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by Yvette KOSMANN-SCHWARZBACH

1. INTRODUCTION

Our point of departure is a very general construction that was introduced in unpublished notes of Koszul [Ko]. From a graded Lie algebra structure and an odd derivation of square 0, this construction produces a graded bracket of opposite parity. While these new brackets, which we call "derived brackets", are not Lie brackets on the algebra itself, they are Lie brackets on certain of its subspaces or quotients. In fact, the construction of derived brackets makes sense, more generally, in the case of what we call Loday algebras, *i.e.*, Leibniz algebras in the sense of Loday [L1], [L2], and their graded generalizations [A]. Loday algebras are non-commutative analogues of Lie algebras, defined by a bilinear bracket, which is no longer skew-symmetric, while an appropriate form of the Jacobi identity is satisfied (see formulas (2.1) and (2.1′)). We have changed the accepted term "Leibniz algebra" [L1], [L2] to "Loday algebra" in order to reserve the term "Leibniz rule" for the interplay between a bracket and another multiplication. The derived brackets of graded Loday brackets are indeed graded Loday brackets (proposition 2.1), but they are not, in general, graded Lie brackets, even when the initial bracket is a graded Lie bracket. However, when restricted to the space of co-exact elements (equivalence classes modulo coboundaries), or to certain Abelian subalgebras, these Loday brackets are indeed skew-symmetric and therefore are Lie brackets.

When the Loday algebra is, in addition, equipped with an associative, graded commutative multiplication, and when the initial bracket satisfies

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the graded Leibniz rule (2.3),

$$[a, bc] = [a, b]c + (-1)^{(|a|+|f|)|b|} b[a, c],$$

where $|f|$ is the degree of the bracket, the derived bracket also satisfies the graded Leibniz rule, but for a different $|f|$, *i.e.*, the Leibniz rule of a Gerstenhaber algebra (where $|f|$ is odd) when the initial bracket is a graded Poisson structure (where $|f|$ is even) and vice-versa (proposition 2.2). Actually, the assumption that the multiplication, to be denoted by m, is graded and commutative, and even the assumption that it is associative, can be relaxed. This will prove to be important when we make the connection, in section 6, between our theory and Akman's theory of generalized BV-algebras [A]. These general facts are explained and proved in section 2.

In section 3, we describe fundamental examples of these constructions, which appear in the Poisson calculus [K] [K-SM] [Va]. The Poisson bracket of functions appears as the restriction to the Abelian subalgebra of functions of both the derived Loday bracket on multivectors obtained from the Schouten bracket and the Lichnerowicz-Poisson differential, and the derived Loday bracket on differential forms obtained from the Koszul bracket and the de Rham differential. These derived brackets, even graded brackets extending the Poisson bracket of functions, and their skew-symmetrized relatives (which are not Lie brackets either!) have been introduced by several authors, in diverse contexts, in the search for a "unification" of the various graded brackets occurring in differential geometry, first by Buttin [B], then by Michor [M] and Vinogradov [V] [CV].

In yet another example, which is of importance in the theory of Lie bialgebras [LR] [R] [K-S1], the algebraic Schouten bracket on the exterior algebra of a Lie algebra is shown to be the restriction to this algebra of the derived bracket obtained from the "big bracket" [CNS] [KtS] [K-S1] on the exterior algebra of the direct sum of the underlying vector space and its dual, and from the Lie algebra cohomology operator, which is an interior derivation of the big bracket. This example is described in section 4.

In section 5, we review many facts first proved by Gerstenhaber [G], then by Nijenhuis [N], concerning the structure of the space of cochains on an algebra, with values in the algebra itself. The case of associative algebras is treated in subsection 5.1, while the case of Lie algebras is treated analogously in subsection 5.2. This material is well-known and parts of it have been reviewed many times [N] [GS1], [GS2] and generalized [D-VM], but we chose to include it here for two reasons. The first is that the Frölicher-Nijenhuis bracket on cochains on associative (resp., Lie) algebras is closely related to the derived bracket constructed from the composition (resp., Nijenhuis-Richardson) bracket and the Hochschild (resp., Chevalley-Eilenberg) differential on an associative algebra (resp., a Lie algebra). These two brackets differ only by an exact term (formula (5.16)), and this property yields an immediate proof of the fact that the Hochschild (resp., Chevalley-Eilenberg) coboundary operator is a morphism from the Frölicher-Nijenhuis bracket to the opposite of the composition (resp., Nijenhuis-Richardson) bracket (formula (5.17′), proposition 5.1). The second motivation for developing this theory here is to establish the connection with the generalized BV-algebras (section 6). Indeed formulas (5.7) and (5.21) present a striking analogy, with an intriguing difference in the signs, with the formula defining the Gerstenhaber bracket in a BV-algebra. It was natural to try to interpret them in this spirit.

In section 6, we describe the notion of BV-algebra and various generalizations. In a BV-algebra, the Gerstenhaber bracket is the defect in the derivation property of an operator with respect to an associative multiplication, in other words it is given by the Hochschild coboundary of that operator. There has been a surge of interest in the study of BV-algebras because of the appearance of BV-brackets on the BRST cohomology of topological field theories [LZ] [PS] and on the semi-infinite cohomology of some W-algebras [BMP] [BP]. The term BV-algebra (BV stands for Batalin and Vilkovisky) comes from quantum field theory, where there is a quantum correction of the form $\hbar\Delta$ to the classical BRST operator. Witten [W] explained the relation between the operator Δ and the graded Lie bracket on the space of functions over phase-space tensored with an algebra of the form $\Lambda(F\oplus F^*)$, the space of "ghosts" and "anti-ghosts". This bracket is actually a Gerstenhaber bracket. The identity relating Δ and the Gerstenhaber bracket [W] [PS] [Gt] had actually appeared much earlier, in Koszul's work on the Schouten bracket and calculus on Poisson manifolds [K], as we observed in [K-S2]. More recently, Akman [A] has presented a straightforward but far-reaching generalization of BV-algebras, in which she called attention to the connection with what we have here called Loday algebras. We recall Akman's definition, we give the interpretation of the BV-brackets in terms of generalized Hochschild differentials, and, in Proposition 6.2, we reformulate one of the principal results of [A] in terms of the Loday-Gerstenhaber algebras introduced at the beginning of section 2:

second-order differential operators of square 0 on a graded algebra, not assumed to be commutative nor even associative, give rise to generalized Loday-Gerstenhaber brackets. We then analyze to what extent the cup-brackets on cochains on algebras, discussed in section 5, are special cases of further generalizations of the notion of a BV-algebra, involving a non-associative multiplication of non-zero degree, such as the composition or hook product. This question will be the subject of subsequent research.

The consideration of brackets defined by means of a differential is also central to the algebraic formulation of non-commutative geometry by Gelfand, Daletskii and Tsygan [GDT] and to recent work by Daletskii and Kushnirevitch [DK1], [DK2], of which we became aware while this paper was circulating as a preprint.

2. DERIVED LODAY BRACKET ON A DIFFERENTIAL LIE ALGEBRA

2.1. Loday brackets.

Let $A = \bigoplus_{i\in\mathbb{Z}} A^i$ be a $\mathbb{Z}$-graded vector space over a field k of characteristic not equal to 2. In order to define various graded structures on A, we consider a map $f : A \to \operatorname{End} A$, of degree $|f|$, where $\operatorname{End} A$ is the space of graded endomorphisms of A. We shall denote the graded commutator of endomorphisms of A by $\big[\, , \,\big]$. If $u, v \in \operatorname{End} A$, then, by definition, $\big[u, v\big] = u \circ v - (-1)^{|u||v|} v \circ u$. For $a, b \in A$, we set

$$[a, b] = f(a)b,$$

and we say that $[\, , \,]$ is the bracket defined by f. Since f is of degree $|f|$, then $[A^i, A^j] \subset A^{i+j+|f|}$. If f satisfies the graded Jacobi identity,

$$\big[f(a), f(b)\big] = f(f(a)b), \tag{2.1}$$

for $a, b \in A$, then the bracket $[\, , \,]$ defined by f will be called a *graded Loday bracket* of degree $|f|$. If moreover f satisfies the graded skew-symmetry identity

$$f(a)b = -(-1)^{(|a|+|f|)(|b|+|f|)} f(b)a, \tag{2.2}$$

for $a \in A^{|a|}$, $b \in A^{|b|}$, then $[\, , \,]$ is called a *graded Lie bracket* of degree $|f|$.

Let us now assume that (A, m) is an algebra, where $m(A^i \times A^j) \subset A^{i+j}$. Let us consider the following property, called the graded Leibniz identity,

(2.3) $f(a)$ is a graded derivation of (A, m) of degree $|a| + |f|$, for each $a \in A^{|a|}$.

When (A, m) is associative and graded commutative, and when f is of even (resp., odd) degree and satisfies (2.1), (2.2) and (2.3), then $(A, m, [\, , \,])$ is called a *Poisson* (resp., *Gerstenhaber*) *algebra.*

A bracket defined by f of even (resp., odd) degree satisfying (2.1), (2.2) and (2.3) when (A, m) is an algebra which is not necessarily associative nor graded commutative, will be called a *generalized Poisson* (resp., *Gerstenhaber*) *bracket.*

A bracket defined by f of even (resp., odd) degree satisfying (2.1) and (2.3) will be called a *generalized Loday-Poisson* (resp., *Loday-Gerstenhaber*) *bracket.*

An explicit form of (2.1) is

$$[a, [b, c]] = [[a, b], c] + (-1)^{(|a|+|f|)(|b|+|f|)}[b, [a, c]], \tag{2.1'}$$

for $a \in A^{|a|}, b \in A^{|b|}, c \in A$.

