ANNALES DE L'INSTITUT FOURIER

KARSTEN BUECKER

Congruences between Siegel modular forms on the level of group cohomology

Annales de l'institut Fourier, tome 46, nº 4 (1996), p. 877-897

<http://www.numdam.org/item?id=AIF_1996__46_4_877_0>

© Annales de l'institut Fourier, 1996, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (http://annalif.ujf-grenoble.fr/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Ann. Inst. Fourier, Grenoble
46, 4 (1996), 877–897

CONGRUENCES BETWEEN SIEGEL MODULAR FORMS ON THE LEVEL OF GROUP COHOMOLOGY

by Karsten BUECKER

Introduction.

In this paper we study congruences occurring between Siegel modular forms of degree two for non-parallel changes in weight. Such congruences are central to the questions of attaching Galois representations to modular forms and computing special values of L-functions, and they have been studied in similar contexts by Hida (see e.g. [Hi1] to [Hi3]) and others. Our work extends Hida's cohomological methods and complements results of R. Taylor and Tilouine & Urban. Our results should lead to corresponding applications, in particular an analogue of Hida's theory of Λ-adic forms for the symplectic group.

General vector-valued Siegel modular forms of degree g are holomorphic functions from the Siegel upper half-space (a subset of complex $g \times g$ matrices) to a finite-dimensional complex representation of $GL(g)$. They transform under symplectic transformations with an automorphy factor which is described by an integral g-component "weight" vector $\vec{k} = (k_1, \ldots, k_g)$ (see §1). $g = 1$ gives classical elliptic modular forms; our objects of study are forms with degree $g = 2$.

Now it is known that the space $S_{k_1,k_2}(\Gamma)$ of Siegel cusp forms embeds in a cohomology group $H^3(\Gamma, V_{k_1-3,k_2-3})$, where $V_{m,n}$ is the irreducible

Key words: Vector-valued Siegel modular forms – Congruences of Hecke eigenvalues – Group cohomology.
Math. classification: 11F46 – 11F33.

representation of the symplectic group $Sp_4(\mathbb{C})$ with "highest weight" (m,n), plus a suitable action of the centre. The embedding respects the action of the Hecke operators. We lose track of the full set of Fourier coefficients of a modular form, but in many applications, for example to L-functions, Hecke eigenvalues are all one needs. The approach of this paper is based on this natural embedding.

One motivation for the study of congruences in this context are the results of Ash and Stevens for the degree one case, which build on older work of Shimura and others. They describe how systems of mod p Hecke eigenvalues occurring on spaces of weight k and level N modular forms (N prime to p) can also occur in weight 2 and level Np, with a twist by a character. In [AS] it is also shown how to translate this into congruences between special values of L-functions associated to the corresponding modular forms.

The theory of Λ-adic families of modular forms was invented by Hida in the 1980s in the $GL(2)$ case. Such a family consists of a q-expansion with coefficients in the Iwasawa algebra $\Lambda \cong \mathbb{Z}_p[[X]]$, which specialises to *ordinary* modular forms of weight k, level Np^r and character χ at prime ideals of Λ of the form $(X - \chi(1+p)\,(1+p)^k + 1)$ (where χ is a character of $1 + p\mathbb{Z}_p$ of conductor p^r). Here "ordinary" refers to the subspace of forms on which the Hecke operator T_p acts invertibly. The point is that whilst Eisenstein series are "easy" to interpolate thanks to our knowledge of their Fourier coefficients, for cusp forms one needs to resort to more abstract methods. Hida defined a "universal Hecke algebra" as a limit of ordinary Hecke rings of increasing levels, and his Λ-adic forms are the dual of this Hecke ring. In order to recover spaces of modular forms of finite levels and arbitrary weights from this space, one needs three ingredients: the independence of weight of the universal space, a study of its Λ-module structure, and a control theorem allowing one to lower the levels.

One can then associate Galois representations into $GL_2(\Lambda)$ (or finite extensions of Λ) to such p-adic families of modular forms by patching together the representations coming from the specialisations of the family at different weights (see for example the exposition in Chapter 7 of [Hi3]). This has proved an extremely useful tool in associating Galois representations to modular forms, and verifying certain predictions of the Langlands program. For example, the techniques have been adapted by Wiles, in [W], to attach Galois representations to ordinary Hilbert modular forms and check their local behaviour at a decomposition group at p.

In the degree two Siegel case, we encounter several notions of ordinarity, because there are now two Hecke operators at p: $T_p = [\Gamma g_1 \Gamma]$ and $R_p = [\Gamma g_2 \Gamma]$, where $g_1 = \mathrm{diag}(p,p,1,1)$ and $g_2 = \mathrm{diag}(p,p^2,p,1)$. Let M_i be the centraliser of g_i in $GSp(4)$. Let B be the minimal parabolic subgroup consisting of 4×4 matrices which are upper triangular in block form, whose top left block is lower triangular and whose bottom right block is upper triangular. Its conjugacy class is uniquely determined. Then R_p and T_p correspond to two different maximal parabolic subgroups P_1, P_2, via $P_i = M_i B$. Depending on with respect to which Hecke operator we demand ordinarity, one obtains a Λ-adic family interpolating cohomology groups whose coefficient modules have weights $\vec{k}_0 + (a,b)$, where (a,b) run through the dominant (*i.e.* $b \geq a \geq 0$) weights of characters of M_i. Thus in [Tayl], R. Taylor worked with the operator T_p, making P_i the Siegel parabolic of matrices which are upper triangular in block form, and M_i its Levi subgroup of matrices which just have two 2×2 blocks on the diagonal. All characters of $M_1 \cap Sp_4(\mathbb{Q})$ are powers of the determinant on the top left block, so the weights (a,b) take values (λ,λ) with $\lambda \in \mathbb{N}$—the case we call parallel weight change. In that case, the necessary congruences could be obtained by multiplication by complex-valued Eisenstein series congruent to 1 modulo p, although one still needed a bound on the ordinary components to be able to recover congruent Hecke eigenforms. This bound was obtained by cohomological means.

In this paper, we consider forms that are ordinary with respect to R_p, and consequently obtain weight changes in the direction $(0,1)$. Since we are now dealing with vector-valued forms, we can no longer simply multiply by Eisenstein series. Instead we need to refine the cohomological methods of [Tayl].

Finally, in [TU], Tilouine and Urban impose p-ordinarity with respect to both T_p and R_p, so their theory corresponds to the Borel subgroup B. Consequently they obtain families interpolating all weights in a cone $b \geq a \geq 0$. They prove a control theorem under conditions on the order of the torsion subgroup in the cohomology of degrees one to four. Using recent results of Weissauer on the existence of Galois representations attached to Siegel modular forms, and of Louise Nyssen and R. Taylor on the theory of pseudorepresentations (to carry out the "patching"), Tilouine and Urban can construct a Galois representation into $GSp_4(\Lambda)$ lifting a given representation into $GSp_4(\mathbb{Z}_p)$.

The results of this paper give the independence of weight of the analogous universal Hecke algebra which is ordinary with respect to R_p. However, the control theorem is not unconditional because of an error term arising from H^2 (we can show that the ordinary part of the H^1 error, with torsion coefficients, is zero, and we hope to publish a more concise proof than we have at present in a future paper). The remainder of the theory would be a fairly straightforward algebraic consequence of the control theorem. Because of its incompleteness, we have not included this work here.

Our method is basically an abstraction of earlier works. For example, in [AS] Ash & Stevens use multiplication by a polynomial θ between various symmetric powers of $\mathbb{F}_l^2$ to carry over systems of mod l Hecke eigenvalues, in analogy with the classical Hasse invariant. Similarly Hida uses simple maps between different symmetric powers to prove his congruences in [Hi1] (Theorem 4.4).

We construct a map j between (the mod p^r reductions of) lattices in $V_{m,n}$ and $V_{m,n+p^{r-1}(p-1)}$ and by applying functoriality, we show that j induces a Hecke equivariant map on cohomology groups, which is in fact an isomorphism on the ordinary (with respect to R_p) components (Theorem 4.2). Thus we obtain congruences modulo p^r between systems of eigenvalues occurring in different weights.

