ANNALES DE L'INSTITUT FOURIER

TODOR GRAMCHEV
GEORGI POPOV

Nekhoroshev type estimates for billiard ball maps

Annales de l'institut Fourier, tome 45, nº 3 (1995), p. 859-895
<http://www.numdam.org/item?id=AIF_1995__45_3_859_0>

Ann. Inst. Fourier, Grenoble
45, 3 (1995), 859-895

NEKHOROSHEV TYPE ESTIMATES FOR BILLIARD BALL MAPS

by T. GRAMCHEV(1) and G. POPOV(2)

1. Introduction.

This paper is concerned with the effective stability of the billiard flow near the boundary of a strictly convex bounded domain Ω in $\mathbf{R}^{n+1}$, $n \geq 1$, with an analytic boundary $\partial\Omega$. The billiard flow in Ω, called as well generalized geodesic flow, is described by a particle moving with unit velocity along straight lines inside Ω and reflecting at the boundary by the law of the geometric optics "angle of reflection equals angle of incidence". The boundary is invariant with respect to the billiard flow, and if the particle is on it with velocity tangent to $\partial\Omega$, then it travels with unit speed along the corresponding geodesic of $\partial\Omega$. We are interested in effective stability estimates for the billiard flow near the boundary, i.e. in stability for finite but exponentially long time intervals.

The billiard flow induces a discrete dynamical system at the boundary described by the billiard ball map

$$B: J \to J, \quad J = \{(x,\xi) \in T^*\mathbf{R}^{n+1} : \; x \in \partial\Omega, \; |\xi| = 1\},$$

(see Sect. 2) which is easier to deal with. The map B is analytic in J and it coincides with the identity mapping on the glancing manifold

(1) Partially supported by funds MURST 40 %, Italy, by GNAFA of the CNR, Italy and by grant MM-48/91 with MES, Bulgaria.

(2) Supported by the Alexander von Humboldt Foundation and partially supported by grant MM-8/91 with MES, Bulgaria.

Key words : Billiard ball map – Nekhoroshev type estimate – Glancing hypersurfaces.
Math. classification : 58F.

$$K = \{(x, \xi) \in J : \ \langle \xi, n(x) \rangle = 0\},$$

$n(x)$ being the inward unit normal to $\partial\Omega$ at x.

The main problem we are concerned with, is to find an upper bound for the rate of diffusion of the iterates

$$B^j(\varrho), \ j = 1, 2, ...,$$

from the glancing manifold K for ϱ close to K. The plane case is trivial, because there is a large set of smooth invariant curves of B accumulating at K [16], [21], which trap any orbit starting close to K inside an annulus. The corresponding caustics (the envelope of the rays issued from the invariant circles) are smooth and strictly convex curves situated an a neighborhood of $\partial\Omega$ inside Ω [16]. The picture is completely different when $n \geq 2$. It was shown by Berger [4] that if $n = 2$ and $\partial\Omega$ admits a smooth caustic inside Ω, then Ω is an ellipsoid. Generically, even if there exist invariant tori of B in J, they are at least of codimension two, and the orbits could escape any fixed neighborhood of K.

The first effective stability result for perturbations of a completely integrable Hamiltonian was obtained by Nekhoroshev. The Nekhoroshev theorem [22], [23], states that the variation of the action of each orbit of an analytic Hamiltonian H_ε, close to a completely integrable one H_0, remains stable in a finite but exponentially large time interval

$$0 \leq t \leq T \exp(\varepsilon^{-a}),$$

if H_0 satisfies certain generic steepness conditions. Since then a number of new results about effective stability have appeared. Sharper estimates on the stability exponent a have been proved recently by Benettin, Gallavotti, Galgani, Giorgilli and others in the convex case (H_0 is convex) in order to investigate stability problems in Celestial and Statistical mechanics [2], [3], [9]. A new approach to the effective stability of convex Hamiltonians, based on an analysis near the worst resonances of the system, has been recently proposed by Lochak [17]. The best exponent $a = 1/2n$ in the quasi-convex case was found by Lochak and Neishtadt [18] and by Pöschel [25] who added new geometric ideas to the traditional proof. Exponential stability for time dependent potentials has been proved by Giorgilli and Zehnder [10]

Effective stability results for iterates of a symplectic mapping P_ε close to a completely integrable one P_0 have been proved by Kuksin and Pöschel

[15]. The main idea there, is to write P_ε as a time-one-shift of the flow of a 1-periodic analytic Hamiltonian H_ε which is ε - close to a completely integrable one. Bazzani, Marmi and Turchetti have obtained Nekhoroshev estimates for isochronous non resonant symplectic maps [1].

The main difference between this paper and the results cited above is that the billiard ball map is in general far from a completely integrable one for $n \geq 2$. Nevertheless, there exists a smooth "approximate" first integral ζ of B defining K, which means that $\zeta \in C^\infty(J)$, $\zeta = 0$ and $d\zeta \neq 0$ on K, and

$$\zeta(B(x,\xi)) = \zeta(x,\xi) + r(x,\xi),\ (x,\xi) \in J, \tag{1.1}$$

where the function $r \in C^\infty(J)$ is flat at K. Hereafter, we say that a function $r \in C^\infty(J)$ is flat at K, if r has a zero of infinite order at K, i. e. for any smooth vector field Y in J and any integer $k \geq 0$ the function $Y^k r$ vanishes at K. A natural candidate for ζ is the approximate interpolating Hamiltonian of B which has been introduced in a little bit different context for the corresponding boundary maps $\delta^\pm$ by Marvizi and Melrose in [19] (see also [12]). In our case J is equipped with an analytic two-form ω_0 given by the pull-back of the canonical two-form in $T^*\mathbf{R}^{n+1}$ via the natural inclusion map $J \to T^*\mathbf{R}^{n+1}$. Note that the two-form ω_0 is symplectic in $J \setminus K$ but it is degenerate at K. More precisely, taking $n-1$ times the exterior product of ω_0 by itself we get

$$\omega_0^n = \langle \xi, n(x)\rangle dv,\ (x,\xi) \in J,$$

where dv is a volume form in J and $d\langle \xi, n(x)\rangle \neq 0$ at K since $\partial\Omega$ is strictly convex (see Sect. 2). In particular, for any $\phi \in C^\infty(J)$ satisfying $d\phi(\varrho) = 0$, $\forall \varrho \in K$, the Hamiltonian vector field H_ϕ of ϕ with respect to ω_0 is well defined by the inner product $\imath(H_\phi)\omega_0 = -d\phi$, and H_ϕ is smooth in J. We will call $\zeta \in C^\infty(J)$ an approximate interpolating Hamiltonian of B if $\zeta = 0$ and $d\zeta \neq 0$ on K, and for any $f \in C^\infty(J)$ the function

$$R(\varrho) = f(B(\varrho)) - f\left(\exp\left(\zeta(\varrho)H_{\zeta^2}\right)(\varrho)\right),\ \varrho \in J, \tag{1.2}$$

is flat at K. Here $t \to \exp(tH_{\zeta^2})$ stands for the one-parameter group of diffeomorphisms of the smooth vector field H_{ζ^2} of ζ^2 with respect to ω_0. Making use of the normal forms of glancing hypersurfaces obtained by Melrose [20], one can find an approximate interpolating Hamiltonian ζ of B as in [12] and [19].

The main goal in this paper is to prove the existence of an approximate interpolating Hamiltonian ζ of B of Gevrey class $G^2(J)$ (for a definition of Gevrey classes see Sect. 2) provided that the boundary $\partial\Omega$ is analytic and strictly convex. Then the function R in (1.2), respectively r in (1.1), will be in G^2 as well, and since r is flat at K we will get the estimate

$$|r(x,\xi)| \leq C_1 \exp\left(-\frac{C}{|\zeta(x,\xi)|}\right), \quad (x,\xi) \notin K,$$

with some C, $C_1 > 0$, which leads to effective stability at the boundary. Note that $\zeta(x,\xi)$ measures the "distance" in J from a given point $(x,\xi) \in J$ to the glancing manifold K.

For any $\varrho \in J$ and any positive integer k we denote by $T_k(\varrho)$ the length of the broken geodesic arc issuing from ϱ and having k points of reflection at $\partial\Omega$, i.e.

$$T_k(\varrho) \;=\; \sum_{j=0}^{k-1} \left|x^{j+1} - x^j\right|,$$

where $\varrho = (x^0,\xi^0)$, $\varrho^j = (x^j,\xi^j) = B^j(\varrho)$, $j = 1,\ldots,k$, and $\left|x^{j+1} - x^j\right|$ stands for the usual distance in $\mathbf{R}^{n+1}$.

Our main result is :

THEOREM 1. — *Let Ω be a strictly convex bounded domain in $\mathbf{R}^{n+1}$, $n \geq 1$, with an analytic boundary. Then there exists an approximate interpolating Hamiltonian $\zeta \in G^2(J)$ of the billiard ball map B. Moreover, there exist positive constants δ and C such that for any $0 < \varepsilon \leq 1$, and any $\varrho = (x,\xi) \in J$, $0 < |\zeta(\varrho)| < \delta$, we have*

$$\left|\zeta(B^k(\varrho)) - \zeta(\varrho)\right| \;<\; \varepsilon\zeta(\varrho)^2$$

provided that

$$0 \;\leq\; T_k(\varrho) \;\leq\; \varepsilon|\zeta(\varrho)|^5 \exp(C|\zeta(\varrho)|^{-1}).$$

More generally, we prove in Section 2 that Theorem 1 holds for the billiard ball map associated with any pair of analytic glancing hypersurfaces having a compact glancing manifold K. In particular, we obtain effective stability estimates near K for the billiard ball map of any compact real-analytic Riemannian manifold whose boundary is strictly geodesically convex. The main idea in the proof is to find simultaneously a local normal form

$$\sigma_0 = 2\xi_1 d\xi_1 \wedge dx_1 + \sum_{j=2}^{n} d\xi_j \wedge dx_j \tag{1.3}$$

for ω_0 and an approximate local normal form for the billiard ball map in G^2

$$B(x,\xi) = (x_1 + \xi_1, x_2, \dots, x_n, \xi) + R(x,\xi), \tag{1.4}$$

where $K = \{\xi_1 = 0\}$, the function R belongs to the Gevrey class G^2 and it is flat at K. When the boundary is strictly convex and C^∞ smooth, Melrose [20] has found smooth local coordinates (x, ξ) such that (1.3) and (1.4) hold with $R \equiv 0$. In particular, the billiard ball maps of any two strictly convex domains with smooth boundaries are locally equivalent to each other in the C^∞ category. More generally, Melrose proved that any two pairs of glancing hypersurfaces are locally symplectically equivalent in the C^∞ case. As it was observed by Oshima [24], this is not true in the analytic case. In fact, the example of Oshima shows even that there exist pairs of analytic glancing hypersurfaces which are not locally symplectically equivalent in the Gevrey classes G^s, $1 \le s < 2$, (see Remark 2.4). Oshima's example suggests that $s = 2$ is the best Gevrey regularity for the approximate interpolating Hamiltonian one can hope for.

The existence of the normal forms (1.3) and (1.4) of ω_0 and B is influenced by the construction of the normal forms in [11] and [20]. The novelty in this paper is, that we analyse rather precisely the formal power series arising in the traditional proof. These series do not converge in the usual sense but they do converge in suitable "Gevrey" spaces of formal series. Similar idea (to estimate the rate of divergence of formal series arising in normal forms) has been used in [9] to study the effective stability for a hamiltonian system in the vicinity of an elliptic equilibrium point. Our approach is different from those in [9], it is based on certain techniques which come from the calculus in Gevrey classes (see [5], [6], [13], [26]). The relationship of the Gevrey classes with this type of problems was pointed out by Lochak [17] as well. We would like to mention that our method could be used to treat Gevrey normal forms for the billiard ball maps of pairs of non analytic Gevrey glancing hypersurfaces as well.

