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RELATIVE DISCRETE SERIES OF LINE BUNDLES OVER BOUNDED SYMMETRIC DOMAINS

by A.H. DOOLEY, B. ØRSTED and G. ZHANG

0. Introduction.

The explicit Plancherel formula for a L^2-space on a Riemannian symmetric space G/K has been obtained by using the theory of spherical functions of Harish-Chandra [He]. However the Plancherel formula in the vector bundle case is less well understood. When G/K is a Hermitian symmetric space there is a family of line bundles and the corresponding Plancherel formula is studied in [LP], [PPZ] and [Zh2] for rank one-symmetric spaces and more recently in [Sh] for all Hermitian symmetric spaces. In that case there are relative discrete series entering into the Plancherel formula. One of the main result in [Sh] states that all the relative discrete series are equivalent to holomorphic discrete series. In most analysis problems concerning the spectral property of the Plancherel formula it is important to have explicit formulas (e.g. orthogonal bases, reproducing kernels) for the discrete series. In this paper we study a concrete realization of the relative discrete series.

We take the simplest case $D = G/K$ to be the unit disk in the complex plane. The L^2 space of the sections of a line bundle over D can be realized as the L^2-space of functions that are square integrable with respect

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to certain weighted measure and the group action is the regular action with a multiplier (or a cocycle). The relative discrete series appearing in the L^2-space can be characterized by certain invariant Cauchy-Riemann operators and are equivalent to holomorphic discrete series (see [PPZ] and [LP]). The holomorphic discrete series have their standard realizations as weighted Bergman spaces of holomorphic functions. The unitary intertwining operators from the weighted Bergman spaces into the relative discrete series are obtained in [Zh1]; they are certain Bol's type operators and are related to the so called "transvectant" in classical invariant theory [JP]. The functions in the relative discrete series can be expressed as holomorphic polynomials whose coefficients are anti-holomorphic polynomials of a certain degree. This indicates that they can be realized in the tensor product of a holomorphic discrete series with a finite dimensional representation of the anti-holomorphic polynomials. The purpose of this paper is to make the above observation precise and prove that it is true for all bounded symmetric domains. In particular we find the highest weight vectors of certain relative discrete series; see Theorem 3.5.

The main results are summarized in Theorems 3.3, 3.4 and 3.5. In §1 we fix some notation and in §2 we identify the space of L^2-functions on G transforming according to a one-dimensional representation of K with the space of L^2-functions on the bounded symmetric domain with a certain weighted measure.

It would also be interesting to realize the relative discrete series using the invariant Cauchy-Riemann operator as in [PPZ] and to find the highest weight vectors of the relative discrete series. Some work in this direction has been done in [P].

To avoid some technical difficulties involving universal coverings of groups we consider here only line bundles whose parameter ν satisfies some integral conditions (see Lemma 1.1 and Proposition 2.1). We note that the main results (Theorems 3.3 and 3.4) still hold for a general $\nu > p-1$.

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1. Preliminaries.

We briefly recall the bounded realization of a Hermitian symmetric space, see [FK], [Sa], [W] and [L].

Let $\mathfrak{g}$ be a simple Lie algebra of Hermitian type and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be its Cartan decomposition. Thus $\mathfrak{k}$ has one-dimensional center and $\mathfrak{k}_s = [\mathfrak{k}, \mathfrak{k}] \neq \mathfrak{k}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{k}$. Let Φ^+ be the set of non-compact roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ which are positive with respect to some ordering. A basis of root vectors $\{e_\alpha\}$ can be chosen so that $\tau e_\alpha = -e_{-\alpha}$, $[e_\alpha, e_{-\alpha}] = h_\alpha$ and $[h_\alpha, e_\alpha] = 2e_\alpha$, where τ is the conjugation with respect to the real form $\mathfrak{k} + i\mathfrak{p}$. So we have the Harish-Chandra decomposition of $\mathfrak{g}^{\mathbb{C}}$,

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^- + \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^+,$$

where $\mathfrak{p}^-$ and $\mathfrak{p}^+$ are sum of the negative and positive non-compact root spaces. Now an element $x \in \mathfrak{k}$ is in $\mathfrak{k}_s$ if and only if $\operatorname{tr}_{\mathfrak{p}^+} x = 0$. Here $\operatorname{tr}_{\mathfrak{p}^+} x$ is the trace of $\operatorname{ad} x$ on $\mathfrak{p}^+$.

Let $G^{\mathbb{C}}$ be the adjoint group of $\mathfrak{g}^{\mathbb{C}}$ and P^+, P^-, G and K the connect subgroups with Lie algebras $\mathfrak{p}^+$, $\mathfrak{p}^-$, $\mathfrak{g}$ and $\mathfrak{k}$ respectively. Now $P^+K^{\mathbb{C}}P^-$ is a dense subset of $G^{\mathbb{C}}$. Let $g \in G^{\mathbb{C}}, z \in \mathfrak{p}^+$ we let $\mathcal{K}(g : z)$ be the $K^{\mathbb{C}}$ component of $g\exp(z)$. Namely

$$g\exp(z) = \exp(g \cdot z)\mathcal{K}(g : z)p_- \tag{1.1}$$

for some $p_- \in P^-$. Under the above action the G-orbit $D = G \cdot 0$ of $z = 0 \in \mathfrak{p}^+$ is a bounded domain in $\mathfrak{p}^+$ and K is the isotropy subgroup of 0. This is the Harish-Chandra realization of G/K.