In Section 1 we give the definitions of Siegel modular forms in our context. In Section 2 we set up the Hecke algebra most suitable to our needs: big enough to contain most interesting operators but small enough to make our method work. In Section 3 we explain the representation theory which lies at the core of our argument. In Section 4 we use the setup of §3 and a formal cohomological lemma to compare the ordinary components of the cohomology groups in question. The idea here is to use an intermediate coefficient module $V'_{m,n}$ which embeds into both $V_{m,n}$ and $V_{m,n+1}$. Finally, in Section 5, we mention the corresponding (unpublished) results obtained in [Tayl]. We also explain an interesting computation on the eigenvalues of R_p occurring on the L-packet at p.

I would like to express my deep gratitude to Richard Taylor for the time and effort he has expended on introducing me to this subject, and for his manifold assistance. This paper is based on parts of my thesis ([Buck]) which was completed under his direction. I would also like to thank Kevin Buzzard for countless enlightening discussions, and Fred Diamond for many useful suggestions on later drafts of this paper.

I have enjoyed the financial support of SERC, Trinity College Cambridge, and an EC Human Capital and Mobility grant. I also thank the Institut Fourier in Grenoble for their hospitality during the writing of this paper.

Notation. — We employ little more than what is standard usage. $M_n(R)$ denotes $n \times n$ matrices over the commutative ring R, I_n the $n \times n$ identity matrix, and for a matrix X, X_{ij} refers to the (i,j) entry of X.

1. Preliminaries.

We begin with a review of the theory of Siegel modular forms. Our definitions follow those in Shimura [Shim]. We repeat these here since the vector-valued version of modular forms is perhaps less well-known.

For any commutative ring R and integer $g \in \mathbb{N}$, we consider the group of symplectic matrices

$$GSp_{2g}(R) = \left\{ M \in GL_{2g}(R) : \ M \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} M^T = \nu(M) \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}, \quad \nu(M) \in R^* \right\}$$

and define $Sp_{2g}(R)$ to be the kernel of the multiplier map ν. Also let $GSp_{2g}(\mathbb{R})^+$ be the matrices in $GSp_{2g}(\mathbb{R})$ with positive multiplier ν. A matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ (where A, B, C, D are $g \times g$ blocks) lies in Sp_g if and only if

(1) $\quad A^T C$ is symmetric, $B^T D$ is symmetric, and $A^T D - C^T B = I_g$

(or equivalently DC^T is symmetric, AB^T is symmetric, and $DA^T - CB^T = I_g$).

We have the Siegel upper half space of degree g

$$\mathcal{Z}_g = \{ Z \in M_g(\mathbb{C}) : Z^T = Z, \operatorname{Im}(Z) > 0 \}$$

where for a hermitian matrix S we write $S \geq 0$ or $S > 0$ according to whether S is positive semi-definite or positive definite, respectively. $\mathcal{Z}_g$ is a convex, simply connected, symmetric domain. The group $GSp_{2g}(\mathbb{R})^+$ acts transitively on $\mathcal{Z}_g$ by the rule

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \ : \ Z \mapsto (AZ + B)(CZ + D)^{-1}.$$

Now let ρ be an arbitrary finite-dimensional representation of $GL_g(\mathbb{C})$ on a complex vector space W, and let f be a function from $\mathcal{Z}_g$ to W. For $\gamma \in GSp_4(\mathbb{R})^+$ we define an action on f:

$$(f|_\rho \gamma)(Z) = \rho(\, J(\gamma, Z)\,)^{-1} f(\gamma Z) \qquad (Z \in \mathcal{Z}_g)$$

where the automorphic factor J is defined as usual by $J((\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}), Z) = CZ + D$ (again A, B, C, D are $g \times g$ blocks).

For a congruence subgroup $\Gamma \subset GSp_{2g}(\mathbb{Q})^+$, we write $M_\rho(\Gamma)$ for the vector space of holomorphic functions $f : \mathcal{Z}_g \to W$ which satisfy $f|_\rho \gamma = f$ for all $\gamma \in \Gamma$ and which are finite at cusps (by the Koecher principle, this finiteness condition is automatic for $g > 1$). If in addition

$$\lim_{\lambda \to \infty} (f|\gamma)\left(\begin{pmatrix} z & 0 \\ 0 & i\lambda \end{pmatrix}\right) = 0 \qquad \forall \gamma \in GSp_{2g}(\mathbb{Q})^+,\ z \in \mathcal{Z}_{g-1}$$

we call f a cusp form, $f \in S_\rho(\Gamma)$.

Notice that the representation $W = \mathbb{C}$, $\rho(X) = (\det X)^k \in GL_1(\mathbb{C})$ ($k \in \mathbb{Z}$) returns us to the classical situation of scalar-valued Siegel modular forms.

In the case of $g = 2$, which is the setting of our results, the irreducible representations of $GL_2(\mathbb{C})$ are given by an irreducible representation of $SL_2(\mathbb{C})$ twisted by some power of the determinant. If $W \cong \mathrm{Sym}^{n-m}\,\mathbb{C}^2$ with the centre λI_2 acting as λ^{n+m} ($n \geq m \geq 0$), we speak of $M_\rho(\Gamma)$ as "modular forms of weight (m, n)" and write $M_{(m,n)}(\Gamma)$. In particular, parallel weights (k, k) correspond to scalar-valued weight $2k$ Siegel modular forms.

There is also an alternative definition of Siegel modular forms in terms of automorphic representations of $GSp_{2g}(\mathbb{A})$, which is equivalent to that given here. We will say a little about this in §5.

Our method relies on the following result. We do not know to whom it is originally due, but a convenient reference is Falting's paper [Falt].

Recall from the representation theory of Lie groups (see eg. [Hmph]) that the weights of a representation are the characters of a maximal torus occurring in the representation. Further, the roots of a Lie group are the nonzero weights occurring in the adjoint representation of the group on its Lie algebra. If we fix $T \subset Sp_4$ to be the diagonal matrices, we have $\mathrm{Hom}_{\mathrm{alg}}(T(\mathbb{C}), \mathbb{C}^*) \cong \mathbb{Z}^2$ via $\left(\mathrm{diag}(\alpha, \beta, \alpha^{-1}, \beta^{-1}) \longmapsto \alpha^m \beta^n\right) \leftrightarrow (m, n)$. The Lie algebra $\mathfrak{sp}_4$ is ten-dimensional and the roots of $Sp_4(\mathbb{C})$ are given by $(\pm 1, \pm 1); (\pm 2, 0); (0, \pm 2) \in \mathbb{Z}^2$. Choose the Weyl chamber $n \geq m \geq 0$, corresponding to the Borel subgroup of $Sp_4(\mathbb{C})$ consisting of elements of the form

$$\begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}.$$

Then the irreducible representations of $Sp_4(\mathbb{C})$ are parametrised by their highest weight in this Weyl chamber; let us denote the unique irreducible representation with highest weight (m, n) by $V_{m,n}$. The other weights of $V_{m,n}$ which occur are precisely the ones congruent to the highest weight modulo the root lattice, and which lie in the convex hull of the images of (m, n) under the Weyl group. (In this case one obtains an octagon centred on the origin, whose top right face joins (m, n) and (n, m).)

Now let $\Gamma \subset Sp_4(\mathbb{R})$ be a discrete subgroup. An irreducible $Sp_4(\mathbb{C})$-module $V_{m,n}$ as above is automatically a Γ-module, and from a standard construction we can define cohomology groups $H^*(\Gamma, V_{m,n})$. Then it follows from Theorem 10 in [Falt] (see also [Tayl], §2.3) that there is a natural embedding

$$S_{m,n}(\Gamma) \hookrightarrow H^3(\Gamma, V_{m-3,n-3}). \tag{2}$$

Here "natural" signifies that the embedding respects the action of the *Hecke operators* (see below) on each side.

