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# JACOBI-EISENSTEIN SERIES AND $p$ -ADIC INTERPOLATION OF SYMMETRIC SQUARES OF CUSP FORMS

by Pavel I. GUERZHOY

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## 1. INTRODUCTION AND STATEMENT OF THE MAIN THEOREM

Let

$$f = \sum_{\substack{n \in \mathbf{Z} \\ n \geq 0}} a(n)e(n\tau) \quad (e(x) = e^{2\pi ix})$$

be a cusp Hecke eigenform of even integral weight  $k$  on the full modular group  $SL_2(\mathbf{Z})$ . We denote the space of all such forms of weight  $k$  by  $S_k$  and the space of all modular forms of weight  $k$  by  $M_k$ . Let  $M$  be an integer,  $3 \leq M \leq k-1$ , and  $\chi$  be a Dirichlet character modulo  $r$ ,  $\chi(-1) = (-1)^{M+1}$ . The special values of symmetric squares of the cusp form  $f$  are defined by the following:

$$(1) \quad D_f(M, \chi) = \sum_{n \geq 1} \frac{a(n^2)\chi(n)}{n^{k+M-1}}.$$

The values (1) are known to become algebraic numbers after multiplication by an appropriate constant. Below we extend in a natural way this definition for  $f$  being an Eisenstein series.

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*Key words:* Jacobi forms – Eisenstein series – Symmetric square – Modular forms –  $p$ -adic interpolation – Rankin’s method.

*Math. classification:* 11F67 – 11F85 – 11F55.

Let  $\{f_j \mid j = 1, \dots, \dim M_k\}$  be a basis of the linear space  $M_k$  of modular forms of weight  $k$ . This basis consists of the normalized (i.e.  $a(1) = 1$ ) Hecke eigenforms. The modular form

$$(2) \quad F(k, M, \chi) = \sum_{j=1}^{\dim M_k} \frac{f_j}{\langle f_j, f_j \rangle} D_{f_j}(M, \chi)$$

is the kernel function for the special values of the symmetric square with respect to the Petersson scalar product  $\langle \cdot, \cdot \rangle$  in the following sense:

$$\langle F(k, M, \chi), f_j \rangle = D_{f_j}(M, \chi).$$

We construct a generating function from the modular forms  $F(k, M, \chi)$  and their derivatives.

**MAIN THEOREM.** — *Let  $M \geq 1$  be a fixed natural number,  $\chi$  be a fixed Dirichlet character modulo  $r$ , and  $t = (1 - \chi(-1))/2$ .*

Then

$$(3) \quad \sum_{\nu \geq 0} z^{2\nu+t} \sum_{0 \leq \mu \leq \nu} \Lambda(\mu, \nu) F^{(\mu)}(2\nu - 2\mu + M + t + 1, M, \chi) = E_{M+1}^{\chi}(\tau, z),$$

with the Jacobi-Eisenstein series  $E_{M+1}^{\chi}$  (of weight  $M + 1$  and index  $r$  on  $SL_2(\mathbf{Z})$ ) as explained below, where  $\Lambda(\mu, \nu)$  is given by the following:

$$\begin{aligned} \Lambda(\mu, \nu) &= 2^{1-4M-3t-2\nu} \pi^{1/2-M-t} r^{2\nu+t} \Gamma(M + 1/2)^{-1} \\ &\times \sum_{0 \leq \mu \leq \nu} (-1)^{\nu-\mu} (2\pi i)^{\mu} \frac{\Gamma(M + 2\nu - 2\mu + t + 1) \Gamma(2M + 2\nu - 2\mu + t)}{\Gamma(M + 2\nu - \mu + t + 1) \Gamma(\mu + 1) \Gamma(2\nu - 2\mu + t + 1)}. \end{aligned}$$

We must say a few words about the Jacobi-Eisenstein series  $E_{M+1}^{\chi}(\tau, z)$  occurring in (3). Almost all necessary facts concerning Jacobi forms one can find in [2]. Taking into account that our notations are slightly different from those given in this work, we shall briefly recall some definitions and propositions of this theory. Hereafter the letter  $\mathbf{H}$  denotes the complex upper-half plane,  $\mathbf{C}$  denotes the whole complex plane, the letter  $\mathbf{Z}$  denotes the set of integers. For  $\tau \in \mathbf{H}$  and  $\Gamma \in SL_2(\mathbf{Z})$  we assume  $\gamma(\tau) = \frac{a\tau + b}{c\tau + d}$ . The formulas

$$\begin{aligned} \left( \phi \mid_{k,r} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) (\tau, z) &= (c\tau + d)^{-k} e \left( \frac{-cz^2}{c\tau + d} \right) \phi \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right), \\ \left( \phi \mid_r (\lambda, \mu) \right) (\tau, z) &= e(r(\lambda^2\tau + 2\lambda z)) \phi(\tau, z + \lambda\tau + \mu) \end{aligned}$$

define the action of Jacobi group  $\Gamma^J$  (i.e. the semi direct product of  $SL_2(\mathbf{Z})$  and  $(\mathbf{Z} \times \mathbf{Z})$ ) in the space of holomorphic functions  $\phi(\tau, z)$  of two variables

( $\tau \in \mathbf{H}, z \in \mathbf{C}$ ). Let  $k$  and  $r$  be positive integers. A function  $\phi$  is referred to as Jacobi form of weight  $k$  and index  $r$  if it satisfies the following conditions:

$$\phi|_{k,r} \xi(\tau, z) = \phi(\tau, z) \text{ for every element } \xi \text{ of } \Gamma^J,$$

$$\phi(\tau, z) = \sum_{n \geq 0} \sum_{\substack{m \in \mathbf{Z} \\ m^2 \leq 4rn}} c(n, m)e(n\tau + mz).$$

Denote as  $J_{k,r}$  the finite-dimensional linear space of Jacobi forms of weight  $k$  and index  $r$ . For an integer  $k > 2$  and any integer  $s$  the Eisenstein series  $E_{k,r,s}$  in the space  $J_{k,r}$  is defined, as in [2], p. 25, by the following:

$$E_{k,r,s}(\tau, z) = \sum_{\gamma \in \Gamma_\infty^J \backslash \Gamma^J} e(as^2\tau + 2absz)|_{k,r} \gamma,$$

where

$$\Gamma_\infty^J = \left\{ \gamma \in \Gamma^J : 1|_{k,r} \gamma = 1 \right\} = \left\{ \left( \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, (0, \mu) \right) \mid (n, \mu) \in \mathbf{Z} \right\},$$

and where we use  $a, b$  for the unique natural numbers such that  $r = ab^2$  and  $a$  is square-free. This series depends only on the residue of  $s$  modulo  $b$ . A Jacobi form is referred to as a cusp form if

$$\phi(\tau, z) = \sum_{n \geq 0} \sum_{\substack{m \in \mathbf{Z} \\ m^2 < 4rn}} c(n, m)e(n\tau + mz).$$

The Eisenstein series in (3) is now given by

$$E_k^\chi(\tau, z) = (4\pi i)^{-t} 1/2 \sum_{s \bmod r} \chi(s) E_{k,r,s}(\tau, z).$$

The idea to construct generating functions connected with special values of  $L$ -functions associated with modular forms appeared in [9]. In this paper a generating function associated with the period polynomials of modular forms was constructed and this generating function was calculated in terms of Jacobi theta function.