Let $\gamma_1 < \cdots < \gamma_r$ be the Harish-Chandra strongly orthogonal roots with the corresponding root vectors $e_j = e_{\gamma_j}$. We put $h_j = h_{\gamma_j}$, and $e_{-j} = e_{-\gamma_j}$. Thus we have $\gamma_j(h_k) = 2\delta_{j,k}$. Let

$$e = e_1 + \ldots + e_r,\ e_- = e_{-1} + \ldots + e_{-r}.$$

We let

$$\xi_j = e_j + e_{-j}.$$

Thus $\{\xi_j\} \subset \mathfrak{p}$ and they span a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$.

Let $\mathfrak{h}^-$ be the subalgebra of $\mathfrak{h}$ generated by the elements ih_j, $j = 1, \ldots, r$. Then $\mathfrak{h} = \mathfrak{h}^- \oplus \mathfrak{h}^+$ and $\gamma_k = 0$ on $(\mathfrak{h}^+)^{\mathbb{C}}$. We let

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} + \mathfrak{n}_K^+ + \mathfrak{n}_K^-$$

be the root space decomposition of $\mathfrak{k}^{\mathbb{C}}$. Thus

$$\mathfrak{g}^{\mathbb{C}} = (\mathfrak{p}^+ + \mathfrak{n}_K^+) + \mathfrak{h}^{\mathbb{C}} + (\mathfrak{p}^- + \mathfrak{n}_K^-)$$

is the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$. By highest weight module of $\mathfrak{g}^{\mathbb{C}}$ we will always mean one with respect to the positive root space $\mathfrak{p}^+ + \mathfrak{n}_K^+$.

The Cayley transform is defined by

$$\gamma_e = \operatorname{Ad}\exp\Big(i\frac{\pi}{4}(e+e_-)\Big).$$

Then $\gamma_e^{-1}\mathfrak{h}$ is a maximal abelian subspace of $\mathfrak{p}$ and $\gamma_e^{-1}(h_j) = i(e_{-j} - e_j)$.

Let $\beta_j = (\gamma_e^{-1})^*(\gamma_j)$, where $(\gamma_e^{-1})^*$ is the adjoint of γ_e^{-1}. Thus $\beta_j(\xi_{e_k}) = 2\delta_{jk}$. Moreover

$$0 < \beta_1 < \beta_2 < \cdots < \beta_r$$

is an ordering of $\mathfrak{a}^*$. The root system $\Sigma^+(\mathfrak{g},\mathfrak{a})$ consists of β_j, $\dfrac{\beta_j \pm \beta_k}{2}$, $j > k$, and $\dfrac{\beta_j}{2}$ with multiplicities 1, a, and $2b$ respectively. See [L]. Thus the longest root is $\beta_1 + \ldots + \beta_r$ and the Killing form in $\mathfrak{g}$ can be calculated by the root system as

$$B(\xi_{e_1},\xi_{e_1}) = \operatorname{tr}(\operatorname{ad}{\xi_{e_1}}^2) = 2(2(r-1)a + 4 + 2b) = 4p$$

where

$$p = (r-1)a + 2 + b$$

is called the genus of D. Now $\beta_j(\xi_{e_j}) = 2$ thus the length of the longest roots β_j is

$$|\beta_j| = \frac{1}{p^{1/2}}.$$

One also easily checks that $\operatorname{tr}_{\mathfrak{p}^+} H_j = p$.

Since $(\gamma_e^{-1})^*(\gamma_j) = \beta_j$ the length of γ_j is also $\dfrac{1}{p^{1/2}}$.

We let Z_0 be the element in the center of $\mathfrak{k}$ which defines the complex structure on $\mathfrak{p}^+ = V$, that is $Z_0 = i$ on $\mathfrak{p}^+$ and $Z_0 = -i$ on $\mathfrak{p}^-$. We let $n = \dim_{\mathbb{C}} V$ and

$$Z = \frac{p}{n} Z_0.$$

Note that the Killing form $B(Z, Z_0)$ in $\mathfrak{g}^{\mathbb{C}}$ is

$$B(Z,Z_0) = 2\frac{p}{n}\operatorname{tr}_{\mathfrak{p}^+} Z_0^2 = -2p = -\frac{2}{|\gamma_1|^2}. \tag{1.2}$$

We let $Z^* \in \mathfrak{z}^*$ be defined such that $Z^*(w) = B(z,w)$ for all w.

Let K^s the analytic subgroup of K with Lie algebra $\mathfrak{k}_s$ and $K = \exp(\mathbb{R}Z)K^s$. Clearly for each $x \in \mathfrak{k}^s$ we have $\operatorname{tr}_{\mathfrak{p}^+} x = 0$. Thus for each $k \in K^s$ we have $\det_{\mathfrak{p}^+\mathfrak{k}} k = 1$. Also $\exp(\mathbb{R}Z) = \mathbb{T}$, the one-dimensional torus. For each $\nu \in p\mathbb{Z}$ we consider the representation τ_ν of K defined by $\tau_\nu(\exp(tZ)) = e^{it\nu}$ and τ_ν is trivial on K^s.