This is, of course, an extension of the well-known Eichler-Shimura isomorphism for $SL_2(\mathbb{C})$. Unfortunately we do not have an explicit identification of the image of the embedding (2) as in the parabolic cohomolgy of H^1.

2. Hecke operators.

We will be using a Hecke algebra defined as follows. Let p be a prime, r a positive integer and N an integer coprime to p, and put:

– for $q \nmid Np$, let $U_q = GSp_4(\mathbb{Z}_q)$ and $D_q = M_4(\mathbb{Z}_q) \cap GSp_4(\mathbb{Q}_q)$;

– for $q \mid N$, choose $U_q \subset GSp_4(\mathbb{Z}_q)$ to be any subgroup such that $GSp_4(\mathbb{Z}_q) \ni \gamma \equiv I_4 \pmod{N} \implies \gamma \in U_q$, $\nu(U_q) = \mathbb{Z}_q^*$, and also $\mathrm{diag}(p^r, p^{2r}, p^r, 1) \in U_q$, (*i.e.* basically a congruence subgroup of level N); let $D_q = U_q$;

– for $q = p$,

$$U_p = \left\{ g \in GSp_4(\mathbb{Z}_p) : g \equiv \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \ (p^r),\ p^{2r} | g_{42} \right\}$$

(recall that g_{42} denotes the $(4,2)$ matrix entry of g), and D_p is the set of all matrices of the form $\alpha.u$ with $u \in U_p$ and α lying in the set

$$X = \left\{ \begin{pmatrix} a_1 & 0 & 0 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ 0 & 0 & d_1 & d_2 \\ 0 & 0 & 0 & d_4 \end{pmatrix} \in M_4(\mathbb{Z}_p) \cap GSp_4(\mathbb{Q}_p) \ : \right.$$
$$\left. \operatorname{ord}_p(a_4) = 2\operatorname{ord}_p(a_1) = 2\operatorname{ord}_p(d_1)\,; \quad d_4 \in \mathbb{Z}_p^* \right\}.$$

The idea here is that our only Hecke operator at p should be

$$[U_p \operatorname{diag}(p^r, p^{2r}, p^r, 1)\, U_p].$$

Let $U = \prod U_l$ as an open compact subgroup of $GSp_4(\hat{\mathbb{Z}})$ and $D = \prod_q D_q \subset GSp_4(\mathbb{A}_f)$ ($\mathbb{A}_f =$ finite adeles). Further let $\Gamma = \Gamma_U = U \cap Sp_4(\mathbb{Z})$ and $\Delta = D \cap GSp_4(\mathbb{Q})$. One can check that D (and hence Δ) is in fact a semigroup.

Then the Hecke algebra $\mathcal{H}(\Gamma\backslash\Delta/\Gamma)$ (resp. $\mathcal{H}(U\backslash D/U)$) is the free $\mathbb{Z}$-module generated by double cosets $\Gamma g \Gamma$ with $g \in \Delta$ (resp., $U\alpha U$ with $\alpha \in D$). One can define a multiplication law on such double cosets in a standard way (see eg. §2.1.7 in [Pan]), and we extend it to the whole Hecke algebra by linearity.

LEMMA 2.1. — *We have a canonical isomorphism of rings*

$$\mathcal{H}(\Gamma\backslash\Delta/\Gamma) \cong \mathcal{H}(U\backslash D/U)$$

given by the map

$$[\Gamma\alpha\Gamma] \mapsto [U\alpha U]$$

(where $\alpha \in GSp_4(\mathbb{Q})$ is embedded diagonally into $GSp_4(\mathbb{A}_f)$).

Proof. — We appeal to the criterion of Lemma 1.3.8 in [KPS], viz. it suffices to prove

(*a*) $D = \Delta.U$

(*b*) $U\alpha U = U\alpha\Gamma$ for all $\alpha \in \Delta$

(*c*) $U\alpha \cap \Delta = \Gamma\alpha$ for all $\alpha \in \Delta$.

Condition (*a*) is just the Strong Approximation Theorem, which applies because $\nu(U) = \hat{\mathbb{Z}}^*$.

Condition (*c*) is easily seen as $\Gamma = U \cap Sp_4(\mathbb{Z}) = U \cap GSp_4^+(\mathbb{Q})$.

To prove condition (b), we need to show that $U \subset \alpha^{-1}U\alpha\Gamma$, or equivalently, that $U \subset (U \cap \alpha^{-1}U\alpha).GSp_4^+(\mathbb{Q})$. This follows from the Strong Approximation Theorem, *provided* that $\nu(U \cap \alpha^{-1}U\alpha) = \hat{\mathbb{Z}}^*$.

We will now show that $\nu(U_q \cap \alpha^{-1}U_q\alpha) = \mathbb{Z}_q^*$ for all primes q.

For $q \mid N$, we know that $\alpha \in U_q$; but $\nu(U_q) = \mathbb{Z}_q^*$.

For $q \nmid Np$, we can decompose $\alpha = u_1 t u_2$ with $u_1, u_2 \in U_q$ and t diagonal. Then $\nu(U_q \cap \alpha^{-1}U_q\alpha) = \nu(U_q \cap t^{-1}U_q t)$, and given $\lambda \in \mathbb{Z}_q^*$, $\mathrm{diag}(\lambda, \lambda, 1, 1) \in U_q$ commutes with t and hence lies in the intersection. But $\nu(\mathrm{diag}(\lambda, \lambda, 1, 1)) = \lambda$.

For $q = p$, write $\alpha = x.u$ with $x \in X, u \in U_p$ as in the definition. Then

$$\begin{aligned}\nu(U_p \cap \alpha^{-1}U_p\alpha) &= \nu(U_p \cap u^{-1}x^{-1}U_p x u)\\ &= \nu(xuU_pu^{-1}x^{-1} \cap U_p)\\ &= \nu(xU_px^{-1} \cap U_p).\end{aligned}$$

Again, given $\lambda \in \mathbb{Z}_p^*$, consider $\mathrm{diag}(\lambda, \lambda, 1, 1) \in U_p$: then one can calculate directly that for any $x \in X$, we have that $x\,\mathrm{diag}(\lambda, \lambda, 1, 1)\,x^{-1}$ lies in U_p, hence in $xU_px^{-1} \cap U_p$, and has the desired multiplier value λ.

This verifies condition (b), and completes the proof. □

LEMMA 2.2. — *$\mathcal{H}(U\backslash D/U)$ is a commutative ring.*

Proof. — This is performed locally. The only problem is at p, where we will exhibit an anti-automorphism of D_p satisfying

$$\begin{aligned}&(a) \qquad \alpha^{\iota\iota} = \alpha, \quad (\alpha\beta)^\iota = \beta^\iota\alpha^\iota \quad \forall \alpha, \beta \in D_p,\\ &(b) \qquad \gamma \in U_p \Rightarrow \gamma^\iota \in U_p,\\ &(c) \qquad U_p\alpha U_p = U_p\alpha^\iota U_p \qquad \forall \alpha \in D_p.\end{aligned}$$

By Satz IV.1.10 of [Frei], this is enough to prove commutativity. The anti-automorphism ι is defined by

$$\begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{pmatrix}^\iota = \begin{pmatrix} a_1 & p^r a_3 & c_1 & c_3/p^r \\ a_2/p^r & a_4 & c_2/p^r & c_4/p^{2r} \\ b_1 & p^r b_3 & d_1 & d_3/p^r \\ p^r b_2 & p^{2r} b_4 & p^r d_2 & d_4 \end{pmatrix}$$

and one can check that the above conditions are met. □

If M is a Δ-module, then the cohomology groups $H^i(\Gamma, M)$ come equipped with an action of $\mathcal{H}(\Gamma\backslash\Delta/\Gamma)$: if $\Gamma g \Gamma = \coprod_u \alpha_u \Gamma$, define maps in degree zero

$$[\Gamma g \Gamma] : H^0(\Gamma, M) \longrightarrow H^0(\Gamma, M)$$
$$x \longmapsto \sum_u \alpha_u x$$

and extend functorially. (For details see e.g. Brown [B].) In general, suppose $[\phi] \in H^w(\Gamma, M)$ with ϕ a w-cocycle. For each $(\gamma_1, \ldots, \gamma_w) \in \Gamma^w$ we let $\alpha_{u_0} = \alpha_u$ and successively pick $\alpha_{u_{i+1}}$ such that $\alpha_{u_i}^{-1}\gamma_{i+1}\alpha_{u_{i+1}} \in \Gamma$, for $0 \leq i \leq w-1$. Then set

$$(\phi|[\Gamma g \Gamma])(\gamma_1, \ldots, \gamma_w) = \sum_u \alpha_u \, \phi(\alpha_u^{-1}\gamma_1\alpha_{u_1}, \ldots, \alpha_{u_{w-1}}^{-1}\gamma_w\alpha_{u_w}).$$

We denote g_{p^r} the special element $\mathrm{diag}(p^r, p^{2r}, p^r, 1)$ of Δ, and write R_{p^r} for the corresponding Hecke operator $[\Gamma g_{p^r} \Gamma]$.

