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GREGOR FELS

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A DIFFERENTIAL GEOMETRIC CHARACTERIZATION OF INVARIANT DOMAINS OF HOLOMORPHY

by Gregor FELS

1. Introduction.

In this paper(*) we investigate domains and functions on connected complex reductive groups invariant by a compact form.

A **complex reductive group** is a universal complexification of a compact group, (see [Ho]). Examples of such Lie groups are for instance complex tori $(\mathbb{C}^*)^n = ((S^1)^n)^{\mathbb{C}}$ or special and general linear groups $\mathrm{SL}(n,\mathbb{C}) = (\mathrm{SU}(n))^{\mathbb{C}}$ resp. $\mathrm{GL}(n,\mathbb{C}) = (\mathrm{U}(n))^{\mathbb{C}}$.

As a consequence of the Peter and Weyl theorem every compact Lie group K has an embedding in $\mathrm{GL}(n,\mathbb{C})$. The universal complexification $K^{\mathbb{C}}$ of K is simply the minimal affine algebraic set in $\mathrm{GL}(n,\mathbb{C})$ containing K. This is the same as the minimal *complex analytic* set containing K. Furthermore, K is totally real with $\dim_{\mathbb{R}} K = \dim_{\mathbb{C}} K^{\mathbb{C}}$ and K is a maximal compact subgroup of $K^{\mathbb{C}}$. Of course the underlying complex manifold structure of $K^{\mathbb{C}}$ is Stein. In all what follows we fix a maximal compact subgroup K of $G = K^{\mathbb{C}}$ and concern the action

$$K \times G \longrightarrow G \qquad g, x \mapsto xg^{-1}.$$

Let $\pi : G \to G/K =: (M, x_0)$ denote the quotient map and x_0 the point $eK \in G/K$. This map is open and proper.

(*) This paper is an abbreviated version of a part of the author's dissertation.

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The quotient $M = G/K$ carries a natural G–invariant Riemannian structure (see § 2 below). In the abelian case, where $G = (\mathbb{C}^*)^n$, $K = (S^1)^n$ domains Ω in G, which are K-invariant are Reinhardt domains. It is well–known that such a domain is holomorphically convex if and only if its image in M is geodesically convex. In the general non–abelian setting, holomorphic convexity of a K–invariant domain implies geodesic convexity of the corresponding domain in M ([Ro]). The converse also holds for domains which are invariant under *both* the left and right translations by elements of K. In 1985 Loeb constructed a geodesically convex domain in $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ such that the corresponding domain in $\mathrm{SL}(2,\mathbb{C})$ is not Stein ([Lo1]). Thus it becomes clear that, if holomorphic convexity could be characterized by a differential geometric property in M, then, in order to see this, one must analyze the fine structures at hand.

Our main result is a characterization of Stein invariant domains $\Omega \subset K^{\mathbb{C}}$ with smooth boundary in terms of sectional curvature of the boundary $\partial\Omega_M$, (see Theorem 5.4).

We also give a characterization of Stein invariant domain without any boundary condition by using the boundary distance function, (see Theorem 6.3).

Convention. — By Ω we denote a K–invariant domain in $K^{\mathbb{C}}$ and by $\Omega_M \subset M$ the image $\pi(\Omega)$ in M.

Analogously we write $f_M \in C^0(M)$ for the push-forward of a continuous, K–invariant function $f : K^{\mathbb{C}} \to \mathbb{C}$, i.e. $f = f_M \circ \pi$.

Remark. — Every K–orbit in G is a total real maximal submanifold in G. It follows $\mathcal{O}(G)^K = \mathbb{C}$ (identity principle). In contrast notice that the K–invariant *plurisubharmonic* functions on G separate all K–orbits. This remains still true in the context of arbitrary invariant domains in Stein K–spaces; (see [F], prop. 4.15).

2. Riemannian structure on G and M.

The construction of metrics on G and M is classical. We recall it briefly for the convenience of the reader.

Let $\mathfrak{g} = T_eG$ be the Lie algebra of G and $\mathfrak{k}$ the Lie subalgebra of the maximal compact subgroup $K \subset G$. Then we have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus J\mathfrak{k} =: \mathfrak{k} \oplus \mathfrak{p}.$$

Let $\theta \in \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g})$ denote the Cartan involution, i.e. conjugation with respect to $\mathfrak{k}$ and $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}, [\ ,\ ]) \subset \mathrm{GL}_{\mathbb{C}}(\mathfrak{g})$ be the adjoint representation of G on $\mathfrak{g}$.

Choose an Euclidean metric g on $T_eG = \mathfrak{g}$ which is J– and $\mathrm{Ad}\,(K)$–invariant and such that the decomposition $\mathfrak{k} \oplus J\mathfrak{k}$ is orthogonal. Further let g be normalized by the condition

$$(2.1) \qquad g(X,Y) = -\mathrm{re}\, B_{\mathfrak{g}'}(X, \theta(Y)), \qquad X, Y \in \mathfrak{g}'.$$

Here $\mathfrak{g}'$ is the semisimple part of $\mathfrak{g}$ and $B_{\mathfrak{g}}$ denotes the Killing form of the complex Lie algebra $\mathfrak{g}$.

The extension $g : TK^{\mathbb{C}} \oplus TK^{\mathbb{C}} \to \mathbb{R}$ of this Euclidean metric by left translations is $(G \times K)$–invariant Riemannian metric on $K^{\mathbb{C}}$.

Remarks. — This metric is not Kähler.

We call the subspace tangent to the fibers of $\pi : G \to M$ **vertical** and the orthogonal complement **horizontal**. Notice that the complex structure J maps isometrically the horizontal subbundle of TG onto the vertical subbundle. If $TG = G \times \mathfrak{g}$ is the trivialization by left invariant vector fields then the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ corresponds to the decomposition of TG in the horizontal and vertical subbundle.

The projection π induces a fibrewise isomorphic bundle map of the horizontal subbundle of TG onto TM. Since $g : \mathfrak{g} \oplus \mathfrak{g} \to \mathbb{R}$ is $\mathrm{Ad}\,(K)$–invariant this induces a well defined Riemannian metric (also called g) on M.

By construction the projection $\pi : K^{\mathbb{C}} \to M$ is a Riemannian submersion with respect to this metrics on $K^{\mathbb{C}}$ and M. It is well-known that (M, g) is a Riemannian globally symmetric Space with non-positive sectional curvature.

For convenience of the reader we list here some basic facts about symmetric spaces (see e.g. [Hel] and [Wo]).

(2.2) The Levi-Civita connection on G is : ([ChE] p. 64)

$$\nabla^G_X Y = \tfrac{1}{2}\big([X,Y] - \mathrm{ad}^*_X(Y) - \mathrm{ad}^*_Y(X)\big).$$

Here $X, Y \in \mathfrak{g}$ are left invariant vector fields on G and ad_X^* denotes the adjoint endomorphism of ad_X which acts on $\mathfrak{g}$ via the identification $\mathfrak{g} \cong \mathfrak{g}^*$ induced by the metric.

(2.3) If θ is the Cartan involution, then $\mathrm{ad}_X^*(Y) = -\mathrm{ad}_{\theta(X)}(Y)$. In particular $\nabla_X^G X = [\theta(X), X] = 2[X^{\mathfrak{k}}, X^{\mathfrak{p}}]$,

(2.4) Every geodesic line in M has the form $t \mapsto g\exp(tP)x_0 \quad P \in \mathfrak{p}$, $g \in G$. Two arbitrary points in M can be connected by a unique geodesic line.