Remarks.

(a) What we call a Loday algebra, is usually called a (left) Leibniz algebra [L1], [L2] in the ungraded case. See also [A] for the graded case.

(b) Gerstenhaber algebras have also been called G-algebras [GS1], [GS2] or Schouten algebras [K-SM].

(c) In section 6, we shall also consider generalized BV-algebras in the sense of [A]. They are generalized Loday-Gerstenhaber algebras which are exact in a sense that we shall specify.

(d) Generalized Poisson (resp., Gerstenhaber) algebras in which (A, m) is associative but not necessarily commutative are particular cases of Leibniz pairs (L, A) in the sense of [FGV], in which $L = A$ as vector spaces. There they are called left Poisson superalgebras (resp., left Gerstenhaber algebras).

Conventions.

(i) Henceforth all objects will be of the graded category, so we shall drop the adjective "graded". It is to be so understood whenever relevant.

(ii) When no confusion can arise, we shall simply write a for the degree of $a \in A$, and f for the degree of $f \in \operatorname{End} A$.

2.2. Derived brackets.

DEFINITION 2.1. — *A derivation of the Loday bracket defined by f on a vector space A is an endomorphism d of A, such that*

$$[d, f(a)] = f(da) \tag{2.4}$$

for each $a \in A$, where $[\,,\,]$ denotes the (graded) commutator in End A.

It is clear that for $a_0 \in A$, $f(a_0) = [a_0, .]$ is a derivation of degree $|a_0| + |f|$ of the Loday bracket defined by f. This statement is a reformulation of (2.1). Derivations of the form $f(a_0)$, for $a_0 \in A$, are called *interior derivations.*

We now consider a derivation d of degree $|d|$ of the Loday algebra $(A, [\,,\,])$ defined by f, and we assume that d is of square 0,

$$d^2 = 0. \tag{2.5}$$

We consider $f_d : A \to \operatorname{End} A$ defined by

$$f_d(a) = [f(a), d] \tag{2.6}$$

for $a \in A$, and we set

$$[a, b]_d = f_d(a)(b).$$

More explicitly, using (2.4),

$$f_d(a) = [f(a), d] = (-1)^{d(a+f)+1}[d, f(a)] = (-1)^{d(a+f)+1} f(da). \tag{2.7}$$

Thus the map f_d defines a new bracket, which we call the *derived bracket* and which we denote by $[\,,\,]_d$. It follows from the definition that

$$[a, b]_d = (-1)^{d(a+f)+1}[da, b]. \tag{2.8}$$

Let us now describe the properties of this bracket, $[\,,\,]_d$. First we observe that it is of degree $|f| + |d|$, so, if d is odd, the parity of $[\,,\,]_d$ is opposite to that of $[\,,\,]$.

PROPOSITION 2.1. — *Let d be a derivation of degree $|d|$ and of square 0 of the Loday bracket $[\,,\,]$ of degree $|f|$. Then $(A,[\,,\,]_d)$, where $[\,,\,]_d$ is defined by (2.8), is a Loday bracket of degree $|f|+|d|$. Moreover, d is a derivation of the Loday bracket $[\,,\,]_d$.*

Proof. — Using (2.7) and (2.1), we compute

$$\begin{aligned}\big[f_d(a), f_d(b)\big] - f_d(f_d(a)b) &= \big[[f(a),d],[f(b),d]\big] - \big[f([f(a),d]b),d\big]\\ &= (-1)^{d(a+b)}\big[f(da),f(db)\big] + (-1)^{d(a+f)}\big[f(f(da)b),d\big]\\ &= (-1)^{d(a+b)}f([f(da),d]b) = 0,\end{aligned}$$

since, by (2.4) and (2.5), $\big[d,f(da)\big] = f(dda) = 0$.

We now show that d is a derivation of the bracket $[\,,\,]_d$. In fact, by (2.5), $\big[d,d\big] = 0$ and therefore $\big[d, f_d(a)\big] = \big[d,\big[f(a),d\big]\big] = f_d(da)$. □

Remark. — Using (2.4), we can write

$$[a,b]_d = [a,db] + (-1)^{d(a+f)+1}d[a,b]. \tag{2.9}$$

We shall now see whether imposing conditions on the original bracket $[\,,\,]$ implies additional properties for the bracket $[\,,\,]_d$.

If (A,m) is an algebra, we denote $m(a,b)$ simply by ab, for $a,b\in A$.

PROPOSITION 2.2. — *Assume that d is odd. If $(A,m,[\,,\,])$ is a generalized Loday-Poisson algebra, then $(A,m,[\,,\,]_d)$ is a generalized Loday-Gerstenhaber algebra, and vice-versa.*

Proof. — A computation using (2.8) and (2.3) shows that if $f(a)$ is a derivation of (A,m) of degree $|a|+|f|$, then $f_d(a)$ is a derivation of (A,m) of degree $|a|+|f|+|d|$. □

We now assume that $[\,,\,]$ is a Lie bracket of degree $|f|$, *i.e.*, that non only conditions (2.1), but also (2.2) (skew-symmetry) are satisfied. In general, this does not imply that $[\,,\,]_d$ is a Lie bracket of degree $|f|+|d|$. In fact, using (2.2) and (2.8), we obtain

$$\begin{aligned}&f_d(a)b + (-1)^{(a+f+d)(b+f+d)}f_d(b)a\\ &\qquad = (-1)^{d(a+f)+1}\Big(f(da)b + (-1)^{d(a+f+d)+1}f(a)db\Big).\end{aligned}$$

Again assuming that d is odd and using (2.4), we obtain

$$f_d(a)(b) + (-1)^{(a+f+1)(b+f+1)}f_d(b)a = (-1)^{a+f+1}d(f(a)b). \tag{2.10}$$

Thus, in general, $[\,,\,]_d$ is still only a Loday bracket when $[\,,\,]$ is a Lie bracket.

Example. — If a_0 is an element in a Loday algebra $(A,[\,,\,])$ such that $|a_0|+|f|$ is odd and such that $[a_0,a_0]=0$, then $f(a_0)$ is an interior derivation of degree $|a_0|+|f|$, and of square 0 of $(A,[\,,\,])$. We shall denote $[\,,\,]_{f(a_0)}$ simply by $[\,,\,]_{a_0}$. Then

$$(2.11)\quad [a,b]_{a_0}=(-1)^{a+f+1}[[a_0,a],b]=[a,[a_0,b]]+(-1)^{a+f+1}[a_0,[a,b]].$$

If moreover $[\,,\,]$ is skew-symmetric, then

$$(2.12)\qquad [a,b]_{a_0}=[[a,a_0],b].$$

2.3. Skew-symmetrized derived brackets.

Let us assume again that d is an odd derivation of a Loday algebra $(A,[\,,\,])$, and let us consider the following bracket of degree $|f|+|d|$, which is obviously skew-symmetric,

$$(2.13)\qquad [a,b]_d^{\sim}=\frac{1}{2}\left([a,b]_d-(-1)^{(a+f+1)(b+f+1)}[b,a]_d\right).$$

Assuming now that $[\,,\,]$ is a Lie bracket, we can use (2.10) to write

$$(2.14)\qquad [a,b]_d^{\sim}=[a,b]_d+\frac{1}{2}(-1)^{a+f}d[a,b],$$

$$(2.15)\qquad [a,b]_d^{\sim}=\frac{1}{2}\left([a,db]+(-1)^{a+f+1}[da,b]\right).$$

However, this skew-symmetric bracket is not, in general, a Lie bracket.

Example. — We continue the example of section 2.2, where a_0 is an odd element of square 0 in a Lie algebra $(A,[\,,\,])$ defined by a map f of even degree. Then, from (2.11) and (2.14),(2.15), we obtain

$$(2.16)\qquad \begin{aligned}[a,b]_{a_0}^{\sim}&=[a,[a_0,b]]+\frac{1}{2}(-1)^{a}[a_0,[a,b]]\\&=\frac{1}{2}\left([a,[a_0,b]]+(-1)^{a+1}[[a_0,a],b]\right).\end{aligned}$$

2.4. The Vinogradov bracket.

Now let us assume that A itself is the (graded) vector space of (graded) endomorphisms of a (graded) vector space E, and let us consider the (graded) commutator on $A = \text{End}\, E$, which is a Lie bracket of degree 0. We write, as above,

$$f(a)b = [a, b],$$

for $a, b \in A = \text{End}\, E$. Now let d_0 be a differential on E, *i.e.*, a linear map of degree 1 and of square 0. Let us consider $f(d_0) = [d_0, .]$, which is an interior derivation of degree 1 and of square 0 of $(\text{End}\, E, [\, , \,])$. Let us introduce the notation $\mathcal{L}_a^{d_0} = [a, d_0]$. Then $\mathcal{L}_a^{d_0} = (-1)^{a+1} f(d_0)a$. The brackets that are naturally defined on End E have been denoted $[\, , \,]_{d_0}$ and $[\, , \,]_{d_0}^{\sim}$ in the examples of section 2.2 and 2.3. By formula (2.12), for $a, b \in \text{End}\, E$,

$$[a, b]_{d_0} = [[a, d_0], b] = [\mathcal{L}_a^{d_0}, b],$$

and we know that $[\, , \,]_{d_0}$ is a Loday bracket, but it is not in general skew-symmetric. By (2.16),

$$\begin{aligned} [a, b]_{d_0}^{\sim} &= \frac{1}{2}([a, [d_0, b]] + (-1)^{a+1}[[d_0, a], b]) \\ &= \frac{1}{2}([\mathcal{L}_a^{d_0}, b] + (-1)^{b+1}[a, \mathcal{L}_b^{d_0}]), \end{aligned}$$

and

$$\begin{aligned} [a, b]_{d_0}^{\sim} &= [[a, d_0], b] + \frac{1}{2}(-1)^a [d_0, [a, b]] \\ &= [\mathcal{L}_a^{d_0}, b] + \frac{1}{2}(-1)^{b+1}\mathcal{L}_{[a,b]}^{d_0}. \end{aligned}$$

The bracket $[\, , \,]_{d_0}^{\sim}$ is skew-symmetric by definition, but it does not in general satisfy the Jacobi identity. It coincides with Vinogradov's bracket [V] [CV]. Thus the Vinogradov bracket appears as a particular case of the general construction of skew-symmetrized derived brackets.