LEMMA 1.1. — *If $\nu \in p\mathbb{Z}$ then τ_ν is a well-defined character on K.*

Proof. — We need only to show that if $\exp(tZ) \in K_s$ then $\tau_\nu(\exp(tZ)) = 1$. However for every element k in K_s we have $\det_{\mathfrak{p}^+} k = 1$. Thus $\exp(tZ) \in K_s$ implies that $e^{ipt} = \det_{\mathfrak{p}^+} \exp(tZ) = 1$ and that $\tau_\nu(\exp(tZ)) = e^{it\nu} = (e^{itp})^{\nu/p} = 1$. □

Remark. — It follows from (1.2) that the element Z is the same as that in [Sch], §3.

2. Line bundle over D.

Let $\nu > p - 1$ and suppose that ν/p is an integer. Let $L^2(G, \tau_\nu)$ be the space of functions in $L^2(G)$ which transform according to

$$F(gk) = \tau_\nu(k)F(g).$$

Then G acts unitarily on $L^2(G, \tau_\nu)$ by the left regular action

$$gF(x) = F(g^{-1}x).$$

This is an induced representation of G from τ_ν of K. There is another realization of the induced representation as space of functions on D.

We let $h(z)$ be the K-invariant polynomial on $\mathfrak{p}^+$ whose restriction on $\mathbb{R}e_1 + \cdots + \mathbb{R}e_r$ is given by

$$h\Big(\sum_{j=1}^{r} a_j e_j\Big) = \prod_{j=1}^{r}(1 - a_j^2).$$

Let $h(z, w)$ be its polarization. Consider, for each real number $\alpha > -1$, the Hilbert space $L^2(D, d\mu_\alpha)$ of square integrable functions with respect to the measure $d\mu_\alpha(z) = C_\alpha h(z, z)^\alpha dm(z)$ on D, where $dm(z)$ is the Euclidean measure on $\mathfrak{p}^+ \supset D$ induced from the Killing form, and $\mu_\alpha(D) = 1$. We put

$$\nu = \alpha + p. \tag{2.1}$$

There is a unitary representation of G on $L^2(D, d\mu_\alpha)$ given by the formula

$$U_g^{(\nu)} : f(z) \mapsto f(g^{-1}(z))(J_g^{-1}(z))^{\frac{\nu}{p}} \qquad (g \in G) \tag{2.2}$$

where J_g stands for the Jacobian of the transformation g.

PROPOSITION 2.1. — *Assume that $\alpha > -1$ is such that ν/p is an integer. Then realizing the symmetric space G/K as the bounded symmetric domain D we have*

$$I : F(g) \mapsto f(z) = F(g)\tau_\nu(\mathcal{K}(g : o)),\ z = g \cdot 0 \in D$$

is a unitary G-intertwining operator from $L^2(G, \tau_\nu)$ to $L^2(D, d\mu_\alpha)$.

Proof. — First we see that if F is in $L^2(G, \tau_\nu)$ then $F(g)\tau_\nu(\mathcal{K}(g : o))$ is right K-invariant and thus f is well-defined. We now check the intertwining relation. Let $h \in G$. Then for $z = g \cdot 0 \in D$

$$I(h \cdot F)(z) = F(h^{-1}g)\tau_\nu(\mathcal{K}(g : o)).$$

However we recall from [Sa] that the Jacobian $J_g(z)$ of $g \in D$ on D defines a character on $K^{\mathbb{C}}$

$$\mathcal{K}(h^{-1}g : 0) = \mathcal{K}(h^{-1} : g0)\mathcal{K}(g : 0)$$

and

$$J_g(z) = \tau_p(\mathcal{K}(g : z)).$$

In particular

$$J_k(z) = \tau_p(k),\ k \in K.$$

Thus

$$\tau_\nu(\mathcal{K}(h^{-1}g : 0)) = \tau_\nu(\mathcal{K}(h^{-1} : g0))\tau_\nu(\mathcal{K}(g : 0)) = (J_{h^{-1}}(z))^{\frac{\nu}{p}}(J_g(0))^{\frac{\nu}{p}}$$

or

$$(J_{h^{-1}}(z))^{\frac{\nu}{p}} = \tau_{-\nu}(\mathcal{K}(h^{-1}g : 0))\tau_{-\nu}(\mathcal{K}(g : 0))^{-1}.$$

Thus

$$\begin{aligned} I(h \cdot F)(z) &= F(h^{-1}g)\tau_\nu(\mathcal{K}(g : 0)) \\ &= F(h^{-1}g)\tau_\nu(\mathcal{K}(h^{-1}g : 0))\tau_{-\nu}(\mathcal{K}(h^{-1}g : 0)) \\ &= IF(h^{-1} \cdot z)(J_{h^{-1}}(z))^{\frac{\nu}{p}} \\ &= U^\nu(h)IF(z). \end{aligned}$$

It is clear that I is onto. In fact for any $f \in L^2(D, d\mu_\alpha)$ we let

$$F(g) = f(g \cdot 0)\tau_\nu(k(g : o))^{-1},$$

we have $IF = f$. Using the Iwasawa decomposition of G and the corresponding integral formula ([He], Proposition 5.1, Ch. I) we can easily prove that I is an isometry; we omit the details. □

The irreducible decomposition of $L^2(G, \tau_\nu)$ under G is given in [Sh]. One of the main results there states that the relative discrete series appearing in the decomposition are all holomorphic discrete series, which we recall here. First we introduce some notation.