Section 2 is devoted to the proof of the Main Theorem. In section 3 we shall derive explicit formulas for the Fourier coefficients of the series  $E_k^\chi(\tau, z)$  (cf. Theorem 2). In section 4 we shall use our Main Theorem and these formulas to prove the existence of a  $p$ -adic analytic function such that its special values coincide with those of the symmetric square of a  $p$ -ordinary cusp form (cf. Theorem 3). The main idea for this is to use the well-known  $p$ -adic interpolation properties of the special values of Dirichlet  $L$ -function. These special values appear in the formulas for

Fourier coefficients of our Jacobi-Eisenstein series. We construct the  $p$ -adic analytic function in question as the non-Archimedean Mellin transform of a bounded  $C_p$ -valued measure. The existence of this measure is proved using the abstract Kummer congruences.

A similar result on  $p$ -adic interpolation of symmetric squares one can find in [5]. Our method to prove it differs from those of [5]: we use Jacobi forms instead of non-holomorphic modular forms.

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## 2. PROOF OF THE MAIN THEOREM

To prove this theorem we must recall some facts concerning Jacobi forms and the Rankin's method of calculating the symmetric square special values.

### 2.1. Differential operators acting in the space of modular forms.

For two smooth functions  $f$  and  $g$ , a natural number  $\nu$  and real positive  $k_1$  and  $k_2$  Cohen [1] defined smooth functions  $F_\nu^{k_1, k_2}$  by the formulas

$$(4) \quad \begin{aligned} F_\nu^{k_1, k_2}(f, g) &= \sum_{0 \leq \mu \leq \nu} (-1)^{\nu-\mu} \binom{\nu}{\mu} \frac{\Gamma(k_1+\nu)\Gamma(k_2+\nu)}{\Gamma(k_1+\mu)\Gamma(k_2+\nu-\mu)} f^{(\mu)} g^{(\nu-\mu)} \\ &= \sum_{0 \leq \mu \leq \nu} (-1)^\mu \binom{\nu}{\mu} \frac{\Gamma(k_1+\nu)\Gamma(k_1+k_2+2\nu-\mu-1)}{\Gamma(k_1+\nu-\mu)\Gamma(k_1+k_2+\nu-1)} (fg^{(\nu-\mu)})^{(\mu)}. \end{aligned}$$

It is known that if  $f$  and  $g$  are modular forms on some group  $H \subseteq SL_2(\mathbf{Z})$ , with weights  $k_1$  and  $k_2$ , then  $F_\nu^{k_1, k_2}(f, g)$  is a modular form on  $H$  of weight  $k_1 + k_2 + 2\nu$ .

A function  $\tilde{f}^\ell(\tau, z)$  of two variables was constructed in [1] for a real positive number  $\ell$  and a smooth function  $f$ :

$$\tilde{f}^\ell(\tau, z) = \sum_{\nu \geq 0} \frac{(2\pi i)^\nu \Gamma(\ell)}{\Gamma(\nu+1)\Gamma(\ell+\nu)} f^\nu(\tau) z^{2\nu}.$$

Both these operators are tightly connected:

$$\tilde{f}_1^{k_1}(\tau, z) \tilde{f}_2^{k_2}(\tau, iz) = \sum_{\nu \geq 0} z^{2\nu} \frac{(2\pi i)^{2\nu}}{\Gamma(\nu+1)\Gamma(k_1+\nu)\Gamma(k_2+\nu)} F_\nu^{k_1, k_2}(f_1, f_2).$$

**2.2. Rankin’s method for symmetric squares.**

These operators can be used for the calculation of the symmetric square special values.

Let  $\text{Tr}_1^{4r^2} : M_k(\Gamma_0(4r^2)) \rightarrow M_k$  be the trace operator as in [4].

PROPOSITION 1 ([10]). — *Let  $\nu$  and  $M$  be natural numbers. Let  $\chi$  be a Dirichlet character modulo  $r$ ,  $t = (1 - \chi(-1))/2$ , and  $\chi(-1) = (-1)^{M+1}$ . Then there exists an Eisenstein series  $S = S(2\nu + M + t + 1)$  in the space of modular forms of weight  $2\nu + M + t + 1$  on  $SL_2(\mathbf{Z})$  such that*

$$(5) \quad S + F^c(2\nu + M + t + 1, M, \chi) = (2\pi i)^{-\nu} \frac{(4\pi)^{2\nu + M + t} \Gamma(M + 1/2)}{\Gamma(M + 2\nu + t) \Gamma(M + \nu + 1/2)} \times \text{Tr}_1^{4r^2} F_\nu^{t+1/2, M+1/2}(h_\chi, E_{M+1/2}^\chi),$$

where

$$F^c(k, m, \chi) = \sum_{j=1}^{\dim S_k} \frac{f_j}{\langle f_j, f_j \rangle} D_{f_j}(M, \chi),$$

and the sum is carried out through all normalized cusp Hecke eigenforms of weight  $k$ . Functions  $h_\chi$  and  $E_{M+1/2}^\chi$  are the modular forms of half integral weight introduced in [8]:

$$h_\chi(\tau) = 1/2 \sum_{n \in \mathbf{Z}} \chi(n) n^t e(n^2 \tau),$$

$$E_{M+1/2}^\chi = \sum_{\substack{(c, d)=1 \\ c \equiv 0 \pmod{4r}} } \frac{\chi(d) \left(\frac{-1}{d}\right) \left(\frac{c}{d}\right) \varepsilon_d^{-2t-1}}{(c\tau + d)^{M+1/2}} |c\tau + d|^{-2s} \Big|_{s=0}.$$

Now we define the special value of symmetric square  $D_f(M, \chi)$  when  $f$  is not necessary a cusp form by deleting the “addition member”  $S$  in (5). Assuming this definition one can rewrite (5):

$$(6) \quad F(2\nu + M + t + 1, M, \chi) = (2\pi i)^{-\nu} \frac{(4\pi)^{2\nu + M + t} \Gamma(M + 1/2)}{\Gamma(M + 2\nu + t) \Gamma(M + \nu + 1/2)} \times \text{Tr}_1^{4r^2} F_\nu^{t+1/2, M+1/2}(h_\chi, E_{M+1/2}^\chi).$$