The paper is organized as follows : In Sect. 2 we consider pairs of analytic glancing hypersurfaces F, G, in an analytic symplectic manifold

M and the corresponding analytic involutions $\mathcal{J}_F$ and $\mathcal{J}_G$ in $J = F \cap G$ having a common fixed point manifold K. The billiard ball map is given by $B = \mathcal{J}_F \circ \mathcal{J}_G$, and the analytic two-form ω_0 is invariant with respect to both the involutions and it is degenerate at K. Theorem 1 follows from Theorem 2.3 which provides simultaneously a G^2 normal form for the two-form ω_0 and the involution $\mathcal{J}_F$ and gives at the same time an approximate normal form of $\mathcal{J}_G$ modulo an error term of Gevrey class G^2 which is flat at K. Theorem 2.3 is proved in two steps. First we obtain an approximate normal form for the pair of involutions in G^2 (see Theorem 3.1) paying no attention to the form ω_0. To put ω_0 in a normal form keeping fixed the normal forms of the pair of involutions given by Theorem 3.1 one could adapt the proof of Theorem 21.4.4 in [11]. Instead, we give a new proof which is based on the deformation argument of Moser-Weinstein, exploring the invariance of the two-form ω_0 under the involutions. Theorem 3.1 is proved in Sect. 4. The Appendix contains technical lemmas concerning some estimates in Gevrey classes.

Using the results of this paper we can show that the billiard ball maps of any two strictly convex domains with analytic boundaries are G^3 equivalent to each other. This represents a loss of Gevrey regularity with respect to the approximate interpolating Hamiltonian of B which has G^2 Gevrey regularity. The corresponding result is a subject of another paper [8].

2. Glancing hypersurfaces.

The billiard ball map in a strictly convex bounded domain can be associated to a pair of transversally intersecting glancing hypersurfaces. First we recall certain facts about pairs of glancing hypersurfaces which can be found in [11], [20]. First we consider a strictly convex domain Ω in $\mathbf{R}^{n+1}$ with a real analytic boundary $\partial\Omega$, $n \geq 1$. Denote by

$$\omega = d\xi_1 \wedge dx_1 + \ldots + d\xi_{n+1} \wedge dx_{n+1}$$

the canonical two-form in $T^*\mathbf{R}^{n+1}$. Let f be a real analytic function in $\mathbf{R}^{n+1}$ such that $f(x) = 0$, $df(x) \neq 0$, on $\partial\Omega$, and $f(x) > 0$ inside Ω. Set $f(x,\xi) = f(x)$ and $g(x,\xi) = 1 - |\xi|^2$, and denote by X_f and X_g the correponding Hamiltonian vector fields with respect to ω. Consider the pair of analytic hypersurfaces

$$F = \{(x,\xi) \in T^*\mathbf{R}^{n+1} : f(x,\xi) = 0\}, \ \ G = \{(x,\xi) \in T^*\mathbf{R}^{n+1} : g(x,\xi) = 0\}.$$

Since $\partial\Omega$ is strictly convex, F and G form a pair of transversal glancing hypersurfaces [11], [20], i.e. F and G intersect transversally at $J = F \cap G$, and for any $\varrho \in J$, either the integral curve of X_f (X_g) passing through ϱ intersects G (F) transversally or it is simply tangent to G (F). More precisely, the equality

$$\{f,g\}(\varrho) = 0, \ \varrho = (x,\xi) \in J, \tag{2.1}$$

(i.e. $\langle \xi, n(x)\rangle = 0$) implies

$$\{f,\{f,g\}\}(\varrho) < 0, \quad \{g,\{g,f\}\} < 0. \tag{2.2}$$

Indeed, the first term in (2.2) is equal to $-2|f_x(x)|^2 < 0$ and the second one is $4\langle f_{xx}(x)\xi,\xi\rangle < 0$. Here $\{\cdot,\cdot\}$ stands for the Poisson brackets in $T^*\mathbf{R}^{n+1}$ corresponding to ω while $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product in $\mathbf{R}^{n+1}$. Because of (2.2), the glancing set

$$K = \{f = g = \{f,g\} = 0\}$$

is a submanifold of J. Indeed, the differentials df, dg, and $d\{f,g\}$ are linearly independent at any $\varrho \in K$, since df and dg are linearly independent at ϱ and

$$df(X_f) = dg(X_f) = 0, \ d\{f,g\}(X_f) \neq 0,$$

at ϱ according to (2.2). There are two analytic involutions

$$\mathcal{J}_F : J \longrightarrow J, \ \mathcal{J}_G : J \longrightarrow J,$$

defined as follows. For any ϱ in J outside K, $\mathcal{J}_F(\varrho)$ ($\mathcal{J}_G(\varrho)$) is the second point of intersection with J of the integral curve of X_f (X_g) passing through ϱ, and $\mathcal{J}_F$ ($\mathcal{J}_G$) coincides with the identity mapping on the glancing manifold K. The billiard ball map B is defined by

$$B = \mathcal{J}_F \circ \mathcal{J}_G. \tag{2.3}$$

Note that J is a manifold of dimension $2n$ equipped with a two-form ω_0 which is the pull-back of ω via the natural inclusion mapping $J \to T^*\mathbf{R}^{n+1}$. The two-form ω_0 is symplectic in $J\backslash K$ but it is degenerate on K, indeed, in view of Lemma 21.4.7 in [11] we have

$$\omega_0^n = m\,v, \tag{2.4}$$

where v is a volume form in J, $m(x,\xi) = \{f,g\}(x,\xi)$, and the differential of $m(\varrho)$ does not vanish on K according to (2.2). Moreover, ω_0 is invariant with respect to the involutions $\mathcal{J}_F$ and $\mathcal{J}_G$. The set of fixed points of both $\mathcal{J}_F$ and $\mathcal{J}_G$ coincides with K and their differentials are linearly independent at any point of K.

More generally, we consider an analytic symplectic manifold (M,ω) where M is an analytic manifold of dimension $2n+2$, $n \geq 1$, and ω is an analytic symplectic two-form on it. Denote by $\{\cdot,\cdot\}$ the Poisson brackets in M corresponding to ω. We consider a pair of analytic transversal glancing hypersurfaces F, G, in M of the form

$$F = \{\varrho \in M : \ f(\varrho) = 0\}, \ G = \{\varrho \in M : \ g(\varrho) = 0\}, \tag{2.5}$$

where f and g are smooth functions in M and analytic in a neighborhood of $J = F \cap G$, such that $df \neq 0$ on F, $dg \neq 0$ on G, and the differentials df and dg are linearly independent at J. As above we suppose that equality (2.1) implies (2.2) and we denote by ω_0 the pull-back of ω via the inclusion map $J \to M$. Moreover, we assume that the glancing manifold $K = \{\varrho \in J : \{f,g\}(\varrho) = 0\}$ is compact. It is easy to see that the involutions $\mathcal{J}_F$ and $\mathcal{J}_G$ are well-defined and analytic in a neighborhood J_0 of K in J, and we define $B : J_0 \to J$ by (2.3). Moreover, the function $t(\varrho)$, $\varrho \in J_0$, defined by

$$\exp\left(t(\varrho)X_g\right)(\varrho) = \mathcal{J}_G(\varrho)$$

is analytic in J_0 in view of the implicit function theorem (see (2.10)), and we have $\pm t(\varrho) > 0$ in

$$J_\pm = \{\varrho \in J_0 : \ \pm\{g,f\}(\varrho) > 0\}.$$

On the other hand, for any positive integer k there is a neighborhood U of K in J_0 such that $B^j(J_\pm \cap U) \subset J_\pm$ for every $0 \leq j \leq k$. Then we set $T_0(\varrho) = 0$, and define

$$T_j(\varrho) = \sum_{p=0}^{j-1} |t(B^p(\varrho))|, \ \varrho \in J_\pm \cap U, \ 1 \leq j \leq k.$$

The broken bicharacteristic $\Phi^t(\varrho)$ of G issuing from a point ϱ in $J_\pm \cap U$ and propagating for $\pm t \in [0, T_{k+1}(\varrho))$ in the domain

$$G_0 = \{(x,\xi) \in G : \ f(x,\xi) \geq 0\}$$

will be defined as follows :

$$\Phi^t(\varrho) = \exp(tX_g)(B^j(\varrho)), \quad T_j(\varrho) \le \pm t < T_{j+1}(\varrho), \quad 0 \le j \le k.$$

Let us take for example $M = T^*\mathbf{R}^{n+1}$ equipped with the standard symplect two-form ω. Let $f = f(x)$ be independent of ξ, and assume that the domain

$$\Omega = \{x \in \mathbf{R}^{n+1} : \ f(x) \ge 0\}$$

has a compact boundary. Suppose that g has the form

$$g(x,\xi) = E - H(x,\xi) - V(x)$$

in a neighborhood of J where $H(x,\xi) = \sum_{i,j=1}^{n+1} g^{ij}(x)\xi_i\xi_j$ is the Hamiltonian corresponding to an analytic Riemannian metric in a neighborhood of $\partial\Omega$. Then the first inequality in (2.2) is equivalent to

$$\sum_{i,j=1}^{n+1} g^{ij}(x)N_i(x)N_j(x) > 0, \ \forall x \in \partial\Omega,$$

where $N(x) = \mathrm{grad} f(x)$. On the other hand, if $V = 0$ and $E > 0$, the second inequality in (2.2) means that Ω is strictly geodesically convex with respect to the geodesic flow associated to the Hamiltonian g.

Before formulating the main results we recall certain basic facts about the Gevrey classes. If X is an open domain in $\mathbf{R}^n$ and $\sigma \ge 1$, we denote by $G^\sigma(X)$ the space of all Gevrey functions in X of index σ, namely $f \in G^\sigma(X)$ iff $f \in C^\infty(X)$ and for every compact subset Y of X there exists $C > 0$ such that

$$\sup_{x\in Y} |\partial_x^\alpha f(x)| \le C^{|\alpha|+1}(\alpha!)^\sigma, \ \alpha \in \mathbf{Z}_+^n.$$

Evidently $G^1(X)$ coincides with the space of all real analytic functions in X, while for $\sigma > 1$ there are nonzero compactly supported G^σ functions, namely $G^\sigma(X) \cap C_0^\infty(X) \ne \{0\}$. We point out that that for any G^σ function, $\sigma > 1$, which is flat at the hypersurface $x_1 = 0$, and any compact Y, there exist two constants C and c such that for every $\alpha \in \mathbf{Z}_+^n$ the following estimate holds :

$$|\partial_x^\alpha f(x)| \le C^{|\alpha|+1}(\alpha!)^\sigma \exp\left(-c|x_1|^{-1/(\sigma-1)}\right), \ x_1 \ne 0, x \in Y.$$

For more details on Gevrey classes we refer to [17] and [26].

We formulate now a general result which yields in particular effective stability estimates for the billiard ball map of a real-analytic manifold with a strictly geodesically convex boundary.

THEOREM 2.1. — *Let F, G, be a pair of transversal glancing hypersurfaces in an analytic symplectic manifold (M, ω). Suppose that F and G are analytic in a neighborhood of the glancing manifold K and assume that K is compact. Then there exists an approximate interpolating Hamiltonian ζ of Gevrey class G^2 for the billiard ball map B such that $\pm\zeta > 0$ in $J_\pm$. Moreover, there are positive constants δ and C such that for any $0 < \varepsilon \leq 1$ and any $\varrho \in J$ with $0 < |\zeta(\varrho)| < \delta$, we have*

$$\left|\zeta(B^k(\varrho)) - \zeta(\varrho)\right| \ < \ \varepsilon\zeta(\varrho)^2,$$

provided that

$$0 \ \leq \ T_k(\varrho) \ \leq \ \varepsilon|\zeta(\varrho)|^5 \exp(C|\zeta(\varrho)|^{-1}).$$

For any $\varrho \in G$ we define the "distance" to the gliding manifold K by

$$d(\varrho) = |f(\varrho)| + |\{f, g\}(\varrho)|.$$

As a consequence of Theorem 2.1 we obtain

THEOREM 2.2. — *Suppose that the assumptions of Theorem 2.1 hold. Then there exist positive constants δ and C, and some $0 < C_1 < 1 < C_2$ such that for any $\varrho \in J_\pm$ with $d(\varrho) < \delta$ we have*

$$C_1 d(\varrho) \leq d(\Phi^t(\varrho)) \leq C_2 d(\varrho),$$

provided that

$$0 \leq \pm t \leq d(\varrho)^5 \exp\left(\frac{C}{d(\varrho)}\right).$$

The results formulated above are based on an approximate normal form for a pair of symplectic involutions.