(2.5) Parallel displacement $\mathcal{T}^s_{g\exp(tP)} : T_{gx_0}M \to T_{g\exp(sP)x_0}M$ along the geodesic line $g\exp(tP)x_0$ is given by push-forward via the map $M \to M \quad x \mapsto g\exp(sP)g^{-1}x$.

(2.6) The curvature tensor $R^M(X,Y)Z := \nabla_X^M \nabla_Y^M Z - \nabla_Y^M \nabla_X^M Z - \nabla_{[X,Y]}^M Z$ is of the following form :

$$R(X,Y)Z = -[[X,Y],Z] \qquad \forall\, X,Y,Z \in \mathfrak{p} = T_{x_0}M.$$

3. Levi form of a function.

Let X be a complex manifold and (TX, J) the (real) tangent bundle with complex structure J. The formal complexification decomposes

$$T^{\mathbb{C}}X := TX \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}X \oplus T^{0,1}X := \mathrm{Eig}(J^{\mathbb{C}}, i) \oplus \mathrm{Eig}(J^{\mathbb{C}}, -i).$$

The projection $\pi^{1,0} : TX \to T^{1,0}X \quad X \mapsto \frac{1}{2}(X - iJX)$ yields a canonical identification of TX and $T^{1,0}X$ which will often be used without explicit mention.

Let $(z^1, \dots, z^n)$ be local holomorphic coordinates on X. The Levi form of a function ϕ can be defined as the Hermitian matrix

$$L_\phi^\alpha(p) := \Big(\frac{\partial^2 \phi}{\partial z^j \partial \bar{z}^k}(p)\Big).$$

For our consideration we need an intrinsic description of L_ϕ.

Definition 3.1. — *Let $\phi : X \to \mathbb{R}$ be a C^2-function. The* **Levi–Form** *of ϕ is the $\mathbb{C}$-bilinear mapping :*

$$\begin{aligned} L_\phi : T^{1,0}X \oplus T^{0,1}X &\to \mathbb{C} \\ (Z \quad , \quad W) &\mapsto Z(\tilde{W}\phi). \end{aligned}$$

Here $\tilde{W}$ denotes an arbitrary local antiholomorphic extension of $W \in T^{0,1}X$.

Remarks 3.2. — Since $[\Gamma_{\mathcal{O}}(X,T^{1,0}X),\Gamma_{\overline{\mathcal{O}}}(X,T^{0,1}X)] = 0$, it follows that $\tilde{Z}\tilde{W}\phi = \tilde{W}\tilde{Z}\phi$, where $\tilde{Z}$ denotes a local holomorphic extension of Z. Hence the definition of L_ϕ does not depends on the choice of extensions $\tilde{Z},\tilde{W}$.

In all what follows we are concerned with the quadratic form (also called Levi form)

$$\ell_\phi : TX \to \mathbb{R} \qquad \ell_\phi(v) := L_\phi(v,v) = L_\phi(v^{1,0},v^{0,1}) \tag{3.3}$$

rather than the sesquilinear form $L_\phi : TX \oplus TX \to \mathbb{C}$.

It is easy to compute $\ell_\phi(Z)$. Let $Z \in TX$ be a tangent vector and $\tilde{Z}$ be a local holomorphic extension i.e. a local section in TX such that the projection $\pi^{1,0}\tilde{Z}$ is a local holomorphic section in $T^{1,0}X$. Then

$$\begin{aligned} 4\ell_\phi(Z) &= (\tilde{Z} - iJ\tilde{Z})(\tilde{Z} + iJ\tilde{Z})\phi \\ &= \tilde{Z}(\tilde{Z}\phi) + J\tilde{Z}(J\tilde{Z}\phi) + i[\tilde{Z},J\tilde{Z}]\phi \\ &= Z(\tilde{Z}\phi) + JZ(J\tilde{Z}\phi). \end{aligned} \tag{3.4}$$

In our case, where $X = K^{\mathbb{C}}$, let $\phi : K^{\mathbb{C}} \to \mathbb{R}$ be a K–invariant, smooth function, and let $Z = T + P \in \mathfrak{k} \oplus J\mathfrak{k} = T_gG$ be the decomposition in the horizontal and vertical part. Let T^L, P^L denote the left invariant vector fields with $T^L(g) = T, P^L(g) = P$. Then we have the following formula :

LEMMA 3.5.

$$4\ell_\phi(Z) = P(P^L\phi) + JT(JT^L\phi) + 2[T,P]\phi.$$

Proof. — The left- and right invariant vector fields $Z \in \Gamma(G,TG)$ are holomorphic. Equivalently,

$$L_Z \circ J = J \circ L_Z.$$

Keeping in mind that ϕ is a K–invariant function, substitute $Z = T + P$ in formula 3.4 :

$$\begin{aligned} \ell_\phi(T+P) &= (T^L + P^L)^2\phi + (JT^L + JP^L)^2\phi \\ &= TT^L\phi + TP^L\phi + PT^L\phi + PP^L\phi + \\ &\qquad + JTJT^L\phi + JTJP^L\phi + JPJT^L\phi + JPJP^L\phi \\ &= PP^L\phi + TP^L\phi + JTJT^L\phi + JPJT^L\phi \\ &= PP^L\phi + [T,P]\phi + JTJT^L\phi + [JP,JT]\phi \\ &= PP^L\phi + JTJT^L\phi + 2[T,P]\phi. \end{aligned}$$

□

We would like to formulate the above condition in terms of the Riemannian geometry on the quotient M.

The first step will be a construction of a $\mathbb{R}$–bilinear operator $\mathcal{K}$: $TM \oplus TM \to TM$. It gives an adequate description of the "Lie bracket term" in the formula 3.5.

Let $X_1, X_2 \in T_xM$, $x = gx_0$ be two tangent vectors and $P_1, P_2 \in T_gK^{\mathbb{C}}$ the horizontal lifts of X_1, X_2 respectively at $g \in \pi^{-1}(x)$. Notice that the Lie bracket $[JP_1^L, P_2^L](g)$ of the corresponding left invariant extensions is a horizontal vector.

DEFINITION 3.6. — *The operator*

$$\mathcal{K} : TM \oplus TM \to TM \qquad \mathcal{K}(X_1, X_2) := \pi_*([JP_1^L, P_2^L](g))$$

is called the directional curvature on M.

Remarks 3.7. — A short computation shows that $\mathcal{K}$ is well defined i.e. it does not depend on the choice of a point g in the fibre $\pi^{-1}(x)$. Further $\forall\, X_1, X_2 \in TM$ we have the following fact :

$$\mathcal{K}(X_1, X_2) = -\mathcal{K}(X_2, X_1) \tag{3.7a}$$

$$g(\mathcal{K}(X_1, X_2), X_j) = 0 \quad , \quad j = 1, 2 \tag{3.7b}$$

$$\mathcal{K}(X_1, X_2)f = [JP_1, P_2]_g(f \circ \pi) = \pi_*([JP_1, P_2]_g)f, \quad f \in C^2(M). \tag{3.7c}$$

Notation. — Let $X \in T_xM$ be a tangent vector. We denote by γ_X the unique geodesic determined by $\dot{\gamma}(0) = X$.

Recall that all geodesic lines in the global symmetric space M have the following form (see 2.6) :

$$\gamma(t) = g \exp(tP) \cdot x_0, \quad P \in \mathfrak{p} = i\mathfrak{k} \subset \mathfrak{g}.$$

Hence we can take the m^{th} derivative of a function in the direction of X :

$$X^m f := \frac{d^m}{dt^m}\Big|_{t=0} (f \circ \gamma_X).$$

Further let $\Delta(X, Y) := X^2 + Y^2 = \Delta(Y, X)$, $X, Y \in T_pM$, denote the 2–dimensional Laplace operator at $x \in M$.