3. Representation theory.

For this section, fix an irreducible representation $\rho : GSp_4(\mathbb{C}) \longrightarrow GL(V)$ with $V = V_{m,n}$ a finite-dimensional complex vector space having highest weight (m, n) (as an $Sp_4(\mathbb{C})$-module) and the action of the centre $\lambda I_4 \subset GSp_4(\mathbb{C})$ being λ^n. We have $n \geq m \geq 0$ and assume throughout that m is *even*.

V decomposes as a direct sum

$$V = \bigoplus_{x,y} V_{m,n}^{x,y},$$

so that an element $\mathrm{diag}(\alpha, \beta, \nu/\alpha, \nu/\beta)$ acts as $\nu^{n/2-x/2-y/2}\alpha^x\beta^y$ on a weight space $V_{m,n}^{x,y}$.

ρ gives rise to a representation ρ' of the Lie algebra $\mathfrak{sp}_4$ on $V_{m,n}$, determined by

$$\rho'(X) = \left[\frac{d}{dt}\rho(e^{tX})\right]\Big|_{t=0} \quad (X \in \mathfrak{sp}_4).$$

There is an exponential map taking $\mathfrak{sp}_4$ to $Sp_4(\mathbb{C})$, with

$$\rho(\exp X)(v) = v + \rho'(X)(v) + \frac{1}{2}\,(\rho'(X))^2(v) + \cdots \quad (X \in \mathfrak{sp}_4, v \in V).$$

$\mathfrak{sp}_4$ has a Chevalley basis consisting of the matrices

$$\mathrm{X}_{2,0} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{X}_{-2,0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Y_{-1,1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y_{1,1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y_{0,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Z_{1,-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad Z_{-1,-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad Z_{0,-2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

(we will use the letter $W_{\vec{t}}$ to denote an arbitrary one of these eight) together with the elements $\mathrm{diag}(1,-1,-1,1)$ and $\mathrm{diag}(1,1,-1,-1)$. The action of the $W_{\vec{t}}$ on weight spaces $V^{x,y}$ is then easy to understand; for example if $v \in V^{x,y}$ then $\rho'(Y_{0,2})\, v \in V^{x,y+2}$, *etc.*

Denote by $\mathcal{U}$ the universal enveloping algebra of $\mathfrak{sp}_4$. By the Poincaré-Birkhoff-Witt Theorem, $\mathfrak{sp}_4$ embeds in $\mathcal{U}$, and if we identify the elements $W_{\vec{t}}$ with their images in $\mathcal{U}$, then $\mathcal{U}$ has a $\mathbb{C}$-basis consisting of vectors $W_{\vec{t}_1}^{r_1} \dots W_{\vec{t}_k}^{r_k}$ with the $W_{\vec{t}}$'s in a fixed order. Further let $\mathcal{U}_{\mathbb{Z}}$ be the $\mathbb{Z}$-subalgebra of $\mathcal{U}$ generated by elements of the form $W_{\vec{t}}^{r_{\vec{t}}}/(r_{\vec{t}}!)$: this will allow us to construct a useful lattice in $V_{m,n}$, as required in order to find congruence properties. $\mathcal{U}_{\mathbb{Z}}$ inherits an action on $V_{m,n}$.

A $\mathbb{Z}$-lattice $L \subset V$ is called admissible if it is preserved by $\mathcal{U}_{\mathbb{Z}}$. Then it is known (see e.g. [Hmph] §27) that any finite-dimensional irreducible $\mathfrak{sp}_4(\mathbb{C})$-module contains an admissible lattice, for example $\mathcal{U}_{\mathbb{Z}}\, v$ for any lowest weight vector v, and moreover that any admissible lattice is equal to the sum of its intersections with the weight spaces $V^{x,y}$.

Let R stand for $\mathbb{Z}$ or $\mathbb{Z}/N\mathbb{Z}$. We note the following facts:

(a) The set $S(R) = \left\{ \begin{pmatrix} * & 0 & * & 0 \\ 0 & 1 & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in Sp_4(R) \right\}$ is generated by $\exp(X_{2,0})$ and $\exp(X_{-2,0})$.

(b) The set $P(R) = \left\{ \begin{pmatrix} * & 0 & * & * \\ * & 1 & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in Sp_4(R) \right\}$ is generated by the $\exp(X_{\vec{t}})$ and the $\exp(Y_{\vec{t}})$.

(c) The set $Sp_4(\mathbb{Z})$ is generated by all eight $\exp(W_{\vec{t}})$.

We prove these using row and column operations. We employ the obvious notation: R_n stands for the n^{th} row, C_m for the m^{th} column, and a dash $'$ denotes the corresponding new terms after the operation in question.

Consider the matrices

$$A = \exp(Y_{0,2}) \exp(-Z_{0,-2}) \exp(Y_{0,2}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix};$$

$$B = \exp(Z_{0,-2}); \qquad C = \exp(Y_{-1,1});$$

$$D = \exp(Y_{-1,1}) \exp(-Z_{1,-1}) \exp(Y_{-1,1}) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

These have the following effects:

$$\begin{aligned}
A:&\quad R_2' = R_4,\ R_4' = -R_2 \quad \text{(premultiplying)};\\
&\quad C_2' = -C_4,\ C_4' = C_2 \quad \text{(postmultiplying)};\\
B:&\quad R_4' = R_4 + R_2 \quad \text{(premultiplying)};\\
&\quad C_2' = C_2 + C_4 \quad \text{(postmultiplying)};\\
C:&\quad C_1' = C_1 + C_2,\ C_4' = C_4 - C_3 \quad \text{(postmultiplying)};\\
D:&\quad C_1' = C_2,\ C_2' = -C_1,\ C_3' = C_4,\ C_4' = -C_3 \quad \text{(postmultiplying)}.
\end{aligned}$$

Now suppose we are given $\gamma \in Sp_4(\mathbb{Z})$; premultiplying by a suitable sequence of A and B, we can ensure $\gamma_{41} = 0$. Then use postmultiplication by A and B to make $\gamma_{42} = 0$, and postmultiplication by C and D to further ensure $\gamma_{43} = 0$. But then it follows from the conditions (1) for symplecticity that in fact, $\gamma_{32} = \gamma_{12} = 0$ and $\gamma_{22} = \gamma_{44} = \pm 1$. Thus we have reduced (c) to (b). (b) may be reduced to (a) by successive postmultiplication by $\exp(Y_{1,1})$, $\exp(Y_{-1,1})$ and $\exp(Y_{0,2})$, again using the relations (1). But statement (a) is easy.

As a consequence, $Sp_4(\mathbb{Z})$ will preserve any admissible lattice.