**2.3. Taylor expansions of Jacobi forms.**

PROPOSITION 2. — Let  $\phi \in J_{k,r}$  be a Jacobi form. We denote by  $X_\nu(\phi)(\tau)$  the Taylor expansion coefficients of the function  $\phi$  on  $z$ :

$$\phi(\tau, z) = \sum_{\nu \geq 0} X_\nu(\phi)(\tau) z^\nu.$$

a) The function

$$\xi_\nu^r(\phi)(\tau) = \sum_{0 \leq \mu \leq \nu/2} \frac{(-2\pi i r)^\mu \Gamma(k + \nu - \mu - 1)}{\Gamma(k + \nu - 1) \Gamma(\mu + 1)} X_{\nu - 2\mu}^{(\mu)}(\phi)(\tau)$$

is a modular form of weight  $k + \nu$  on  $SL_2(\mathbf{Z})$ . In other words, one can define operators  $\xi_\nu^r : J_{k,r} \rightarrow M_{k+\nu}$ .

b) The following identities take place:

$$X_\nu(\phi)(\tau) = \sum_{0 \leq \mu \leq \nu/2} \frac{(2\pi i r)^\mu \Gamma(k + \nu - 2\mu)}{\Gamma(k + \nu - \mu) \Gamma(\mu + 1)} (\xi_{\nu - 2\mu}^r)^{(\mu)}(\phi)(\tau).$$

It means that the set of modular forms  $\xi_\nu^r(\phi)(\tau)$  defines the Jacobi form  $\phi$  uniquely.

c) Let  $SL_2(\mathbf{Z}) = \bigcup_j H \sigma_j$  be a finite coset decomposition. Then

$$\sum_j \xi_\nu^r(\phi) |_{k,r} \sigma_j = \xi_\nu^r \left( \sum_j \phi | \sigma_j \right).$$

In other words, the operators  $\xi_\nu^r$  commute with the trace operator.

d) One can construct the operators  $\xi_\nu^r$  using the Fourier coefficients  $c(n, m)$  of Jacobi form  $\phi$ . If

$$\phi(\tau, z) = \sum_{n \geq 0} \sum_{\substack{m \in \mathbf{Z} \\ m^2 \leq 4r^2 n}} c(n, m) e(n\tau + mz) \in J_{k,r^2}$$

then

$$\begin{aligned} (2\pi)^{-2\nu-t} \xi_{2\nu+t}^{r^2}(\phi) &= \sum_{n \geq 0} \sum_{m \in \mathbf{Z}} \sum_{0 \leq \mu \leq \nu} (-1)^\mu \\ &\times \frac{\Gamma(2\nu+t+k-\mu-1)}{\Gamma(\mu+1) \Gamma(2\nu+t-2\mu+1) \Gamma(2\nu+k+t-1)} m^{2\nu+t-2\mu} r^{2\mu} n^\mu c(n, m) e(n\tau). \end{aligned}$$

Here one must take  $t$  equal to 0 or 1 to make the number  $k + t$  even.

Parts a), b) and d) of this proposition are contained in [2], Theorem 3.2. To prove c) we consider firstly the case when  $k$  is even. Consider the space  $M_{k,r}$  of holomorphic functions  $\phi$  of two variables with the property

$$\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k e\left(\frac{rcz^2}{c\tau + d}\right) \phi(\tau, z)$$

for every element  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  of  $SL_2(\mathbf{Z})$ . The differential operator

$$L_k = 8\pi ir \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial z^2} - \frac{2k-1}{z} \frac{\partial}{\partial z}$$

maps an element  $\phi$  of  $M_{k,r}$  to an element  $L_k\phi$  of  $M_{k+2,r}$ . It is easy to see that

$$L_k(\phi|_{k,r}\sigma) = (L_k)|_{k+2,r}\sigma \text{ for all } \sigma \in SL_2(\mathbf{Z}), \phi \in M_{k,r},$$

$$\xi_{2\nu}^r(\phi)(\tau) = (L_{k+2\nu-2} \circ L_{k+2\nu-4} \circ \dots \circ L_k\phi)(\tau, 0).$$

Part c) of the Proposition 2 in the case of even  $k$  follows immediately from these formulas. To prove it in the case when  $k$  is odd one must consider the function

$$\phi_1(\tau, z) = z\phi(\tau, z) \in M_{k+1,r}.$$

It has the same Fourier coefficients as  $\phi$ , the number  $k+1$  is even and it is enough to apply part a) to finish the proof.

### 2.4.

Let  $\chi : \mathbf{Z}/r\mathbf{Z} \rightarrow \mathbf{C}$  be a Dirichlet character;  $t = 0$  or  $1$ ,  $\chi(-1) = (-1)^t$ . We denote by  $\theta_\chi$  the theta-function associated with character  $\chi$ :

$$\theta_\chi(\tau, z) = 1/2(4\pi i)^{-t} \sum_{m \in \mathbf{Z}} \chi(m) e(m^2\tau + 2mrz).$$

LEMMA 1. — Let  $SL_2(\mathbf{Z}) = \bigcup_j \Gamma_0(4r^2)\sigma_j$  be a right coset decomposition. Then for a natural number  $k \geq 2$

$$E_{M+1}^\chi(\tau, z) = \sum_j (\theta_\chi(\tau, z) E_{k-1/2}^\chi(\tau)) \Big|_{k,r^2} \sigma_j.$$

To prove this lemma we use the following assertion connected with the action of elements of  $\Gamma_0(4r^2)$  on the function  $\theta_\chi$ .



LEMMA 2. — Let  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4r^2)$ , and let  $\chi$  be a primitive Dirichlet character modulo  $r$ . Then

$$\theta_\chi \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = \chi(d) \left( \frac{-1}{d} \right)^t \left( \frac{c}{d} \right) \varepsilon_d^{-1} (c\tau + d)^{1/2} e \left( \frac{cr^2 z^2}{c\tau + d} \right) \theta_\chi(\tau, z),$$

where  $\varepsilon_d = 1$  or  $i$  according as  $d \equiv 1$  or  $3 \pmod{4}$ .

This lemma follows immediately from the modular properties of the function  $h_\chi$  (cf. [8]) and the following three propositions.

PROPOSITION 3 ([8], Proposition 2.2, p. 457). — If  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4r^2)$ , then

$$h_\chi(\gamma(\tau)) = \chi(d) \left( \frac{-1}{d} \right)^t (c\tau + d)^{t+1/2} h_\chi(\tau).$$

PROPOSITION 4. — The following identity holds true:

$$\theta_\chi(\tau, z) = (rz)^t \tilde{h}_\chi^{1/2+t}(\tau, rz).$$

PROPOSITION 5. — If  $f$  is a smooth function,  $\ell$  a natural number, and  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$  then

$$\tilde{f}^\ell \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = (c\tau + d)^\ell e \left( \frac{cr^2 z^2}{c\tau + d} \right) \left[ \widetilde{(c\tau + d)}^{-\ell} f \left( \frac{a\tau + b}{c\tau + d} \right) \right]^\ell.$$

The Cohen's operator in the right-hand side of this equation acts on the function in the square parentheses. Now we turn to the proof of the propositions.