A word on the decomposition in horizontal and vertical directions : Let $Z = T + P =: JQ + P \in T_gG$ be such a decomposition. Here P and Q

are both horizontal vectors. Then we can define the corresponding tangent vectors at M :

$$X := \pi_* P_g, \; Y := \pi_* Q_g \in T_{gx_0} M.$$

On the other hand let $X, Y \in T_x M$ and $g \in G$ with $\pi(g) = x$. By $P_X, P_Y \in T_g K^{\mathbb{C}}$ we denote the horizontal liftings of X, Y at $g \in G$. Then we associate to every pair $X, Y \in T_x M$ the tangent vector $Z = JP_Y + P_X \in T_g G$.

Now we give an explicit description of the Levi form of a K–invariant function in terms of the two dimensional Laplacian and the operator $\mathcal{K}$:

BASIC FORMULA 3.8. — *Let $\phi \in C^2(U)$ denote a K–invariant function which is defined in a K–invariant neighborhood U of $gK \subset G$. Then for all $Z \in TU$:*

$$4\, \ell_\phi(Z) = \Delta(X, Y)\phi_M + 2\, \mathcal{K}(X, Y)\phi_M.$$

Proof. — Recall formula 3.5 :

$$4\ell_\phi(Z) = 4\, \ell_\phi(T + P) = P(P^L\phi) + JT(JT^L\phi) + 2[T, P]\phi.$$

The terms $P(P^L\phi)$ and $JT(JT^L\phi)$ can be described as a second derivative along a geodesic :

Let $\gamma_P(t) := g\exp(tP)$ be the 1-PSG of P^L in G at g. Then $\gamma_X(t) = g\exp(tP)x_0$ is a geodesic in M and we have :

(3.8a)

$$(P^L)^2\phi = \frac{d^2}{dt^2}(\phi \circ \gamma_P) = \frac{d^2}{dt^2}(\phi_M \circ \pi \circ \gamma_P) = \frac{d^2}{dt^2}(\phi_M \circ \gamma_X) \qquad \forall\, t \in (-\varepsilon, \varepsilon).$$

An analogous computation for $JT_g(JT^L\phi)$, evaluated at $t = 0$ together with (3.8a), yields

$$P_g(P^L\phi) + JT_g(JT^L\phi) = X^2\phi_M + Y^2\phi_M. \tag{3.8b}$$

The remaining term $2[T, P]$ can be described by the operator $\mathcal{K}$:

$$\begin{aligned} 2\,[T, P]_g\phi &= 2\,[T, P]_g(\phi_M \circ \pi) = 2\,\pi_*([T, P]_g)\phi_M \\ &= 2\,\pi_*([JQ, P]_g)\phi_M = 2\,\mathcal{K}(Y, X)\phi_M. \end{aligned} \tag{3.8c}$$

The claim follows by putting 3.8b and 3.8c together. □

As an immediate consequence, it is possible to give a description of plurisubharmonic K–invariant functions on $K^{\mathbb{C}}$ by an inequality formulated on M.

PROPOSITION 3.9. — *A real valued and K–invariant function $\phi \in C^2(G)$ is plurisubharmonic if and only if the following two conditions are fulfilled :*

(i) ϕ_M *is geodesic convex, i.e. for every geodesic segment* $\gamma : [0,1] \to M$ *it holds :*

$$\phi_M(\gamma(t)) \leqslant (1-t)\phi_M(\gamma(0)) + t\phi_M(\gamma(1)) \qquad \forall\, t \in [0,1]$$

(ii) *For all* $p \in M \quad X, Y \in T_pM$

$$\Delta(X,Y)\phi_M \geqslant 2\,\mathcal{K}(X,Y)\phi_M.$$

Remarks. — Condition (i) of the previous proposition follows from the (much stronger) condition (ii). We formulate condition (i) explicitly, in order to underline a question of Rothaus ([Ro]) :

$$\Omega_M \text{ geodesic convex } \overset{?}{\Longrightarrow} \Omega \subset G \text{ is Stein.}$$

A counterexample to this was discovered by Loeb ([Lo1]). We will discuss this in detail later.

Notice that a function ϕ is *strongly* plurisubharmonic if and only if $\forall\, p \in M$ and $(X,Y) \in T_pM \times T_pM \setminus (0,0)$ it follows that

$$\Delta(X,Y)\phi_M > 2\,\mathcal{K}(X,Y)\phi_M.$$

□

4. Invariant Stein domains with smooth boundary : The sectional curvature.

Let $\Omega \subset K^{\mathbb{C}}$ be a K–invariant domain with C^2–boundary. It is well-known, see [DoGr] that Ω is Stein iff the Levi form of $\partial\Omega$ is non–negative definite.

In this section we will prove that the Levi condition on the boundary can be equivalently formulated in terms of the sectional curvature of the boundary of Ω_M (Theorem 5.4).

We begin with some preparations. Let $r : U \to \mathbb{R}$ be a local defining function of $\partial\Omega_M$. Then the induced function ρ on a K–invariant neighborhood in $K^{\mathbb{C}}$, i.e. $\rho := r \circ \pi$, is a K–invariant defining function of $\partial\Omega$.

The first step will be a reformulation of the Levi condition (3.8). Fixing X_1, X_2, we compute $\inf\{\Delta(Y_1, Y_2)f;\ Y_1, Y_2 \in E;\ \mathcal{K}(Y_1, Y_2) = \mathcal{K}(X_1, X_2)\}$ in terms of the Hessian form $H_f : TM \oplus TM \to \mathbb{R}$ of f.

Recall that for *Riemannian* manifolds it is possible to define the Hessian form globally (see e.g. [F], Appendix).

LEMMA 4.1.

(i) *Let $f \in C^2(M)$ be a function and X_1, X_2 be two tangent vectors which span a plane $E \subset T_xM$. We assume $H_f|_E \geqslant 0$ and $\mathcal{K}(X_1, X_2) \neq 0$. Then*

$$\inf\{\Delta(Y_1, Y_2)f \mid Y_1, Y_2 \in E,\ \mathcal{K}(Y_1, Y_2) = \mathcal{K}(X_1, X_2)\}$$
$$= 2\sqrt{H_f(X_1, X_1)H_f(X_2, X_2) - H_f(X_1, X_2)H_f(X_1, X_2)}.$$

(ii) *Under the additional assumption $H_f|_E > 0$ for a fixed basis $X_1, X_2 \in E$, there exist tangent vectors $\overline{X}_1, \overline{X}_2 \in E$ having the following property :*

$$\mathcal{K}(\overline{X}_1, \overline{X}_2) = \mathcal{K}(X_1, X_2)$$
$$\Delta(\overline{X}_1, \overline{X}_2)f = \inf\{\Delta(Y_1, Y_2)f \mid Y_1, Y_2 \in E,\ \mathcal{K}(Y_1, Y_2) = \mathcal{K}(X_1, X_2)\}.$$

Of course the choice of $\overline{X}_1, \overline{X}_2$ depends on f.