2.5. Morphisms.

The notion of *morphism of Loday algebras* is clear: it is a linear map, u, from a Loday algebra $(A, [\, , \,])$ to a Loday algebra $(A', [\, , \,]')$ such that

$$[ua, ub]' = u[a, b],$$

for $a, b \in A$. With this definition, we can state the following simple, yet important proposition.

PROPOSITION 2.3. — *Let d be a derivation of square 0 of a Loday algebra $(A, [\,,\,])$. Then d is a morphism of Loday algebras from $(A, [\,,\,]_d)$ to $(A, [\,,\,])$.*

Proof. — By (2.4), (2.5) and (2.9),

$$d([a, b]_d) = d[a, db] = [da, db]\,. \qquad \square$$

It is clear from formula (2.14) that if moreover $[\,,\,]$ is a Lie bracket, then

$$d([a, b]_d^{\sim}) = [da, db].$$

As a particular case of this result, we obtain property (ii) of the Vinogradov bracket ([CV], p. 78).

Let us now consider two Loday algebras $(A, [\,,\,])$ and $(A', [\,,\,]')$ of the same parity, $|f| = |f'|$, and let d and d' be derivations of square 0, of the same parity, $|d| = |d'|$. Let u be a morphism of degree 0 of Loday algebras, $u : A \longrightarrow A'$ such that

$$d' \circ u = u \circ d\,. \tag{2.17}$$

When, in particular, d and d' are of degree 1, we call a morphism satisfying (2.17) a *morphism of differential Loday algebras.* When (2.17) holds, u is also a morphism of Loday algebras from $(A, [\,,\,]_d)$ to $(A'[\,,\,]'_{d'})$. In particular,

PROPOSITION 2.4. — *A morphism of differential Loday algebras from $(A, [\,,\,], d)$ to $(A', [\,,\,]', d')$ is also a morphism of Loday algebras from $(A, [\,,\,]_d)$ to $(A', [\,,\,]'_{d'})$.*

2.6. The Lie algebra of co-exact elements.

We assume again that d is a derivation of square 0 of the Loday algebra $(A, [\,,\,])$. We consider the linear space $\overline{A}_d = \bigoplus\limits_{i \in \mathbb{Z}} A^i/d(A^{i-|d|})$, where $d(A^{i-|d|})$ is the image of $A^{i-|d|}$ in A^i. We denote the equivalence class of $a \in A^{|a|}$, modulo $d(A^{|a|-|d|})$, by $\bar{a}$. The linear space $\overline{A}_d$ can be called the *space of co-exact elements* in (A, d). From (2.8) and (2.9), it is clear that the equivalence class of $[a,b]_d$ depends only upon the equivalence classes of a and b. Thus we can set

$$[\bar{a},\bar{b}]_d = \overline{[a,b]_d}\,.$$

Obviously, the bracket thus defined on $\overline{A}_d$ is a Loday bracket because, by proposition 2.1, $[\,,\,]_d$ is a Loday bracket. But, when d is odd and $[\,,\,]$ is a Lie bracket, then it follows from formula (2.10) that the bracket on $\overline{A}_d$ is in fact skew-symmetric, and thus a Lie bracket. We remark that $[\,,\,]_d$ and $[\,,\,]_d^{\sim}$ give rise to the same Lie bracket on the space of co-exact elements. The parities of the Lie algebras $(A,[\,,\,])$ and $(\overline{A}_d,[\,,\,]_d)$ are opposite. However, if $da=0$ or $db=0$, then the equivalence class of $[a,b]_d$ is clearly 0, so both brackets $[\,,\,]_d$ and $[\,,\,]_d^{\sim}$ induce the trivial bracket on the cohomology of (A,d).

Because it is of square 0, the derivation $d: A \to A$ induces a well-defined map from $\overline{A}_d$ to A, which we denote by the same letter. By proposition 2.3, the mapping, d, is a morphism of Lie algebras of opposite parity from $(\overline{A}_d,[\,,\,]_d)$ to $(A,[\,,\,])$. Thus

PROPOSITION 2.5. — *To any differential Lie algebra $(A,[\,,\,],d)$ is associated a Lie algebra of opposite parity $(\overline{A}_d,[\,,\,]_d)$, where $\overline{A}_d$ is the space of co-exact elements, and d is a morphism from the latter to the former.*

2.7. Even (resp., odd) brackets on Abelian subalgebras of odd (resp., even) Lie algebras.

Let us again consider the case where $(A,[\,,\,])$ is a Lie algebra. Let $\mathfrak{a}$ be an Abelian subalgebra of $(A,[\,,\,])$ such that $[d\mathfrak{a},\mathfrak{a}]\subset\mathfrak{a}$. Then $[\,,\,]_d$ can be restricted to $\mathfrak{a}$, and $(\mathfrak{a},[\,,\,]_d)$ is a Lie algebra, and by (2.9) for $a,b\in\mathfrak{a}$, $[a,b]_d=[a,db]$. If $(A,m,[\,,\,])$ is a generalized Poisson (resp., Gerstenhaber) algebra and if $\mathfrak{a}$ is also a subalgebra of (A,m), then for d odd, $(\mathfrak{a},[\,,\,]_d)$ is a generalized Gerstenhaber (resp., Poisson) algebra. For instance, d can be an interior derivation $[a_0,\cdot]$, where $a_0\in A$ and $[a_0,a_0]=0$. We remark also that, if A is a Lie algebra, it follows from formula (2.14) that the restrictions of $[\,,\,]_d$ and $[\,,\,]_d^{\sim}$ to any Abelian subalgebra coincide. Summarizing, we have proved

PROPOSITION 2.6. — *Let $\mathfrak{a}$ be an Abelian subalgebra of a Poisson (resp., Gerstenhaber) algebra and let d be an odd derivation of square 0* *such that $[d\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{a}$. Then the restriction of $[\,,\,]_d$ to $\mathfrak{a}$ is a Gerstenhaber (resp., Poisson) bracket.*

3. EXAMPLES AND APPLICATIONS IN POISSON GEOMETRY

There are important examples in Poisson geometry of the preceding constructions, which we shall now examine. In the two examples in this section, the original bracket is odd and the derived bracket is even.

3.1. The Poisson bracket of functions as a derived bracket.

Let (M, P) be a smooth Poisson manifold. All fields and sections are assumed to be smooth. Let $V(M) = \bigoplus_{i \geqslant 0} V^i$, where $V^i = \Gamma(\wedge^i(TM))$, the space of fields of multivectors on M, and let $\Omega(M) = \bigoplus_{i \geqslant 0} \Omega^i$, where $\Omega^i = \Gamma(\wedge^i(T^*M))$, the space of differential forms on M. The subspace V^0 of $V(M)$ and the subspace Ω^0 of $\Omega(M)$ both coincide with the space of functions $C^\infty(M)$.

Both $V(M)$ and $\Omega(M)$ with the exterior multiplication, $\wedge$, are associative, graded commutative algebras. On $(V(M), \wedge)$ (resp., $(\Omega(M), \wedge)$) the Schouten bracket $[\,,\,]$ (resp., the Koszul bracket, $[\,,\,]_P$) is a Gerstenhaber bracket of degree -1. The space $V^0 = C^\infty(M)$ is both a subalgebra of $(V(M), \wedge)$ and of $(\Omega(M), \wedge)$. It is also both an Abelian subalgebra of $(V(M), [\,,\,])$ and of $(\Omega(M), [\,,\,]_P)$. We recall that the Koszul bracket of forms on a Poisson manifold (M, P) satisfies

$$[df, dg]_P = d(P(df, dg)),$$
$$[df, g]_P = P(df, dg).$$

See [K], [K-SM], [Va].

3.1.1. The derived bracket of the Schouten bracket.

Let d_P be the Lichnerowicz-Poisson differential. It is a derivation of degree 1 of both $(V(M), \wedge)$ and $(V(M), [\,,\,])$, of square 0. In fact

$$d_P(Q) = [P, Q], \tag{3.1}$$

for $Q \in V(M)$. In particular, for $f \in V^0 = C^\infty(M)$,

$$d_P f = -\underline{P}(df) \in V^1.$$

Here and below, $\underline{P} : \Omega^1 \to V^1$ is defined by $P(\alpha, \beta) = \langle \beta, \underline{P}\alpha \rangle$, for $\alpha, \beta \in \Omega^1$. Since d_P is of degree 1 and $[\, , \,]$ is of degree -1,

$$[d_P V^i, V^j] \subset V^{i+j}$$

and, in particular, $[d_P V^0, V^0] \subset V^0$. Now let the bracket $[\, , \,]_{d_P}$ on V be defined by (2.8). In the present notation,

$$[Q, Q']_{d_P} = (-1)^i [d_P Q, Q'], \tag{3.2}$$

for $Q \in V^i$, $Q' \in V(M)$. By (2.9), we see that

$$[Q, Q']_{d_P} = [Q, d_P Q'] + (-1)^i d_P [Q, Q']. \tag{3.3}$$

It is clear that $[Q, Q']_{d_P}$ is a Loday-Poisson bracket of degree 0 on $V(M)$, but it is not a (graded) Lie bracket because it is not (graded) skew-symmetric. We can also consider $[\, , \,]^{\sim}_{d_P}$ defined by (2.15),

$$[Q, Q']^{\sim}_{d_P} = \frac{1}{2} \left([Q, d_P Q'] + (-1)^i [d_P Q, Q'] \right), \tag{3.4}$$

which is (graded) skew-symmetric, but is not a (graded) Lie bracket because it does not satisfy the (graded) Jacobi identity.