*Proof of Proposition 4.* — After differentiation and changing the order of summation one has:

$$\tilde{h}_\chi^{1/2+t}(\tau, z) = \frac{\Gamma(t + 1/2)}{2} \sum_{n \in \mathbf{Z}} \chi(n) e(n^2 \tau) \sum_{\nu \geq 0} \frac{(2\pi iz)^{2\nu} n^{t+2\nu}}{\Gamma(\nu + 1) \Gamma(t + 1/2 + \nu)}.$$

To finish the proof of Proposition 4, it is sufficient to use the Legendre formulas for  $\Gamma$ -function and to observe that

$$\sum_{n \in \mathbf{Z}} \chi(n) e(n^2 \tau) \sum_{\nu \geq 0} \frac{(4\pi inz)^{2\nu + (1-t)}}{\Gamma(2\nu + (1-t) + 1)} = 0.$$

*Proof of Proposition 5.* — We consider a function  $E(\tau) = 1/(\tau - \bar{\tau})$  on the upper-half plane. The bar denotes the complex conjugation. This function satisfies the following functional equation:

$$E\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E(\tau) - c(c\tau + d)$$

for every  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$ . We construct a two-variable function  $G_{f,\ell}(\tau, z)$  associated with the function  $f(\tau)$ :

$$G_{f,\ell}(\tau, z) = \exp(z^2 E(\tau)) \sum_{\nu \geq 0} \frac{z^{2\nu}}{\Gamma(\nu + 1)\Gamma(t + \ell)} f^{(\nu)}(\tau).$$

From the following identities proved by Cohen ([1], p. 281) one gets the statement of Proposition 3:

$$G_{f,\ell}(\tau, z\sqrt{2\pi i}) = e(z^2 E(\tau)) \frac{1}{\Gamma(\ell)} \tilde{f}^\ell(\tau, z),$$

$$G_{f,\ell}\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^{-\ell} G_{f|_{\ell\gamma}, \ell}(\tau, z).$$

Now we turn to the proof of Lemma 1. We claim that for every integer  $k \geq 2$

$$E_k^X(\tau, z) = \sum_{(c,d)} \theta_{\chi|_{k,r^2}} \begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau, z),$$

where for each pair of integers  $(c, d)$  the numbers  $a$  and  $b$  are such that  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$  and the sum is carried out through all such pairs  $(c, d)$  for which an appropriate pair  $(a, b)$  exists. One has:

$$\begin{aligned} & \sum_{(c,d)} \theta_{\chi|_{k,r^2}} \begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau, z) \\ &= \sum_{(c,d)} (c\tau + d)^{-k} e\left(-\frac{cr^2 z^2}{c\tau + d}\right) \theta_{\chi}\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) \\ &= \frac{1}{2}(4\pi i)^{-t} \sum_{(c,d)} (c\tau + d)^{-k} e\left(-\frac{cr^2 z^2}{c\tau + d}\right) \\ & \quad \times \sum_{m \in \mathbf{Z}} \chi(m) e\left(m^2 \frac{a\tau + b}{c\tau + d}\right) e\left(2mr \frac{z}{c\tau + d}\right) \\ &= \frac{1}{2}(4\pi i)^{-t} \sum_{(c,d)} \sum_{\lambda \in \mathbf{Z}} \chi(\lambda) (c\tau + d)^{-k} e\left(\lambda^2 \frac{a\tau + b}{c\tau + d} + 2r\lambda \frac{z}{c\tau + d} - r^2 \frac{cz^2}{c\tau + d}\right) \\ &= E_k^X(\tau, z). \end{aligned}$$

On the other side one can apply Lemma 2:

$$\begin{aligned}
 E_k^\chi(\tau, z) &= \sum_{(c,d)} \theta_\chi|_{k,r^2} \begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau, z) \\
 &= \sum_j \left( \sum_{\substack{(c,d) \\ c \equiv 0 \pmod{4r^2}}} \theta_\chi|_{k,r^2} \begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau, z) \right) \Big|_{k,r^2} \sigma_j \\
 &= \sum_j \left( \theta_\chi(\tau, z) \sum_{\substack{(c,d) \\ c \equiv 0 \pmod{4r^2}}} \frac{\chi(d) \left(\frac{-1}{d}\right) \left(\frac{c}{d}\right) \varepsilon_d^{-2t-1}}{(c\tau + d)^{k-1/2}} \right) \Big|_{k,r^2} \sigma_j \\
 &= \sum_j (\theta_\chi(\tau, z) E_{k-1/2}(\tau)) \Big|_{k,r^2} \sigma_j.
 \end{aligned}$$

Lemma 1 is proved.

### 2.5. Proof of the Main Theorem.

It is enough to calculate the Taylor expansion coefficients of the Jacobi-Eisenstein series  $E_{M+1}^\chi$ . Taking into account Proposition 4 we have:

$$\begin{aligned}
 (7) \quad \theta_\chi(\tau, z) E_{M+1/2}(\tau) &= \Gamma(t + 1/2) \sum_{\nu \geq 0} (2\pi i)^\nu \frac{(rz)^{2\nu+t}}{\Gamma(\nu + 1)\Gamma(t + 1/2 + \nu)} h_\chi^{(\nu)} E_{M+1/2}.
 \end{aligned}$$

We see from (7) that

$$X_{2\nu+t}(\theta_\chi(\tau, z) E_{M+1/2}(\tau)) = \frac{\sqrt{\pi}}{2^t} r^{2\nu+t} \frac{(2\pi i)^\nu}{\Gamma(\nu+1)\Gamma(t+1/2+\nu)} h_\chi^{(\nu)} E_{M+1/2}.$$

Using assertion a) of Proposition 2 and (4) we get

$$\begin{aligned}
 (8) \quad \xi_{2\nu+t}(\theta_\chi(\tau, z) E_{M+1/2}(\tau)) &= \frac{\sqrt{\pi}}{2^t} r^{2\nu+t} (2\pi i)^\nu \\
 &\quad \times \frac{\Gamma(M + t + \nu)}{\Gamma(\nu + 1)\Gamma(M + 2\nu + t)\Gamma(t + 1/2 + \nu)} \mathbf{F}_\nu^{t+1/2, M+1/2}(h_\chi, E_{M+1/2}).
 \end{aligned}$$