Let $E(\Lambda)$ be a K-module with highest weight Λ. Let $N(\Lambda)$ be the highest weight module of $\mathfrak{g}^{\mathbb{C}}$

$$N(\Lambda) = U(\mathfrak{g}^{\mathbb{C}}) \otimes_{\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^+} E(\Lambda).$$

As $\mathfrak{k}^{\mathbb{C}}$-module

$$N(\Lambda) \cong S(\mathfrak{p}^-) \otimes E(\Lambda). \tag{2.3}$$

Let $\alpha > -1$ and let ν be as in (2.1). We define

$$l = \begin{cases} \frac{\alpha+1}{2} - 1 = \frac{\nu-p-1}{2} & \text{if } \alpha \text{ is an odd integer} \\ \left[\frac{\alpha+1}{2}\right] = \left[\frac{\nu-p+1}{2}\right] & \text{otherwise.} \end{cases} \tag{2.4}$$

Here $[t]$ stands for the integer part of $t \in \mathbb{R}$.

Denote

$$D_\nu = \left\{ \underline{m} = \sum_{j=1}^{r} m_j \gamma_j, 0 \leq m_1 \leq \cdots \leq m_r \leq l, m_j \in \mathbb{Z} \right\}.$$

Shimeno proved in [Sh], Theorem 5.10 that relative discrete series (if any) in $L^2(G, \tau_{-\nu})$ are equivalent to a holomorphic discrete series. We reformulate this result in the following.

THEOREM 2.2 (Shimeno [Sh]). — *The relative discrete series representations appearing in $L^2(G, \tau_{-\nu})$ are all holomorphic discrete series of the form $N(\Lambda)$, with*

$$\Lambda \Big|_{(\eta^-)^{\mathbb{C}}} = \underline{m} - \frac{\nu}{2} \sum_{j=1}^{r} \gamma_j, \ \ \underline{m} \in D_\nu$$

and $\Lambda(\eta^+ \cap \mathfrak{k}_s) = 0$, $\Lambda(iZ) = -\nu$ in case $D = G/K$ is non-tube domain.

Proof. — The highest weights Λ of the relative discrete series are determined in [Sh], Theorem 5.10. They are determined by $\Lambda|_{\eta^+ \cap \mathfrak{k}_s} = 0$ and $\Lambda|_{\eta^-} = (\nu - \rho) \circ \gamma_e$ where $\nu \in D_l$ (see (6.13), loc. cit.), ρ is the half trace for the root system of $\mathfrak{a}$ in $\mathfrak{g}$ and c is the Cayley transform. Each $\nu \in D_l$ can be written as (using also the identification in (1.8), (1.9), loc. cit.)

$$\nu = \rho - \nu + \sum_{j=1}^{r} m_j \beta_j$$

with $0 \leq m_1 \leq \cdots \leq m_r$ are integers Since $c\gamma_j = \beta_j$ our result follows. □

Remark. — The reason why we define Λ on iZ is that when D is non-tube type then $Z \notin \eta^-$, the restriction of Λ on η^- is not sufficient to characterize Λ. However all the highest weights involved in this paper vanish on $\eta^+ \cap \mathfrak{k}_s$ and it is easy to calculate the action of the center. Thus we will only consider the restriction of highest weights on η^-.

Similarly to the trivial line bundle case one can develop the theory of τ_ν-spherical functions and generalized Harish-Chandra c-functions [Sh]. Roughly speaking, the relative discrete series appear at the most singular poles of the c-functions. The above result is proved in [Sh] by using the results Casselman and Milićič [CM] and by finding those holomorphic discrete series which contains the $\tau_{-\nu}$-type.

The aim of the present paper is to give an explicit embedding of the holomorphic discrete series. We will use a completely different approach based on the idea of diagonal operator (see Lemma 3.2 below). In particular we will not use the general theory of $\tau_{-\nu}$-spherical functions.