Remark 3.1. — Let R again denote $\mathbb{Z}$ or $\mathbb{Z}/N\mathbb{Z}$. Let L be an admissible lattice in V, and define $G_L(R)$ to be the subgroup of $GL(L \otimes R)$ generated by elements of the form $\exp X_{\vec{t}}$, $\exp Y_{\vec{t}}$ and $\exp Z_{\vec{t}}$. Denote by $P_L(R)$ the subgroup of $G_L(R)$ generated by the $\exp X_{\vec{t}}$ and the $\exp Y_{\vec{t}}$, and by $S_L(R)$ the one generated by the $\exp X_{\vec{t}}$. It is known (see [St]) that if L_1 and L_2 are two admissible lattices in different representations with $\mathrm{Ker}\ \rho_1 \subset \mathrm{Ker}\ \rho_2$, then there is a unique map $G_{L_1}(R) \to G_{L_2}(R)$ taking each $\exp W_{\vec{t}} \in G_{L_1}(R)$ to $\exp W_{\vec{t}} \in G_{L_2}(R)$, so it maps $P_{L_1}(R)$ to $P_{L_2}(R)$ and $S_{L_1}(R)$ to $S_{L_2}(R)$. In particular we see that $G_L(R)$, $P_L(R)$ and $S_L(R)$ depend only on $\mathrm{Ker}\ \rho$ up to canonical isomorphism.

Now let $L = L_{m,n} = \mathcal{U}_{\mathbb{Z}}\, v \quad (v \in V_{m,n}^{-m,-n})$ be an admissible lattice. We set $L_{(p)} = L \otimes \mathbb{Z}_{(p)}$ $(\mathbb{Z}_{(p)} = \mathbb{Q} \cap \mathbb{Z}_p)$. Then Δ will preserve $L_{(p)}$: indeed, if T denotes the diagonals of GSp_4, the elementary divisor theory for the symplectic group (see e.g. [Frei] Hilfssatz 4.1.12) tells us

$$GSp_4^+(\mathbb{Q}) \cap M_4(\mathbb{Z}) \subseteq Sp_4(\mathbb{Z})\, T(\mathbb{Z})\, Sp_4(\mathbb{Z}) \tag{3}$$

and each factor preserves $L_{(p)}$.

Now consider the projection map $j : V \to V'$, where $V' = \bigoplus_x V_{m,n}^{x,-n}$ is the bottom line of the weight diagram of V. Using the general theory, one sees that all the weight spaces in V' are one-dimensional. V' is isomorphic to the unique irreducible representation of $SL_2(\mathbb{C})$ of highest weight m, via

$$\begin{aligned} SL_2(\mathbb{C}) &\hookrightarrow Sp_4(\mathbb{C}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Let $L' = j(L)$. Our aim is to compare cohomology with coefficients in L and in L'. Of course, L' is no longer preserved by $Sp_4(\mathbb{Z})$; however, we shall see that after reducing mod p^r, it *is* preserved by Γ. This is the motivation for our congruence restrictions at p.

$Sp_4(\mathbb{Z})$ inherits an action on $L \otimes \mathbb{Z}/p^r\mathbb{Z} = L_{(p)} \otimes \mathbb{Z}/p^r\mathbb{Z}$ by $\mathbb{Z}$-linearity.

LEMMA 3.2. — *The $Sp_4(\mathbb{Z})$-action on $L \otimes \mathbb{Z}/p^r\mathbb{Z}$ factors modulo p^r.*

Proof. — We apply Remark 3.1 with ρ_1 the standard faithful four-dimensional representation of $Sp_4(\mathbb{C})$ and $\rho_2 = \rho$. For ρ_1 the result is certainly true. Thus the vertical map in the following commutative diagram exists and is unique:

$$\begin{array}{ccccc} Sp_4(\mathbb{Z}) & \longrightarrow & G_{L_1} = Sp_4(\mathbb{Z}/p^r\mathbb{Z}) & \subset & \mathrm{End}\,(\mathbb{Z}/p^r\mathbb{Z})^4 \\ & \bar{\rho} \searrow & \downarrow & & \\ & & G_{L_2}(\mathbb{Z}/p^r\mathbb{Z}) & \subset & \mathrm{End}\,(L \otimes \mathbb{Z}/p^r\mathbb{Z}), \end{array}$$

i.e. for $\gamma \in Sp_4(\mathbb{Z})$, $\bar{\rho}(\gamma)$ only depends on the reduction of γ modulo p^r. □

But after reducing modulo p^r, Γ can be factorised into $\exp(X_?)$ and $\exp(Y_?)$; each of these preserve Ker j. Furthermore, the action of Δ also factorises modulo p^r (using (3) and the known action of diagonals), whence one checks from the definition of Δ that it also preserves Ker j: we can factorise any element of Δ into diagonals with multiplier coprime to p, a power of g_p, and some factors of $\exp(X_?)$ and $\exp(Y_?)$.

We therefore obtain an action of Δ on $L' \otimes \mathbb{Z}/p^r\mathbb{Z}$ from that on $L \otimes \mathbb{Z}/p^r\mathbb{Z}$. $\Gamma \subset \Delta$ acts through the projection

$$\begin{array}{ccc} \Gamma & \longrightarrow & SL_2(\mathbb{Z}/p^r\mathbb{Z}) \\ \begin{pmatrix} a & 0 & b & * \\ * & 1 & * & * \\ c & 0 & d & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \pmod{p^r} & \longmapsto & \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \end{array}$$

and diagonals $\mathrm{diag}(\alpha, \beta, \nu/\alpha, \nu/\beta)$ will act as $\nu^{-x/2}\, \alpha^x\, (\nu/\beta)^n$.

4. Change of coefficient module.

We are now able to form cohomology groups $H^i(\Gamma, L_{m,n} \otimes \mathbb{Z}/p^r\mathbb{Z})$ and $H^i(\Gamma, L_{m,n}' \otimes \mathbb{Z}/p^r\mathbb{Z})$ which are equipped with an action of the Hecke algebra $\mathcal{H}(\Gamma\backslash\Delta/\Gamma)$. The projection j covariantly induces a map

$$j_* : H^i(\Gamma, L_{m,n} \otimes \mathbb{Z}/p^r\mathbb{Z}) \longrightarrow H^i(\Gamma, L_{m,n}' \otimes \mathbb{Z}/p^r\mathbb{Z}).$$

Note that j restricts to an isomorphism $g_{p^r}(L \otimes \mathbb{Z}/p^r\mathbb{Z}) \longrightarrow g_{p^r}(L' \otimes \mathbb{Z}/p^r\mathbb{Z})$.

We have thus satisfied the conditions of a group cohomological lemma of Richard Taylor ([Tayl] Lemma 1.1; see also [W] Theorem 1.2.2):

LEMMA 4.1. — *Let Δ be a semigroup, $\Gamma \subset \Delta$ a subgroup, $g \in \Delta$ with Γ and $g\Gamma g^{-1}$ commensurable, M and N $<\Gamma, g>$-modules, and $j : M \to N$ a $<\Gamma, g>$-morphism such that $j : gM \xrightarrow{\sim} gN$. Then there exists $I : H^i(\Gamma, N) \to H^i(\Gamma, M)$ such that*

$$I \circ j_* = [\Gamma g \Gamma]$$
$$\text{and} \quad j_* \circ I = [\Gamma g \Gamma].$$

Proof (sketch). — Let g_* be the map on cohomology induced from conjugation by g on $\Gamma \cap g\Gamma g^{-1}$ and let i_* be the map induced by $i : gM \to M$. Then one can factorise the Hecke operator $[\Gamma g \Gamma]$ as $\mathrm{cor} \circ i_* \circ g_*$ (where cor is the corestriction operator). Defining $I = \mathrm{cor} \circ i_* \circ (j|_{gM})_*^{-1} \circ g_*$, for the first part one checks that the diagram

$$\begin{array}{ccccccc} H^i(\Gamma, M) & \xrightarrow{g_*} & H^i(\Gamma \cap g\Gamma g^{-1}, gM) & \xrightarrow{i_*} & H^i(\Gamma \cap g\Gamma g^{-1}, M) & \xrightarrow{\mathrm{cor}} & H^i(\Gamma, M) \\ \downarrow j_* & & & & & & \uparrow \mathrm{cor} \\ H^i(\Gamma, N) & \xrightarrow{g_*} & H^i(\Gamma \cap g\Gamma g^{-1}, gN) & \xleftarrow{j_*} & H^i(\Gamma \cap g\Gamma g^{-1}, gM) & \xrightarrow{i_*} & H^i(\Gamma \cap g\Gamma g^{-1}, M) \end{array}$$

commutes, i.e. that i_* commutes with cor and j_* commutes with g_*.