But we can apply the construction of section 2.7 to the Abelian subalgebra $V^0 = C^\infty(M)$ of $(V(M), [\, , \,])$. The restrictions of $[\, , \,]_{d_P}$ and $[\, , \,]^{\sim}_{d_P}$ to $C^\infty(M)$ coincide and we obtain a Poisson bracket on $C^\infty(M)$, which, by (3.3), is equal to

$$[f, g]_{d_P} = [f, d_P g] = -[f, \underline{P} dg] = \langle df, \underline{P} dg \rangle = -P(df, dg), \tag{3.5}$$

for $f, g \in C^\infty(M)$. So the bracket $[\, , \,]_{d_P}$, restricted to the space of functions on M, is nothing but the usual Poisson bracket defined by the Poisson structure, P, up to sign.

3.1.2. The derived bracket of the Koszul bracket.

Let d be the de Rham differential of forms. It is a derivation of degree 1 of both $(\Omega(M), \wedge)$ and $(\Omega(M), [\, , \,]_P)$, of square 0, and

$$[d\Omega^i, \Omega^j]_P \subset \Omega^{i+j},$$

in particular $[d\Omega^0, \Omega^0]_P \subset \Omega^0$. Now let bracket $[\, , \,]_{P,d}$ on $\Omega(M)$ be defined by (2.8). In the present notation,

$$[\alpha, \beta]_{P,d} = (-1)^i [d\alpha, \beta]_P, \tag{3.6}$$

for $\alpha \in \Omega^i$, $\beta \in \Omega(M)$. By (2.9), we see that

$$[\alpha, \beta]_{P,d} = [\alpha, d\beta]_P + (-1)^i d[\alpha, \beta]_P. \tag{3.7}$$

This bracket is a (graded) Loday-Poisson bracket of degree 0 on $\Omega(M)$, but not a (graded) Lie bracket. We can also consider

$$[\alpha, \beta]^{\sim}_{P,d} = \frac{1}{2}\left([\alpha, d\beta]_P + (-1)^i [d\alpha, \beta]_P\right), \tag{3.8}$$

which is a (graded) skew-symmetric bracket but does not satisfy the (graded) Jacobi identity.

We can repeat the discussion of 3.1.1. Both these brackets restrict to $\Omega^0 = C^\infty(M)$, and coincide there, and this restriction is a Poisson bracket. In fact, for $f, g \in C^\infty(M)$,

$$[f, g]_{P,d} = [df, g]_P = P(df, dg). \tag{3.9}$$

So, again the bracket $[\, ,\,]_{P,d}$, restricted to $C^\infty(M)$, is just the usual Poisson bracket of functions.

Remark. — When $\underline{P}$ is invertible, *i.e.*, the Poisson manifold (M, P) is symplectic, then [K-SM], (6.12) the de Rham differential is in fact the interior derivation, $[\omega, \cdot]_P$, where ω is the symplectic 2-form such that, for each $\alpha, \beta \in \Omega^1$,

$$\omega(\underline{P}\alpha\ ,\ \underline{P}\beta) = P(\alpha, \beta). \tag{3.10}$$

In this case, formula (3.6) becomes

$$[\alpha, \beta]_{P,d} = (-1)^i [[\omega, \alpha]_P, \beta]_P, \tag{3.6$'$}$$

and this formula is obviously dual to the formula

$$[Q, Q']_{d_P} = (-1)^i [[P, Q], Q'], \tag{3.2$'$}$$

valid for any Poisson structure P.

Remark (Koszul bracket and Vinogradov bracket). — In the notation of section 2.4, we assume that $E = V(M)$, where M is a manifold, and we let $d_0 = d_P$, where P is a Poisson structure on M. (More generally, E can be the space of multiderivations of a commutative algebra, and P a

Poisson algebra structure.) For (graded) endomorphisms u, v of E, we set, as in section 2.4,

$$[u,v]_{d_P} = [[u,d_P],v]]$$

and

$$[u,v]^{\sim}_{d_P} = [[u,d_P],v] + \frac{1}{2}(-1)^{|u|}[d_P,[u,v]].$$

Now let $u = i_\alpha, v = i_\beta$, where $\alpha, \beta \in \Omega(M)$ and i_α denotes the interior product with the form α. It is proved in section 6 of [Kr2] that

i) $[i_\alpha, i_\beta]_{d_P} = [i_\alpha, i_\beta]^{\sim}_{d_P} = [[i_\alpha, d_P], i_\beta]$,

ii) $[i_\alpha, i_\beta]_{d_P}$ is of the form $i_{[\alpha,\beta]_P}$, for some element $[\alpha,\beta]_P$ in E,

iii) the bracket $[\,,\,]_P$ thus defined coincides with the Koszul bracket.

3.2. The even bracket on co-exact forms.

By the construction of section 2.6, we know that $[\,,\,]_{P,d}$ and $[\,,\,]^{\sim}_{P,d}$ both define the same graded Lie bracket, which we denote by the symbol $[\,,\,]_{P,d}$, on the space $\overline{\Omega}(M)$ of co-exact forms in $(\Omega(M), d)$. By (3.7) and (3.8), $[\bar{\alpha}, \bar{\beta}]_{P,d}$ is the class of $[\alpha, d\beta]_P$, which is the same as the class of $\frac{1}{2}\left([\alpha, d\beta]_P + (-1)^i[d\alpha, \beta]_P\right)$. For $f, g \in \Omega^0$, we know that this bracket reduces to the Poisson bracket of functions, $\{\,,\,\}_P$. Moreover we know that d is a morphism of (graded) Loday algebras from $(\Omega(M), [\,,\,]_{P,d})$ to $(\Omega(M), [\,,\,]_P)$. Therefore d induces a morphism of (graded) Lie algebras from $(\overline{\Omega}(M), [\,,\,]_{P,d})$ to $(\Omega(M), [\,,\,]_P)$. In particular, for $f, g \in C^\infty(M)$,

$$d(\{f,g\}_P) = [df, dg]_P. \tag{3.10}$$

While the restriction of $[\,,\,]_P$ to $C^\infty(M)$ vanishes, the restriction of $[\,,\,]_{P,d}$ to $C^\infty(M)$ is the Poisson bracket of functions, and $[\,,\,]_{P,d}$ on $\overline{\Omega}(M)$ can be seen as the prolongation of the Poisson bracket into an even bracket.

3.3. The even bracket on d_P-co-exact multivectors.

We can dualize the preceding construction. Let us consider the space $\overline{V}(M)$ of co-exact fields of multivectors in $(V(M), d_P)$. There is a (graded) Lie bracket $[\,,\,]_{d_P}$ on $\overline{V}(M)$ where, by (3.3) and (3.4), $[\overline{Q}, \overline{Q}']_{d_P}$ is the class of $[Q,[P,Q']]$, which is the same as the class of $\frac{1}{2}\left([Q,[P,Q']] + (-1)^i[[P,Q],Q']\right)$. This bracket reduces to the opposite of the Poisson bracket of functions on $V^0 = C^\infty(M)$. Moreover d_P induces a morphism of (graded) Lie algebras from $(\overline{V}(M), [\, ,\,]_{d_P})$ to $(V(M), [\, ,\,])$. In particular,

$$d_P(\{f,g\}_P) = -[d_P f, d_P g]\,. \tag{3.11}$$

3.4. The Hamiltonian mapping.

For each differential from $\alpha \in \Omega(M)$, we consider the endomorphism X^P_α of $\Omega(M)$ defined by

$$X^P_\alpha = f_{P,d}(\alpha) = \big[f_P(\alpha), d\big],$$

where $\big[\, ,\, \big]$ is the (graded) commutator, and

$$f_P(\alpha) = [\alpha, .]_P,$$

is the map defining the Koszul bracket. From proposition 2.2, it follows that X^P_α is a derivation of degree $|\alpha|$ of $(\Omega(M), \wedge)$. Thus by definition and by formulas (3.6), (3.7), for $\alpha \in \Omega^{|\alpha|}, \beta \in \Omega(M)$,

$$\begin{aligned} X^P_\alpha(\beta) &= [\alpha, \beta]_{P,d} = (-1)^{|\alpha|}[d\alpha, \beta]_P \\ &= [\alpha, d\beta]_P + (-1)^{|\alpha|} d[\alpha, \beta]_P. \end{aligned}$$

Each derivation X^P_α commutes with d, and therefore is defined by its restriction to $\Omega^0 = C^\infty(M)$, which is a vector-valued form on M which we denote by the same symbol. For $\alpha = f_0 df_1 \wedge df_2 \wedge \ldots \wedge df_k \in \Omega^k$ and $g \in C^\infty(M)$,

$$X^P_\alpha(g) = \sum_{i=0}^{k} (-1)^i \{f_i, g\} df_0 \wedge df_1 \wedge \ldots \wedge \widehat{df_i} \wedge \ldots \wedge df_k\,.$$

If, in particular, $\alpha = f \in \Omega^0$, then X^P_f is a vector field on M such that

$$X^P_f(g) = [df, g]_P = \{f, g\}_P,$$

the usual Hamiltonian vector field associated with the function f. Thus, the map $\alpha \in \Omega(M) \to X^P_\alpha \in \mathrm{Der}\ (\Omega^0, \Omega(M))$ extends the usual Hamiltonian mapping,

$$f \in \Omega^0 \to X^P_f \in \mathrm{Der}(\Omega^0, \Omega^0).$$

Since $f_{P,d}$ satisfies (2.1), it follows that for $\alpha, \beta \in \Omega(M)$,

$$X^P_{[\alpha,\beta]_{P,d}} = \big[X^P_\alpha, X^P_\beta\big],$$

where $[\, , \,]$ is the (graded) commutator of derivations of $\Omega(M)$. When these derivations commuting with d are identified with vector-valued forms, the bracket on the right-hand side becomes the Frölicher-Nijenhuis bracket. (See subsection 5.2.4 below.)