For $\nu \in \mathbb{R}$ let $\mathcal{A}_K^\nu(D)$ the space of K-finite functions in $L^2(D, d\mu_\alpha)$ with $\nu = \alpha + p$ as (2.1). We consider (2.2) and its induced action on $\mathcal{A}_K^\nu(D)$

$$u_\nu(X)f(z) = -(Xf)(z) - \frac{\nu}{p} j_X(z) f(z) \tag{2.5}$$

where

$$\begin{aligned}(Xf)(z) &= \left(\frac{d}{dt}\right)_0 f(\exp(tX)\cdot z)\\ &= \left\langle \partial f(z), \left(\frac{\partial(\exp(tX)\cdot z)}{\partial t}\right)_0 \right\rangle + \left\langle \bar\partial f(z), \left(\frac{\partial\overline{(\exp(tX)\cdot z)}}{\partial t}\right)_0 \right\rangle\end{aligned}$$

and

$$j_X(z) = \left(\frac{d}{dt}\right)_0 J_{\exp tX}(z)$$

and $J_g(z)$ is the complex Jacobian $\det dg(z)$ of $g \in G$ (as a holomorphic mapping on D).

We also consider the actions π_ν of $\mathfrak{g}^{\mathbb{C}}$ on the subspace of *holomorphic polynomials* on D, defined by the same formula as

$$\begin{aligned}\pi_\nu(X)f(z) &= -(Xf)(z) - \frac{\nu}{p} j_X(z) f(z)\\ &= \left(\frac{d}{dt}\right)_0 f(\exp tX\cdot z) - \frac{\nu}{p} j_X(z) f(z)\\ &= \left\langle \partial f(z), \left(\frac{\partial(\exp tX\cdot z)}{\partial t}\right)_0 \right\rangle - \frac{\nu}{p} j_X(z) f(z).\end{aligned} \tag{2.6}$$

The next result is proved by Faraut-Koranyi [FK], see also [W]; the first part is a known theorem of W. Schmid; see also [J], [Up].

THEOREM 2.3 (Faraut-Koranyi [FK], Theorem 5.4). — *The space $\mathcal{P}$ of holomorphic polynomials on $\mathfrak{p}^+$ decomposes into irreducible subspaces under* $\mathrm{Ad}(K)$, *with multiplicity one as:*

$$\mathcal{P} \cong \sum_{\underline{m} \geq 0} P_{\underline{m}}.$$

Each $P_{\underline{m}}$ is of lowest weight $-\underline{m} = -(m_1\gamma_1 + \cdots + m_r\gamma_r)$ with $0 \leq m_1 \leq \ldots \leq m_r$. With the action (2.5) for $\nu > \frac{a}{2}(r-1)$ the space

$$\mathcal{P}_\nu(D) = \mathcal{P}(D) \quad \text{under } \pi_\nu$$

forms a highest weight module of $\mathfrak{g}^{\mathbb{C}}$ with highest weight $-\frac{\nu}{2}\sum_{j=1}^{r}\gamma_j$ and highest weight vector the constant holomorphic polynomial 1. *For $\nu \leq 0$ an integer, the space*

$$\mathcal{P}_\nu(D) = \sum_{m_r \leq \nu} P_{\underline{m}} \tag{2.7}$$

forms a (finite-dimensional) highest weight module of $\mathfrak{g}^{\mathbb{C}}$ with highest weight $-\frac{\nu}{2}\sum_{j=1}^{r}\gamma_r$ and highest weight vector 1.

All the results above except (2.7) are stated explicitly in [FK], Theorem 5.4. It follows now from [FK], Theorem 5.4 that when $\nu \leq 0$ is an integer M_0 is a (finite-dimensional) highest weight module of $\mathfrak{g}^{\mathbb{C}}$, with highest weight $-\frac{\nu}{2}\sum_{j=1}^{r}\gamma_r$ and highest weight vector 1. However their M_0 is just our P_ν in (2.7) above, by [FK], Theorem 3.8.

We denote by $\overline{\pi_\nu}$ the representation on the space of anti-holomorphic polynomials obtained by taking the complex conjugate of the above formula.

3. Intertwining operator into relative discrete series.

In this section we will consider the embedding of the holomorphic discrete series into the relative discrete series.

By Theorem 2.3, the lowest weights appearing in $\mathcal{P}_{-l}(D)$ are

$$-\sum_{j=1}^{r} m_j\gamma_j + \frac{l}{2}\sum_{j=1}^{r}\gamma_j,\ 0 \le m_1 \le \cdots \le m_r, m_r \le l.$$

Thus we have the following.

LEMMA 3.1. — *Under the $\overline{\pi_{-l}}$-action restricted to K, $\overline{\mathcal{P}_{-l}(D)}$ decomposes as*

$$\overline{\mathcal{P}_{-l}(D)} = \sum_{0\le m_1\le\cdots\le m_r\le l} \overline{\mathcal{P}^{(l,\underline{m})}}$$

where $\overline{\mathcal{P}^{(l,\underline{m})}}$ is of lowest weight-$\underline{m} = -\sum_{j=1}^{r} m_j\gamma_j + \frac{l}{2}\sum_{j=1}^{r}\gamma_j$ with $0 \le m_1 \le m_2 \le \cdots \le m_r \le l$.

The tensor product $\mathcal{P}_{\nu-l}(D)\otimes\overline{\mathcal{P}_{-l}(D)}$ can be realized as the space of polynomials $F(z,w) = \sum_j f_j(z)\overline{g_j(w)}$ where f_j are holomorphic polynomials of z, and $\overline{g_j}$ in $\overline{\mathcal{P}_{-l}(D)}$ are anti-holomorphic polynomial in w. We define the operator R by

$$R : F(z,w) \mapsto F(z,z)h(z)^{-l}.$$

The key fact about R is the following (see also [PZ] where the tensor product of two discrete series are considered).