For the second part one has the diagram

$$\begin{array}{ccccccc} H^i(\Gamma,N) & \xrightarrow{g_*} & H^i(\Gamma\cap g\Gamma g^{-1},gN) & \xrightarrow{i_*} & H^i(\Gamma\cap g\Gamma g^{-1},N) & \xrightarrow{\mathrm{cor}} & H^i(\Gamma,N) \\ \downarrow g_* & & & & & & \uparrow j_* \\ H^i(\Gamma\cap g\Gamma g^{-1},gN) & \xleftarrow{j_*} & H^i(\Gamma\cap g\Gamma g^{-1},gM) & \xrightarrow{i_*} & H^i(\Gamma\cap g\Gamma g^{-1},M) & \xrightarrow{\mathrm{cor}} & H^i(\Gamma,M) \end{array}$$

and we have to check that j_* commutes with cor and with i_*. All these verifications are straightforward. □

In our application, $M = L\otimes\mathbb{Z}/p^r\mathbb{Z}$, $N = L'\otimes\mathbb{Z}/p^r\mathbb{Z}$, and $g = g_{p^r}$. Therefore we obtain a map $I : H^i(\Gamma, L_{m,n}{}'\otimes\mathbb{Z}/p^r\mathbb{Z}) \longrightarrow H^i(\Gamma, L_{m,n}\otimes\mathbb{Z}/p^r\mathbb{Z})$ satisfying $I\circ j_* = [\Gamma g_{p^r}\Gamma] = R_{p^r}$ on $H^i(\Gamma, L_{m,n}\otimes\mathbb{Z}/p^r\mathbb{Z})$, and $j_*\circ I = [\Gamma g_{p^r}\Gamma] = R_{p^r}$ on $H^i(\Gamma, L_{m,n}{}'\otimes\mathbb{Z}/p^r\mathbb{Z})$.

It is well known that these cohomology groups are finitely generated abelian groups, so we can associate a Hida idempotent e to R_p (and hence to R_{p^r}, since we can show $R_{p^r} = R_p^r$). Recall that this is an element of the endomorphism ring End $(H^3(\Gamma,X))$ such that R_p is invertible on $e\,H^3(\Gamma,X)$, and topologically nilpotent on $(1-e)\,H^3(\Gamma,X)$. It may be constructed as $e = \lim_{k\to\infty} R_p^{k!}$. Then e is a projector onto ordinary modular forms. For general facts on these projectors see e.g. [MW] Chapter 2.

We have a commutative diagram

$$\begin{array}{ccc} H^i(\Gamma, L_{m,n}\otimes\mathbb{Z}/p^r\mathbb{Z}) & \xrightarrow{j_*} & H^i(\Gamma, L_{m,n}{}'\otimes\mathbb{Z}/p^r\mathbb{Z}) \\ \Big\downarrow R_p & \swarrow I & \Big\downarrow R_p \\ H^i(\Gamma, L_{m,n}\otimes\mathbb{Z}/p^r\mathbb{Z}) & \xrightarrow{j_*} & H^i(\Gamma, L_{m,n}{}'\otimes\mathbb{Z}/p^r\mathbb{Z}). \end{array}$$

On the ordinary components, the vertical maps are isomorphisms. Hence so are the horizontal ones.

One can check explicitly that all other Hecke operators $T\in\mathcal{H}(\Gamma\backslash\Delta/\Gamma)$ commute with j_*, because j is a map of Δ-modules in degree zero.

Now consider changing the highest weight of our representation from (m,n) to $(m,n+1)$. $V'_{m,n}$ and $V'_{m,n+1}$ (defined in the obvious way) are both just isomorphic to the unique irreducible $SL_2(\mathbb{C})$-module with highest weight m, hence $L'_{m,n}\otimes\mathbb{Z}/p^r\mathbb{Z} \cong L'_{m,n+1}\otimes\mathbb{Z}/p^r\mathbb{Z}$, preserving the action of $X_{\bar{t}}$ and Γ.

On the other hand, from the description at the end of §3 one can see that the action of the diagonals differs: altering n to $n+1$ changes the action of $g\in\Delta$ by a factor $\chi(g)$, where $\chi:\ \Delta\to(\mathbb{Z}/p^r\mathbb{Z})^*$ is the character given by $M\mapsto M_{44}(\mathrm{mod}\,p^r)$.

We need to change the action of the centre on V by letting λI_4 act as λ^{n+m} instead of as λ^n, in order to obtain the correct Hecke action carried over from $S_{m,n}(\Gamma)$. However, this simply twists the action of the Hecke operators $[\Gamma g\Gamma]$ on cocycles by a scalar $\nu(T)^{m/2}$ (consider the factorisation $[\Gamma g\Gamma] = \mathrm{cor} \circ i_* \circ g_*$), and m is fixed throughout. Our only Hecke operator at p is R_p so this will not interfer with our congruences.

Putting our results together, and repeating them sufficiently often to remove the twist by χ, we obtain

THEOREM 4.2. — *We have an isomorphism of $\mathcal{H}(\Gamma\backslash\Delta/\Gamma)$-modules*
$$e\,H^i(\Gamma, L_{m,n} \otimes \mathbb{Z}/p^r\mathbb{Z}) \cong e\,H^i(\Gamma, L_{m,n+p^{r-1}(p-1)} \otimes \mathbb{Z}/p^r\mathbb{Z}).$$

Remark 4.3. — Unfortunately, one cannot simply lift the above result to characteristic zero because of torsion in $H^4(\Gamma, L_n(\mathbb{Z}_p))$. In the SL_2 case, as long as $\Gamma \subset SL_2(\mathbb{Z})$ is torsion-free, one has $H^2(\Gamma, L_n(\mathbb{Z}_p)) = 0$ and thus $H^1(\Gamma, L_n \otimes \mathbb{Z}/p^r\mathbb{Z}) \cong H^1(\Gamma, L_n(\mathbb{Z}_p)) \otimes \mathbb{Z}/p^r\mathbb{Z}$. We can, however, deduce the following result:

COROLLARY 4.4. — $\dim eS_{k_1,k_2}(\Gamma)$ *is bounded independently of* k_2.

Proof. — By (2), it suffices to show that $\dim eH^3(\Gamma, V_{m,n})$ is bounded independently of n. It is enough to show that $\dim eH^3(\Gamma, L_{m,n} \otimes \mathbb{Q}_p)$ is so bounded. Now consider the short exact sequence
$$0 \longrightarrow \mathbb{Z}_p \xrightarrow{p^r} \mathbb{Z}_p \longrightarrow \mathbb{Z}/p^r\mathbb{Z} \longrightarrow 0,$$
which induces
$$0 \to eH^i(\Gamma, \mathbb{Z}_p) \otimes \mathbb{Z}/p^r\mathbb{Z} \to eH^i(\Gamma, \mathbb{Z}/p^r\mathbb{Z}) \to eH^i(\Gamma, \mathbb{Z}_p)[p^r] \to 0.$$
We have shown that $\dim eH^i(\Gamma, L_{m,n} \otimes \mathbb{Z}/p^r\mathbb{Z})$ is bounded independently of n. But $\dim eH^3(\Gamma, L_{m,n} \otimes \mathbb{Q}_p) = \dim eH^i(\Gamma, L_{m,n} \otimes \mathbb{Z}_p) \otimes \mathbb{Q}_p$, and we are done. $\square$

Also, if one assumes that a given system of eigenvalues $\lambda{:}\mathcal{H}(\Gamma\backslash\Delta/\Gamma) \to \mathbb{F}_p$ corresponding to an eigenform in $eH^3(\Gamma, L_{m,n}(\mathbb{Z}_p))$ does not occur in cohomology of degrees other than three (i.e. the localisations $e\,H^i(\Gamma, L_{m,n}(\mathbb{Z}_p))_{\mathfrak{m}_\lambda} = 0$ whenever $i \neq 3$, where $\mathfrak{m}_\lambda$ is the maximal ideal of $\mathcal{H}(\Gamma\backslash\Delta/\Gamma)$ corresponding to λ), one can deduce that λ also occurs on $eH^3(\Gamma, L_{m,n+p-1}(\mathbb{Z}_p))$.