3.5. Comparison with other work.

In [M], the space of vector-valued differential forms is considered as a subspace of the space of (graded) derivations of $\Omega(M)$, commuting with d, and is thus equipped with the Frölicher-Nijenhuis bracket. In [V] [CV], the spaces of forms, multivectors and vector-valued forms are considered to be subspaces of the space of graded endomorphisms of $\Omega(M)$. This "unification technique" was discovered by C. Buttin in the early 70's (see [B]), and used to "unify" the Frölicher-Nijenhuis and Schouten brackets. We observe that [M] treats only the symplectic case, while [CV] extend his results to the case of a Poisson manifold. It was Akman [A] who remarked that Michor's bracket $\{\, ,\}^2$ is a Loday bracket. In fact, Michor's $\{\, ,\, \}^2$ coincides with the derived bracket $[\, ,\,]_{P,d}$, and therefore Michor's $\{\, ,\, \}^3$ coincides with $[\, ,\,]^{\sim}_{P,d}$, while $\{\, ,\, \}^1$ and $\{\, ,\, \}^3$ differ by an exact term. Bracket $\langle\, ,\, \rangle$ of Cabras and Vinogradov is neither a Loday bracket nor a skew-symmetric bracket (prop. 4 of [CV]), but of course it induces the same bracket (denoted $\{\, ,\}$) as the derived bracket $[\, ,\,]_{P,d}$ on co-exacts forms. Similarly, bracket $\{\, ,\, \}_\Theta$ of [CV] coincides with the derived bracket $[\, ,\,]_{d_P}$ on d_P-co-exact multivectors.

The Hamiltonian mapping defined in section 3.4 coincides with Michor's "generalized Hamiltonian mapping", $H_\alpha = \rho(d\alpha)$, where ρ is the unique extension of $\underline{P} : \Omega^1 \to V^1$ into a derivation of degree -1 on $\Omega(M)$ with values in the vector-valued forms on M, and with the "generalized Hamiltonian fields" X_α defined by Cabras and Vinogradov. (In [BM], Beltrán and Monterde also introduce the derivation ρ, but what they call the "Hamiltonian graded vector field associated to α" is the derivation $f_P(\alpha) = [\alpha, .]_P$ of $\Omega(M)$, and is therefore different from X^P_α.)

3.6. Vinogradov's diagram.

We know [K] [Kr1] [K-SM] that $\wedge(\underline{P}) = \underset{i\geqslant 0}{\oplus} \wedge^i \underline{P}$ is a morphism of Gerstenhaber algebras from $(\Omega(M), \wedge, [\, ,\,]_P)$ to $(V(M), \wedge, [\, ,\,])$ and that

$\wedge(-\underline{P})$ interwines d_P and d,

$$(\wedge^{i+1}\underline{P})(d\alpha) + d_P((\wedge^i\underline{P})\alpha) = 0. \tag{3.20}$$

It is clear, using proposition 2.3 and the results of 3.2 and 3.3, that the following diagram ((29) in [CV]) is a commuting diagram of morphisms of (graded) Lie algebras:

$$\begin{array}{ccc}
(\overline{\Omega}(M), [\,,\,]_{P,d}) & \xrightarrow{\Lambda(\underline{P})} & (\overline{V}(M), [\,,\,]_{d_P}) \\
d \downarrow & & \downarrow d_P \\
(\Omega(M), [\,,\,]_P) & \xrightarrow{\Lambda(\underline{P})} & (V(M), [\,,\,])
\end{array}$$

Here we have also denoted by $\Lambda(\underline{P})$ the mapping induced by $\Lambda(\underline{P})$ on co-exact forms. On the top row are the Poisson algebras of d-co-exact forms and d_P-co-exact multivectors (both extending the Poisson algebra of functions) and on the bottom row the Gerstenhaber algebras of forms (with the Koszul bracket) and multivectors (with the Schouten bracket).

4. THE BIG BRACKET AND THE ALGEBRAIC SCHOUTEN BRACKET

Another bracket that can be obtained by the general construction described in section 2, is the algebraic Schouten bracket. Here the original bracket is even and the derived bracket is odd.

Let F be a finite-dimensional vector space over k. On $\Lambda(F \oplus F^*)$ there exists a unique Poisson bracket, $[\,,\,]$, of degree -2, such that

$$[a, b] = \langle a, b\rangle \quad \text{for} \quad a \in F, b \in F^*,$$

$$[a, b] = 0 \quad \text{otherwise, if } |a| \leqslant 1 \text{ and } |b| \leqslant 1.$$

This bracket was introduced in [CNS] [Kt] [KtS]. See [LR] [R]. We shall call it the big bracket as we did in [K-S1]. An element M of degree 3 in $\Lambda(F \oplus F^*)$, defining a derivation $d_M = [M, .]$, of degree 1, is a sum $\varphi+\gamma+\mu+\psi$, where $\varphi \in \Lambda^3 F$, $\gamma \in \Lambda^2 F \otimes F^*$, $\mu \in F \otimes \Lambda^2 F^*$, $\psi \in \Lambda^3 F^*$. Let us assume that the element M is of square 0, $[M, M] = 0$. Then by definition, M defines a proto-Lie-bialgebra structure on F. When $\varphi = \psi = 0$, then $M = \mu + \gamma$ is a Lie bialgebra structure on F. When $\gamma = \varphi = \psi = 0$, then $M = \mu$ is just a Lie algebra structure on F. When $\mu = \varphi = \psi = 0$, then $M = \gamma$ is a Lie coalgebra structure on F, *i.e.*, a Lie algebra structure on F^*.

We can consider the bracket $[\,,\,]_M$ on $\Lambda(F \oplus F^*)$ defined by (2.8), where d is the derivation of degree 1 and square $0, d_M = [M, \cdot]$, namely

$$[a, b]_M = (-1)^{a+1}[[M, a], b],$$

for $a \in \Lambda^{|a|}(F \oplus F^*)$, $b \in \Lambda(F \oplus F^*)$. By propositions 2.1 and 2.2, it is a Loday-Gerstenhaber bracket on $\Lambda(F \oplus F^*)$, of degree -1. The interesting question is when does $[\,,\,]_M$ restrict to a Gerstenhaber bracket on a subalgebra of $\Lambda(F \oplus F^*)$. In order to apply proposition 2.6, we observe that ΛF and ΛF^* are both Abelian subalgebras of $(\Lambda(F \oplus F^*), [\,,\,])$ and subalgebras of $(\Lambda(F \oplus F^*), \wedge)$, while $d_M = [M, \cdot]$ is a derivation of degree 1 and square 0 of $(\Lambda(F \oplus F^*), [\,,\,])$. Now if $M = \mu$, then $[d_\mu(\Lambda F), \Lambda F] \subset \Lambda F$ and

$$[a, b]_\mu = [a, [\mu, b]],$$

for $a, b \in \Lambda F$. If μ is a Lie algebra structure on F, there is a unique Gerstenhaber bracket of degree -1 on ΛF which vanishes on $\Lambda^0 F = k$ and extends the Lie bracket defined on F by μ. (See [K], [D].) This bracket is called the *algebraic Schouten bracket* because it coincides with the restriction of the Schouten bracket to fields of left-invariant multivectors on a Lie group with Lie algebra (F, μ).

PROPOSITION 4.1 [R] [K-S1]. — *Assume that μ is a Lie algebra structure on F. Then $[\,,\,]_\mu$ is a Gerstenhaber bracket on ΛF which coincides with the algebraic Schouten bracket on the exterior algebra of the Lie algebra (F, μ).*

Proof. — It is enough to observe that μ is a Lie algebra structure on F if and only if $[\mu, \mu] = 0$, and to apply proposition 2.2. In addition, bracket $[a, b]_\mu$ vanishes if a or $b \in k = \Lambda^0 F$, but if $a, b \in F$, then

$$[a, b]_\mu = (-1)^{a+1}[[\mu, a], b] = [a, [\mu, b]] = \mu(a, b). \qquad \square$$

When μ is a Lie algebra structure on F, $d_\mu = [\mu, \cdot]$ is equal, up to a sign, to the Lie algebra cohomology operator on cochains on F with values in ΛF considered as an F-module under the adjoint action. It is a derivation of degree -1, of square 0, of $(\Lambda F \otimes \Lambda F^*, [\,,\,])$. Then

$$[a, b]_\mu = (-1)^{a+1}[[\mu, a], b] = [a, [\mu, b]] - (-1)^a[\mu, [a, b]]$$

is a Loday-Gerstenhaber bracket on $\Lambda F \otimes \Lambda F^*$, and $d_\mu = [\mu, \cdot]$ is a Loday algebra morphism of degree 1 from $(\Lambda F \otimes \Lambda F^*, [\,,\,]_\mu)$ to $(\Lambda F \otimes \Lambda F^*, [\,,\,])$.