LEMMA 3.2. — *The operator R is a G-intertwining operator from $\mathcal{P}_{\nu-l}(D)\otimes\overline{\mathcal{P}_{-l}(D)}$ to $L^2(D, d\mu_\alpha)$.*

Proof. — First we show that R indeed maps $\mathcal{P}_{\nu-l}(D)\otimes\overline{\mathcal{P}_{-l}(D)}$ into $L^2(D, d\mu_\alpha)$. For any $f \in \mathcal{P}_{\nu-l}(D)$ and $\overline{g} \in \overline{\mathcal{P}_{-l}(D)}$ we have

$$R(f\otimes\overline{g})(z) = f(z)\overline{g(z)}h^{-l}.$$

Now $f(z)$ and $g(z)$ are polynomials and thus there exist a constant C such that $|f(z)||g(z)|^2 \le C$ for all $z \in D$. So,

$$\begin{aligned}\int_D |R(f\otimes\overline{g})(z)|^2 d\mu_\alpha(z) &= \int_D |f(z)\overline{g(z)}|^2 h^{-2l}h(z)^\alpha dm(z)\\ &\le C\int_D h^{\alpha-2l}(z)dm(z)\\ &< \infty.\end{aligned}$$

The last integral is finite because $\nu - p - 2l = \alpha - 2l > -1$ (see [FK]).

Next we prove the intertwining relation. For any $x \in G$,

$$\begin{aligned} U^{\nu}(x)R(f\otimes \overline{g})(z) &= f(x^{-1}z)\overline{g(x^{-1}z)}h^{-l}(x^{-1}z)J_{x^{-1}}(z)^{\frac{\nu}{p}} \\ &= f(x^{-1}z)\overline{g(x^{-1}z)}h^{-l}(z)J_{x^{-1}}(z)^{\frac{-l}{p}}\overline{J_{x^{-1}}(z)^{\frac{-l}{p}}}J_{x^{-1}}(z)^{\frac{\nu}{p}} \\ &= R\pi_{\nu-l}(x)\otimes\overline{\pi_{-l}}(x)(f\otimes g)(z). \end{aligned}$$

This finishes the proof. □

The next results proves that the tensor product $\mathcal{P}_{\nu-l}(D)\otimes\overline{\mathcal{P}_{-l}(D)}$ is decomposed into highest weight modules, and each is of multiplicity one. It is, roughly speaking, similar to the fact that in the tensor product decomposition of two modules with highest weights λ_1 and λ_2, the modules with larger highest weight $\lambda_1+\lambda_2$ occurs with multiplicity one.

THEOREM 3.3. — *As $(\mathfrak{g}^{\mathbb{C}},K)$ module we have*

$$\mathcal{P}_{\nu-l}(D)\otimes\overline{\mathcal{P}_{-l}(D)} = \sum_{\underline{m},m_r\le l} A^{\nu,\underline{m}}(D)$$

where $A^{\nu,\underline{m}}(D)$ is a highest weight module of $\mathfrak{g}^{\mathbb{C}}$ with highest weight $\sum_{j=1}^{r} m_j\gamma_j - \frac{\nu}{2}\sum_{j=1}^{r}\gamma_j$ with $0\le m_1\le\cdots\le m_r\le l$.

We note that $A^{\nu,\underline{m}}$ is a holomorphic discrete series of $\mathfrak{g}^{\mathbb{C}}$ since $\nu-2l>p-1$.

The theorem will be proved by carefully checking the multiplicity of certain weights appearing in the tensor product. First we consider the highest weights appearing in $\overline{\mathcal{P}_{-l}(D)}$. We define an ordering of weights of the form $\alpha Z^*+m_1\gamma_1+m_2\gamma_2+\cdots+m_r\gamma_r$ by saying that

$$\alpha Z^*+m_1\gamma_1+m_2\gamma_2+\cdots+m_r\gamma_r \ge \alpha Z^*+m_1'\gamma_1+m_2'\gamma_2+\cdots+m_r'\gamma_r$$

if $\sum_{j=1}^{r} m_j > \sum_{j=1}^{r} m_j'$ or if $\sum_{j=1}^{r} m_j = \sum_{j=1}^{r} m_j'$ and $m_j=m_j'$ for $j=1,\cdots,k-1$ and $m_k>m_k'$ for some $1\le k\le r$.

Proof of Theorem 3.3. — The idea of the proof is similar to that of [Re], though only tensor products of holomorphic discrete series are studied there. The $\mathfrak{g}^{\mathbb{C}}$-module $\mathcal{P}_{\nu-l}(D)$ has an K-irreducible decomposition

$$\mathcal{P}_{\nu-l}(D) = \sum_{\underline{m}} P^{\underline{m}}_{\nu-l}(D)$$

where each $P^{\underline{m}}_{\nu-l}(D)$ is a K-module of K-lowest weight $-\underline{m}-\frac{\nu-l}{2}\sum_{j=1}^{r}\gamma_j$, and $\mathcal{P}_{\nu-l}(D)$ is a highest weight module of $(\mathfrak{g}^{\mathbb{C}},K)$ with highest weight vector 1. Since $\mathcal{P}_{-l}(D)$ is a highest weight module, $\overline{\mathcal{P}_{-l}(D)}$ is a lowest weight module. Moreover being finite dimensional it is also a highest weight module with highest weight $\sum_{j=1}^{n} l\gamma_j - \frac{l}{2}\sum_{j=1}^{r}\gamma_j$. Let v_l be the corresponding highest weight vector.