THEOREM 2.3. — *Let J be an analytic manifold of dimension $2n$ and let ω_0 be a closed analytic two-form on it satisfying (2.4) where $dm \neq 0$ on $K = \{\varrho \in J : m(\varrho) = 0\}$. Let $\mathcal{J}_j$, $j = 1, 2$, be a pair of analytic involutions such that $\mathcal{J}_j^*\omega_0 = \omega_0$. Suppose that the sets of fixed points of*

$\mathcal{J}_j$, $j = 1,2$, coincide with K, and that $d\mathcal{J}_j$ are linearly independent over K. Then for any $\rho_0 \in K$ there exists a diffeomorphism χ of Gevrey class G^2 mapping a neighborhood of the origin W in $\mathbf{R}^{2n}$ to a neighborhood of ϱ_0 in J, $\chi(0) = \varrho_0$, such that

$$\chi^*(\omega_0) = \sigma_0 \equiv 2\xi_1 d\xi_1 \wedge dx_1 + \sum_{j=2}^{n} d\xi_j \wedge dx_j,$$

while the involutions $\mathcal{J}_j^0 = \chi^{-1} \circ \mathcal{J}_j \circ \chi$, $j = 1,2$, become

$$\mathcal{J}_1^0(x,\xi) = (x, -\xi_1, \xi'),$$

$$\mathcal{J}_2^0(x,\xi) = (x_1 + \xi_1, x', -\xi_1, \xi') + R(x,\xi),\ (x,\xi) \in W,$$

where R belongs to $G^2(W)$ and it is flat at $\{(x,\xi) \in W :\ \xi_1 = 0\}$.

Remark 2.4. — Making use of the example of Oshima of analytic glancing hypersurfaces which are not analytically equivalent (see (7), [24]), one can show that the Gevrey regularity G^2 in the theorem above is optimal. Indeed, the estimates in [24], p. 57, can be used to prove that the pair of analytic involutions associated to the glancing hypersurfaces (7) are not locally symplectically equivalent to the pair $\mathcal{J}_1^0$, $\mathcal{J}_2^0$, in any Gevrey class G^s, $1 \le s < 2$.

Proof of Theorem 2.1. — The first statement in Theorem 2.1 follows from Theorem 2.3 taking $\mathcal{J}_1 = \mathcal{J}_F$, $\mathcal{J}_2 = \mathcal{J}_G$, and then choosing local coordinates $(x(\varrho), \xi(\varrho)) = \chi^{-1}(\varrho) \in W$ where $\varrho \in U = \chi(W)$. Then we have

$$\begin{aligned}(\chi^{-1} \circ B \circ \chi)(x,\xi) &= (x_1 + \xi_1, x', \xi) + R(x,\xi)\\ &= \exp\left(\xi_1 H_{\xi_1^2}\right)(x,\xi) + R(x,\xi),\ (x,\xi) \in W,\end{aligned}$$

where $H_{\xi_1^2}$ is the Hamiltonian vector field of ξ_1^2 with respect to σ_0, R belongs to $G^2(W)$, and $R \approx 0$. Hereafter, we say that $R \approx 0$ if R is flat at $\{\xi_1 = 0\}$. Since $\chi^*(\omega_0) = \sigma_0$, the G^2 function $\zeta(\varrho) = \xi_1(\varrho)$ is an approximate interpolating Hamiltonian of B in U. Moreover, changing eventually ξ_1 with $-\xi_1$ we can assume that $\zeta > 0$ in J_+. We are going to show that ζ is uniquely determined in U modulo a flat function.

Let $\chi_0 : W_0 \to U$ be another G^2 diffeomorphism given by Theorem 2.3. Denote by $(y(\varrho), \eta(\varrho)) = \chi_0^{-1}(\varrho) \in W_0$ the corresponding G^2 local coordinates in U, and set $\zeta_0(\varrho) = \eta_1(\varrho)$ where $\zeta_0 > 0$ in J_+.

LEMMA 2.5. — *The function $\zeta_0(\varrho) - \zeta(\varrho)$ is flat at K.*

Proof. Set $\psi = \chi_0^{-1} \circ \chi : W \to W_0$ and denote $\phi(x,\xi) = \zeta_0(\chi(x,\xi)) = (\eta_1 \circ \psi)(x,\xi)$. Then $\pm\phi(x,\xi) > 0$ in $\{(x,\xi) \in W : \pm\xi_1 > 0\}$. We are going to show that $\phi(x,\xi) - \xi_1 \approx 0$. Set

$$B_0(x,\xi) = (x_1 + \xi_1, x', \xi).$$

Then we have $\psi^{-1} \circ B_0 \circ \psi - B_0 \approx 0$, hence,

$$\exp\left(\phi H_{\phi^2}\right) - B_0 = \psi^{-1} \circ \exp\left(\eta_1 H_{\eta_1^2}\right) \circ \psi - B_0 \approx 0,$$

and for any smooth function f and any integer $N \geq 1$ we obtain the equality

$$f(x_1 + \xi_1, x', \xi) \approx f\left(\exp\left(\phi H_{\phi^2}\right)(x,\xi)\right)$$

$$= f(x,\xi) + \sum_{k=1}^{N} \frac{\phi(x,\xi)^k}{k!} H_{\phi^2}^k f(x,\xi) + O\left(|\phi(x,\xi)|^{N+1}\right).$$

Set

$$H_{\phi^2} = \sum_{j=1}^{n} \left(\alpha_j(x,\xi)\frac{\partial}{\partial x_j} + \beta_j(x,\xi)\frac{\partial}{\partial \xi_j}\right).$$

Taking $f(x,\xi) = \xi_j$, $1 \leq j \leq n$, we get $\beta_j \approx 0$. In the same way we obtain $\alpha_1 \approx 1$ and $\alpha_j \approx 0$ for $2 \leq j \leq n$. Hence, $d\phi^2 = -\imath(H_{\phi^2})\sigma_0 \approx d\xi_1^2$, and we obtain $\phi \approx \xi_1$ since ϕ and ξ_1 have the same sign. □

Patching together in G^2 the local approximate interpolating Hamiltonians obtained above, we get an approximate interpolating Hamiltonian ζ of B in a neighborhood of K of the Gevrey class G^2.

We are going to prove the second statement in Theorem 2.1.

LEMMA 2.6. — *There exist positive constants C, C_1, and δ such that*

$$(2.6)\qquad \left|\zeta(B^j(\varrho)) - \zeta(\varrho)\right| \leq 2C_1 j \exp(-C|\zeta(\varrho)|^{-1}) \leq 2\varepsilon C_1|\zeta(\varrho)|^3$$

for any $0 < |\zeta(\varrho)| \leq \delta$ and $0 < \varepsilon \leq 1$ provided that

$$(2.7)\qquad 0 \leq j \leq \varepsilon|\zeta(\varrho)|^3 \exp(C|\zeta(\varrho)|^{-1}).$$

Proof. — Taking $f = \zeta$ in (1.2), and using the equality

$$\zeta(\exp(\zeta H_{\zeta^2})(\varrho)) = \zeta(\varrho),$$

we show that the G^2 function $\zeta(B(\varrho)) - \zeta(\varrho)$ is flat at K. Hence, there exist positive constants C, C_1, and δ such that the estimate

$$|\zeta(B(\varrho)) - \zeta(\varrho)| \leq C_1 \exp(-C|\zeta(\varrho)|^{-1}) \tag{2.8}$$

holds if $0 < |\zeta(\varrho)| \leq 3\delta$. We have shown that (2.8) is valid for $j = 1$. Fix j such that (2.7) holds. Suppose that (2.6) is valid for some $1 \leq k < j$. Then the inductive assumption implies

$$\left|\zeta(B^k(\varrho)) - \zeta(\varrho)\right| \leq 2C_1 k \exp(-C|\zeta(\varrho)|^{-1}) \leq 2\varepsilon C_1 |\zeta(\varrho)|^3 < \delta, \tag{2.9}$$

if $\delta^2 < (2C_1)^{-1}$. Now using (2.8) we get

$$\begin{aligned}\left|\zeta(B^{k+1}(\varrho)) - \zeta(\varrho)\right| &\leq \left|\zeta(B^{k+1}(\varrho)) - \zeta(B^k(\varrho))\right| + 2C_1 k \exp(-C|\zeta(\varrho)|^{-1}) \\ &\leq C_1 \exp(-C|\zeta(B^k(\varrho))|^{-1}) + 2C_1 k \exp(-C|\zeta(\varrho)|^{-1}).\end{aligned}$$

On the other hand, for $0 < \delta < (4C_1)^{-1}$ estimate (2.9) implies

$$|\zeta(B^k(\varrho))|^{-1} \geq |\zeta(\varrho)|^{-1} - 4\varepsilon C_1 |\zeta(\varrho)|,$$

which yields for $\delta < (4CC_1)^{-1} \ln 2$ the inequality

$$\exp(-C|\zeta(B^k(\varrho))|^{-1}) \leq 2 \exp(-C|\zeta(\varrho)|^{-1}),$$

and using (2.7) we get (2.6) for $B^{k+1}(\varrho)$. The proof of the lemma is complete. □

LEMMA 2.7. — *There exists $a \in C^\infty(J)$, $a > 0$, such that*

$$t(\varrho) = a(\varrho)\zeta(\varrho)$$

in a neighborhood of K in J_0.

Proof. — The manifold K is defined in J by $\{g, f\}(\varrho) = 0$, $(d\{g, f\} \neq 0$ on $K)$ and we have $\zeta(\varrho) = 0$, $d\zeta(\varrho) \neq 0$, $\varrho \in K$. Hence,

$$\{g, f\}(\varrho) = a_1(\varrho)\zeta(\varrho), \ \varrho \in J_0,$$

and we can suppose that $a_1 > 0$ since $\zeta > 0$ in J_+. Applying Taylor's formula to

$$f(\exp(t(\varrho)X_g)(\varrho)) = f(\varrho) = 0, \ \varrho \in J_0,$$

we obtain

$$\{g,f\}(\varrho)+\frac{t(\varrho)}{2}\{g,\{g,f\}\}(\varrho)+O(t(\varrho)^2)=0, \tag{2.10}$$

which proves the assertion. □

LEMMA 2.8. — *There exists $\delta>0$ such that (2.7) holds if $0<\varepsilon\le 1$, $0<|\zeta(\varrho)|\le\delta$, and*

$$T_j(\varrho)\le\varepsilon|\zeta(\varrho)|^5\exp(C|\zeta(\varrho)|^{-1}). \tag{2.11}$$

Proof. — Let $0\le k<j$ and suppose that T_j satisfies (2.11) and

$$k\le\varepsilon|\zeta(\varrho)|^3\exp(C|\zeta(\varrho)|^{-1}).$$

Clearly Lemma 2.6 and Lemma 2.7 lead to

$$\begin{aligned}T_{k+1}(\varrho)&=\sum_{p=0}^{k}|t(B^p(\varrho))|\ge C_0\sum_{p=0}^{k}|\zeta(B^p(\varrho))|\\&\ge(k+1)C_0|\zeta(\varrho)|(1-2C_1\delta^2)\ge(k+1)\frac{C_0}{2}|\zeta(\varrho)|,\end{aligned}$$

if $\delta^2<1/4C_1$. Now (2.11) yields

$$k+1\le\varepsilon\frac{2}{C_0}\zeta(\varrho)^4\exp(C|\zeta(\varrho)|^{-1})\le\varepsilon|\zeta(\varrho)|^3\exp(C|\zeta(\varrho)|^{-1})$$

for $\delta<C_0/2$, which proves the assertion. □

Making use of Lemma 2.6 and Lemma 2.8 we complete the proof of Theorem 2.1. □

To prove Theorem 2.2 we note that

$$d(\varrho)=|\{f,g\}(\varrho)|=a_1(\varrho)|\zeta(\varrho)|,\ \varrho\in J_0,$$

and use Theorem 2.1.