Moreover, d_μ restricts to a morphism of (graded) Lie algebras from $(\Lambda F, [\ ,\ ]_\mu)$ to $(\Lambda F \otimes F^*, [\ ,\ ])$. The latter (graded) Lie algebra is the space of vector-valued forms on F^*, equipped with the big bracket, which coincides, up to sign, with the Nijenhuis-Richardson bracket. (See [K-S1].)

5. THE COMPOSITION BRACKET, CUP BRACKET AND FRÖLICHER-NIJENHUIS BRACKET

5.1. Cochains on associative algebras.

Let E be a vector space on a field of characteristic different from 2, and let $M(E) = \bigoplus_{a \geqslant -1} M^a$, where $M^a = L^{a+1}(E,E)$, and $L^i(E,E)$ denotes the space of i-linear maps from E^i to E, for $i \geqslant 0$. Elements of $M(E)$ are called cochains on E. The composition product $\bar{\circ}$ is defined by

(5.1)

$$(a \bar{\circ} b)(x_0, \ldots, x_{|a|+|b|}) = \sum_{k=0}^{|a|} (-1)^{|b||k|} a(x_0, \ldots, b(x_k, \ldots, x_{k+|b|}), \ldots x_{|a|+|b|}),$$

for $a \in M^{|a|}$, $|a| \geqslant 0$, $b \in M^{|b|}$, and $a \bar{\circ} b = 0$ if $|a| = -1$. Though it is neither associative nor graded commutative, the composition product satisfies the "graded pre-Lie ring property" [G], called the "commutative-associative law" in [N], and elsewhere, a "right-symmetric algebra" law,

$$(a \bar{\circ} b) \bar{\circ} c - a \bar{\circ} (b \bar{\circ} c) = (-1)^{|b||c|}((a \bar{\circ} c) \bar{\circ} b - a \bar{\circ} (c \bar{\circ} b)), \tag{5.2}$$

for $a \in M(E)$, $b \in M^{|b|}$, $c \in M^{|c|}$. See also [dWL].

5.1.1. The composition bracket.

It follows from (5.2) that the bracket on $M(E)$ obtained by skew-symmetrizing the composition product is a graded Lie bracket of degree 0 on $M(E)$. We shall denote it by $[\ ,\ ]^{\bar{\circ}}$ and we call it the composition bracket. Thus,

$$[a, b]^{\bar{\circ}} = a \bar{\circ} b - (-1)^{|a||b|}\, b \bar{\circ} a, \tag{5.3}$$

for $a \in M^{|a|}, b \in M^{|b|}$. Let $m \in M^1 = L^2(E,E)$ be a multiplication on E. Then $d_m = [m, .]^{\bar{\circ}}$ is a derivation of degree 1 of $(M(E), [\ ,\ ]^{\bar{\circ}})$. We also consider the operator δ_m of degree 1, defined by

$$\delta_m a = (-1)^{|a|} d_m a.$$

Multiplication m is associative if and only if the following equivalent conditions are satisfied:

$$m\bar{\circ}m = 0 \; ; [m,m]^{\bar{\circ}} = 0 \; ; (d_m)^2 = 0 \; ; (\delta_m)^2 = 0.$$

If m is associative, δ_m coincides with the Hochschild coboundary operator on cochains on the associative algebra (E, m).

5.1.2. The cup bracket of cochains on an associative algebra.

We can now define the cup product and the cup bracket on $M(E)$. Given an element m in $M^1 = L^2(E,E)$, for $a \in M^{|a|} = L^i(E,E)$, $b \in M^{|b|} = L^j(E,E)$, where $i = |a|+1$, $j = |b|+1$, we define the cup product as

$$(5.4) \qquad (a \cup_m b)(x_1, \cdots, x_{i+j}) = m(a(x_1, \cdots, x_i), b(x_{i+1}, \cdots, x_{i+j})),$$

and the cup bracket, which is obviously skew-symmetric, as

$$(5.5) \qquad [a,b]^{\cup}_m = a \cup_m b - (-1)^{ij} b \cup_m a.$$

Then

$$(5.6) \qquad [a,b]^{\cup}_m = (-1)^{ij+i}((m\bar{\circ}a)\bar{\circ}b - m\bar{\circ}(a\bar{\circ}b)).$$

See the proof in [G] or [N]. (However the sign in the first formula of [N], p. 475, has to be changed into its opposite, cf. (5.4) of [N].) It follows from (5.6) that

$$(5.7) \qquad [b,a]^{\cup}_m = (-1)^i(d_m(a\bar{\circ}b) - d_m a\bar{\circ}b - (-1)^{i+1} a\bar{\circ}d_m b).$$

This formula expresses the fact that the cup product is homotopy commutative. It will be interpreted in terms of generalized BV-algebras in section 6. It follows from (5.7) that

$$(5.8) \qquad d_m[a,b]^{\cup}_m = (-1)^{ij+i} d_m(d_m a\bar{\circ}b) + (-1)^{ij+1} d_m(a\bar{\circ}d_m b)$$

while

$$(5.9) \qquad [d_m a, b]^{\cup}_m = (-1)^{ij+i+j} d_m(d_m a\bar{\circ}b) + (-1)^{ij+j+1} d_m a\bar{\circ}d_m b,$$

$$(5.10) \qquad [a, d_m b]^{\cup}_m = (-1)^{ij+1} d_m(a\bar{\circ}d_m b) + (-1)^{ij} d_m a\bar{\circ}d_m b.$$

(We observe that formula (5.9) appears on page 416 of [R].) Formulas (5.8-10) yield

$$(5.11) \qquad d_m[a,b]^{\cup}_m = (-1)^j [d_m a, b]^{\cup}_m + [a, d_m b]^{\cup}_m.$$

Introducing the operator δ_m, we obtain

$$\delta_m[a,b]^{\cup}_m = [\delta_m a, b]^{\cup}_m + (-1)^i[a, \delta_m b]^{\cup}_m. \tag{5.12}$$

Thus, while d_m is a derivation of degree 1 of $(M(E) = \bigoplus_{a\geqslant 1} M^a, [\,,\,]^{\bar{\circ}})$, the operator δ_m is a derivation of degree 1 of $(\bigoplus_{i\geqslant 0} L^i(E,E), [\,,\,]^{\cup}_m)$.

If m is associative, so is the cup product, and the cup bracket is therefore a Lie bracket, while on the Hochschild cohomology the cup product induces a (graded) commutative product of degree 0 and the cup bracket vanishes. The composition bracket induces a Lie bracket of degree 1 on the Hochschild cohomology of (E,m), and this bracket is a Gerstenhaber bracket with respect to the cup product ([G], whence the terminology).

There is a simple interpretation of the composition bracket by means of the (graded) commutator of derivations. Assume, for simplicity, that the vector space E is finite-dimensional, with dual vector space E^*. Every $(i+1)$-cochain a can be considered as a linear map form E^* to $(E^*)^{\otimes(i+1)}$, and admits a unique extension j_a as a derivation of degree i of the tensor algebra of E^*. Now the composition bracket satisfies $j_{[a,b]^{\bar{\circ}}} = [j_a, j_b]$.

5.1.3. The derived bracket of the composition bracket.

We shall now consider the derived bracket $[\,,\,]^{\bar{\circ}}_{d_m}$ on $M(E)$, defined by

$$[a,b]^{\bar{\circ}}_{d_m} = (-1)^{a+1}[[m,a]^{\bar{\circ}},b]^{\bar{\circ}} = [a,[m,b]^{\bar{\circ}}]^{\bar{\circ}} + (-1)^{a+1}[m,[a,b]^{\bar{\circ}}]^{\bar{\circ}}, \tag{5.13}$$

where $a \in M^{|a|} = L^{|a|+1}(E,E), b \in M(E)$. By proposition 2.1, we know that if m is associative, the derived bracket is a Loday bracket of degree 1 on $M(E)$. However that bracket is not in general a Lie bracket. Its skew-symmetrized version is

$$[a,b]^{\bar{\circ}\sim}_{d_m} = \frac{1}{2}([a,[m,b]^{\bar{\circ}}]^{\bar{\circ}} + (-1)^{a+1}[[m,a]^{\bar{\circ}},b]^{\bar{\circ}}). \tag{5.14}$$

When we consider $\overline{M}(E)$, the quotient of $M(E)$ by the image of d_m, we obtain a Lie bracket of degree 1. (This follows from the results of section 2.6.) Since m is associative, this bracket is trivial on the Hochschild cohomology of (E,m) with values in E, as is the cup-bracket!

We also know that $[\,,\,]^{\bar{\circ}}_{d_m}$ restricts to a bracket on $M^{-1} = E$. We can show that for $a,b \in M^{-1}$,

$$[a,b]^{\bar{\circ}}_{d_m} = [[m,a]^{\bar{\circ}},b]^{\bar{\circ}} = m(a,b) - m(b,a),$$

so the restriction of $[\ ,\ ]^{\bar{\circ}}_{d_m}$ to E is just the bracket on E defined by the multiplication m, and is a Lie bracket since m is associative.