Step 1. The vector $1\otimes v_l$ is annihilated by $\mathfrak{p}^+ + \mathfrak{n}_K^+$ and thus generates a highest weight module of $\mathfrak{g}^{\mathbb{C}}$, say

$$A^{\nu,(l,\cdots .l)} = \pi_{\nu-l}\otimes\overline{\pi_{-l}}(U(\mathfrak{g}^{\mathbb{C}}))(1\otimes v_l) \cong S(\mathfrak{p}^-)\otimes E(l,\cdots .l)$$

where $E(l,\cdots,l)$ denote a module of K with highest weight $\sum_{j=1}^{r} l\gamma_j - \frac{\nu}{2}\sum_{j=1}^{r}\gamma_j$.

Step 2. Continue this process and go down to smaller highest weights. Let $0\le m_1\le\cdots\le m_r\le l$. We claim that there exists exactly one highest weight module $A^{\nu,\underline{m}}$ of $\mathfrak{g}^{\mathbb{C}}$ with highest weight $\sum_{j=1}^{r} m_j\gamma_j - \frac{\nu}{2}\sum_{j=1}^{r}\gamma_j$.

We use induction. This is true for $\sum l\gamma_j - \frac{\nu}{2}\sum\gamma_j$ from Step 1. Suppose it is true for all highest weight $\sum_{j=1}^{r} n_j\gamma_j - \frac{\nu}{2}\sum_{j=1}^{r}\gamma_j$ that are bigger than $\sum_{j=1}^{r} m_j\gamma_j - \frac{\nu}{2}\sum_{j=1}^{r}\gamma_j$.

We observe that the weight $\sum_{j=1}^{r} m_j\gamma_j - \frac{\nu}{2}\sum_{j=1}^{r}\gamma_j$ will not appear in submodule with highest weight $\sum_{j=1}^{r} n_j\gamma_j - \frac{\nu}{2}\sum_{j=1}^{r}\gamma_j$ if $\sum_{j=1}^{r} n_j\gamma_j < \sum_{j=1}^{r} m_j\gamma_j$. In fact the $\mathfrak{g}^{\mathbb{C}}$-module with highest weight module $\sum_{j=1}^{r} n_j\gamma_j - \frac{\nu}{2}\sum_{j=1}^{r}\gamma_j$ is isomorphic to $S(\mathfrak{p}^-)\otimes E\Big(\sum_{j=1}^{r} n_j\gamma_j\Big)$ where $E\Big(\sum_{j=1}^{r} n_j\gamma_j\Big)$ is a module of $\mathfrak{k}^{\mathbb{C}}$ with highest weight $\sum_{j=1}^{r} n_j\gamma_j - \frac{\nu}{2}\sum_{j=1}^{r}\gamma_j$; see (2.3). Thus, by theorem 2.3, all the $\mathfrak{k}^{\mathbb{C}}$-highest weights appearing are of the form $\sum_{j=1}^{r} n_j\gamma_j + \sum_{j=1}^{r} n'_j\gamma_j - \frac{\nu}{2}\sum_{j=1}^{r}\gamma_j$ with $n'_1,\ldots,n'_r$ being nonpositive integers, all of which are smaller than $\sum_{j=1}^{r} n_j\gamma_j - \frac{\nu}{2}\sum_{j=1}^{r}\gamma_j$, and thus smaller than $\sum_{j=1}^{r} m_j\gamma_j - \frac{\nu}{2}\sum_{j=1}^{r}\gamma_j$. In other words, the weight $\sum_{j=1}^{r} m_j\gamma_j - \frac{\nu}{2}\sum_{j=1}^{r}\gamma_j$ will not appear in the submodule with highest weight $\sum_{j=1}^{r} n_j\gamma_j - \frac{\nu}{2}\sum_{j=1}^{r}\gamma_j$.

Thus the multiplicity of $\sum_{j=1}^{r} m_j\gamma_j - \frac{\nu}{2}\sum_{j=1}^{r}\gamma_j$ appearing as a highest weight in the tensor product is (see [Re], Theorem 1)

$$n(\underline{m}) - \sum_{\underline{\lambda}>\underline{m}} n_{\underline{\lambda}}(\underline{m})$$

where $n(\underline{m})$ is the multiplicity of $lX^* + \sum_{j=1}^{r} m_j\gamma_j - \frac{\nu}{2}\sum_{j=1}^{r}\gamma_j$ appearing in the tensor product, and $n_{\underline{\lambda}}(\underline{m})$ is the multiplicity of $\underline{m}$ appearing in $\underline{\lambda}$.