Proof. — Let $E := ((X_1, X_2)) \subset T_xM$ be a 2–dimensional subspace spanned by X_1, X_2. We have seen that the operator $\mathcal{K}$ is bilinear and skew symmetric, (see 3.7). For $A \in \mathrm{GL}(E)$

$$\mathcal{K}(A(X_1), A(X_2)) = \det A \cdot \mathcal{K}(X_1, X_2).$$

Our next step is the computation of $\Delta(A(X_1), A(X_2))f$ as a function of A. First, let $\widehat{X}_1, \widehat{X}_2$ denote the parallel extension of X_1, X_2 along the geodesic segments centered in x and let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be the matrix representation of A w.r.t. the basis X_1, X_2 of E. It is easy to see that

(4.1a)

$$\begin{aligned}
\Delta(AX_1, AX_2)f &= (a^2+b^2)\widehat{X}_1^2 f + 2(ac+bd)\widehat{X}_1\widehat{X}_2 f + (c^2+d^2)\widehat{X}_2^2 f \\
&= \operatorname{tr} {}^t\!\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} H_f(X_1,X_1) & H_f(X_1,X_2) \\ H_f(X_2,X_1) & H_f(X_2,X_2) \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&=: \operatorname{tr} {}^t\!A\, H_f^{X_1X_2} A.
\end{aligned}$$

The last equation is a consequence of the following identity for the Hessian form :

$$\nabla_{X_j} df\ (X_k) = H_f(X_j, X_k) = X_j\widehat{X}_k f = X_k \widehat{X}_j f.$$

Now we compute $\inf\{\Delta(AX_1, AX_2)f \mid \det A = 1\} = \inf\{\operatorname{tr}\ {}^t\!A\ H_f^{X_1X_2} A \mid \det A = 1\}$.

Let Y_1, Y_2 be a basis of E, which arises from the old one after an orthogonal transformation such that $H_f^{Y_1Y_2}$ is diagonal. Obviously $\Delta(X_1, X_2)f = \Delta(Y_1, Y_2)f$ and $\det H_f^{X_1X_2} = \det H_f^{Y_1Y_2}$.

Let α, β be the (nonnegative) eigenvalues of $H_f^{Y_1Y_2}$. Keeping our computation as easy as possible we use then the Iwasawa decomposition of A :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{pmatrix} \cdot k \qquad k \in \mathrm{SO}(E).$$

First we investigate the case "$H_f^{X_1X_2} > 0$" :

$$\begin{aligned}
\inf\{\Delta(AX_1, AX_2)f \mid \det A = 1\} &= \inf\left\{\operatorname{tr}\ {}^t\!\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid \det A = 1\right\} \\
&= \inf\left\{\operatorname{tr}\left(\begin{smallmatrix} \lambda & 0 \\ \mu & \lambda^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)\left(\begin{smallmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{smallmatrix}\right) \mid \lambda > 0\right\} \\
&= \inf\{\alpha(\lambda^2+\mu^2) + \beta\lambda^{-2} \mid \lambda > 0\} \\
&= \inf\{\alpha\lambda^2 + \beta\lambda^{-2} \mid \lambda > 0\} = 2\sqrt{\alpha\beta}.
\end{aligned}$$

The last equation follows from the fact that the function $\psi(\lambda) := \alpha\lambda^2 + \beta\lambda^{-2}$ has its global minimum at $\lambda = \sqrt[4]{\beta/\alpha}$.

Now we construct the vectors $\overline{X}_1, \overline{X}_2$ with the claimed property (ii).

Since

$$\begin{aligned}
&\inf\{\Delta(AX_1, AX_2)f \mid \det A = 1\} \\
&\qquad = \operatorname{tr}\begin{pmatrix} \sqrt[4]{\beta/\alpha} & 0 \\ 0 & \sqrt[4]{\alpha/\beta} \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} \sqrt[4]{\beta/\alpha} & 0 \\ 0 & \sqrt[4]{\alpha/\beta} \end{pmatrix},
\end{aligned}$$

we have

$$\overline{X}_1 := \sqrt[4]{\frac{\beta}{\alpha}}\, Y_1 \qquad \text{and} \qquad \overline{X}_2 := \sqrt[4]{\frac{\alpha}{\beta}}\, Y_2.$$

Finally we consider the remaining cases.

For "$H_f^{X_1X_2} = 0$" there is nothing to proof.

Suppose $H_f^{X_1X_2}$ is semipositive but non positive definite. We can assume $\alpha = 0$. From (4.1a) it follows

Claim 4.1b.

$$\inf\{\Delta(gX_1, gX_2)f \mid \det g = 1\} = 0 = \det H_f^{X_1X_2}.$$

Thus the infimum will *not* be achieved. □

Motivated by the above lemma we define the two dimensional Laplace and directional curvature operators so that they depend on the plane E and not on the generating vectors.

DEFINITION 4.2.

$$\begin{aligned}\Delta(E,f) :&= \frac{2}{|X_1 \vee X_2|}\sqrt{H_f(X_1,X_1)H_f(X_2,X_2) - \big(H_f(X_1,X_2)\big)^2}\\ &= 2\sqrt{\det \mathbf{H}_f^E}\\ \mathcal{K}(E) :&= \pm\frac{\mathcal{K}(X_1,X_2)}{|X_1 \vee X_2|}.\end{aligned}$$

Here $|X_1 \vee X_2|$ denote the area of the parallelogram spanned by X_1 and X_2.

Remarks 4.3.

(i) The operator $\mathcal{K}(E)$ is defined only modulo sign. In fact $\mathcal{K}(E)f$ denotes 2 tangent vectors $\pm\mathcal{K}(X,Y)f$ (for an orthonormal basis X, Y).

(ii) Justifying the name "directional curvature", $-\|\mathcal{K}(E)\|^2 = K^M(E)$.

Proof. — Let X, Y be an orthonormal basis of $E \subset T_xM$. We may assume $x = x_0$, because $\mathcal{K}$ is invariant by isometries from G. Recall the identification $T_{x_0}M = \mathfrak{p} \subset \mathfrak{g}$. Using (2.6), it follows that

$$\begin{aligned} K^M(E) \equiv K^M(X,Y) &= g(R(X,Y)Y,X) \\ &= -\mathrm{re}B_{\mathfrak{g}}(-[[X,Y],Y],-X) = -B_{\mathfrak{g}}([[X,Y],Y],X) \\ &= B_{\mathfrak{g}}([X,Y],[X,Y]) = B_{\mathfrak{g}}(J[X,Y],-J[X,Y]) \\ &= -g(J[X,Y],J[X,Y]) = -g(\mathcal{K}(X,Y),\mathcal{K}(X,Y)). \end{aligned}$$

5. The fundamental form of a Riemannian hypersurface.

For convenience we recall some elementary facts about the fundamental form of a hypersurface in a Riemannian manifold (see [GHL], p. 216–226 for more details).

Let $S \subset M$ be a hypersurface in a Riemannian manifold (M,g), which is endowed with the induced metric g. We denote by ∇^S the Levi–Civita connection on (S,g) and by NS the normal subbundle of S in M. Locally there exists a defining function $f : U \to \mathbb{R}$ of S (i.e. $S \cap U = \{f = 0\}$ and $df \neq 0$ on S). We can use this function to define a (local) normal vector field on S :

$$\bar{n} : U_S \to NS|_U \qquad x \mapsto \|\mathrm{grad}\, f\|^{-1}\, \mathrm{grad}\, f.$$

Let $II_S(X,Y) := \nabla_X Y - \nabla^S_X Y, \quad \forall X,Y \in TS$, be the NS–valued second fundamental form of S and

$$q_S : TS \oplus TS \to \mathbb{R} \qquad \text{with} \qquad q_S(X,Y)\,\bar{n}_S = -II_S(X,Y) \tag{5.1a}$$

the corresponding real valued second fundamental form.

