal}_S} \chi(x)\mathcal{N}_p(x)^r\, d\bar{\mu}(x)\right).
\end{aligned}$$

Summarizing, we have found that for every positive integer N and all integral ideals $\mathfrak{n}$, $\mathfrak{m}$ with $S(\mathfrak{m}) = \mathfrak{m}_0$ there exist p-integral elements $\alpha_i = \alpha_i(N, \mathfrak{n}, \mathfrak{m}) \in \overline{\mathbb{Q}}$, fractional ideals $\mathfrak{a}_i = \mathfrak{a}_i(N, \mathfrak{n}, \mathfrak{m})$, and $\mathcal{O}_p$-valued measures μ_i on Gal_S such that for every Hecke character χ of finite order with $\mathfrak{c}(\chi)|\mathfrak{m}$ the following congruence holds:

$$\begin{aligned}
i_p\Big(C(\mathfrak{n}, G^*(\mathbf{f}, g, \chi, r))\Big) &\equiv \sum_i \alpha_i\chi^*(\mathfrak{a}_i)\mathcal{N}(\mathfrak{a}_i)^r \int_{\mathrm{Gal}_S} \chi\mathcal{N}_p^r\, d\mu_i \\
&\equiv \sum_i \alpha_i \int_{\mathrm{Gal}_S} \chi(x\mathfrak{a}_i)\mathcal{N}_p^r(x\mathfrak{a}_i)\, d\mu_i(x) \qquad \mod p^N.
\end{aligned}$$

This, together with (38), now implies the abstract Kummer congruences as well as the relation between the different distributions. □

Let us finish with a few remarks on the conditions we imposed on g. In the integral case every Hilbert automorphic form can be written as $\mathbf{f} = \sum \alpha_i \mathbf{f}_i | \mathfrak{a}_i$ with primitive Hilbert automorphic forms $\mathbf{f}_i$, complex numbers α_i, and integral ideals $\mathfrak{a}_i$. Moreover, the primitive Hilbert automorphic forms $\mathbf{f}_i$ have the following important properties: They are characterized by the fact that both $\mathbf{f}_i$ and $\mathbf{f}_i | J_{\mathfrak{c}(\mathbf{f}_i)}$ are simultaneous Hecke eigenforms, cf. Corollary 4.6.22 of [M], and if $k_1 \equiv \ldots \equiv k_n$, then the Fourier coefficients $C(\mathfrak{m}, \mathbf{f}_i)$ are algebraic integers by Proposition 2.2 of [S4]. Therefore, it is most natural when investigating convolutions of Hilbert automorphic forms of integral weight as in [P], to assume that both forms are primitive. In the half-integral case the analog of the Atkin-Lehner theory has so far only been established for the field $F = \mathbb{Q}$, and only for the case $2 \nmid \mathfrak{b}\mathfrak{b}'$ and a quadratic character ψ, cf. [Koh], [MRV] or [U]. There, a multiplicity-1 theorem is only established for certain subspaces of $\mathcal{M}_k((2)^{-1}, 4\mathfrak{b}\mathfrak{b}')$, e.g. the Kohnen +-space, but our inverter may not respect these subspaces. If g is a simultaneous Hecke eigenform and $\mathbf{g}$ denotes the Shimura lift of g, then $C(\mathfrak{p}, \mathbf{g}) = \mathcal{N}(\mathfrak{p})^{\frac{k_0}{2}} c(\mathfrak{p}, \mathbf{g})$, (20) and Proposition 3.2 show that for $\xi \in \mathfrak{o}$ with (ξ) squarefree the quotient $\lambda_g(\xi, \mathfrak{m})/\lambda_g(\xi, \mathfrak{o})$ is p-integral for $\mathfrak{m}$ prime to p, but in this way we cannot derive any integrality properties for $\lambda_g(\xi, \mathfrak{o})$ itself. However, the algebraicity condition on g is very natural: By Proposition 8.1 of [S7] the space $\mathcal{M}_l(\mathfrak{c}, \phi)$ is spanned by its $\overline{\mathbb{Q}}$-rational elements and by Proposition 8.9 of [loc. cit.] the same is true for the space $\mathcal{S}_l(\mathfrak{c}, \phi, \mathbf{h})$ of Hilbert cusp forms of half integral weight l whose Shimura lift is the primitive Hilbert automorphic form $\mathbf{h}$ of integral weight $2l - \mathbf{1}$. Since the results on "primitive" half-integral forms and integrality of their Fourier coefficients are not yet as complete as in the integral case, we have formulated the theorem with the above conditions on g. For $F = \mathbb{Q}$, an example of a modular form g satisfying our assumptions can be obtained as follows: Let

$$\Theta(z) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad F(z) = \sum_{n>0, 2\nmid n} \sigma_1(n) q^n$$

with $q = e^{2\pi i z}$ and $\sigma_1(n) = \sum_{d|n} d$. By Proposition IV.4 of [Kob], $\mathcal{M}_{\frac{9}{2}}((2)^{-1}, (2), \varepsilon)$ is spanned by $F^2\Theta$, $F\Theta^5$ and Θ^9. At the cusps ∞, $-\frac{1}{2}$ and 0 the forms F and Θ^4 take the values 0, $\frac{1}{16}$, $-\frac{1}{64}$ and 1, 0, $-\frac{1}{4}$, respectively (cf. Problem III.3.10 and III.3.11 of [loc. cit.]). The cusp $-\frac{1}{2}$ being irregular for the weight $\frac{9}{2}$ by Problem IV.1.3 of [loc. cit.], it follows that $\mathcal{S}_{\frac{9}{2}}((2)^{-1},(2),\varepsilon)$ is one-dimensional and spanned by $g' := 16F^2\Theta - F\Theta^5$. We can therefore take $g(z) := g'(\frac{1}{2}z) \in \mathcal{S}_{\frac{9}{2}}((1),\varepsilon_2)$.

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