5. Other results.

As was pointed out in the introduction, Theorem 4.2 and Corollary 4.4 complement results for parallel weight changes. We take this opportunity to record two unpublished results of R. Taylor which are obtained by similar cohomological methods. We thank him for his consent to including them here.

PROPOSITION 5.1 ([Tayl] Proposition 2.1). — *Let*

$$\Gamma_1(p^r) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_4(\mathbb{Z}) : C \equiv 0 \pmod{p^r}, \det(A) \equiv 1 \pmod{p^r} \right\},$$

and let e_T be the Hida idempotent associated to the Hecke operator $[\Gamma_1(p^r)\,\mathrm{diag}(p,p,1,1)\,\Gamma_1(p^r)]$. Suppose that $k_2 \geq k_1 \geq 0$. Then there is a constant C such that for all $\lambda \geq 0$,

$$\dim e_T S_{k_1+\lambda,k_2+\lambda}(\Gamma_1(p^r)) < C.$$

One can construct a family of cusp forms congruent modulo p^r in $S_{k_1+\lambda,k_2+\lambda}(\Gamma_1(p^r))$, for λ a multiple of $(p-1)p^{r-1}$, simply by multiplication by a suitable theta series. Then using Proposition 5.1, one can apply a Fitting ideal argument to recover a family of *eigenforms* which have congruent Hecke eigenvalues. However, in doing so one again loses control of the Fourier coefficients which do not appear as Hecke eigenvalues.

The next result employs the standard notations for *elliptic* modular forms.

PROPOSITION 5.2 ([Tayl] Theorem 1.1). — *Fix a prime p and an extension of the p-adic valuation on $\mathbb{Q}$ to the algebraic closure $\overline{\mathbb{Q}}$. Fix also an integer N and a constant D. Then the sum of the dimensions of the eigenspaces in $S_k(\Gamma_1(N))$ for the Hecke operator $\left[\Gamma_1(N)\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\Gamma_1(N)\right]$ for which the corresponding eigenvalue has p-adic valuation less than D (i.e. eigenspaces with "slope" $\leq D$) is bounded independently of k.*

Because we have no analogue to the ordinary projector e for forms of positive slope, this is insufficient to construct families of modular forms as in the work of Coleman ([Cole]).

We finish by giving an interesting criterion for ordinarity in the sense of the previous sections.

Define another congruence subgroup

$$\Gamma_0(p) = \left\{ \gamma \in Sp_4(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \pmod p \right\}.$$

PROPOSITION 5.3. — *Let Γ be a congruence subgroup of level N prime to p. Suppose $f \in S_{m,n}(\Gamma)$ is an eigenvector for $T_l = [\Gamma\,\mathrm{diag}(1,1,l,l)\,\Gamma]$, $R_l = [\Gamma\,\mathrm{diag}(l,l^2,l,1)\,\Gamma]$, and $S_l = [\Gamma\, l I_4\, \Gamma]$ for all primes $l \nmid N$, say $f|T = \lambda(T)\, f$. Let $Q_p(X)$ be the Hecke polynomial*

$$Q_p(X) = X^4 - T_pX^3 + (pR_p + p(p^2+1)S_p)X^2 - p^3T_pS_pX + p^6S_p^2.$$

Let the roots of $\lambda(Q_p(X))$ be $\alpha, \beta, \gamma, \delta$, labelled so that $\alpha\delta = \beta\gamma$, and suppose these are distinct and the ratio of no two of them is p.

Then there are eigenforms $f_1, \cdots, f_4 \in S_{m,n}(\Gamma \cap \Gamma_0(p))$, satisfying $f_i|T = \lambda(T)\, f_i$ for all $T = T_l, R_l, S_l$ with $l \nmid Np$, and with eigenvalues $\frac{\alpha\beta}{p}$, $\frac{\gamma\delta}{p}$, $\frac{\alpha\gamma}{p}$, and $\frac{\beta\delta}{p}$, respectively, under R_p. In particular, if any of these has p-adic valuation m, then the corresponding f_i lies in $e\, S_{m,n}(\Gamma \cap \Gamma_0(p))$, where e is the Hida idempotent associated to the Hecke operator $p^{-m}R_p$ on modular forms, and we have $e\, f \neq 0$.

Proof. — We use the automorphic setting for modular forms. So recall that the space of automorphic forms of weight $\vec{k}$ is a direct sum $\oplus\sigma$ of admissible irreducible representations $\sigma = \otimes\sigma_p$ of $GSp_4(\mathbb{A}_f)$. The local factors σ_p may be found as irreducible subquotients of principal series representations.

The unramified principal series $\pi = \pi_{\chi_1,\chi_2,\psi}$ corresponding to a triplet (χ_1, χ_2, ψ) of unramified characters on $\mathbb{Q}_p^*$ (i.e. they are trivial on $\mathbb{Z}_p^*$) is defined as follows. Give the triplet an action on a minimal parabolic subgroup of $GSp_4(\mathbb{Q}_p)$ by

$$(\chi_1, \chi_2, \psi) : \begin{pmatrix} \lambda & 0 & * & * \\ * & \mu & * & * \\ 0 & 0 & \nu/\lambda & * \\ 0 & 0 & 0 & \nu/\mu \end{pmatrix} \mapsto \chi_1(\lambda)\,\chi_2(\mu)\,\psi(\nu)$$

and define a character $\delta := |\lambda^2\mu^4\nu^{-3}|_p$ on such a matrix.

Then π has as underlying space the set $I_{\chi_1,\chi_2,\psi}$ of maps $\phi : GSp_4(\mathbb{Q}_p) \to \mathbb{C}$ which are locally constant and satisfy

$$\phi(bg) = ((\chi_1, \chi_2, \psi)\delta^{1/2})(b)\,\phi(g)$$

whenever $b \in B(\mathbb{Q}_p)$, and the (left) action is given by right translation:

$$\pi : \ GSp_4(\mathbb{Q}_p) \longrightarrow GL(I_{\chi_1,\chi_2,\psi})$$
$$(\pi(g)\phi)(h) = \phi(hg).$$

Let U be an open compact subgroup of $\prod GSp_4(\mathbb{Z}_l)$ such that $U \cap GSp_4(\mathbb{Q})^+ = \Gamma$. Then $f \in S_{m,n}(\Gamma)$ corresponds to an automorphic form $\phi \in \mathcal{S}_{\vec{k}}(U)$, and we may assume $\phi \in {\sigma_p}^U$, where σ_p is a subquotient of $\pi = \pi_{\chi_1,\chi_2,\psi}$ as above.

First we compute the roots of $\lambda(Q_p(X))$ in terms of the characters χ_1, χ_2, and ψ. As $p \nmid N$, we have $\sigma_p^U = \pi^{GSp_4(\mathbb{Z}_p)}$. Then the Iwasawa decomposition $GSp_4(\mathbb{Q}_p) = B(\mathbb{Q}_p). GSp_4(\mathbb{Z}_p)$ ensures that the space $\pi^{GSp_4(\mathbb{Z}_p)}$ is one-dimensional, spanned over $\mathbb{C}$ by Θ, say, where

$$\Theta(bk) = ((\chi_1, \chi_2, \psi)\delta^{1/2})(b) \qquad (b \in B(\mathbb{Q}_p),\ k \in GSp_4(\mathbb{Z}_p)).$$

Then we can read off the eigenvalues of the Hecke operators from their action on Θ.