3. Normal forms of pairs of analytic involutions.

We are going to find a G^2 normal form of the pair of analytic involutions $\mathcal{J}_1$ and $\mathcal{J}_2$ given in Theorem 2.3. If $n\ge 3$ and $y\in\mathbf{R}^n$, we set $y=(y_1,y')=(y_1,y'',y_n)$, $y''=(y_2,\dots,y_{n-1})$. We have :

THEOREM 3.1 — *Let f and g be two real analytic involutions in a neighborhood of the origin in $\mathbf{R}^n$, $n \geq 2$. Suppose that f and g coincide with the identity on $K = \{y_n = 0\}$ and that the differentials df and dg are linearly independent at the origin. Then there exists a diffeomorphism u of Gevrey class G^2 in a neighborhood U of the origin such that the maps $f^0 = u \circ f \circ u^{-1}$ and $g^0 = u \circ g \circ u^{-1}$ have the form*

$$f^0(y) = (y_1, y'', -y_n), \quad g^0(y) = (y_1 + y_n, y'', -y_n) + R(y),$$

where R belongs to the Gevrey class $G^2(U)$ and it is flat at K.

The proof of Theorem 3.1 will be given in Section 4.

Proof of Theorem 2.3. — According to Theorem 3.1 there exist G^2 coordinates (x, ξ) in a neighborhood U of $(0,0)$ in $\mathbf{R}^n \times \mathbf{R}^n$ such that

$$\mathcal{J}_1(x,\xi) = (x, -\xi_1, \xi'), \quad \mathcal{J}_2(x,\xi) = \mathcal{J}_0(x,\xi) + R(x,\xi), \quad (x,\xi) \in U,$$

where $\mathcal{J}_0$ stands for the involution

$$\mathcal{J}_0(x,\xi) = (x_1 + \xi_1, x', -\xi_1, \xi'),$$

$R \in G^2(U)$, and R is flat at $K = \{(x,\xi) \in U : \xi_1 = 0\}$. Using the invariance of ω_0 with respect to the involutions given above, we are going to show that it has a quite simple form which allows us to use the deformation argument of Moser - Weinstein.

First we introduce the following notations : We set $\tilde{\xi}_1 = \xi_1^2$ and $\tilde{\xi}' = \xi'$, $\xi = (\xi_1, \xi') \in \mathbf{R}^n$. Next, for given $r \in C^\infty(U)$ we say as above that $r \approx 0$ if r is flat at K. More generally, for a given k-form ω in $T^*(\mathbf{R}^n)$ we say that $\omega \approx 0$, if all the coefficients of ω are flat functions at K.

Now, using the equality $\mathcal{J}_1^*\omega_0 = \omega_0$ we write the two-form ω_0 as follows

$$\omega_0 = \sum_{1 \leq i,j \leq n} a_{ij}(x,\xi) d\tilde{\xi}_i \wedge dx_j + \sum_{1 \leq i<j \leq n} b_{ij}(x,\xi) dx_i \wedge dx_j + \sum_{1 \leq i<j \leq n} c_{ij}(x,\xi) d\tilde{\xi}_i \wedge d\tilde{\xi}_j, \tag{3.1}$$

where the functions $a_{ij}(x,\xi)$, $b_{ij}(x,\xi)$, and $c_{ij}(x,\xi)$ belong to $G^2(U)$ and they are even with respect to ξ_1, i.e.

$$\mathcal{J}_1^* a_{ij} = a_{ij}, \quad \mathcal{J}_1^* b_{ij} = b_{ij}, \quad \mathcal{J}_1^* c_{ij} = c_{ij}. \tag{3.2}$$

Now, condition (2.4) reads

$$\omega_0^n = \xi_1 dv, \tag{3.3}$$

where dv is a volume form in U and we have taken n times the exterior product of ω_0.

PROPOSITION 3.2. — *Let ω_0 be a closed two-form in a neighborhood of the origin in $T^*(\mathbf{R}^n)$ with G^2 coefficients satisfying (3.3). Suppose that $\mathcal{J}_j^*\omega_0 = \omega_0$, $j = 1, 2$. Then there exists a local diffeomorphism $\psi \in G^2(U)$ in a neighborhood U of $\varrho^0 = (0,0)$ such that*

$$\psi \circ \mathcal{J}_1 = \mathcal{J}_1 \circ \psi, \ \ \psi \circ \mathcal{J}_0 = \mathcal{J}_0 \circ \psi, \ \text{ in } U,$$

and

$$\psi^*\omega_0 = \sum_{1\le i,j\le n} a_{ij}^0(y',\eta) d\tilde{\eta}_i \wedge dy_j + \sum_{1\le i<j\le n} b_{ij}^0(y',\eta) dy_i \wedge dy_j \tag{3.4}$$

$$+ \sum_{1\le i<j\le n} c_{ij}^0(y',\eta) d\tilde{\eta}_i \wedge d\tilde{\eta}_j + \tilde{\omega},$$

where $\tilde{\omega} \approx 0$, $\tilde{\eta} = (\eta_1^2, \eta')$, and

$$a_{11}^0(y',\eta) \equiv 1, \ a_{i1}^0(y',\eta) \equiv 0, \ b_{1i}^0(y',\eta) \equiv 0, \ 2 \le i \le n. \tag{3.5}$$

We postpone for a while the proof of the proposition. Now we can apply the deformation argument of Moser - Weinstein to the form $\sigma_1 = \psi^*\omega_0$.

LEMMA 3.3. — *Let σ_1 be a closed two-form in a neighborhood of the origin in $\mathbf{R}^{2n}$ with G^2 coefficients given by (3.4) and satisfying (3.3) and (3.5). Suppose that $\mathcal{J}_j^*\omega_0 = \omega_0$, $j = 1, 2$. Then there exists a diffeomorphism $\phi \in G^2(U)$ in a neighborhood U of $\varrho^0 = (0,0)$ such that*

$$\phi^*\sigma_1 = \sigma_0 \equiv 2\eta_1 d\eta_1 \wedge dy_1 + \sum_{j=2}^n d\eta_j \wedge dy_j,$$

and

$$\phi \circ \mathcal{J}_1 = \mathcal{J}_1 \circ \phi, \ \ \phi \circ \mathcal{J}_2 \approx \mathcal{J}_2 \circ \phi \text{ in } U.$$

Proof. — After performing a linear change of the variables

$$z_1 = y_1 + \sum_{j=2}^{n} \left(a^0_{1j}(0) y_j + c^0_{1j}(0)\eta_j \right), \ \ \zeta_1 = \eta_1,$$

$$(z', \zeta') = A(y', \eta'), \ \det(A) \neq 0,$$

which commutes with both $\mathcal{J}_1$ and $\mathcal{J}_0$ we can suppose that $\sigma_1 = \sigma_0 + \sigma_2$, where σ_2 has the form (3.4), its coefficients satisfy (3.2) ($\mathcal{J}_1^* \sigma_2 = \sigma_2$), and

$$a^0_{i1}(y', \eta) \equiv 0, \ b^0_{1i}(y', \eta) \equiv 0, \ 1 \le i \le n, \tag{3.6}$$

$$a^0_{ij}(0) = b^0_{ij}(0) = c^0_{ij}(0) = 0, \ 1 \le i, j \le n.$$

We interpolate σ_1 and σ_0 by the family

$$\sigma_t = \sigma_0 + t(\sigma_1 - \sigma_0), \ 0 \le t \le 1.$$

We are looking for a time dependent vector field X_t such that

$$(\phi^t)^* \sigma_t = \sigma_0, \tag{3.7}$$

where ϕ^t is defined by

$$\frac{d}{dt}\phi^t = X_t(\phi^t), \ \phi^0 = \mathrm{Id}.$$

Differentiating (3.7) with respect to t we get

$$\mathcal{L}_{X_t}\sigma_t + \sigma_1 - \sigma_0 = 0,$$

where $\mathcal{L}_{X_t} = \imath_{X_t} \circ d + d \circ \imath_{X_t}$ is the Lie derivative, $\imath_{X_t}$ being the inner product by X_t. Since σ_t is closed, we arrive at the equality

$$d(\imath_{X_t}\sigma_t) = \sigma_0 - \sigma_1. \tag{3.8}$$

We have $\sigma_0 - \sigma_1 = d\alpha, \ \alpha(\varrho^0) = 0$, where α can be chosen in the form

$$\alpha \approx \sum_{j=2}^{n} \alpha_j(y', \eta) dy_j + \sum_{j=1}^{n} \beta_j(y', \eta) d\eta_j, \tag{3.9}$$

in view of (3.4) and (3.6). We can suppose that α_j and β_j are G^2 functions. Moreover, taking $(\mathcal{J}_1^*\alpha + \alpha)/2$ instead of α we can arrange $\mathcal{J}_1^*\alpha = \alpha$ still keeping (3.9), which implies $\beta_1(y', \eta) = \eta_1 \gamma(y', \eta)$ with γ in G^2. Set

$B_0 = \mathcal{J}_1 \circ \mathcal{J}_0$. In view of (3.9) we have $B_0^*\alpha \approx \alpha$ which implies $\mathcal{J}_0{}^*\alpha \approx \alpha$ since $\mathcal{J}_0 = \mathcal{J}_1 \circ B_0$.

Now (3.8) leads to the equation $\imath_{X_t}\sigma_t = \alpha$ which has a unique solution

$$X_t = \sum_{j=1}^{n} p_j(y,\eta)\frac{\partial}{\partial y_j} + \sum_{j=1}^{n} q_j(y,\eta)\frac{\partial}{\partial \eta_j}.$$

Indeed, since $\beta_1(y',\eta) = \eta_1\gamma(y',\eta)$ and the coefficient of dy_1 in α is flat at $\eta_1 = 0$, taking into account (3.6) we obtain an (algebraic) non-degenerate linear system for p_j and q_j with G^2 coefficients wich has an unique solution in G^2.

We are going to show that $\phi = \phi^1$ is the desired diffeomorphism. We have to prove that ϕ commutes with $\mathcal{J}_1$. Setting

$$Y_t = \mathcal{J}_1^* X_t \equiv (d\mathcal{J}_1)^{-1} X_t \circ \mathcal{J}_1$$

we get

$$\alpha = \mathcal{J}_1^*\left(\imath_{X_t}\sigma_t\right) = \imath_{Y_t}\left(\mathcal{J}_1^*\sigma_t\right) = \imath_{Y_t}\sigma_t,$$

which implies $\mathcal{J}_1^* X_t = X_t$. Hence, $\mathcal{J}_1 \circ \phi^t = \phi^t \circ \mathcal{J}_1$. In the same way we show the relation $\mathcal{J}_0 \circ \phi^t \approx \phi^t \circ \mathcal{J}_0$ which implies $\mathcal{J}_2 \circ \phi^t \approx \phi^t \circ \mathcal{J}_2$. The proof of Lemma 3.3 is complete. □

Now, Theorem 2.3 follows immediately from Theorem 3.1, Proposition 3.2 and Lemma 3.3.