We list now some basic properties of q_S and the Gaussian curvature κ.

$$q_S(X,Y) = g(Q_S X, Y), \tag{5.1b}$$

where the symmetric operator Q_S is defined by $Q_S(X) := \nabla_X \bar{n}$.

The Gaussian curvature is defined by

$$\kappa(E) = \kappa(X,Y) := \frac{q_S(X,X)q_S(Y,Y) - q_S^2(X,Y)}{g(X,X)g(Y,Y) - g^2(X,Y)}. \tag{5.1c}$$

It is well-known, (see [GHL])

$$K^S(E) = K^M(E) + \kappa(E), \tag{5.1d}$$

where

$$K(E) = K(X,Y) = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g^2(X,Y)} \tag{5.1e}$$

denotes the sectional curvature of the plane E. The index refers to the corresponding Riemannian manifold S (resp. M).

The following lemma relates the (real valued) second fundamental form of S to the Hessian form H_r of a (local) defining function $r : M \to \mathbb{R}$ of $S \subset M$.

LEMMA 5.2.

$$q_S(X,Y) = \frac{1}{\|\operatorname{grad} r\|} H_r(X,Y) \qquad \forall\, X,Y \in TS.$$

Proof. — For arbitrary $X, Y \in T_pS$ let $\tilde{Y}$ denote an extension of Y in TS and let ∇ be the Levi–Civita covariant derivation of (M,g). From the definition of ∇ it follows that $\nabla g = 0$, i.e.

$$0 = X\, g(\operatorname{grad} r,\ \tilde{Y}) = g(\nabla_X \operatorname{grad} r, \tilde{Y}) + g(\operatorname{grad} r, \nabla_X \tilde{Y}).$$

Thus

$$-g(\operatorname{grad} r, \nabla_X \tilde{Y}) = g(\nabla_X(\operatorname{grad} r), \tilde{Y}). \tag{5.2a}$$

Further we have

$$\begin{aligned} -dr(\nabla_X \tilde{Y}) &= -g(\operatorname{grad} r, \nabla_X \tilde{Y}) = g(\nabla_X \operatorname{grad} r, Y) \\ &= \|\operatorname{grad} r\|\ g(\nabla_X \bar{n}, Y) + (X\|\operatorname{grad} r\|)\ g(\bar{n}, Y) \\ &= \|\operatorname{grad} r\|\ g(\nabla_X \bar{n}, Y) \overset{5.1b}{=} \|\operatorname{grad} r\|\ q_S(X,Y). \end{aligned} \tag{5.2b}$$

Recall the definition of the dual connection ∇^* : for $X, Y \in TS$,

$$\nabla^*_X(dr)(Y) = X(\tilde{Y} r) - dr(\nabla_X \tilde{Y}) = -dr(\nabla_X \tilde{Y}). \tag{5.2c}$$

Summarizing the above we obtain

$$H_r(X,Y) = \nabla^*_X(dr)(Y) = \|\operatorname{grad} r\|\ q_S(X,Y). \qquad \square$$

We use the above result to prove the equivalence of two inequalities which will be of use in our context.

LEMMA 5.3. — *Let $S \subset M$ be a hyperplane and r a defining function of S. We assume $X^2 r \geqslant 0$, $\forall\, X \in TS$. Then the following two inequalities*

(i) $\Delta(E,r) \geqslant 2\, |\mathcal{K}(E)r|, \qquad \forall\, E \subset T_pS,\ p \in S,$

(ii) $\kappa(E) \geqslant |\mathcal{K}(E)r|^2 = |g(\mathcal{K}(E), \bar{n}_S)|^2, \quad \forall\, E \subset T_pS,\ p \in S$

are equivalent.

Proof. — Let Y_1, Y_2 be an orthonormal basis of E. Lemma 4.1(i) implies that the inequality (i) is equivalent to

$$
\begin{aligned}
& 2\sqrt{\det \mathbf{H}_r} \geqslant 2\,|\mathcal{K}(E)r| \\
\iff\ & \det \mathbf{H}_r \geqslant |\mathcal{K}(E)r|^2 \\
\iff\ & H_r(Y_1,Y_1)H_r(Y_2,Y_2) - H_r(Y_1,Y_2)^2 \geqslant |\mathcal{K}(E)r|^2 \\
\overset{\text{Lemma 5.2}}{\iff}\ & q_S(Y_1,Y_1)q_S(Y_2,Y_2) - q_S(Y_1,Y_2)^2 \geqslant \frac{|\mathcal{K}(E)r|^2}{\|\operatorname{grad} r\|^2} \\
\overset{(5.1\text{c})}{\iff}\ & \kappa(E) \geqslant |\mathcal{K}(E)r|^2 = |g(\mathcal{K}(E), \bar{n}_S)|^2 \ .
\end{aligned}
$$

□

Now we are able to prove the main result of this paper. Throughout we will use the notation *local defining function for* $S = \partial\Omega_M$ for a function $r : U \to \mathbb{R}$ with $S \cap U = \{r = 0\}$, $U \cap \Omega_M = \{r < 0\}$, $dr|_{\partial\Omega_M} \neq 0$ and normalized by $\|\operatorname{grad} r\| = 1$ on S. Let $\rho := r \circ \pi$ be the corresponding local defining function of $\partial\Omega$. Notice that ρ is also of C^2.

THEOREM 5.4. — *Let $\Omega \subset G = K^{\mathbb{C}}$ be a K–invariant domain with a C^2–boundary and r and ρ be local defining function for $\partial\Omega_M$ and $\partial\Omega$ respectively.*

The domain Ω is Stein if and only if the two following conditions are fulfilled :

(i) *Ω_M is geodesic convex.*

(ii) *The (smooth) boundary $S := \partial\Omega_M$ satisfies one of the following equivalent conditions, for all two dimensional planes $E \subset TS$:*

(iia) $\kappa(E) \geqslant |g(\bar{n}_S, \mathcal{K}(E))|^2$

(iib) $K^S(E) \geqslant K^M(E) + |g(\bar{n}_S, \mathcal{K}(E))|^2 = -\|\mathcal{K}(E)\|^2 + |g(\bar{n}_S, \mathcal{K}(E))|^2$

(iic) $\Delta(E, r) \geqslant 2\ \mathcal{K}(E)r$ (*resp.* $\Delta(X,Y)r \geqslant 2\,\mathcal{K}(X,Y)r \quad \forall\ X, Y \in TS$).

The condition : "$\partial\Omega$ is strongly Levi convex" is equivalent to the corresponding conditions on the curvature of $\partial\Omega_M$, i.e. in this case in (ii) "$\geqslant$" can be replaced by "$>$".

The proof of this theorem breaks into several lemmas. From Lemma 5.3 and (5.1d) it follows that the curvature conditions (iia),(iib) and (iic) are equivalent.

First we show that, if for an invariant domain $\Omega \subset K^{\mathbb{C}}$ the conditions (i) and (ii) hold, then Ω is Stein. By a theorem of Docquier and Grauert, (see [DoGr]) it is enough to show that the boundary of Ω is Levi convex.