Applying to (8) Lemma 1 and assertion c) of Proposition 2:

$$\begin{aligned}
 (9) \quad \xi_{2\nu+t}(E_{M+1}^\chi) &= \frac{\sqrt{\pi}}{2^t} r^{2\nu+t} (2\pi i)^\nu \\
 &\quad \times \frac{\Gamma(M + t + \nu)}{\Gamma(\nu + 1)\Gamma(M + 2\nu + t)\Gamma(t + 1/2 + \nu)} \mathbf{Tr}_1^{4r^2} \mathbf{F}_\nu^{t+1/2, M+1/2}(h_\chi, E_{M+1/2}).
 \end{aligned}$$

Now one can rewrite the right-hand side of (9) in accordance with (6):

$$\xi_{2\nu+t}(E_{M+1}^x) = r^{2\nu+t} 2^{1-4M-3t+2\nu} \pi^{1/2-M-t} (-1)^\nu \times \frac{\Gamma(2M+t+2\nu)}{\Gamma(1+2\nu+t)\Gamma(M+1/2)} F(2\nu+M+t+1, M, \chi).$$

Application of assertion b) of the Proposition 2 finishes the proof of the Main Theorem.

### 3. CALCULATION OF THE JACOBI-EISENSTEIN SERIES FOURIER

In the simplest case of the Jacobi-Eisenstein series  $E_{k,1}$  of index one, these coefficients were calculated in [2]. To prove the  $p$ -adic interpolation theorem for symmetric squares of cusp forms we need some information about the Fourier coefficients  $e_k^\psi(n, m)$  of the Jacobi-Eisenstein series

$$E = (4\pi i)^t E_k^\psi(\tau, z) = \sum_{n,m} e_k^\psi(n, m) e(n\tau + mz),$$

where  $\psi$  is a primitive Dirichlet character modulo  $r = p^L$ ,  $L \geq 0$ ,  $p$  is a fixed odd prime number. We denote by  $G(\psi)$  the Gauss sum associated with the character  $\psi$  and by  $L(s, \chi)$  the Dirichlet  $L$ -function associated with the character  $\chi$ :

$$G(\psi) = \sum_{m \pmod r} \psi(m) e(m/r),$$

$$L(s, \chi) = \sum_{m \geq 0} \chi(m) m^{-s} \quad (\Re s > 1).$$

We set  $G(\psi) = 1$  if the character  $\psi$  is trivial.

**THEOREM 2.** — *In the above notations one has:*

- a)  $e_k^\psi(n, m) = 0$  if  $m^2 > 4r^2n$ .
- b)  $e_k^\psi(n, m) = \psi(m/2r) = \psi(\sqrt{n})$  if  $m^2 = 4r^2n$ .
- c) If  $m^2 < 4r^2n$  then

$$(10) \quad e_k^\psi(n, m) = i^k \frac{\pi^{k-1/2}}{2^{k-2}\Gamma(k-1/2)} r^{2-2k} G(\psi) \frac{L(k-1, \xi_D \psi)}{L(2k-2, \psi^2)} g_1 Y.$$

In (10)  $D = m^2 - 4r^2n < 0$ ,  $\xi_D$  is the Dirichlet character associated with the imaginary quadratic field  $\mathbf{Q}(\sqrt{D})$ ,

$$Y = Y(\psi, m, n) = \prod_{q|D} Y_q,$$

where the product is taken over all prime numbers  $q$  dividing  $D$  and  $Y_q$  is a polynomial in the variable  $\{\psi(q)q^{1-k}\}$ ,  $g_1 = 1$  if  $\psi$  is the trivial character (i.e.  $L = 0$ ). If  $L > 0$  then

$$(11) \quad g_1 = g_1(\psi, m, n) = \sum_{\ell \geq 0} p^{\ell(1-k)-L} (p/(p-1))^{\delta_{\ell,0}} \\ \times \sum_{\substack{\lambda \bmod p^{L+\ell} \\ \lambda^2 r - \lambda m + rn \equiv 0 \pmod{p^\ell}}} \psi(L) \bar{\psi}((\lambda^2 r - \lambda m + rn)/p^\ell),$$

where  $\delta_{\ell,0} = 1$  or  $0$  according as  $\ell = 0$  or  $\ell > 0$ .

d) Let

$$H(\ell) = \sum_{\substack{\lambda \bmod p^{\ell+L} \\ Q(\lambda) \equiv 0 \pmod{p^\ell}}} \psi(\lambda) \bar{\psi}(Q(\lambda)/p^\ell),$$

be the internal sum in (11),  $Q(\lambda) = \lambda^2 r - \lambda m + rn$ ;  $r = p^L$ ,  $m = p^a \hat{m}$ ,  $a \geq 0$ ,  $p \nmid \hat{m}$ ;  $n = p^b \hat{n}$ ,  $b > 2L > 0$ ,  $p \nmid \hat{n}$ .

Then  $H(\ell) \neq 0$  implies  $\ell = a$ , and  $\ell$  is equal to  $0$  or  $1$ .

*Remark.* — One can prove that the summation over  $\ell$  in (11) is finite also in the case when  $b \leq 2L$ , but we do not need this fact for our purposes.

The assertions a) and b) of Theorem 2 are contained in [2] and are almost evident. The proof of part c) is similar to the Fourier coefficients calculation in the case when the character  $\psi$  is trivial. This calculation is contained in [2], Theorem 2.1. We will prove now the assertion d). One can rewrite the condition  $Q(\lambda) \equiv 0 \pmod{p^\ell}$  as:

$$(12) \quad p^L \lambda^2 - p^a \hat{m} \lambda + p^{b+L} \hat{n} \equiv 0 \pmod{p^\ell}.$$

PROPOSITION 6.

- a) If  $a > L$  then  $H(\ell) = 0$ .  
 b) If  $a \leq L$  and  $\ell \neq a$  then  $H(\ell) = 0$ .

To prove part a) of the proposition, we consider three cases:  $0 \leq \ell < L$ ;  $\ell = L$ ;  $\ell > L$ .

If  $0 \leq \ell < L$  then (12) is true for any  $\lambda$  but  $Q(\lambda)/p^\ell \equiv 0 \pmod{p}$  yields  $\psi(Q(\lambda)/p) = 0$ .

If  $\ell = L$  then

$$\begin{aligned} H(\ell) &= \sum_{\lambda \bmod p^{2L}} \psi(\lambda) \bar{\psi}(\lambda^2 - \hat{m}\lambda p^{a-L} + \hat{n}p^b) \\ &= \sum_{\substack{\lambda \bmod p^{2L} \\ p|\lambda}} \bar{\psi}(\lambda - \hat{m}p^{a-L}) \\ &= \sum_{\lambda \bmod p^{2L}} \bar{\psi}(\lambda - \hat{m}p^{a-L}) = 0. \end{aligned}$$

If  $\ell > L$  then (12) implies  $\lambda^2 \equiv 0 \pmod p$  and  $\psi(\lambda) = 0$ .