5.1.4. The Frölicher-Nijenhuis bracket of cochains on an associative algebra.

Let $A = (E, m)$ be an algebra as above. We shall consider the graded skew-symmetric bracket on $M(E) = \underset{i\geqslant 0}{\oplus} L^i(E, E)$, defined by

$$[a,b]^{Fr-Nij}_m = [a,b]^{\cup}_m - b\bar{\circ}d_m a + (-1)^{ij} a\bar{\circ}d_m b, \tag{5.15}$$

for $a \in L^i(E,E), b \in L^j(E,E)$. See [N], (5.6). By (5.7),

$$[a,b]^{Fr-Nij}_m = (-1)^{ij+i}([d_m a, b]^{\bar{\circ}} - d_m(a\bar{\circ}b)),$$

and, by definition (5.13) of the derived bracket, $[\ ,\ ]^{\bar{\circ}}_{d_m}$,

$$[a,b]^{Fr-Nij}_m = (-1)^{ij}[a,b]^{\bar{\circ}}_{d_m} + (-1)^{i(j+1)+1} d_m(a\bar{\circ}b). \tag{5.16}$$

Thus, up to sign, the Frölicher-Nijenhuis bracket and the derived bracket differ by a d_m-exact term. Skew-symmetrizing this relation, we obtain

$$[a,b]^{Fr-Nij}_m = (-1)^{ij}([a,b]^{\bar{\circ}\sim}_{d_m} + \frac{1}{2}(-1)^{i+1} d_m[a,b]^{\bar{\circ}}).$$

If m is associative, we obtain

$$d_m[a,b]^{Fr-Nij}_m = (-1)^{ij}[d_m a, d_m b]^{\bar{\circ}} = -[d_m b, d_m a]^{\bar{\circ}} \tag{5.17}$$

from (5.16), using proposition 2.3, or in terms of the Hochschild coboundary δ_m,

$$\delta_m[a,b]^{Fr-Nij}_m = (-1)^{ij+1}[\delta_m a, \delta_m b]^{\bar{\circ}} = [\delta_m b, \delta_m a]^{\bar{\circ}}. \tag{5.17$'$}$$

Thus δ_m and $-d_m$ are morphisms of degree 1 from the Frölicher-Nijenhuis bracket to the opposite of the composition bracket. (This agrees with the result of Nijenhuis [N], p. 482 since his bracket $[\ ,\ ]^{\circ}$ is the opposite of $[\ ,\ ]^{\bar{\circ}}$.)

When m is associative, the Frölicher-Nijenhuis bracket is a Lie bracket [N].

5.2. Vector-valued forms on Lie algebras.

Let E be a vector space on a field of characteristic 0. We now consider $A(E) = \underset{a\geqslant -1}{\oplus} A^a$, where $A^a = \widetilde{L}^{a+1}(E,E)$, and $\widetilde{L}^i(E,E)$ denotes the space of skew-symmetric i-linear maps from E^i to E, for $i \geqslant 0$. Elements of $A(E)$ are called (skew-symmetric) cochains or vector-valued forms on E. One now considers the hook product defined by $a \bar{\wedge} b = 0$ if $a \in A^{-1}$ and $b \in A(E)$, and

$$(5.18) \quad (a\bar{\wedge}b)(x_0, \cdots, x_{|a|+|b|}) = \sum_{\sigma} (-1)^{|\sigma|} a(b(x_{\sigma(0)}, \cdots, x_{\sigma(|b|)}), \\ x_{\sigma(|b|+1)}, \cdots, x_{\sigma(|a|+|b|)}),$$

for $a \in A^{|a|}, |a| \geqslant 0$, $b \in A^{|b|}$, where σ is a shuffle of $0, \cdots, |a|+|b|$, *i.e.*, a permutation such that $\sigma(0) < \cdots < \sigma(|b|)$, $\sigma(|b|+1) < \cdots < \sigma(|a|+|b|)$, and $(-1)^{|\sigma|}$ is the signature of σ. The hook product is obtained from the composition product by alternation. It satisfies the graded pre-Lie ring property, as in (5.2).

5.2.1. The Nijenhuis-Richardson bracket.

The bracket on $A(E)$, obtained by skew-symmetrizing the hook product is a graded Lie bracket of degree 0 on $A(E)$, called the Nijenhuis-Richardson bracket, which we denote by $[\ ,\ ]^{\bar{\wedge}}$. Thus

$$(5.19) \qquad [a,b]^{\bar{\wedge}} = a\bar{\wedge}b - (-1)^{|a||b|} b\bar{\wedge}a.$$

For example, if $a, b \in A^{-1} = E$ and $\mu \in A^1 = \widetilde{L}^2(E,E)$, $[\mu, a]^{\bar{\wedge}}(b) = [\mu, a]^{\bar{\wedge}} \bar{\wedge} b = \mu(a,b)$, and $a \bar{\wedge} [\mu, b]^{\bar{\wedge}} = 0$.

Let μ be an element of $A^1 = \widetilde{L}^2(E,E)$. Then $d_\mu = [\mu, .]^{\bar{\wedge}}$ is a derivation of degree 1 of $(A(E), [\ ,\ ]^{\bar{\wedge}})$. Moreover μ is a Lie algebra structure on E if and only if

$$[\mu, \mu]^{\bar{\wedge}} = 0.$$

If μ is a Lie algebra structure, the derivation d_μ is of square 0, and it coincides, up to sign, with the Chevalley-Eilenberg cohomology operator δ_μ on cochains of the Lie algebra (E, μ) with values in E, considered as an E-module under the adjoint action. More precisely, it is known that for $a \in A^{|a|}, \delta_\mu = (-1)^{|a|} d_\mu$. See [NR].

Example. — If E is the algebra of functions on a smooth manifold M, the Nijenhuis-Richardson bracket, restricted to the skew-symmetric multilinear maps on E which are derivations in each argument, coincides with the Schouten bracket of fields of multivectors. We thus recover the fact that a Poisson algebra structure on $E = C^\infty(M)$ is defined by a field of bivectors whose Schouten bracket with itself vanishes.

5.2.2. The cup bracket of vector-valued forms on a Lie algebra.

Given an element μ in A^1, the cup bracket is defined by

$$[a,b]^{\cup}_{\mu}(x_1,\cdots,x_{i+j}) = \sum_{\sigma}(-1)^{\sigma}\mu(a(x_{\sigma(1)},\cdots,x_{\sigma(i)}),b(x_{\sigma(i+1)},\cdots,x_{\sigma(i+j)})),$$

where the sum is taken over all shuffles, and it satisfies

$$[a,b]^{\cup}_{\mu} = (-1)^{ij+i}((\mu\bar{\wedge}a)\bar{\wedge}b - \mu\bar{\wedge}(a\bar{\wedge}b)), \tag{5.20}$$

for $a \in \widetilde{L}^i(E,E), b \in \widetilde{L}^j(E,E)$. And therefore,

$$[b,a]^{\cup}_{\mu} = (-1)^i(d_\mu(a\bar{\wedge}b) - d_\mu a\bar{\wedge}b - (-1)^{i+1}a\bar{\wedge}d_\mu b). \tag{5.21}$$

If μ is a Lie algebra structure, then the cup bracket $[\ ,\ ]^{\cup}_{\mu}$ is a graded Lie bracket of degree 0 on $\bigoplus_{i\geqslant 0}\widetilde{L}^i(E,E)$, of which δ_μ is a derivation. Obviously, the cup bracket vanishes in cohomology.

5.2.3. The derived bracket of the Nijenhuis-Richardson bracket.

The derived bracket $[\ ,\ ]^{\bar{\wedge}}_{d_\mu}$ is defined by

$$[a,b]^{\bar{\wedge}}_{d_\mu} = (-1)^{a+1}[[\mu,a]^{\bar{\wedge}},b]^{\bar{\wedge}}.$$

On $M^{-1} = E$, it restricts to the given bracket μ. (Here elements of E are viewed as 0-forms on E with values in E.) If μ is a Lie algebra structure, the derived bracket is a Loday bracket, while the skew-symmetrized derived bracket is not.

5.2.4. The Frölicher-Nijenhuis bracket of vector-valued forms on a Lie algebra.

Let (E,μ) be a Lie algebra. As in [N], we set

$$[a,b]^{Fr-Nij}_{\mu} = [a,b]^{\cup}_{\mu} - b\bar{\wedge}d_\mu a + (-1)^{ij}a\bar{\wedge}d_\mu b \tag{5.22}$$

for $a \in \widetilde{L}^i(E,E), b \in \widetilde{L}^j(E,E)$. Then we obtain relations analogous to (5.16) and (5.17), whence

PROPOSITION 5.1. — *The maps δ_μ and $-d_\mu$ are morphisms of degree 1 from the Frölicher-Nijenhuis bracket to the opposite of the Nijenhuis-Richardson bracket.*

The Frölicher-Nijenhuis bracket is a Lie bracket [N]. If $a,b \in \widetilde{L}^0(E,E) = E$, then $[a,b]^{Fr-Nij}_{\mu} = [a,b]^{\cup}_{\mu} = \mu(a,b)$, so that the Frölicher-Nijenhuis bracket is a prolongation of the Lie bracket μ on E, considered as the space of vector-valued 0-forms on E, to all vector-valued forms on E.

If $a, b \in \widetilde{L}^1(E,E) = L^1(E,E)$, then

$$[a,b]_\mu^{Fr-Nij}(x,y) = \mu(ax,by) + \mu(bx,ay) - a(\mu(bx,y) + \mu(x,by)) \\ -b(\mu(ax,y) + \mu(x,ay)) + ab\mu(x,y) + ba\mu(x,y).$$

If, in particular, $a = b$, we recover the usual formula for (twice) the *Nijenhuis torsion* of $a \in L^1(E,E)$.

Formula (5.22) agrees with ([D-VM] 5.7 (2)) since $\delta_\mu a = (-1)^{i+1} d_m a = -[a,m]^\wedge$. But the expression given in [R], p. 415, does not coincide with (5.22), and does not reduce, for an endomorphism of a Lie algebra, to a multiple of the Nijenhuis torsion.