Proof of Proposition 3.2. — To put ω_0 into a normal form, keeping the involutions $\mathcal{J}_1$ and $\mathcal{J}_0$ fixed, we use the following

LEMMA 3.4. — *Let $p \in C^\infty(U)$, $\mathcal{J}_1{}^*p = p$, and $\mathcal{J}_0{}^*p \approx p$. Then $\partial p/\partial x_1 \approx 0$.*

Proof. — Consider the Taylor expansion at $(x,0,\xi')$ of the smooth function

$$p(x_1+\xi_1,x',\xi) - p(x,\xi) = B_0^*p(x,\xi) - p(x,\xi) \approx 0,$$

where $B_0 = \mathcal{J}_1 \circ \mathcal{J}_0$. For any integer $\beta \geq 0$ the coefficint of $\xi_1^{\beta+1}$ is equal to

$$\sum_{\gamma=0}^{\beta}\sum_{\alpha=1}^{\beta+1-\gamma}\frac{1}{\alpha!\gamma!}\partial_{y_1}^{\alpha}\partial_{\xi_1}^{\gamma}p(x,0,\xi') = 0.$$

By inductive arguments with respect to β we get

$$\partial_{x_1}\partial_{\xi_1}^{\beta}p(x,0,\xi')=0,$$

which proves the assertion. $\square$

The equality $B_0^*\omega_0 \approx \omega_0$ yields the identities

$$B_0^* a_{11} \approx a_{11},\ B_0^* a_{ij} \approx a_{ij},\ 2\le i\le n,\ 1\le j\le n,$$

$$B_0^* b_{ij} \approx b_{ij},\ 1\le i<j\le n,$$

$$B_0^* c_{ij} \approx c_{ij},\ 2\le i<j\le n.$$

$$B_0^* a_{1j} + \frac{1}{2\xi_1}B_0^* b_{1j} \approx a_{1j},\ 2\le j\le n,$$

$$B_0^* c_{1j} - \frac{1}{2\xi_1}B_0^* a_{j1} \approx c_{1j},\ 2\le j\le n.$$

Now Lemma 3.4 implies

$$a_{11}(x,\xi) \approx a_{11}^0(x',\xi),$$

$$a_{ij}(x,\xi) \approx a_{ij}^0(x',\xi),\ 2\le i\le n,\ 1\le j\le n;$$

$$b_{ij}(x,\xi) \approx b_{ij}^0(x',\xi),\ 1\le i<j\le n,$$

$$c_{ij}(x,\xi) \approx c_{ij}^0(x',\xi),\ 2\le i<j\le n.$$

A similar argument has been used by Melrose [20] in a little bit different context. Applying Lemma 3.4 to $\partial_{x_1}a_{1j}$ and $\partial_{x_1}c_{1j}$ we obtain

$$a_{1j}(x,\xi) \approx a_{1j}^0(x',\xi)+x_1a_{1j}^1(x',\xi),\ 2\le j\le n$$

$$c_{1j}(x,\xi) \approx c_{1j}^0(x',\xi)+x_1c_{1j}^1(x',\xi),\ 2\le j\le n$$

where

(3.10)

$$a_{1j}^1(x',\xi) = -\frac{1}{2\xi_1}b_{1j}(x',\xi)\in G^2 \quad \text{since} \quad b_{1j}(x',0,\xi') = 0,\ 2\le j\le n,$$

(3.11)

$$c_{1j}^1(x',\xi) = \frac{1}{2\xi_1}a_{j1}(x',\xi)\in G^2 \quad \text{since} \quad a_{j1}(x',0,\xi') = 0,\ 2\le j\le n.$$

Moreover, we can suppose that

$$a_{11}^{0}(x', 0, \xi') = 1. \tag{3.12}$$

Indeed, we have

LEMMA 3.5. — *There exists a diffeomorphism $(x, \xi) = \psi(y, \eta)$ in a neighborhood of $\varrho^0 = (0, 0)$ having the form*

$$x_1 = Y(y', \eta)y_1,\ x' = y',\ \xi_1 = X(y', \eta),\ \xi' = \eta',$$

where X and Y are G^2 functions,

$$Y(\varrho^0) > 0,\ X(y', 0, \eta') = 0,\ dX(\varrho^0) \neq 0,$$

such that $\mathcal{J}_1 \circ \psi = \psi \circ \mathcal{J}_1$, $B_0 \circ \psi = \psi \circ B_0$, and the coefficient of $d\eta_1^2 \wedge dy_1$ in $\sigma = \psi^* \omega_0$ is equal to 1.

Proof. — Taking into account (3.3), (3.10), and (3.11), we can suppose that $a_{11}^0(0) > 0$. The coefficient of $d\eta_1^2 \wedge dy_1$ in σ is equal to

$$a_{11}^0(\psi(y, \eta))Y(y', \eta)\frac{\partial X}{\partial \eta_1}(y', \eta) = 1.$$

Moreover, ψ commutes with B_0 iff $X(y', \eta) = Y(y', \eta)\eta_1$, and we obtain the equation

$$a_{11}^0(y', X, \eta')X\frac{\partial X}{\partial \eta_1}(y', \eta) \ = \ \eta_1,\ X(y', 0, \eta') = 0.$$

The solution of this equation is given by

$$2\int_0^X a_{11}^0(y', t, \eta')tdt = \eta_1^2. \tag{3.13}$$

Set

$$p(y', u, \eta') = 2\int_0^u a_{11}^0(y', t, \eta')tdt.$$

Then $p(y', u, \eta') = u^2 q(y', u, \eta')^2$, $q(0) = \sqrt{a_{11}^0(0)}$, where q is a G^2 function, and we obtain the equation $Xq(y', X, \eta') = \eta_1$, which has a unique solution $X(y', \eta) = \eta_1 T(y', \eta)$, where T is a G^2 function and $T(y', 0, \eta') > 0$, in view of the implicit function theorem in Gevrey classes. Moreover, $\mathcal{J}_1{}^* p = p$, and we obtain $\mathcal{J}_1{}^* T = T$ which implies $\mathcal{J}_1 \circ \psi = \psi \circ \mathcal{J}_1$. □

From now on we suppose that (3.12) holds for the form $\sigma = \psi^*\omega_0$. We have $\sigma = \sigma_1 + y_1\sigma_2 + \sigma_3$ in U, where

$$\sigma_1 = \sum_{1\le i,j\le n} a^0_{ij} d\tilde{\eta}_i \wedge dy_j + \sum_{1\le i<j\le n} b^0_{ij} dy_i \wedge dy_j + \sum_{1\le i<j\le n} c^0_{ij} d\tilde{\eta}_i \wedge d\tilde{\eta}_j,$$

$$\sigma_2 = \sum_{j=2}^{n} a^1_{1j} d\eta_1^2 \wedge dy_j + \sum_{j=2}^{n} c^1_{1j} d\eta_1^2 \wedge d\eta_j, \tag{3.14}$$

$\tilde{\eta} = (\eta_1^2, \eta')$, and $\sigma_3 \approx 0$. We are going to explore the two-form σ_2. Since σ is closed and the coefficients of σ_1 do not depend on y_1 we have $d\sigma_2 \approx 0$. Then $\sigma_2 = d\alpha + a(y', \eta)d\eta_1$, where the coefficients of the one-form α do not depend on y_1, and α does not contain multiples of dy_1 and $d\eta_1$. In other words, one can consider α as one-form in (y', η') depending on the parameter η_1. Now (3.14) implies $d_{y',\eta'}\alpha = 0$, hence $\alpha = d\gamma + p(y', \eta)d\eta_1$, where $\gamma, p \in G^2(U)$, and we get

$$\sigma_2 = dp \wedge d\eta_1.$$

Next we write $\sigma = \sigma_1^0 + \sigma_2^0 + \sigma_3$, where $\sigma_3 \approx 0$,

$$\sigma_2^0 = \left(\sum_{i=1}^{n} a^0_{i1}(y', \eta) d\tilde{\eta}_i - \sum_{j=2}^{n} b^0_{1j}(y', \eta) dy_j \right) \wedge dy_1 + y_1 dp(y', \eta) \wedge d\eta_1,$$

while the two-form σ_1^0 does not contain multiples of dy_1, and the coefficients of σ_1^0 are functions of (y', η) only. Our aim is to show that $\sigma_2^0 \approx d\eta_1^2 \wedge dy_1$. Notice that $d\sigma_2^0 \approx 0$ which implies

$$\sum_{i=1}^{n} da^0_{i1} \wedge d\tilde{\eta}_i - \sum_{j=2}^{n} db^0_{1j} \wedge dy_j + dp \wedge d\eta_1 \approx 0.$$

Therefore,

$$\sum_{i=1}^{n} a^0_{i1} d\tilde{\eta}_i - \sum_{j=2}^{n} b^0_{1j} dy_j \approx dr - p d\eta_1,$$

where $r \in G^2(U)$, and we have

$$\sigma_2^0 = dr \wedge dy_1 - p d\eta_1 \wedge dy_1 - y_1 d\eta_1 \wedge dp.$$

On the other hand, since σ_1^0 is invariant with respect to B_0, we get $B_0^*\sigma_2^0 \approx \sigma_2^0$, which gives

$$r(y', \eta) \approx -p(y', \eta)\eta_1 + q(\eta_1), \ q \in G^2(\mathbf{R}),$$

while (3.12) yields

$$\frac{\partial r}{\partial \eta_1}(y',\eta) - p(y',\eta) = 2\eta_1.$$

Hence, r satisfies the equation

$$\eta_1 \frac{\partial r}{\partial \eta_1} + r \approx q(\eta_1) + 2\eta_1^2.$$

Then $v(y',\eta) = r(y',\eta) - r(0,\eta_1,0)$ should satisfy the equation

$$\eta_1 \frac{\partial v}{\partial \eta_1} + v \approx 0,$$

and we obtain $v \approx 0$. Therefore,

$$\sigma_2^0 \approx d\eta_1^2 \wedge dy_1,$$

which completes the proof of the proposition. □

4. Proof of Theorem 3.1.

Consider the real analytic involutions f and g. As in [11] we can suppose after a local analytic change of the variables that

$$f(y) = (y', -y_n) \quad \text{for } y \in B^n(\delta),\ 0 < \delta \ll 1,$$

where $B^n(\delta) = \{z \in \mathbf{R}^n : |z_j| < \delta,\ j = 1, \dots, n\}$.

Then as $g(y)$ is analytic in $B^n(\delta)$ we can write $g(y) = (g'(y), g_n(y))$, $y \in B^n(\delta)$ in the following form :

$$g'(y) = y' + \sum_{j=1}^{\infty} y_n^j a^j(y'),\ a^j(y') = (a_1^j(y'), \dots, a_{n-1}^j(y')), \tag{4.1}$$

$$g_n(y) = -y_n + \sum_{j=1}^{\infty} y_n^{j+1} A_j(y'). \tag{4.2}$$

Moreover, there is $C > 0$ such that

$$\sup_{y' \in B^{n-1}(\delta)} |\partial_{y'}^{\alpha} q^j(y')| \le C^{j+|\alpha|}\alpha!,\ \alpha \in \mathbf{Z}_+^{n-1},\ j = 1, 2, \dots,$$

where q^j stands either for a_r^j, $r = 1, \dots, n-1$, or for A_j. The requirement that the differentials of f and g are linearly independent on their fixed set yields

$$a^1(y') \neq 0 \ \text{ for } \ y' \in B^{n-1}(\delta).$$

Under the assumptions above Theorem C.4.6 in [11] provides a C^∞ smooth change of the variables $u = u(y)$, $u(0) = 0$, which preserves f and transforms g into $u^{-1} \circ g \circ u = g_0$, where

$$g_0(y_1, y'', y_n) = (y_1 + y_n, y'', -y_n).$$

We are concerned here with the problem whether u can be found in a suitable Gevrey class G^σ provided that the involutions are analytic. Fix an integer $p \geq 2$. As in the proof of Theorem C.4.6 in [11], making an analytic change of the variables, we can assume from the very beginning that

$$a^1(y') = (1, 0, \dots, 0), \ A_j(y') = 0, \ 1 \leq j \leq p-1. \tag{4.3}$$

First we consider $u(y)$ as formal power series in y_n

$$u(y', y_n) = \sum_{j=0}^{\infty} y_n^j (u^{j\prime}(y'), y_n u_n^j(y')), \ u^{0\prime}(y') = y', \ u_n^0(y') = 1,$$

with the subsequent compositions obeying the rules of the calculus for formal power series expansions.