To prove part b) we assume that  $a < L$  and consider the cases  $0 \leq \ell < a$  and  $\ell > a$ . If  $0 \leq \ell < a$ , then  $Q(\lambda)/p^\ell \equiv 0 \pmod p$  yields  $\bar{\psi}(Q(\lambda)/p^\ell) = 0$ . If  $\ell > a$  then (12) implies  $\lambda^2 \equiv 0 \pmod p$  and  $\psi(\lambda) = 0$ .

Now we assume that  $a = L$  and consider three cases:  $0 \leq \ell < L$ ,  $L < \ell \leq 2L$  and  $\ell > 2L$ .

If  $0 \leq \ell < L$  then  $Q(\lambda)/p^\ell \equiv 0 \pmod p$  and  $\bar{\psi}(Q(\lambda)/p^\ell) = 0$ .

If  $L < \ell \leq 2L$ , then

$$\begin{aligned} H(\ell) &= \sum_{\substack{\lambda \bmod p^{\ell+L} \\ Q(\lambda) \equiv 0 \pmod{p^\ell}}} \psi(\lambda) \bar{\psi}((\lambda^2 - \hat{m}\lambda + \hat{n}p^b)/p^{\ell-L}) \\ &= \sum_{\substack{\lambda \bmod p^{\ell+L} \\ \lambda \equiv \hat{n} \pmod{p^{\ell-L}}} } \psi(\lambda) \bar{\psi}((\lambda^2 - \hat{m}\lambda)/p^{\ell-L}) \\ &= \sum_{\substack{\lambda \bmod p^{\ell+L} \\ \lambda = \hat{n} \pmod{p^{\ell-L}}} } \bar{\psi}((\lambda - \hat{m})/p^{\ell-L}) \\ &= \sum_{\alpha \bmod p^{2L}} \bar{\psi}(\alpha) = 0. \end{aligned}$$

Here we have done the variable change  $\lambda = \hat{m} + \alpha p^{\ell-L}$ .

If  $\ell > 2L$ , then  $Q(\lambda + p^\ell)/p^\ell \equiv Q(\lambda)/p^\ell \pmod{p^L}$  implies

$$H(\ell) = p^L \sum_{\substack{\lambda \bmod p^\ell \\ Q(\lambda) \equiv 0 \pmod{p^\ell}}} \psi(\lambda) \bar{\psi}((\lambda^2 - \hat{m}\lambda + \hat{n}p^b)/p^{\ell-L}).$$

Let  $\lambda = \alpha + \beta p^{\ell-L}$ ;  $\alpha \bmod p^{\ell-L}$ ;  $\beta \bmod p^L$ . After this variable change we

have

$$\begin{aligned} H(\ell) &= p^L \sum_{\substack{\alpha \bmod p^{\ell-L} \\ Q(\alpha) \equiv 0 \bmod p^\ell}} \sum_{\beta \bmod p^L} \psi(\alpha + \beta p^{\ell-L}) \bar{\psi}((\alpha^2 - \hat{m}\alpha + \hat{n}p^b)/p^{\ell-L} \\ &\quad + \beta(2\alpha - \hat{m}) + \beta^2 p^{\ell-L}) \\ &= p^L \sum_{\substack{\alpha \bmod p^{\ell-L} \\ Q(\alpha) \equiv 0 \bmod p^\ell}} \sum_{\beta \bmod p^L} \bar{\psi}((\alpha^2 - \hat{m}\alpha + \hat{n}p^b)/p^{\ell-L} + \beta(2\alpha - \hat{m})). \end{aligned}$$

The condition  $Q(\alpha) \equiv 0 \bmod p^\ell$  yields  $\alpha^2 - \alpha\hat{m} + \hat{n}p^b \equiv 0 \bmod p^{\ell-L}$ . It takes place only if  $\alpha \equiv 0$  or  $\hat{m} \bmod p$ . In both cases  $H(\ell) = 0$ .

Now Proposition 6 is proved and we are able to finish the proof of part d) of Theorem 2. One can deduce from Proposition 6 that the Fourier expansion coefficient (10) may become non-zero only if  $a \leq L$  and  $\ell = a$ . If  $1 < a < L$ , then

$$\begin{aligned} H(\ell) &= \sum_{\substack{\lambda \bmod p^{a+L} \\ Q(\lambda) \equiv 0 \bmod p^a}} \psi(\lambda) \bar{\psi}(Q(\lambda)/p^a) \\ &= \sum_{\lambda \bmod p^{a+L}} \psi(\lambda) \bar{\psi}(p^{L-a}\lambda^2 - \hat{m}\lambda + \hat{n}p^{b+L-a}) \\ &= \sum_{\substack{\lambda \bmod p^{a+L} \\ p|\lambda}} \psi(p^{L-a}\lambda - \hat{m}) \\ &= p^L \bar{\psi}(-\hat{m}) \sum_{\substack{\lambda \bmod p^a \\ p|\lambda}} \psi(p^{L-a}\lambda + 1) \\ &= p^L \bar{\psi}(-\hat{m}) \left( \sum_{\lambda \bmod p^a} \psi(p^{L-a}\lambda + 1) - \sum_{\lambda \bmod p^{a-1}} \psi(p^{L-a}\lambda + 1) \right) = 0, \end{aligned}$$

because for a primitive Dirichlet character  $\psi$ , both sums in the parentheses are equal to zero if  $a > 1$ . It remains to consider only one case:  $a = L = \ell$ .

Then

$$\begin{aligned} H(\ell) &= \sum_{\lambda \bmod p^{2L}} \psi(\lambda) \bar{\psi}(\lambda^2 - \hat{m}\lambda + \hat{n}p^b) \\ &= \sum_{\substack{\lambda \bmod p^{2L} \\ p|\lambda}} \psi(\lambda - \hat{m}) = p^L \bar{\psi}(-\hat{m}) \sum_{\lambda \bmod p^{L-1}} \psi(p\lambda + 1) = 0. \end{aligned}$$

### 4. $p$ -ADIC INTERPOLATION OF SYMMETRIC SQUARE SPECIAL VALUES

In this section we use the results of the two previous sections to construct the  $p$ -adic interpolation of the symmetric squares special values of cusp forms. We fix an odd prime number  $p$  and an embedding  $i_p : \overline{\mathbf{Q}} \rightarrow \mathbf{C}_p$  of the algebraic closure of the field of rational numbers  $\mathbf{Q}$  into Tate's field. We shall make no difference between  $\mathbf{Q}$  and its image under  $i_p$  and omit symbol  $i_p$  in formulas. One can construct a  $\mathbf{C}_p$ -analytical function on  $X_p = \text{Hom}_{\text{contin}}(\mathbf{Z}_p^*, \mathbf{C}_p^*)$  as a non-archimedean Mellin transform of some bounded  $p$ -adic measure  $\mu$  on  $\mathbf{Z}_p^*$ :

$$L_\mu(x) = \int_{\mathbf{Z}_p^*} x d\mu.$$

We identify the elements of the torsion subgroup of  $X_p^{\text{tors}} \subset X_p$  with primitive Dirichlet characters modulo powers of  $p$ .