Example. — If E is the linear space of vector fields on a smooth manifold M, then the space of vector-valued differential forms on M is a subspace of $\widetilde{L}(E) = \bigoplus_{i \geqslant 0} \widetilde{L}^i(E,E)$. If μ is the Lie bracket of vector fields, then the Frölicher-Nijenhuis bracket defined above coincides with the bracket first considered by these authors in a geometric context [FN]. In fact, to a vector-valued differential form a of degree i, one associates a derivation $\hat{a}$ of the exterior algebra of differential forms on M, by the conditions

$$[\hat{a}, d] = 0,$$
$$(\hat{a}f)(x_1, \cdots, x_i) = \langle df, a(x_1, \cdots, x_i)\rangle,$$

where d is the de Rham differential of forms. It is well-known that $\hat{a} = [i_a, d]$, where i_a is the interior product of forms with the vector-valued form a. Then we must show that the image of $[a,b]_\mu^{Fr-Nij}$ under this map is the graded commutator $[\hat{a}, \hat{b}]$. It is clear that it is a derivation of $\Omega(M)$ which commutes with d. Furthermore one must show that

$$\langle df, [a,b]_\mu^{Fr-Nij}(x_1, \cdots, x_{i+j})\rangle = [\hat{a}, \hat{b}](f)(x_1, \cdots, x_{i+j}).$$

This is obviously true if $a, b \in \widetilde{L}^0(E,E) = E$, and can be proven in general. (See [N].) More generally, E may be the linear space of derivations of a not necessarily commutative ring. The space of differential forms on the manifold is then replaced by skew-symmetric cochains on E with values in the ring. (See [D-VM]). In all these cases, the fact that $a \mapsto \hat{a}$ is injective and is a morphism from the Frölicher-Nijenhuis bracket to the graded commutator furnishes a proof of the Jacobi identity for the Frölicher-Nijenhuis bracket.

6. BV-ALGEBRAS AND GENERALIZED BV-ALGEBRAS

6.1. Exact Gerstenhaber algebras or BV-algebras.

A differential operator of order p on an associative, graded commutative algebra is a well-known concept. That a second-order differential operator has the following property was proved by Koszul [K] and rediscovered by Penkava and Schwarz [PS] and by Getzler [Gt].

PROPOSITION 6.1. — *Let $(A = \underset{i\in\mathbb{Z}}{\oplus} A^i, m, 1)$ be an associative, graded commutative algebra with unit. If Δ is a differential operator of order* 2, *of degree* -1 *and of square* 0, *vanishing on the unit, then*

$$[a,b] = (-1)^i(\Delta(ab) - (\Delta a)b - (-1)^i a\Delta b), \tag{6.1}$$

where $a \in A^i, b \in A$, defines a Gerstenhaber bracket on A.

Here the product $m(a,b)$ is simply denoted ab. The multiplication, m, is assumed to be of degree 0, and therefore the bracket defined by formula (6.1) is of degree -1. It is said to be generated by the operator Δ. This bracket is a 2-cochain on A with values in A, which is, up to sign, equal to the graded Hochschild coboundary (with respect to m) of the 1-cochain $\Delta \in L^1(A,A)$, defined by

$$(\delta_m\Delta)(a,b) = (-1)^i a\Delta b - \Delta(ab) + (\Delta a)b.$$

For this reason, a Gerstenhaber algebra whose bracket is defined by formula (6.1) for some operator Δ was called a coboundary Gerstenhaber algebra in [LZ] and an exact Gerstenhaber algebra in [K-S2]. The use of the term *Batalin-Vilkovisky algebra* or for short, BV-*algebra*, for this object, has since become widespread. Examples of BV-algebra structures are to be found both in geometry and in field theory. In particular, on a Poisson manifold (M,P), the Koszul bracket of differential forms is generated by the Poisson homology operator $[i_P, d]$, so that the algebra of differential forms has a BV-algebra structure. (See [K], [K-S2].)

An important connection between the theory of derived brackets and that of BV-algebras has been recently pointed out to me by I. Krasilshchik. For any odd endomorphism Δ of A, of square 0, we can consider the derived bracket on End A, obtained as in (2.12) from the graded commutator and the interior derivation defined by Δ. The elements of A, acting on A by left multiplication, form a subalgebra of (End A, [,]), which is Abelian if and only if A is graded commutative. A computation shows that, when Δ is a differential operator of order 2, the restriction of this derived bracket to A is the BV-bracket. Thus, proposition 6.1 follows from proposition 2.6, and this proof can be adapted to the case of generalized BV-brackets which we now consider.

6.2. Generalized BV-algebras.

As explained in the introduction, in the generalization of BV-algebras proposed by Akman [A], motivated by considerations drawn from topological field theory, the notion of Loday algebra appears in a natural way. Generalized BV-algebras can be considered as non-commutative versions of the BV-algebras, whose definition we have just recalled.

DEFINITION 6.1. — *A generalized BV-algebra is an algebra with unit* $(A = \bigoplus_{i \in \mathbb{Z}} A^i, m, 1)$, *where* m *is not necessarily associative nor commutative, equipped with a differential operator* Δ, *of order 2, of odd degree, and of square* 0, *which vanishes on the unit.*

Examples. — In a vertex operator algebra with the "normal ordered (Wick) product",

$$a \times_{-1} b = \mathrm{Res}_z z^{-1} a(z) b,$$

where the unit is the vacuum, the mode u_1 of a vertex operator $u(z) = \sum_{n \in \mathbb{Z}} u_n z^{-n-1}$ is a differential operator of order 2 which vanishes on the vacuum [LZ] [A]. Thus one obtains examples of generalized BV-algebras whenever u_1 is of odd degree and of square 0.

Also, while the BRST cohomology of a "string background", *i.e.*, the tensor product of a conformal field theory with a space of ghosts, is a BV-algebra in the usual sense, the space of cochains itself has the structure of a generalized BV-algebra [LZ] [A]. See also theorem 3.11 of [KiSV].

On a generalized BV-algebra, one again considers the bracket defined by (6.1). It is not a Lie bracket in general, but it is a Loday bracket. In fact, recalling the definition of a generalized Loday-Gerstenhaber algebra from section 2, proposition 4.8 of [A] can be reformulated as follows.

PROPOSITION 6.2. — *On a generalized BV-algebra* $(A, m, 1, \Delta)$, *the formula* $[a, b] = (-1)^i(\Delta(ab) - (\Delta a)b - (-1)^i a \Delta b)$, *for* $a \in A^i$, $b \in A$, *defines* *a generalized Loday-Gerstenhaber bracket. This bracket is skew-symmetric if and only if Δ is a derivation of the bracket obtained by skew-symmetrizing the multiplication, m.*

Since m is not necessarily associative, we can no longer speak of the graded Hochschild coboundary operator, however we can still consider the derivation $d_m = [m, .]^{\bar{\circ}}$, and generalized BV-algebras can still be considered as the generalized Loday-Gerstenhaber algebras whose bracket is defined as the image under d_m of an operator Δ.

6.3. A further generalization of BV-algebras.

We now recall formula (5.7) (resp., (5.21)) from the study of the cup-bracket on cochains on associative (resp., Lie) algebras. There, in contrast to what holds in the case of classical or generalized BV-algebras, the multiplication, *i.e.*, the composition product $\bar{\circ}$ (resp., hook product $\bar{\wedge}$) is not of degree 0, but of degree 1, on $\bigoplus_{i\geqslant 0} L^i(E,E)$(resp., $\bigoplus_{i\geqslant 0} \widetilde{L}^i(E,E)$). The formula for the graded Hochschild coboundary of cochains should be modified accordingly. We know that d_m (resp., d_μ) is a derivation of degree 1 of the composition bracket (resp., Nijenhuis-Richardson bracket), but not a derivation of the composition product (resp., hook product). The cup bracket measures the defect of this derivation property with respect to the composition (resp., hook) product. In fact, formulas (5.7) and (5.21) show that the opposite of the cup-bracket appears as a kind of generalized BV-bracket. Comparing formulas (6.1) and (5.7), we see that the change of the factor $(-1)^i$ to the factor $(-1)^{i+1}$ in the last term comes from the fact that in (6.1), the multiplication is of degree 0, while in (5,7) it is of degree 1. The fact that d_m (resp., δ_μ) is a derivation of $[\ ,\ ]^{\bar{\circ}}$ (resp., $[\ ,\ ]^{\bar{\wedge}}$) implies that the cup-bracket is skew-symmetric, as is the bracket of proposition 6.1.

However, the properties of the cup-bracket with respect to the composition product (resp., hook product) are not those of a Gerstenhaber bracket. In fact d_m (resp. d_μ) is not a differential operator of order 2 with respect to $\bar{\circ}$ (resp., $\bar{\wedge}$), and this implies that the cup-bracket does not satisfy the graded Leibniz identity with respect to $\bar{\circ}$ (resp., $\bar{\wedge}$).

But this situation suggests that, using the generalizations of the algebraic brackets to the multigraded case [LMS] [Kr2] (see also [DT]), it is possible to further generalize the notion of BV-algebra to the case of an algebra (A, m) where the multiplication is of nonzero degree $|m|$. We conjecture that

$$(-1)^{i+|m|}(\Delta(ab) - (\Delta a)b - (-1)^{i+|m|}a(\Delta b))$$

defines a generalized Loday-Gerstenhaber bracket of degree $|m| - 1$, if Δ is of degree -1, square 0, vanishes on the unit, and is a differential operator of order 2 in a suitably generalized sense. Interesting examples of such generalized brackets are still to be found!

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