LEMMA 4.1. — *If $u(y)$ satisfies $f \circ u = u \circ f$ then $u(y)$ has only even powers in y_n, namely*

$$u(y', y_n) = \sum_{j=0}^{\infty} y_n^{2j} (u^{j\prime}(y'), y_n u_n^j(y')).$$

Proof. — We have

$$f(u(y)) = (u'(y), -u_n(y)) = \sum_{j=0}^{\infty} (y_n^j u^{j\prime}(y'), -y_n^{j+1} u_n^j(y')),$$

$$u(f(y)) = u(y', -y_n) = \sum_{j=0}^{\infty} ((-y_n)^j u^{j\prime}(y'), (-y_n)^{j+1} u_n^j(y')).$$

Evidently the last two identities imply that $f \circ u = u \circ f$ iff $u^j(y') = (u^{j\prime}(y'), u^j_n(y')) \equiv 0$ for all j odd. $\square$

So we look for $u(y)$ given by the formal power series in Lemma 4.1 and satisfying

$$u(g(y)) = g_0(u(y)). \tag{4.4}$$

First we write explicitly the left and the right hand side of (4.4) as formal power series in y_n. We check easily that formally

$$g_0(u(y)) = \sum_{j=0}^{\infty} y_n^{2j} \left(u_1^j(y') + y_n u_n^j(y'), u^{j\prime\prime}(y'), -y_n u_n^j(y') \right). \tag{4.5}$$

Further we obtain in view of (4.1), (4.2) and Lemma 4.1 that

$$u(g(y)) = \sum_{j=0}^{\infty} (-y_n + \sum_{r=1}^{\infty} y_n^{r+1} A_r(y'))^{2j}$$

$$\times \left(u^{j\prime}(y' + \sum_{s=1}^{\infty} y_n^s a^s(y')), (-y_n + \sum_{r=1}^{\infty} y_n^{r+1} A_r(y')) u_n^j(y' + \sum_{s=1}^{\infty} y_n^s a^s(y')) \right).$$

Next we recall the power series expansions

$$(-y_n + \sum_{r=p}^{\infty} y_n^{r+1} A^r(y'))^j = y_n^j \left((-1)^j + \sum_{\nu=p}^{\infty} A_{j,\nu}(y') y_n^{\nu} \right),$$

$$A_{j,\nu}(y') = \sum \frac{j!}{j_0! j_p! \dots j_\nu!} (-1)^{j_0} (A_p(y'))^{j_p} \dots (A_\nu(y'))^{j_\nu}, \tag{4.6}$$

where the latter sum is taken over the set $\mathbf{Z}_0(j,\nu)$ of all

$$(j_0, j_p, j_{p+1}, \dots, j_\nu) \in \mathbf{Z}^{\nu - p + 2}$$

such that

$$j_0 + j_p + j_{p+1} + \dots + j_\nu = j, \ p j_p + (p+1) j_{p+1} + \dots + \nu j_\nu = \nu.$$

We denote by $\mathbf{Z}_+$ the set of the non-negative integers and by $\mathbf{N}$ the set of the positive ones. We set by convention

$$A_{j,0} = (-1)^j, \ A_{j,\nu} = 0, \text{ for } 1 \le \nu \le p - 1.$$

Moreover,

$$h(y' + \sum_{s=1}^{\infty} y_n^s a^s(y')) = \sum_{\mu=0}^{\infty} y_n^{\mu} \sum_{|\beta|\leq\mu} a_{\mu,\beta}(y') \frac{\partial_{y'}^{\beta} h(y')}{\beta!},$$

$$a_{\mu,\beta}(y') = \sum \frac{\beta!}{\gamma^1! \dots \gamma^{\mu}!} (a^1(y'))^{\gamma^1} \dots (a^{\mu}(y'))^{\gamma^{\mu}}, \tag{4.7}$$

where the latter sum is over the set of multiindices $\mathbf{Z}(\beta, \mu)$ of all $(\gamma^1, \dots, \gamma^{\mu}) \in (\mathbf{Z}_+^{n-1})^{\mu}$, where $\gamma^s \in \mathbf{Z}_+^{n-1}$, $0 \leq s \leq \mu$, and

$$\gamma^1 + \gamma^2 + \dots + \gamma^{\mu} = \beta, \ |\gamma^1| + 2|\gamma^2| + \dots + \mu|\gamma^{\mu}| = \mu.$$

We have used the notations

$$|\gamma^s| = \gamma_1^s + \dots + \gamma_{n-1}^s, \ \gamma^s! = \gamma_1^s! \cdots \gamma_{n-1}^s!, \ \gamma^s = (\gamma_1^s, \dots, \gamma_{n-1}^s),$$

and

$$(a^s(y'))^{\gamma^s} = a_1^s(y')^{\gamma_1^s} \dots a_{n-1}^s(y')^{\gamma_{n-1}^s}.$$

We set by convention $a_{0,0} = 1$, and $a_{\mu,\beta} = 0$ if $|\beta| \leq \mu$ and $\mathbf{Z}(\mu, \beta)$ is empty which happens for example when $\mu \geq 1$ and $\beta = 0$.

LEMMA 4.2. — *Let $\beta = (\beta_1, \dots, \beta_{n-1}) \in \mathbf{Z}_+^{n-1}$, $\beta \neq 0$, and let $\mu = |\beta|$. Then, $a_{\mu,\beta} = 0$ if $\beta \neq (\mu, 0, \dots, 0)$, and $a_{\mu,\beta} = 1$ if $\beta = (\mu, 0, \dots, 0)$.*

Proof. — Let $(\gamma^1, \dots, \gamma^{\mu}) \in \mathbf{Z}(\mu, \beta)$. Then

$$|\gamma^1| + |\gamma^2| + \dots + |\gamma^{\mu}| = |\beta| = \mu = |\gamma^1| + 2|\gamma^2| + \dots + \mu|\gamma^{\mu}|,$$

and we obtain $\gamma^2 = \dots = \gamma^{\mu} = 0$ and $\gamma^1 = \beta$. Now, using (4.3) and (4.7) we prove the assertion. □

The power series expansions given above yield

$$w(y) = u(g(y)) = \sum_{k=0}^{\infty} y_n^k (w^{k\prime}(y'), y_n w_n^k(y')).$$

with

$$w^{k\prime}(y') = \sum_{0\leq j\leq k/2} \sum_{\nu=0}^{k-2j} \sum_{|\beta|\leq k-2j-\nu} A_{2j,\nu}(y') a_{k-2j-\nu,\beta}(y') \frac{\partial_{y'}^{\beta} u^{j\prime}(y')}{\beta!},$$

$$w_n^k(y') = \sum_{0\le j\le k/2}\sum_{\nu=0}^{k-2j}\sum_{|\beta|\le k-2j-\nu} A_{2j+1,\nu}(y')a_{k-2j-\nu,\beta}(y')\frac{\partial_{y'}^{\beta}u_n^j(y')}{\beta!}.$$

Now, taking into account (4.5), we obtain from (4.4) the equalities

$$w^{2k}(y') = (u^{k\prime}(y'), -u_n^k(y')),\ k\in \mathbf{Z}_+,$$

$$w^{2k+1}(y') = (u_n^k(y'), 0),\ k\in \mathbf{Z}_+, \tag{4.8}$$

The resolution of (4.4) in the C^∞ category shows that one has only to solve (4.8) for each k, since the other sequence of identities involving w^{2k} follows from (4.8) taking into account the fact that g is an involution (cf. case (a) in (C.4.3), [11]).

We write the equations (4.8) explicitly, replacing w^k by the corresponding expressions given above. Using (4.3) we obtain

$$\partial_{y_1}u^k(y') \;=\; u_n^k(y')e_1 \;-\; G^k(u), \tag{4.9}$$

where $e_1 = (1,0,\dots,0)$, $G^0 = 0$, and the operator G^k, $k\ge 1$, is given by

$$G^k(u)(y') = \sum_{j=0}^{k-1}\sum_{|\beta|\le 2k+1-2j}\Gamma_{j,\beta}^k(y')\,\frac{\partial_{y'}^{\beta}u^j(y')}{\beta!}.$$

Here

$$\Gamma_{j,\beta}^k(y') \;=\; \begin{pmatrix}\Gamma_{j,\beta}^{k,\prime}(y') & 0\\ 0 & \Gamma_{j,\beta}^{k,n}(y')\end{pmatrix}$$

is $n\times n$ matrix with

$$\Gamma_{j,\beta}^{k,\prime}(y') = \sum_{\nu=0}^{2k+1-2j-|\beta|} A_{2j,\nu}(y')a_{2k+1-2j-\nu,\beta}(y')E_{n-1}, \tag{4.10}$$

$$\Gamma_{j,\beta}^{k,n}(y') = \sum_{\nu=0}^{2k+1-2j-|\beta|} A_{2j+1,\nu}(y')a_{2k+1-2j-\nu,\beta}(y'), \tag{4.11}$$

E_{n-1} being the unit matrix in $\mathbf{R}^{n-1}$.

Before solving equations (4.9) with suitable initial conditions, we rewrite G^k as a sum of two terms, grouping in one of them all $\Gamma_{j,\beta}^k$ with $|\beta| = 2k-2j+1$. Let $\beta = (\beta_1,\dots,\beta_{n-1})\in \mathbf{Z}_+^{n-1}$, $\beta\ne 0$, and let

$|\beta| = 2k - 2j + 1$ for some positive integers k and j. Then $\nu = 0$ in (4.10) and (4.11), and Lemma 4.2 implies that either

$$\Gamma^k_{j,\beta} = 0$$

if $\beta \neq (2k-2j+1, 0, \ldots, 0)$, or

$$\Gamma^k_{j,\beta} = \begin{pmatrix} E & 0 \\ 0 & -1 \end{pmatrix}$$

if $\beta = (2k-2j+1, 0, \ldots, 0)$. Hence,

$$G^k = \mathcal{P}^k + \mathcal{R}^k,$$

where

$$\mathcal{P}^k(u)(y') = \sum_{j=0}^{k-1} \sum_{|\beta| \le 2k-2j} \Gamma^k_{j,\beta}(y') \frac{\partial^\beta_{y'} u^j(y')}{\beta!}, \tag{4.12}$$

and

$$\mathcal{R}^k(u)(y') = \begin{pmatrix} E & 0 \\ 0 & -1 \end{pmatrix} \sum_{j=0}^{k-1} \frac{\partial^{2k-2j+1}_{y_1} u^j(y')}{(2k-2j+1)!}. \tag{4.13}$$

Denote by S the real analytic hypersurface in $B^{n-1}(\delta)$

$$S := \{y' \in B^{n-1}(\delta) : y_1 = 0\},$$

and consider the system of equations (4.9) with initial conditions

$$u^0|_S = (0, y'', 1), \quad u^k|_S = 0, \; k = 1, 2, \ldots . \tag{4.14}$$

They can be solved successively for every $k \in \mathbf{Z}_+$, and the corresponding solution $u^k(y')$ is analytic in $B^{n-1}(\delta)$.

To estimate $\sup_{y' \in B^{n-1}(\delta)} |u^k(y')|$ as $k \to \infty$, we introduce suitable Banach spaces of formal power series which are adapted for the calculus with the Taylor series of Gevrey functions. Similar spaces have been used in [6] to study the calculus of classical formal pseudo-differential operators in Gevrey classes.

Let $FA(\delta, n-1)$ be the set of all sequences

$$h := h(y') = \{h^0(y'), h^1(y'), \ldots\}$$

of analytic functions h^k in $B^{n-1}(\delta)$. For any $T>0,\ k\in\mathbf{Z}_+$, and $\alpha\in\mathbf{Z}_+^{n-1}$ we set

$$M_k^\alpha(h:T)=\frac{T^{3k+|\alpha|}}{(2k+|\alpha|)!}\sup_{y'\in B^{n-1}(\delta)}|\partial_{y'}^\alpha h^k(y')| \tag{4.15}$$

and denote

$$\|h\|_T=\sum M_k^\alpha(h:T) \tag{4.16}$$

where the sum is taken over all $k\in\mathbf{Z}_+$ and $\alpha\in\mathbf{Z}_+^{n-1}$. Let

$$E(T)=\{h\in FA(\delta,n-1):\ \|h\|_T<\infty\}$$

be the corresponding Banach space.

PROPOSITION 4.3. — *There exist constants $\delta>0$ and $C>0$ such that*

$$\sup_{y'\in B^{n-1}(\delta)}|\partial_{y'}^\alpha u^k(y')|\ \le\ C^{k+1+|\alpha|}\,(2k)!\,\alpha!\,,\quad \forall k\in\mathbf{Z}_+\,,\ \forall\alpha\in\mathbf{Z}_+^{n-1}\,.$$

Proof. — To prove Proposition 4.3 we have to show that

$$u(y')=\{u^0(y'),u^1(y'),\dots\}\in E(T)\ \text{ for some } T>0. \tag{4.17}$$

Indeed, we observe that (4.16) and (4.17) imply

$$\sup_{y'\in B^{n-1}(\delta)}|\partial_{y'}^\alpha u^k(y')|\ \le\ C_1\frac{(2k+|\alpha|)!}{T^{3k+|\alpha|}}\ \le\ C^{k+1+|\alpha|}(2k)!\alpha!,$$

where $C_1>1$, $0<T<1$, and $C=C(C_1,n,T)$.