The symbol  $x_p$  will denote the natural embedding  $\mathbf{Z}_p^* \rightarrow \mathbf{C}_p^*$ , so that  $x_p \in X_p$  and all integers  $k$  can be considered as the characters  $x_p^k : y \mapsto y^k$ .

The existence of a special values  $p$ -adic interpolation of some zeta function is equivalent to the existence of a  $p$ -adic measure with given special values ([5]). We shall use the following important fact to prove the existence of these measures.

#### 4.1. The abstract Kummer congruences.

PROPOSITION 7 ([5], [7]). — *Let  $\{f_j\}$  be a family of continuous functions from  $\mathbf{Z}_p^*$  to the ring of integers  $\mathcal{O}_p$  in  $\mathbf{C}_p$ . Assume that the set of finite  $\mathbf{C}_p$ -linear combinations of  $f_j$  is dense in the space of all such functions. Let  $\{a_j\}$  be a family of elements in  $\mathcal{O}_p$ . Then the existence of a measure with the property*

$$\int_Y f d\mu = a_j$$

*is equivalent to the fact that the following statement is true: for every finite set of elements  $b_j \in \mathbf{C}_p$  it follows from  $\left\{ \sum_j b_j f_j(y) \in p^n \mathcal{O}_p \text{ for every } y \in Y \right\}$  that  $\left\{ \sum_j b_j a_j \in p^n \mathcal{O}_p \right\}$ .*



To formulate the theorem we need the definition of  $p$ -ordinary form.

A cusp form

$$f(\tau) = \sum_{n \geq 1} a(n)e(n\tau) \in S_k,$$

normalized by the condition  $a(1) = 1$  which is an eigenform of Hecke algebra is called  $p$ -ordinary if  $|a(p)|_p = 1$ .

We denote the subspace of the  $p$ -ordinary forms of weight  $k$  by  $S_k^0 \subseteq S_k$ .

For a prime number  $q$  we denote by  $\alpha = \alpha(q)$  and  $\beta = \beta(q)$  the roots of the Hecke polynomial  $X^2 - a(q)X + q^{k-1}$ . We define by multiplicativity the numbers  $\alpha(n)$  and  $\beta(n)$  for every natural number  $n$ .

**THEOREM 3.** — *Let  $c > 1$  be a natural number,  $p \nmid c$ . Let  $f$  be a  $p$ -ordinary form of even weight  $k$ . Then there exist a  $\mathbb{C}_p$ -analytic function  $D^c : X_p \rightarrow \mathbb{C}_p$  such that its value  $D^c(x_p^M \chi)$  for  $3 \leq M \leq k - 1$  equals*

$$2^{-4M-2t+1} \pi^{-2M-k+1} r^{k+2M-1} i^{k-M-2} \Gamma(k+M-1) \Gamma(M) G(\chi)^{-2} L(2M, \chi^2) \times (1 - \chi(c)^2 c^{-2M}) \frac{1}{\alpha(r^2)} TD_f(M, \chi),$$

where  $\chi \in X_p^{\text{tors}}$  is a Dirichlet character,  $1 \leq M < r - 1$ ,  $M$  is an integer,

$$T = \begin{cases} 1, & \text{if } \chi \text{ is a non-trivial character } (r > 1) \\ (1 - p^{M-1})(1 - \alpha(p)^{-2} p^{M+k-2})(1 - \alpha(p)^{-2} p^{k-2}), & \text{otherwise.} \end{cases}$$

Here  $r$  is the conductor of  $\chi$ ,  $G(\chi)$  is the Gauss sum, associated with  $\chi$ .

*Proof.* — One can assume that  $\chi$  is primitive. Using the notation

$$\Lambda(k, M, t) = 2^{k-5M-4t} \pi^{1/2-M-t} i^{k-M-t-1} \frac{\Gamma(2M + 2\nu + t)}{\Gamma(2\nu + t + 1) \Gamma(M + 1/2)},$$

$$\Lambda_1(k, M, t) = \frac{(2\pi)^{k-M-1}}{\Gamma(k-M)} \Lambda(k, M, t)^{-1} (4\pi i)^{-t},$$

definition (2), and statement (3) one gets

$$\begin{aligned} \sum_{j=1}^{\dim M_k} \frac{f_j}{\langle f_j, f_j \rangle} D_{f_j}(M, \chi) &= \Lambda_1(k, M, t) \sum_{n \geq 1} \sum_m \sum_{0 \leq \mu \leq (k-M-1)/2} (-1)^\mu \\ (13) \quad &\times \frac{\Gamma(k-\mu-1) \Gamma(k-M)}{\Gamma(\mu+1) \Gamma(k-M-2\mu) \Gamma(k-1)} \\ &\times m^{k-M-2\mu-1} r^{-(k-M-2\mu-1)} n^\mu e_{M+1}^\chi(n, m) e(n\tau). \end{aligned}$$

The idea is to apply some operators to both sides of (13) to get some "good"  $p$ -adic properties in the right-hand side of the obtained identity keeping under control what happens in the left-hand side. We denote by  $V$  and  $U$  the operators

$$f|U(d) = \sum_{n \geq 0} a(dn)e(n\tau) = d^{k/2-1} \sum_{u \pmod d} f|_k \begin{pmatrix} 1 & u \\ 0 & d \end{pmatrix} \in M_k(Nd, \chi),$$

$$f|V(d) = f(d\tau) = d^{-k/2} f|_k \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \in M_k(Nd, \chi),$$

$$V(d) \circ U(d) = \text{id}.$$

It is known [3] that for a natural number  $s$  there exists a limit

$$\mathcal{E}_s = \lim_{v \rightarrow \infty} U(p)^{sp^v}.$$

We apply the operator  $\mathcal{E}_s \circ V(r^2)$  to both sides of (13). To calculate the limit in the left-hand side we consider the modular forms on  $\Gamma_0(p^2)$

$$f_{j,0}(\tau) = f(\tau) - \alpha_j f(p\tau), \quad f_{j,1}(\tau) = f(\tau) - \beta_j f(p\tau).$$

We denote

$$A_j = \lim_{v \rightarrow \infty} \alpha_j^{sp^v}, \quad B_j = \lim_{v \rightarrow \infty} \beta_j^{sp^v}.$$