Considering the decomposition under the action of K of the finite-dimensional $\mathfrak{g}^{\mathbb{C}}$-module $\overline{P_{-l}(D)}$, one sees that $n(\underline{m}) = \sum_{\underline{\lambda}\geq\underline{m}} n_{\underline{\lambda}}(\underline{m})$. From this it follows that

$$n(\underline{m}) - \sum_{\underline{\lambda}>\underline{m}} n_{\underline{\lambda}}(\underline{m}) = 1.$$

This concludes the proof of the claim.

Step 3. The above process stops at the lowest weight $-\frac{l}{2}\sum_{j=1}^{r}\gamma_r$ (with lowest weight vector 1) of $\overline{\mathcal{P}_{-l}(D)}$, and we find modules with a highest weight $l^*X + \sum_{j=1}^{r} m_j\gamma_j$ with $0 \leq m_1 \leq \cdots \leq m_r \leq l$ such that $1\otimes 1$ is in their linear combination. However it follows from [OZ1] that $1\otimes\bar{1}$ is a cyclic vector of $\mathcal{P}_{\nu-l}(D)\otimes\overline{\mathcal{P}_{-l}(D)}$ we have thus

$$\mathcal{P}_{\nu-l}(D)\otimes\overline{\mathcal{P}_{-l}(D)} = \sum_{\underline{m}} S(\mathfrak{p}^-)\otimes E(\underline{m}). \qquad \square$$

Remark. — The above theorem is very similar to the following situation for compact groups. Let V_1 and V_2 be two irreducible representations of a compact group with highest weight λ_1 and λ_2. Then an irreducible submodule of the tensor product $V_1\otimes V_2$ has highest weight $\lambda_1+\lambda_2'$ where λ_2' is a weight of V_2 [Ze]. In particular if all the weights in V_2 are of multiplicity one then $V_1\otimes V_2$ is decomposed into irreducibles with multiplicity at most one.

In view of Proposition 2.1 and Lemma 3.2 we have thus given explicit realization of all the relative discrete series,

$$N(\underline{m}) \cong A^{\nu,\underline{m}} \subset \mathcal{P}_{\nu-l}(D)\otimes\overline{\mathcal{P}_{-l}(D)} \xrightarrow{R} L^2(D,d\mu_\alpha) \xrightarrow{I^{-1}} L^2(G/K,\tau_\nu).$$

Moreover Theorem 2.2 above concludes that all the relative discrete series are realized in this way. We summarize this in the following.

THEOREM 3.4. — *The operator $\mathcal{R}$ maps $A^{\nu,\underline{m}}(D)$ into a relative discrete series of $L^2(D, d\mu_\alpha)$. Moreover all the relative discrete series are realized in this way.*

Note that the explicit decomposition of the tensor product will gives us the embedding of the space $A^{\nu,\underline{m}}$ into $L^2(D, d\mu_\alpha)$. It remains an open question to find the unitary intertwining operator as in [Zh2]. Nevertheless from Theorem 3.3 we can read off the singularity of the reproducing kernel of a relative discrete series in $L^2(D, d\mu_\alpha)$. This might be of help in studying the L^p-properties of the spectral projections onto the discrete series.

We observe that Lemma 3.2 can be generalized as follows. Let k be any integer such that $0 \leq k \leq l$. We consider the operator $R = R(k)$ from $\mathcal{P}_{\nu-k}(D) \otimes \overline{\mathcal{P}_{-k}(D)}$ into the functions on D, given by

$$R : F(z, w) \mapsto F(z, z)h(z)^{-k}.$$

Lemma 3.2 can easily be generalized as follows.

LEMMA 3.2′. — *The operator R is a G-intertwining operator from $\mathcal{P}_{\nu-k}(D) \otimes \overline{\mathcal{P}_{-k}(D)}$ to $L^2(D, d\mu_\alpha)$.*

Now when D is a tube domain then $\overline{\mathcal{P}_{-k}(D)}$ is a highest weight module of $(\mathfrak{g}^{\mathbb{C}}, K)$ with highest weight vector $\overline{\Delta^k(z)}$ where $\Delta(z)$ is the determinant function. Thus $1 \otimes \overline{\Delta^k(z)}$ is a highest weight vector for the tensor product $\mathcal{P}_{\nu-k}(D) \otimes \overline{\mathcal{P}_{-k}(D)}$. Lemma 3.2′ immediately implies the following, which may also be proved using Capelli identity [OZ2].

THEOREM 3.5. — *Suppose D is a tube domain and Let l be as defined in (2.4). Then for any k, $0 \leq k \leq l$ we have that $h(z)^{-k}\overline{\Delta^k(z)}$ is the highest weight vector for a relative discrete series of $L^2(D, d\mu_\alpha)$.*

Finally we remark that if we take $G^{\mathbb{C}}$ to be the simply connected Lie group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and G the connected subgroup with Lie algebra $\mathfrak{g}$, then, by appropriate change in the definition of the Harish-Chandra decomposition (1.1) of $G^{\mathbb{C}}$, we may extend the results in this paper to all $\nu \in \mathbb{R}$.

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