We are going to prove (4.17). To do this, we rewrite the Cauchy problem (4.9) and (4.14) into a system of n integral equations. In this way we obtain an equation

$$u(y')=H(\mathcal{P}(u))(y')+H(\mathcal{R}(u))(y')+U^0(y'),\quad u\in FA(\delta,n-1),$$

where $U^0(y')=\{(O,y'',1),0,\dots,0,\dots\}$ and the operator $H:FA(\delta,n-1)\to FA(\delta,n-1)$ is defined by $H(v)^0=0$ and

$$H(v)^k\ =\ -\left(\int_0^{y_1}\int_0^t v_n^k(s,y'')\,dsdt\right)e_1\ -\ \int_0^{y_1}v^k(t,y'')\,dt,\quad k\in\mathbf{N}.$$

The idea is to prove that the linear operator $H\circ\mathcal{P}+H\circ\mathcal{R}$ is a contraction in $E(T)$, provided that $0<T\ll 1$ and $0<\delta\ll 1$, and then to use the fixed point theorem in the Banach space $E(T)$. First we show that H is a contraction in $E(T)$.

LEMMA 4.4. — *There exist positive constants δ_0, T_0 and C_0, such that*

$$H : E(T) \ni v \longrightarrow H(v) \in E(T)$$

is continuous and

$$\|H(v)\|_T \leq C_0(\delta + T)\|v\|_T\,, \ \forall v \in E(T),\ T \in (0, T_0],\ \delta \in (0, \delta_0].$$

Proof. — For any fixed $\alpha = (\alpha_1, \dots, \alpha_{n-1}) \in \mathbf{Z}_+^{n-1}$ and $k \in \mathbf{N}$, we consider the semi-norm $M_k^\alpha(v : T)$.

There are two possibilities :

i) $\alpha_1 = 0$. In that case $\partial_{y'}^{\alpha} = \partial_{y''}^{\alpha''}$ and the definition of $H(u)$ implies that

$$M_k^\alpha(H(v) : T) \leq \delta M_k^\alpha(v : T).$$

The second possibility is

ii) $\alpha_1 \geq 1$. In that case

$$\partial_{y_1}^{\alpha_1}\partial_{y''}^{\alpha''}\int_0^{y_{n-1}} v^k(t, y'')dt = \partial_{y_1}^{\alpha_1 - 1}\partial_{y''}^{\alpha''}(v^k(y')).$$

Hence, setting $\tilde{\alpha} = (\alpha_1 - 1, \alpha'')$, we obtain

$$M_k^\alpha(H(v) : T) \leq \frac{T}{2k + |\alpha|} M_k^{\tilde{\alpha}}(v : T).$$

The estimates above show that

$$\|H(v)\|_T = \sum M_k^\alpha(H : T) \leq (\delta + T)\|v\|_T,\ v \in E(T),$$

where the sum is taken over all $k \in \mathbf{Z}_+,\ \alpha \in \mathbf{Z}_+^{n-1}$. □

LEMMA 4.5. — *There exist positive constants δ_0 and T_0, such that*

$$H \circ \mathcal{R} : E(T) \ni u \longrightarrow E(T)$$

is continuous, and

$$\|H(\mathcal{R}(v))\|_T \leq 3T\,\|v\|_T,\ \ \forall v \in E(T),\ T \in (0, T_0],\ \delta \in (0, \delta_0].$$

Proof. — Let $k \in \mathbf{N}, \alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbf{Z}_+^{n-1}$, and $v \in E(T)$. According to (4.13) we have

$$H(\mathcal{R}(v))^k(y') = -\begin{pmatrix} E & 0 \\ 0 & -1 \end{pmatrix} \sum_{j=0}^{k-1} \frac{\partial_{y_1}^{2k-2j} v^j(y') - \partial_{y_1}^{2k-2j} v^j(0, y'')}{(2k-2j+1)!}$$
$$+e_1 \sum_{j=0}^{k-1} \frac{\partial_{y_1}^{2k-2j-1} v_n^j(y') - \partial_{y_1}^{2k-2j-1} v_n^j(0, y'') - y_1 \partial_{y_1}^{2k-2j} v_n^j(0, y'')}{(2k-2j+1)!}.$$

Hence,

$$M_k^\alpha(H(\mathcal{R}(v)) : T) \leq 3 \sum_{j=0}^{k-1} \frac{T^{k-j}}{(2k-2j+1)!} \Big(M_j^{(2k-2j+\alpha_1, \alpha'')}(v : T)$$
$$+ T M_j^{(2k-2j+\alpha_1-1, \alpha'')}(v : T) \Big).$$

Summing up with respect to $k \in \mathbf{N}$ and $\alpha \in \mathbf{Z}_+^{n-1}$, and setting $p = k - j$ we obtain

$$\|H(\mathcal{R}(v))\|_T \leq 6 \sum_{p=1}^{\infty} \sum_{j=0}^{\infty} \sum_{\alpha} \frac{T^p}{(2p+1)!} M_j^{(2p+\alpha_1-1, \alpha'')}(v : T)$$
$$\leq 6 \sum_{p=1}^{\infty} \frac{T^p}{(2p+1)!} \|v\|_T \leq 3T \|v\|_T,$$

for all $T \in (0, T_0]$, and $T_0 > 0$ sufficiently small. □

It remains to show that the operator $u \to \mathcal{P}(u)$, given by (4.12) is continuous in $E(T)$.

PROPOSITION 4.6. — *There exist $T_0 > 0$ and $C > 0$ such that the linear map*

$$\mathcal{P} : E(T) \ni v \longrightarrow \mathcal{P}(v) \in E(T)$$

is uniformly continuous for $0 < T \leq T_0$, and

$$\|\mathcal{P}(v)\|_T \leq C \|v\|_T, \ \forall\, T \in (0, T_0). \tag{4.18}$$

Proof. — For any $k \geq 1$ and $\alpha \in \mathbf{Z}^{n-1}$ we have

$$M_k^\alpha(\mathcal{P}(v) : T) \leq \sum_{j=0}^{k-1} \sum_{|\gamma| \leq 2k-2j} \sum_{\rho+\theta=\alpha} \sup_{y' \in B^{n-1}(\delta)} \left| \partial_{y'}^\theta \Gamma_{j,\gamma}^k(y') \right|$$

$$\times \frac{(2j+|\gamma|+|\rho|)!\,(\rho+\theta)!}{(2k+|\rho+\theta|)!\,\rho!\,\theta!} T^{3k-3j+|\theta|-|\gamma|} M_j^{\rho+\gamma}(v_n^j : T). \tag{4.19}$$

To estimate the right hand side of (4.19) we are going to use the following

LEMMA 4.7. — *There exists a constant $C_1 > 0$ such that*

$$\sup_{y' \in B^{n-1}(\delta)} \left|\partial_{y'}^{\theta} \Gamma_{j,\gamma}^k(y')\right| \le C_1^{k-j+|\theta|+1} \frac{(2k-|\gamma|)!}{(2j)!} \theta!, \tag{4.20}$$

for any $\gamma, \theta \in \mathbf{Z}_+^{n-1}$, and $0 \le j \le k-1$, $0 \le |\gamma| \le 2k-2j$.

Lemma 4.7 will be proved in the Appendix. Assume for the moment that (4.20) is valid. Plugging it into (4.19) we arrive at the quantity

$$\frac{(2k-|\gamma|)!\,(2j+|\gamma|+|\rho|)!\,(\theta+\rho)!}{(2j)!\,(2k+|\rho|+|\theta|)!\,\rho!\,\theta!}. \tag{4.21}$$

First we suppose that $|\gamma| = 2k-2j$. Then (4.21) is estimated above by

$$\frac{(2k+|\rho|)!\,(|\rho|+1)\dots(|\rho|+|\theta|)}{(2k+|\rho|+|\theta|)!\,\theta}$$
$$\le \frac{(2k+|\rho|)!\,(2k+|\rho|+1)\dots(2k+|\rho|+|\theta|)}{(2k+|\rho|+|\theta|)!\,\theta!} = \frac{1}{\theta!}.$$

We remind, that for any $z = (z_1, \dots, z_{n-1}) \in \mathbf{Z}_+^{n-1}$ we have $|z| = z_1 + \dots + z_{n-1}$, and that, by convention $0! = 1$.

For $|\gamma| < 2k-2j$, we estimate (4.21) as above by

$$\frac{(2j+|\gamma|+|\rho|+|\theta|)!\,(2j+1)\dots(2j+(2k-2j-|\gamma|))}{(2k+|\rho|+|\theta|)!\,\theta!} \le \frac{1}{\theta!}.$$

Hence, (4.21) is estimated by $\dfrac{1}{\theta!}$, and we obtain

$$M_k^{\alpha}(\mathcal{P}(v) : T) \le C_1 \sum_{j=0}^{k-1} \sum_{|\gamma| \le 2k-2j} \sum_{\rho+\theta=\alpha} \frac{(C_1 T)^{k-j+|\theta|}}{\theta!} M_j^{\rho+\gamma}(v : T).$$

On the other hand, we have

$$\sum_{k=j+1}^{\infty} \sum_{|\gamma| \le 2k-2j} \sum_{\theta \in \mathbf{Z}_+^{n-1}} \frac{(C_1 T)^{k-j+|\theta|}}{\theta!} \le \sum_{p=1}^{\infty} \sum_{\theta \in \mathbf{Z}_+^{n-1}} (2p)^{2n} (C_1 T)^p \frac{(C_1 T)^{|\theta|}}{\theta!} \le 1,$$

for any $T \in (0, T_0]$, and T_0 sufficiently small. Therefore,

$$\|\mathcal{P}(v)\|_T = \sum_{k \in \mathbf{Z}_+} \sum_{\alpha \in \mathbf{Z}_+^{n-1}} M_k^\alpha(\mathcal{P}(v) : T) \leq \|v\|_T,$$

which completes the proof of the proposition. □

Now we pass to the Gevrey realization of the formal change of the variables $u(y)$ and to the proof of the theorem.

Proposition 4.8. — *Let $u^k(y) = (u^{k\prime}(y'), u_n^k(y'))$, $k \in \mathbf{Z}_+$ satisfy the estimates in Proposition 4.3. Then there exists a vector-function $u(y) = (u'(y), u_n(y)) \in G^2(B^n(\delta))$ such that*

$$\partial_{y_n}^{2k} u'(y', 0) = (2k)! u^{k\prime}(y'), \ \partial_{y_n}^{2k+1} u'(y', 0) = 0, \quad k \in \mathbf{Z}_+$$

$$\partial_{y_n}^{2k+1} u_n(y', 0) = (2k+1)! u_n^k(y'), \ \partial_{y_n}^{2k} u_n(y', 0) = 0, \quad k \in \mathbf{Z}_+,$$

i.e. $u(y)$ is a G^2 realization of the formal solution from Proposition 4.3.

Proof. — We observe that in view of Lemma 4.1, $u'(y)$ must be an even function with respect to y_n.

We put $f_{2j+1}(y') = 0$, $f_{2j}(y') = (2j)! u^{j\prime}(y')$, $j \in \mathbf{Z}_+$. Then Proposition 4.3 yields after straightforward calculations that $\{f_j(y')\}_{j=1}^\infty$ satisfy

$$\sup_{y' \in B^{n-1}(\delta)} |\partial_{y'}^\alpha f_k(y')| \leq C^{k+|\alpha|+1} \alpha! (k!)^2, \quad k \in \mathbf{Z}_+, \ \alpha \in \mathbf{Z}_+^{n-1},$$

where $C > 0$.

Now we can use the Whitney extension theorem in Gevrey classes which has been proved in different cases by many authors e.g. [7], [13], [14]. In our case, applying Lemma 1 in [13], or more generally Theorem 4.5 in [14], (see also Theorem 3.9 in [7]) we find a function $u'(y', y_n) \in G^2(B^{n-1}(\delta))$ having as a Taylor expansion in powers of y_n the next formal sum

$$\sum_{j=0}^\infty \frac{y_n^j}{j!} f_j.$$

One deals similarly with $u_n(y)$ (which must be an odd function with respect to y_n) by defining $f_{2j}(y') = 0$, $f_{2j+1}(y') = (2j+1)! u_n^j(y')$, $j \in \mathbf{Z}_+$.