LEMMA 5.5. — *Let $\Omega \subset K^{\mathbb{C}}$ be a K–invariant domain, such that condition* (i) *is fulfilled. If γ_X is a geodesic in M such that $\dot{\gamma}(0) = X \in T_pS$, then for all $X \in TS$*

$$\left.\frac{d^2}{dt^2}\right|_{t=0} (r \circ \gamma_X(t)) = X^2 r \geqslant 0.$$

Proof of the lemma. — Let us assume the claim is false. If $X \in TS$ with $X^2 r < 0$, then

$$\gamma_X(-\epsilon, \epsilon) \cap S = \{p\} \qquad \text{and} \quad \gamma_X(-\epsilon, \epsilon) \subset \overline{\Omega}_M$$

for ϵ small enough. The group of isometries G acts transitively on M. Using a small such isometry, we can move γ_X to a geodesic segment with ends in Ω_M but which is *not* contained in this domain. This gives a contradiction. □

Let $T_{\mathbb{C}}\partial\Omega := T\partial\Omega \cap JT\partial\Omega$ denote the complex tangent bundle on $\partial\Omega$. The following lemma explains the connection between tangent vectors $Z \in T_{\mathbb{C}}\partial\Omega$ and tangent vectors in the (real)bundle $TS = T\partial\Omega_M$.

LEMMA 5.6. — *Let $Z = T + P \in TK^{\mathbb{C}}$ be the decomposition in the vertical and horizontal part.*

$$Z = T + P \in T_{\mathbb{C}}\partial\Omega \quad \Longleftrightarrow \quad \pi_* JT, \pi_* P \in TS.$$

In particular if $Z = T + P$ is contained in $T_{\mathbb{C}}\partial\Omega$, then also $Z' := T - P \in T_{\mathbb{C}}\partial\Omega$.

Proof of the lemma. — The boundary $\partial\Omega$ is K–invariant. This implies :

$$T_z\partial\Omega = \mathfrak{p} \cap T_z\partial\Omega \,\oplus\, \mathfrak{k} \,\subset\, \mathfrak{p} \oplus \mathfrak{k} = T_zG.$$

Since the subspaces $\mathfrak{p} = J\mathfrak{k}$ and $\mathfrak{k}$ are maximally totally real it follows that

$$T_{\mathbb{C},z}\partial\Omega = (T_z\partial\Omega \cap \mathfrak{p}) \oplus (JT_z\partial\Omega \cap J\mathfrak{p}) = (T_{\mathbb{C},z}\partial\Omega \cap \mathfrak{p}) \oplus J(T_{\mathbb{C},z}\partial\Omega \cap \mathfrak{p}).$$

On the other hand we have the following isomorphism : $\pi_* : (T_z\partial\Omega \cap \mathfrak{p}) \to T_xS, \ \ \pi(z) = x$. Hence the claim follows. □

Now we will estimate the Levi form of ρ by using the curvature conditions (ii).

Let $Z = JP_1 + P_2 \in T_{\mathbb{C}}\partial\Omega$ be arbitrary. The above lemma implies that $X_j := \pi_* P_j \in TS$. Our goal is to show that $\partial\Omega$ is Levi convex, i.e. for all $Z \in T_{\mathbb{C}}\partial\Omega$

$$\ell_\rho(Z) \geqslant 0.$$

Consider the basic formula 3.8

$$\ell_\rho(Z) = \Delta(X_1, X_2)r + 2\,\mathcal{K}(X_1, X_2)r.$$

Either we have $\mathcal{K}(X_1, X_2)r \geqslant 0$ in which case $\ell_\rho(Z) \geqslant 0$ follows from Lemma 5.5, or $\mathcal{K}(X_1, X_2)r < 0$. In this case let $\overline{X}_1, \overline{X}_2 \in E := ((X_1, X_2))$ be chosen as in Lemma 4.1. We conclude

$$\begin{aligned} \ell_\rho(Z) &= \Delta(X_1, X_2)r + 2\,\mathcal{K}(X_1, X_2)r \\ &\overset{4.1\mathrm{ii}}{\geqslant} \Delta(\overline{X}_1, \overline{X}_2)r + 2\,\mathcal{K}(\overline{X}_1, \overline{X}_2)r \\ &= |X_1 \vee X_2|\,(\Delta(E, r) + 2\,\mathcal{K}(E)r) \end{aligned}$$

(see also (4.2)), and the Levi convexity then follows from the curvature assumption. Hence Ω must be Stein.

To show the other direction we recall the following well known property of a K–invariant Stein domain $\Omega \subset K^{\mathbb{C}}$ (see [Ro], [Lo1] or [F]) :

Proposition 5.7. — *The corresponding domain $\Omega_M \subset M$ is geodesic convex.*

We finish the proof of the theorem by showing that, for a Stein invariant domain $\Omega \subset K^{\mathbb{C}}$, the boundary $\partial\Omega_M$ fulfilled the curvature condition (iic).

First we remark that, from the geodesic convexity of Ω_M, it follows from Lemma 5.5 that $X^2 r \geqslant 0 \quad \forall\ X \in TS$ i.e. $H_r|_E \geqslant 0$ for all two dimensional planes $E \subset TS$. Here we use the notation explained previous to the statement of the theorem.

Let $X_1, X_2 \in E \subset T_x S$ be an orthonormal basis and $P_1, P_2 \in T_z K^{\mathbb{C}}$ the corresponding horizontal lifts. We can assume $\mathcal{K}(X_1, X_2) \neq 0$ as otherwise the curvature condition follows trivially.

There are two possibilities :

Case 1 : $H_r|_E > 0$. Let $\overline{X}_1, \overline{X}_2 \in E$ be a basis of E as in Lemma 4.1 and $\overline{P}_1, \overline{P}_2 \in \mathfrak{p}$ the corresponding horizontal lifts. Consider the tangent vectors $Z := J\overline{P}_1 + \overline{P}_2$ and $Z' := J\overline{P}_1 - \overline{P}_2$ contained in $T_{\mathbb{C}}\partial\Omega$ (Lemma 5.6). Due to the Levi convexity of $\partial\Omega$ and the basic formula 3.7 we get :

$$(*) \qquad \Delta(\overline{X}_1, \overline{X}_2)r \pm 2\,\mathcal{K}(\overline{X}_1, \overline{X}_2)r = \ell_\rho(Z), \ell_\rho(Z') \geqslant 0,$$

which is equivalent to $\Delta(E,r) \geqslant 2\,\mathcal{K}(E)r$.

Case 2 : $H_r|_E \geqslant 0$ but not $H_r|_E > 0$.

In this situation by Lemma 4.1i $\inf\{\Delta(Y_1,Y_2)r | \mathcal{K}(Y_1,Y_2) = \text{const.}\} = 0$. Choose $X_1^s, Y_1^s \in E$ such that $\mathcal{K}(X_1^s, X_2^s) = \mathcal{K}(X,Y)$ and $\lim_{s\to\infty} \Delta(X_1^s, X_2^s)r = 0 = \Delta(E,r)$. Define $Z_s := JP_1^s + P_2^s$; $Z_s' := JP_1^s - P_2^s$; As usual P_j^s denotes the horizontal lifts of the corresponding vectors X_j^s. Taking the appropriate limit in $(*)$, it follows that $\Delta(E,r) = 2\,\mathcal{K}(E)r$ $(= 0)$. □

Remarks 5.8. — Let Ω be a $K \times K$–invariant domain in $K^{\mathbb{C}}$ i.e. Ω_M is K–invariant in M. It is well-known, (see [Las] or [FH]) that complex analytic properties of such domains can be characterized by the intersection $\Omega \cap T^{\mathbb{C}}$, where $T^{\mathbb{C}}$ is a maximal torus in $K^{\mathbb{C}}$. For example such a domain is Stein if and only if the corresponding domain in M is geodesically convex. The original proof of this fact uses representation theory. For domains with smooth boundary this can be also shown via a straight–forward differential geometric calculation using methods developed in this paper.