It is clear that one of the numbers  $A_j, B_j$  is zero because  $\alpha_j \beta_j = p^{k-1}$ . For a  $p$ -ordinary form  $f_j$  one of them is non-zero. Without loss of generality one can assume that  $A_j \neq 0$  (i.e.  $|\alpha_j|_p = 1$ ). After noticing that

$$f_j = \frac{\alpha_j}{\alpha_j - \beta_j} f_{j,1} + \frac{\beta_j}{\beta_j - \alpha_j} f_{j,0},$$

we can write

$$(14) \quad \Lambda_1(k, M, \chi)^{-1} F(k, M, \chi) |V(r^2)| \mathcal{E}_s \\ = \sum_{j=1}^{\dim S_k^0} \frac{1}{\langle f_j, f_j \rangle} D_{f_j}(M, \chi) \frac{A_j f_{j,1}}{\alpha_j(r^2)} (\alpha_j(p) - \beta_j(p))^{-1}.$$

We denote by  $c_s(k, M, \chi, n)$  the Fourier coefficients of the modular form in the left side of (14):

$$(15) \quad \mathcal{F}(k, M, \chi) = F(k, M, \chi) |V(r^2)| \mathcal{E}_s = \sum_{n \geq 0} c_s(k, M, \chi, n) e(n\tau)$$

and consider the limit in the right side of (15). Now assume that  $\chi$  is non-trivial. Using part d) of Theorem 2 and the notation

$$\Lambda_2(k, \mu, \chi) = i^{M+1} \frac{\pi^{M+1/2}}{2^{M-1} \Gamma(M+1/2)} G(\chi) L(2M, \chi^2)^{-1} r^{-(k+M-1)},$$

we can rewrite (15):

$$\begin{aligned} & \Lambda_2(k, \mu, \chi)^{-1} \Lambda_1(k, M, \chi)^{-1} c_s(k, M, \chi, n) \\ &= \lim_{v \rightarrow \infty} \sum_{\substack{m \not\equiv 0 \pmod p \\ D_1 = 4np^s p^v - m^2 < 0}} D_1^{M-1/2} \bar{\chi}(-m) L(M, \xi_{D_1} \chi) Y(n, m, \chi, M) m^{k-M-1} \\ & \quad - p^{k-2} \sum_{\substack{m \not\equiv 0 \pmod p \\ D_2 = 4np^s p^v - 2 - m^2 < 0}} D_2^{M-1/2} \bar{\chi}(-m) L(M, \xi_{D_2} \chi) Y(n, m, \chi, M) m^{k-M-1}, \end{aligned}$$

and apply the following assertion.

PROPOSITION 8 ([6]). — *Let  $\omega$  be a primitive Dirichlet character modulo  $A$ ,  $(p, A) = 1$ . For an arbitrary integer  $c > 1$  such that  $(c, pA) = 1$ , there exists a  $\mathbf{C}_p$ -valued measure  $\mu^+(c, \omega)$  on  $\mathbf{Z}_p^*$ . This measure is uniquely defined by the following condition:*

$$\begin{aligned} & \int_{\mathbf{Z}_p^*} \bar{\chi} x_p^M d\mu^+(c, \omega) \\ &= (1 - \chi\omega(c)c^{-M}) \frac{A^M r^M}{G(\omega\chi)} L_{pA}(M, \chi\omega) \frac{2i^\delta \Gamma(M) \cos(\pi(M - \delta)/2)}{(2\pi)^M} X, \end{aligned}$$

where

$$X = \begin{cases} 1, & \text{if } \chi \text{ is non-trivial} \\ (1 - \omega(q)q^{M-1})(1 - \omega(q)q^{-M}), & \text{otherwise,} \end{cases}$$

$\delta = 0$  or  $1$ ;  $(-1)^\delta = \chi\omega(-1)$ ;  $M$  a positive integer.

Introducing the notation

$$\Lambda_3(M, \chi) = \frac{1}{2\Gamma(M)} r^{-M} G(\chi) (-2\pi i)^M,$$

one has

$$\begin{aligned} & (1 - \chi^2(c)c^{-2M}) \Lambda_3(M, \chi)^{-1} \Lambda_2(k, \mu, \chi)^{-1} \Lambda_1(k, M, \chi)^{-1} c_s(k, M, \chi, n) \\ &= \lim_{v \rightarrow \infty} \sum_{\substack{m \not\equiv 0 \pmod p \\ D_1 = 4np^s p^v - m^2 < 0}} D_1^{-1/2} m^{k-1} G(\xi_{D_1}) (1 + \chi \xi_{D_1}(c) c^{-M}) \\ & \quad \times (-D_1/Am)^M \bar{\chi}(-D_1/Am) Y(n, m, \chi, M) \int_{\mathbf{Z}_p^*} \bar{\chi} x_p^M d\mu^+(c, \xi_{D_1}) \\ & \quad - p^{k-2} \sum_{\substack{m \not\equiv 0 \pmod p \\ D_2 = 4np^s p^v - 2 - m^2 < 0}} D_2^{-1/2} m^{k-1} G(\xi_{D_2}) (1 + \chi \xi_{D_2}(c) c^{-M}) \\ & \quad \times (-D_2/Am)^M \bar{\chi}(-D_2/Am) Y(n, m, \chi, M) \int_{\mathbf{Z}_p^*} \bar{\chi} x_p^M d\mu^+(c, \xi_{D_2}). \end{aligned}$$

Here we used the fact that  $\xi_D(r) = 1$  and  $\chi(D) = \chi(m^2)$  for  $D = m^2 - 4np^s p^v$  if  $v$  is sufficiently large.

It remains to consider the case when  $\chi$  is trivial. This case is slightly different from previous, but the calculations contain no essentially new ideas. One must apply additionally the operator  $(1 - p^{M+k-2}V(p^2))(1 - p^{k-2}V(p^2))$  to keep the “good” form of the formulas.

Now we notice that in the right-hand side of the obtained equation there are Fourier coefficients of modular forms of weight  $k$  on  $\Gamma_0(p^2)$ . They can be expressed as follows:

$$(16) \quad \sum_j \lambda_j \chi(y_j) y_j^{-M} \int_{\mathbf{Z}_p^*} \bar{\chi} x_p^M d\mu^+(c, \xi_D).$$

The numbers  $\lambda_j$  in (16) are  $p$ -integers, do not depend on  $M$  and  $\chi$  and the sum is finite. Proposition 7 yields this Fourier coefficients to be  $\mathbf{C}_p$ -analytic functions on  $X_p$ . On the other hand the modular forms with these coefficients belong to the finite-dimensional linear space of the modular forms of weight  $k$  on  $\Gamma_0(p^2)$ . It implies the values of each functional on this modular form to be values of some  $\mathbf{C}_p$ -analytic function. To complete the proof of Theorem 3 it remains to consider the linear functional  $\langle \cdot, f_j \rangle$  of the Petersson scalar product with a  $p$ -ordinary form  $f_j$ .

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