The proof of Theorem 3.1 is complete. □

5 Appendix.

A.1. We are going to prove Lemma 4.7. Since $\Gamma^k_{j,\gamma}(y')$ is holomorphic in the polydisk $B^{n-1}(2\delta)$ in $\mathbf{C}^{n-1}$, it is enough to prove (4.16) for $\theta = 0$ (then one uses the Cauchy integral representation formulae in the polydisk $B^{n-1}(2\delta)$ with $y' \in B^{n-1}(\delta)$ for $0 < \delta \ll 1$). We are going to estimate separately $A_{j,\nu}$ and $a_{\mu,\gamma}$. We have

$$A_{j,0} = (-1)^j, \text{ and } A_{j,\nu} = 0, \quad 1 \le \nu \le p-1, \tag{A.1}$$

according to (4.3) and (4.6), where $p \ge 2$ is fixed.

LEMMA A.1. — *Let $\nu \ge p \ge 2$. Then there exists a constant $C > 0$ such that*

$$\sup_{y' \in \mathbf{B}^{n-1}(\delta)} |A_{j,\nu}(y')| \ \le \ C^{\nu+1} \frac{(j+\nu-2)!}{(j-1)!} \tag{A.2}$$

for any $j \in \mathbf{Z}_+$, $j \ge 1$.

Proof. — Since $g(y)$ is analytic, there exists $C > 0$ such that

$$\sup_{y' \in B^{n-1}(\delta)} |A_r(y')| \le C^r, \quad r = 1, 2, \ldots$$

Hence, in view of (4.6) we obtain

$$|A_{j,\nu}(y')| \le \sum_{(j_0, j_p, \ldots, j_\nu) \in \mathbf{Z}_0(j,\nu)} \frac{j!}{j_0! j_p! \ldots j_\nu!} C^\nu,$$

for $y' \in B^{n-1}(\delta)$. Denote by m the minimum of j and the integer part of ν/p. To estimate the right hand side of the inequality above, we consider for any $j \ge 1$ the function

$$h(t) = \left(1 + \frac{t^p}{1-t}\right)^j, \quad t \in (-1,1), \ p \ge 2.$$

Expanding $h(t)$ in power series, we get

$$h(t) = \left(1 + \sum_{r=p}^{\infty} t^r\right)^j = \sum_{\nu=0}^{\infty} \sum_{(j_0, j_p, \ldots, j_\nu) \in \mathbf{Z}_0(j,\nu)} \frac{j!}{j_0! j_p! \ldots j_\nu!} t^\nu.$$

On the other hand,

$$h(t) = 1 + \sum_{s=1}^{j} \binom{j}{s} \frac{t^{sp}}{(1-t)^s}.$$

Hence, for any $\nu \geq p \geq 2$ we have

$$
\begin{aligned}
\sum_{(j_0,j_p,\dots,j_\nu)\in \mathbf{Z}_0(j,\nu)} \frac{j!}{j_0! j_p! \dots j_\nu!} &= \frac{1}{\nu!}\frac{d^\nu h}{dt^\nu}(0) \\
&= \frac{1}{\nu!}\sum_{s=1}^{j}\binom{j}{s}\left(\frac{d}{dt}\right)^{\nu}\left(t^{sp}(1-t)^{-s}\right)|_{t=0} \\
&= \frac{1}{\nu!}\sum_{s=1}^{m}\binom{j}{s}\binom{\nu}{sp}(sp)!\, s(s+1)\dots(s+\nu-sp-1) \\
&\leq \sum_{s=1}^{m} j(j-1)\dots(j-s+1)\,(s+\nu-sp-1)! \\
&\leq \sum_{s=1}^{m} j(j+1)\dots(j+s-1)(s+\nu-sp-1)! \\
&\leq \sum_{s=1}^{m}\frac{(j+\nu-s(p-2)-2)!}{(j-1)!} \\
&\leq \nu\frac{(j+\nu-2)!}{(j-1)!}.
\end{aligned}
$$

Evidently the inequality above yields the desired estimate. □

LEMMA A.2. — *There exists a constant $C > 0$ such that*

$$
\sup_{y'\in \mathbf{B}^{n-1}(\delta)} |a_{\mu,\gamma}(y')| \leq C^{\mu+1} \tag{A.3}
$$

for any $\mu \in \mathbf{Z}_+$, and $\gamma \in \mathbf{Z}_+^{n-1}$, $0 < |\gamma| \leq \mu$.

Proof. — The demonstration is similar to that of Lemma A.1. We have

$$
\sup_{y'\in B^{n-1}(\delta)} |a_{\mu,\gamma}(y')| \leq \sum_{(\gamma_1,\dots,\gamma_\mu)\in \mathbf{Z}(\gamma,\mu)} \frac{\gamma!}{\gamma_1!\dots\gamma_\mu!} C^\mu .
$$

Let h be an analytic function in $\mathbf{R}^{n-1}$. One observes that the right hand-side of the inequality above coincides with the coefficient $\tilde{a}_{\mu,\gamma}(C)$ in the identity

$$
h\left(\frac{Ct}{1-Ct},\dots,\frac{Ct}{1-Ct}\right) = \sum_{\gamma\in \mathbf{Z}_+^{n-1}}\sum_{\mu\geq|\gamma|}^{\infty} \tilde{a}_{\mu,\gamma}(C)t^\mu \frac{h^{(\gamma)}(0)}{\gamma!},
$$

which leads to

$$\begin{aligned}\tilde{a}_{\gamma,\mu}(C) &= \frac{1}{\mu!}\left(\frac{d}{dt}\right)^{\mu}\left((Ct)^{|\gamma|}(1-Ct)^{-|\gamma|}\right)|_{t=0}\\ &= C^{\mu}\frac{(\mu-1)!}{(\mu-|\gamma|)!(|\gamma|-1)!} \leq (2C)^{\mu},\end{aligned}$$

provided $\gamma \in \mathbf{Z}_+^{n-1}$, $0 < |\gamma| \leq \mu$. □

Proof of Lemma 4.7. — Fix γ, k, and $0 \leq j \leq k-1$, so that $|\gamma| \leq 2k-2j$. Using (4.10) - (4.11) as well as (A.1) - (A.3), we obtain

$$\begin{aligned}\left|\Gamma_{j,\gamma}^{k,n}(y')\right| &\leq C^{2k+1-2j} + \sum_{\nu=2}^{2k+1-2j-|\gamma|} C^{2k+1-2j}\frac{(2j+\nu-1)!}{(2j)!}\\ &\leq C^{2k-2j+1}(2k-2j+1-|\gamma|)\frac{(2k-|\gamma|)!}{(2j)!} \leq C_1^{k-j}\frac{(2k-|\gamma|)!}{(2j)!},\end{aligned}$$

with $C_1 > 0$ satisfying $C^{2s+1}(2s+1) \leq C_1^s$, $s \geq 1$. The proof of Lemma 4.5 is complete. □

Acknowledgements. Most of this paper was written while the second author was a fellow of the Alexander von Humboldt foundation at the Institute of Applied Mathematics, Bonn University. He would like to thank the Alexander von Humboldt foundation for the support as well as Rolf Leis for the hospitality. The authors are grateful to Vladimir Lazutkin who drew their attention on the problem of the effective stability for the billiard ball map some years ago.

BIBLIOGRAPHY

[1] A. BAZZANI, S. MARMI, G. TURCHETTI, Nekhoroshev estimate for isochronous non resonant symplectic maps, Celestial Mech. and Dynamical Astronomy, 47(1990), 333-359.

[2] G. BENETTIN, L. GALGANI, A. GIORGILLI, A proof of Nekhoroshev's theorem for the stability times in nearly integrable hamiltonian systems, Celestial Mech., 37 (1985), 1-25.

[3] G. BENETTIN, G. GALLAVOTTI, Stability of motions near resonances in quasi-integrable hamiltonian systems, J. Stat. Phys., 44 (1986), 293-338.

[4] M. BERGER, Sur les caustiques de surfaces en dimension 3. C. R. Acad. Sci. Paris, Série I, 331 (1990), 333-336.

[5] P. BOLLEY, J. CAMUS, G. MÉTIVIER, Régularité Gevrey et itérés pour une classe d'opérateurs hypoelliptiques, Rend. Sem. Mat. Univ. Polit. Torino, 41 (1983), 51-74.

[6] L. BOUTET de MONVEL, P. KREE, Pseudodifferential operators and Gevrey classes, Ann. Inst. Fourier, 17-1 (1967), 295-323.

[7] J. BONET, R. BRAUN, R. MEISE, B. TAYLOR, Whitney's extension theorem for nonquasianalytic classes of ultradifferentiable functions, Studia Math., 99, (1991), 155-184.

[8] T. GRAMCHEV, G.POPOV, Gevrey normal forms of glancing hypersurfaces (in preparation).

[9] A. GIORGILLI, A. DELSHAMS, E. FONTICH, L. GALGANI, C. SIMÓ, Effective stability for hamiltonian system near an elliptic equilibrium point, with an application to the restricted three body problem, J. Diff. Eq., 77 (1989), 167-198.

[10] A. GIORGILLI, E. ZEHNDER, Exponential stability for time dependent potentials, Z. angew. Math. Phys., 43 (1992), 827-855.

[11] L. HÖRMANDER, The Analysis of Linear Partial Differential Operators III, IV, Berlin - Heidelberg - New York, Springer; 1985.

[12] V. KOVACHEV, G. POPOV, Invariant tori for the billiard ball map, Trans. Am. Math. Soc., 317 (1990), 45-81.

[13] H. KOMATSU, Ultradistributions. II. The kernel theorem and ultradistributions with support on submanifold, J. Fac. Sci. Univ. Tokyo, Sect. IA 24 (1977), 607-628.

[14] H. KOMATSU, An analogy of the Cauchy-Kowalevsky theorem for ultradifferentiable functions and a division theorem for ultradistributions as its dual, J. Fac. Sci. Univ. Tokyo, Sect. IA 26 (1979), 239-254.

[15] S. KUKSIN, J. PÖSCHEL, On the inclusion of analytic symplectic maps in analytic hamiltonian flows and its applications, preprint ETH-Zürich (1992).

[16] V. LAZUTKIN, The existence of caustics for a billiard problem in a convex domain, Math. USSR Izv., 7, (1973), 185-214.

[17] P. LOCHAK, Canonical perturbation theory : an approach based on joint approximations (Russian), Uspekhi Mat. Nauk 47, (6), (1992), 59-140; translation in : Russian Math. Surveys 47, (6), (1992), 57-133.

[18] P. LOCHAK, A.I. NEISHTADT, Estimates of stability time in nearly integrable systems with a quasiconvex Hamiltonian, Chaos, 2, (4) (1992), 495-499.

[19] Sh. MARVIZI, R. MELROSE, Spectral invariants of convex planar regions, J. Diff. Geom., 17 (1982), 475-502.

[20] R. MELROSE, Equivalence of glancing hypersurfaces, Inventiones Math., 37 (1976), 165-191.

[21] J. MOSER, On invariant curves of area preserving mappings of an annulus., Nachr. Akad. Wiss. Göttingen Math. Phys. Kl. II, (1962) 1-20.

[22] N. NEKHOROSHEV, Exponential estimate of the stability time of near-integrable hamiltonian systems I, Russ. Math. Surveys, 32 (6) (1977), 1-65.

[23] N. NEKHOROSHEV, Exponential estimate of the stability time of near-integrable hamiltonian systems II, Trudy Sem. Petrovs., 5 (1979), 5-50 (in russian).

[24] T. OSHIMA, On analytic equivalence of glancing hypersurfaces, Sci. Papers College Gen. Ed. Univ. Tokyo, 28 (1) (1978), 51-57.

[25] J. PÖSCHEL, Nekhoroshev estimates for quasi-convex hamiltonian systems, Math. Z., 213 (1993), 187-217.

[26] L. RODINO, Linear partial differential operators in Gevrey spaces, Singapore-New Jersey-London-Hong Kong, World Scientific, 1992.

Manuscrit reçu le 24 février 1994,
accepté le 2 mars 1995.

T. GRAMCHEV,
Dipartimento di Matematica
Università di Cagliari
Via Ospedale 72
09124 Cagliari (Italie).

G. POPOV,
Institute of Mathematics
Bulgarian Academy of Sciences
1113 Sofia (Bulgarie).