Using the double coset decompositions given in [Tayl], one computes

$$\begin{aligned}
\Theta\big|T_p &= p^{3/2}\psi(p)\big(1 + \chi_1(p) + \chi_2(p) + \chi_1\chi_2(p)\big)\,\Theta\\
\Theta\big|S_p &= \chi_1\chi_2\psi^2(p)\,\Theta\\
\Theta\big|R_p &= p^2\psi^2(p)\big(\chi_1(p) + \chi_2(p) + \chi_1\chi_2(p) + \chi_1\chi_2^2(p) + \chi_1^2\chi_2(p)\big)\,\Theta\\
&\quad - \chi_1\chi_2\psi^2(p)\,\Theta
\end{aligned}$$

$$\Longrightarrow \Theta\Big|Q_p(X) = \big(X - p^{3/2}\psi(p)\big)\big(X - p^{3/2}\chi_1\psi(p)\big)\big(X - p^{3/2}\chi_2\psi(p)\big)\big(X - p^{3/2}\chi_1\chi_2\psi(p)\big)$$

i.e. the roots of $\lambda(Q_p(X))$ are $p^{3/2}\psi(p)$, $p^{3/2}\chi_1\psi(p)$, $p^{3/2}\chi_2\psi(p)$ and $p^{3/2}\chi_1\chi_2\psi(p)$. Then the argument of Lemma 2.4 in [Tayl] uses the hypotheses on these roots to show that π is irreducible and so $\pi = \sigma_p$.

Then the Hecke eigenvalues on $\sigma_l^{(U \cap U_0(p))_l}$ will still be given by λ, and our task is to compute the eigenvalues of $\tilde{R}_p$ on $\sigma_p^{(U \cap U_0(p))_p} \cong \pi^{\Gamma_0(p)}$. Here the tilde is merely a reminder that after adding p to the level, the action of R_p changes.

Now we have the decomposition $GSp_4(\mathbb{Q}_p) = \coprod B(\mathbb{Q}_p)\ w_i\ \Gamma_0(p)$, where the w_i for $i = 1$ to 4 are running over representatives for the Weyl group $W_{Sp_4(\mathbb{Z})}/W_{\Gamma_0(p)}$. $f \in \pi^{\Gamma_0(p)}$ implies $f(bw_i\gamma) = ((\chi_1, \chi_2, \psi)\delta^{1/2})(b) f(w_i)$ when $b \in B(\mathbb{Q}_p)$ and $\gamma \in \Gamma_0(p)$, so it suffices to specify $f(w_i)$ to define f.

Hence $\pi^{\Gamma_0(p)}$ has a basis f_1, f_2, f_3, f_4 with $f_i(w_j) = \delta_{ij}$, and to determine the matrix of $\tilde{R}_p = [\Gamma_0(p)\ \mathrm{diag}(p, p^2, p, 1)\ \Gamma_0(p)]$ is to find $f_j|\tilde{R}_p$.

For this we need the coset decomposition $\Gamma_0(p) g_p \Gamma_0(p) = \coprod \alpha_r g_p \Gamma_0(p)$, where the α_r run through

$$\begin{pmatrix} 1 & 0 & 0 & z \\ t & 1 & z & w \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with } \ 0 \le t < p,\ 0 \le z < p,\ 0 \le w < p^2.$$

Then

$$(f_j|\tilde{R}_p)(w_i) \ = \ \Big(f_j\Big|\coprod_r \alpha_r g_p \Gamma_0(p)\Big)(w_i) \ = \ \sum_r f_j\big(w_i \alpha_r g_p\big).$$

To be able to evaluate this sum, we write $w_i \alpha_r g_p = b(i,r)\ w_{k(i,r)}\ \gamma(i,r)$ with $b(i,r) \in B(\mathbb{Q}_p)$ and $\gamma(i,r) \in \Gamma_0(p)$, so that

$$f_j(w_i \alpha_r g_p) = ((\chi_1, \chi_2, \psi)\delta^{1/2})(b(i,r))\ \delta_{jk(i,r)}.$$

Then with respect to the basis f_1, f_2, f_3, f_4, the (i,j) matrix entry of $\tilde{R}_p$ is reduced to $\sum_r ((\chi_1, \chi_2, \psi)\delta^{1/2})(b(i,r))\ \delta_{jk(i,r)}$. After the necessary calculations, one obtains the matrix

$$\psi^2(p) \begin{pmatrix} p^2\chi_1\chi_2^2(p) & p(p-1)\chi_1\chi_2^2(p) & p(p-1)\chi_1\chi_2^2(p) & (p^2-1)\chi_1\chi_2^2(p) \\ 0 & p^2\chi_1^2\chi_2(p) & (p^2-1)\chi_1^2\chi_2(p) & (p-1)\chi_1^2\chi_2(p) \\ 0 & 0 & p^2\chi_2(p) & p(p-1)\chi_2(p) \\ 0 & 0 & 0 & p^2\chi_1(p) \end{pmatrix}.$$

In particular, its eigenvalues are as claimed. The final statement of the proposition follows because f is a linear combination of the f_i. □

BIBLIOGRAPHY

[A] A.N. ANDRIANOV, Quadratic forms and Hecke operators, Grundlehren d. Math. Wiss., 286, Springer, 1987.

[AS] A. ASH and G. STEVENS, Modular forms in characteristic l and special values of their L-functions, Duke Math. Jour., 53, n° 3 (1986), 849–868.

[B] K.S. BROWN, Cohomology of groups, Grad. Texts in Math., 87, Springer, 1982.

[Buck] K. BUECKER, Congruences between Siegel modular forms, Ph.D. Thesis, Cambridge, 1995.

[Cole] R.F. COLEMAN, p-adic Banach spaces and families of modular forms, preprint.

[Falt] G. FALTINGS, On the cohomology of locally symmetric Hermitian spaces, in LNM 1029, Springer (1983), 55–98.

[Frei] E. FREITAG, Siegelsche Modulfunktionen, Grundlehren d. Math. Wiss., 254 Springer, 1983.

[Hi1] H. HIDA, Galois representations into $GL_2(\mathbb{Z}_p[[X]])$ attached to ordinary cusp forms, Invent. Math., 85 (1986), 545–613.

[Hi2] H. HIDA, On p-adic Hecke algebras for GL_2 over totally real fields, Ann. of Math., 128 (1988), 295–384.

[Hi3] H. HIDA, Elementary theory of L-functions and Eisenstein series, LMS Student Texts 26, CUP, 1993.

[Hmph] J.E. HUMPHREYS, Introduction to Lie Algebras and Representation Theory, Graduate Texts in Mathematics 9, Springer, 1972.

[KPS] M. KUGA, W. PARRY, C. SAH, Group cohomology and Hecke operators, in Manifolds and Lie groups, Eds. J. Hano et al., Birkhäuser, 1981.

[MW] B. MAZUR, A. WILES, On p-adic analytic families of Galois representations, Comp. Math., 59 (1986), 231–264.

[Pan] A.A. PANCHISHKIN, Non-Archimedean L-functions of Siegel and Hilbert modular forms, LNM 1471, Springer, 1991.

[Shim] G. SHIMURA, On certain reciprocity laws for theta functions and modular forms, Acta Math., 141 (1978), 35–71.

[St] R. STEINBERG, Lectures on Chevalley Groups, Lecture Notes, Yale (1968).

[Tayl] R.L. TAYLOR, On congruences between modular forms, Ph.D. Thesis, Princeton, 1988.

[TU] J. TILOUINE, E. URBAN, Nearly ordinary cohomology for Sp_4, preprint.

[W] A. WILES, On ordinary λ-adic representations associated to modular forms, Invent. Math., 94 (1988), 529–573.

Manuscrit reçu le 20 février 1996,
accepté le 26 avril 1996.

Karsten BUECKER,
Institut Fourier
Université Joseph Fourier (Grenoble I)
B.P. 74
38402 Saint Martin d'Hères Cedex (France).