The case SL(2,C)

For $K := \mathrm{SU}_2$ and $K^{\mathbb{C}} = \mathrm{SL}_2(\mathbb{C})$, the quotient $M = \mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2$ is isometric equivalent to the 3-dimensional hyperbolic plane

$$\mathbb{H}^3 := \{(x,y,z) \in \mathbb{R}^3 |\ z > 0\} \qquad g = \frac{1}{z^2}\big(dx \otimes dx + dy \otimes dy + dz \otimes dz\big)$$

with constant negative sectional curvature equal to -1.

A hypersurface S in M is two dimensional. Hence the directional curvature $\mathcal{K}(T_xS)$ is parallel to the normal vector field n_S of S, (see 3.7b). The right hand part of the curvature formula 5.4 (iib) is zero.

PROPOSITION 5.9. — *Let $\Omega \subset \mathrm{SL}_2(\mathbb{C})$ be a SU_2–invariant domain with smooth boundary such that the corresponding domain Ω_M in the quotient M is geodesically convex. Then Ω is Stein if and only if it holds :*

$$K(T\partial\Omega_M) \geqslant 0.$$

The boundary $\partial\Omega$ is in a point p semipositive if and only if $K(T_{\pi(p)}\partial\Omega_M) = 0$.

Remarks 5.10. — Berteloot investigated the behavior of plurisubharmonic functions on $\mathrm{SL}(2,\mathbb{C})$ invariant by action of cyclic discrete subgroup Γ and he showed that such functions are invariant also by the Zariski closure $\overline{\Gamma}$. The main step in [B] is the proof of the following fact by using "L^2 - methods" of Hörmander and Skoda :

A $U_{\mathbb{R}} := \left\{ \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \mid x \in \mathbb{R} \right\}$– invariant psh. function is also $U_{\mathbb{C}} := \left\{ \left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right) \mid z \in \mathbb{C} \right\}$– invariant.

The following stronger result can be proved via 5.9 and an elementary computation of the sectional curvature of $\partial\Omega_M$ in M, (see [F]).

A Stein $U_{\mathbb{R}}$–invariant domain is also $U_{\mathbb{C}}$–invariant.

It should be also remarked that the domain $\Omega \subset \mathrm{SL}_2(\mathbb{C})$ for Loeb's counterexample mentioned above corresponds to a domain in $M = \mathbb{H}^3$ which is bounded by a two dimensional totally geodesic submanifold S, which is isometric to $\mathbb{H}^2$. In particular S has everywhere sectional curvature $K = -1$. The Stein holomorphic hull of this $\Omega \subset \mathrm{SL}(2,\mathbb{C})$ is $\mathrm{SL}(2,\mathbb{C})$ itself. In fact, for an arbitrary K–invariant domain in an arbitrary complex reductive Lie group $G = K^{\mathbb{C}}$ it can be shown that the envelope of holomorphy lies in G. (this is true for any invariant domain in an arbitrary complex reductive group G) and is the whole $\mathrm{SL}(2,\mathbb{C})$, see [F].

We conclude this section by observing that in a complex *semisimple* group $K^{\mathbb{C}}$ there exists no Levi flat hypersurface, which is also K–invariant.

PROPOSITION 5.11. — *The Levi form of a K–invariant real hypersurface H in a semisimple group $K^{\mathbb{C}}$ is not identically zero at every point $z \in H$.*

Proof of the proposition. — Let $H \subset K^{\mathbb{C}}$ be a K–invariant hypersurface in $K^{\mathbb{C}}$ and S the corresponding hypersurface in M. Assume the Levi form of H vanish in $e \in H$. From the basic formula 3.7 it follows that for all $X, Y \in T_{x_0}S$

$$\mathcal{K}(X,Y)r = 0.$$

As usual r denotes a local defining function of S. Identifying $T_{x_0}M = \mathfrak{p} = J\mathfrak{k} \subset \mathfrak{k}^{\mathbb{C}}$, if follows from the definition of $\mathcal{K}$ that

there exists a hypersurface $V \subset \mathfrak{p}$ with the property $J[V,V] \subset V$;

here, for instance $V := \ker d\rho \cap \mathfrak{p}, \quad \rho = r \circ \pi$. The following lemma shows that this cannot happen.

LEMMA 5.12. — *Let V be a hypersurface in a compact semisimple Lie algebra $\mathfrak{k}$. Then $\quad [V, V] \not\subset V$.*

Proof of the lemma. — Since K is compact and semisimple, the Killing form $B_{\mathfrak{k}}$ of $\mathfrak{k}$ is negative definite. Assume there exists $V \subset \mathfrak{k}$ with $[V, V] \subset V$. Then let $\mathbb{R}{\cdot}T$ be the orthogonal complement in $\mathfrak{k}$. For all $v, w \in V$ it follows that

$$\forall\, v, w \in V \qquad 0 = B_{\mathfrak{k}}(T, [v, w]) = B_{\mathfrak{k}}([T, v], w),$$

thus $\mathrm{ad}_T(V) \subset \mathbb{R}\, T$. Since $\mathrm{ad}_T(T) = 0$, this implies

$$B_{\mathfrak{k}}(T, T) = \mathrm{tr}\,(\mathrm{ad}_T \circ \mathrm{ad}_T) = \mathrm{tr}\, 0 = 0,$$

in contradiction to the assumption that $B_{\mathfrak{k}}$ is negative definite. □

6. Invariant Stein domains in $K^{\mathbb{C}}$ and -log d.

In this section we study on (M, g) the induced distance function $d : M \times M \to \mathbb{R}_{\geqslant 0}$. As usual we consider

$$d(p, q) := \inf_{c \in \mathcal{L}_{pq}} \int \sqrt{g(\dot{c}(t), \dot{c}(t))}\, dt. \tag{6.1}$$

Here the infimum is taken over all piecewise smooth curves in M connecting p and q. Recall that M is a complete, simply connected Riemannian manifold of non-positive sectional curvature. Thus any two points can be connect by a unique geodesic. The length of such geodesic is equal to the distance between its end points. This metric structure is compatible with the original topological structure on M, (see [Hel]). The isometries of the Riemannian metric g are isometries of the distance d and vice versa.

Example. — Let $d_e : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$ be the Euclidean metric. For a domain $\Omega \subset \mathbb{C}^n$ the boundary distance function d_Ω can be defined as follows :

$$d_\Omega(x) = \inf_{y \in \partial\Omega} d_e(x, y) = \sup\{r \mid B_r(x) \subset \Omega\}.$$

It is a classical result that a domain $\Omega \subset \mathbb{C}^n$ is Stein if and only if $-\log d_\Omega$ is a plurisubharmonic function.

In this section we will prove that an analogous result holds for invariant domains in $K^{\mathbb{C}}$.

The distance function.

Let Ω be a K–invariant domain in $K^{\mathbb{C}}$ without any regularity conditions and let Ω_M be the corresponding domain in M. Define

$$(6.2) \qquad d_\Omega(x) := \inf_{y \in \partial\Omega_M} d(x,y) = \sup\{r \mid B_r(x) \subset \Omega_M\}$$

to be the boundary distance function with respect to $d : M \times M \to \mathbb{R}_{\geqslant 0}$. Here $B_r(x)$ denotes the metric ball in M with radius r and center x. Notice that d_Ω is continuous on Ω_M and $d_\Omega(x_n) \to 0$ if $(x_n) \to \partial\Omega_M$.

THEOREM 6.3. — *A K–invariant domain Ω is Stein if and only if the function $-\log d_{\Omega_M} \circ \pi$ is plurisubharmonic.*

Proof. — ("$\Longleftarrow$")

Let Ω be an invariant domain such that $-\log d_{\Omega_M} \circ \pi =: -\log d_\Omega \circ \pi$ is plurisubharmonic. Since :

$-\log d_\Omega(x_n) \to \infty$ for each sequence $x_n \in \Omega_M$, $x_n \to p \in \partial\Omega_M$, the function $\phi := -\log d_\Omega(x) + d^2(x, x_0)$ is an exhaustion function. The induced function $\phi\circ\pi$ is also an exhaustion function and plurisubharmonic, because $g \mapsto d^2(\pi(g), x_0)$ is also one (see [Lo2]). The domain Ω is contained in a Stein manifold $K^{\mathbb{C}}$ so that, from the affirmative solution of the Levi problem ([DoGr]), it follows that Ω is Stein.

("$\Longrightarrow$")

We must show that $-\log d_\Omega \circ \pi$ is plurisubharmonic. For this we must show that the maps

$$z \mapsto -\log d_\Omega \circ \pi(g \cdot \exp zX)$$

from the disc $\Delta_r := \{z \in \mathbb{C} \mid |z| < r\}$ (r small enough) are subharmonic $\forall\, g \in \Omega$ and $X \in T_eG = \mathfrak{g}$. Here Δ_r; $\Delta \equiv \Delta_1$.

It is well-known (see [N]) that $\phi : U \to \mathbb{R}$, $U \subset \mathbb{C}$ is plurisubharmonic if and only if for every disc $\overline{\Delta}_r(z_0) \subset U$ and every $h \in \mathcal{O}(U)$ the following condition is fulfilled :

$$(*) \qquad \phi \leqslant \operatorname{re} h \text{ on } \partial\overline{\Delta}_r \quad \Longrightarrow \quad \phi \leqslant \operatorname{re} h \text{ on } \overline{\Delta}_r.$$

We will now show that the function $z \mapsto -\log d_\Omega \circ \pi(g \cdot \exp zX)$ satisfies the condition $(*)$.

First we reformulate the inequality in $(*)$:

$$\begin{aligned} & -\log d_\Omega \circ \pi(g \cdot \exp zX) \leqslant \operatorname{re} h(z) \\ (6.3a) \qquad & \Longleftrightarrow d_\Omega \circ \pi(g \cdot \exp zX) \geqslant e^{-\operatorname{re} h(z)} = |e^{-h(z)}| \\ & \Longleftrightarrow \overline{B}_{|e^{-h(z)}|}(\pi(g \cdot \exp zX)) \subset \overline{\Omega}_M. \end{aligned}$$

(By a standard technique of a suitable limit process applied to $h+\epsilon$, we can assume $\overline{B}_{|e^{-h(z)}|}(...) \subset \Omega_M$.)

The idea of the proof is a construction of a suitable Hartogs figure F in Ω. The question on plurisubharmonicity of $-\log d_\Omega \circ \pi$ can then be reduced to the question when the hull of F is also contained in Ω. Of course, for a Stein domain this is clearly the case.

Construction of a Hartogs figure. Let $g \in \Omega$ and $X \in T_gG$ be arbitrarily chosen. Let be $r > 0$ small enough such that $\exp(\overline{\Delta}_r X) \cdot g \subset \Omega$. Further let $h \in \mathcal{O}(U(\overline{\Delta}_r))$ fulfill the inequality

$$-\log d_\Omega \circ \pi(g \cdot \exp zX) < \operatorname{re} h(z) \text{ on } \partial\overline{\Delta}_r.$$

Define

$$(6.3b) \qquad H_P : \overline{\Delta}_r \times \overline{\Delta} \to G \qquad (z,w) \mapsto g \cdot \exp zX \cdot \exp(we^{-h(z)}\,P).$$

Here $P \in \mathfrak{g} = T_pG$ is an arbitrary horizontal tangent vectors of length 1. Notice that H_P is a holomorphic map, because the group theoretical exponential mapping is holomorphic. We assert :

Claim Hartogs :

$$\begin{aligned} H_P(\overline{\Delta}_r, 0) &\subset \Omega \\ H_P(\partial\overline{\Delta}_r, \Delta) &\subset \Omega. \end{aligned}$$

Proof of the claim. — The first inclusion is clear. To show the second inclusion we will study $\pi(g \cdot \exp zX \cdot \exp(we^{-h(z)}P))$.

Due to $[P, JP] = J[P,P] = 0$, we conclude

$$\begin{aligned} \exp(we^{-h(z)}P) &= \exp\big(\operatorname{re}(we^{-h(z)})P + \operatorname{im}(we^{-h(z)})JP\big) \\ &= \exp(\operatorname{re}(we^{-h(z)})P) \cdot \exp(\operatorname{im}(we^{-h(z)})JP). \end{aligned}$$

Thus for all $(z,w) \in \overline{\Delta}_r \times \overline{\Delta}$,

(6.3c) $\pi(g \cdot \exp zX \cdot \exp(we^{-h(z)}P)) = g \cdot \exp zX \cdot \exp(\mathrm{re}\,(we^{-h(z)})P) \cdot x_0.$

From our assumption, it follows that $\overline{B}_{|e^{-h(z)}|}(\pi(g \cdot \exp zX)) \subset \Omega_M$ for $z \in \partial\Delta_r$. This means that for all $T \in \mathfrak{p}$ with $\|T\| \leqslant |e^{-h(z)}|$ the geodesic segment $t \mapsto g \cdot \exp zX \cdot \exp tT\ x_0$, $t \in [0,1]$, must be contained in Ω_M. Since $\|\mathrm{re}\,(we^{-h(z)})P\| \leqslant |e^{-h(z)}|$, it follows from 6.3c that $\pi(g \cdot \exp zX \cdot \exp(we^{-h(z)}P))$ is contained in Ω_M and the claim follows.

Let $F := \overline{\Delta}_r \times \{0\} \cup \partial\Delta_r \times \overline{\Delta}$. be a Hartogs figure. Because of claim (H) there exists a neighbourhood $U(F) \subset \overline{\Delta}_r \times \overline{\Delta}$ such that $U(F) \subset H_P^{-1}(\Omega)$. By assumption, the domain Ω is Stein. Thus, since it contains $H_P(U(F))$, it also contains the hull $H_P(\overline{\Delta}_r \times \overline{\Delta})$ of $H_P(U(F))$ i.e. :

$$
\begin{aligned}
& g \cdot \exp zX \cdot \exp(we^{-h(z)}P) \subset \Omega && \forall\ (z,w) \in \overline{\Delta}_r \times \overline{\Delta} \\
\iff\ & g \cdot \exp zX \cdot \exp(\mathrm{re}\,(we^{-h(z)})P)\ x_0 \subset \Omega_M && \forall\ (z,w) \in \overline{\Delta}_r \times \overline{\Delta}.
\end{aligned}
$$

For a suitable choice of $w \in \overline{\Delta}$, the geodesic $t \mapsto g \cdot \exp zX \cdot \exp(t|e^{-h(z)}|P)\ x_0$ $t \in [0,1]$ must be contained in Ω_M. This holds for all $P \in \mathfrak{p}$, $\|P\| = 1$. Therefore for all $z \in \overline{\Delta}_r$

$$\overline{B}_{|e^{-h(z)}|}(\pi(g \cdot \exp zX)) \subset \Omega_M.$$

Due to $(*)$ and 6.3a, this inclusion implies our claim : $-\log d_\Omega \circ \pi_\Omega \circ \pi$ is plurisubharmonic. □

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Gregor FELS,
Mathematisches Institut
Fachbereich 6
Universität GHS Essen
D-45141 Essen.