ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 45, nº 1 (1995), p. 223-249

<http://www.numdam.org/item?id=AIF_1995__45_1_223_0>

Ann. Inst. Fourier, Grenoble
45, 1 (1995), 223-249

THE SURJECTIVITY OF A CONSTANT COEFFICIENT HOMOGENEOUS DIFFERENTIAL OPERATOR ON THE REAL ANALYTIC FUNCTIONS AND THE GEOMETRY OF ITS SYMBOL

by Rüdiger W. BRAUN

Introduction.

De Giorgi and Cattabriga [10] have conjectured, and Piccinini [20] has shown that the heat operator $\partial^2/\partial x^2 + \partial^2/\partial y^2 - \partial/\partial t$ is not surjective on $A(\mathbb{R}^3)$, the space of all real analytic functions on $\mathbb{R}^3$ with values in $\mathbb{C}$. Hörmander [11] has then characterized the surjective constant coefficient partial differential operators $P(D)$ on $A(\Omega)$, $\Omega \subset \mathbb{R}^N$ a convex domain. To evaluate Hörmander's criterion, one has to decide whether a certain pair of inequalities for plurisubharmonic functions on the variety $V = P_m^{-1}(0)$ of the principal part P_m of P implies a third one. Because of the analogy to a classical theorem, he calls this condition Phragmén-Lindelöf condition. The same phenomenon appears in a more general setting, namely if $A(\mathbb{R}^N)$ is replaced by a Gevrey class $\Gamma^d(\mathbb{R}^N)$, $d \geq 1$; recall that $\Gamma^1(\mathbb{R}^N) = A(\mathbb{R}^N)$ and that $\Gamma^d(\mathbb{R}^N)$ contains test functions if $d > 1$. The surjectivity of constant coefficient partial differential operators on Gevrey classes has been investigated by Cattabriga [8], [9], Zampieri [22], and Braun, Meise, and Vogt [7], who gave, for $\Gamma^d(\mathbb{R}^N)$, the analogous theorem to Hörmander's. For arbitrary convex domains, this result was proved in [4].

Key words : Partial differential operator – Singularities – Phragmén-Lindelöf condition.
Math. classification : 35A30 – 35E10 – 31C10.

If P is a homogeneous polynomial, there is reason to conjecture that the surjectivity of $P(D)$ on $A(\mathbb{R}^N)$ and on $\Gamma^d(\mathbb{R}^N)$, any $d > 1$, are equivalent. It is known by Braun, Meise, and Vogt [7] that, for homogeneous P, surjectivity on $\Gamma^d(\mathbb{R}^N)$ does not depend on $d > 1$ and is implied by surjectivity on $A(\mathbb{R}^N)$. So, for homogeneous operators, it is natural to state necessary conditions in the setting of Gevrey classes.

We present here two geometric conditions which, for homogeneous P, are necessary for the surjectivity of $P(D)$ on $\Gamma^d(\mathbb{R}^N)$ and hence on $A(\mathbb{R}^N)$. We call them "distance condition" and "carry over to the tangent cone". We also extend Hörmander's dimension condition to the case of Gevrey classes.

The dimension condition says that for $\theta \in V$ real, $\theta \neq 0$, we have ${}_{\mathbb{R}}\dim_\theta V \cap \mathbb{R}^N = N - 1$ if $P(D)$ is surjective. Thus the intersection of V with $\mathbb{R}^N$ has maximal dimension.

An example of an operator that satisfies the dimension condition, but not the distance condition is the one with $P(x, y, z) = x^2y - z^3$. The distance condition says roughly that, if $P(D)$ is surjective, then, for all $\theta \in V$, the distance "taken inside V" to the next real point in V is of the same order as $|\operatorname{Im} \theta|$. We replace the notion "taken inside V" by a more practical concept, though. It is clear from Hörmander's work (see 1.4) that it is enough to investigate real, locally irreducible singularities θ of V only. We show in Theorem 3.8 that the existence of a generic locally irreducible singular plane curve in V through some $\theta \in V \cap \mathbb{R}^N$, $\theta \neq 0$, is already an obstruction to the surjectivity of $P(D)$. Note that in contrast to this, the operator $P(D)$ with $P(x, y, z, w) = x^2 + y^2 - z^2$ is surjective, although its variety has real irreducible singularities off the origin.

In the case of three variables, these two conditions completely describe the situation. This leads to a characterization of the surjective operators $P(D)\colon \Gamma^d(\mathbb{R}^3) \to \Gamma^d(\mathbb{R}^3)$, P a homogeneous polynomial, using the language of algebraic geometry.

In four variables, there are operators like the one with $P(x, y, z, w) = x^2w + y^2w + z^3$ that satisfy both dimension and distance condition, but are not surjective. The criterion in these cases is that if a Phragmén-Lindelöf condition holds on a cone V, then it holds on all tangent cones to real points in V, too. This is similar to results of Hörmander [11] and Meise, Taylor, and Vogt [16] about carrying over of Phragmén-Lindelöf conditions from inhomogeneous varieties to their tangent cones at infinity.

To support our impression that we have found all the relevant obstacles to the surjectivity of a constant coefficient partial differential operator on $A(\mathbb{R}^N)$ or on $\Gamma^d(\mathbb{R}^N)$, we end the paper with a discussion of all operators of the form

$$P(D) = A\frac{\partial^n}{\partial x^l \partial w^{n-l}} + B\frac{\partial^n}{\partial y^m \partial w^{n-m}} + C\frac{\partial^n}{\partial z^n}, A, B, C \in \mathbb{C}, l, m, n \in \mathbb{N}_0, \quad l, m \leq n.$$

It turns out that there are only very few choices of the parameters for which $P(D)$ is surjective. In all but one of these cases, the polynomial is locally hyperbolic at every real characteristic, thus Hörmanders's sufficient condition [11], 6.5, applies. The remaining case is solved by an ad hoc argument.

Besides in surjectivity problems, Phragmén-Lindelöf conditions also arise in the investigations of Meise, Taylor, and Vogt [14], [15], concerning the existence of continuous linear right inverses for constant coefficient partial differential operators. Since, for homogeneous varieties, all these conditions are closely related, there are obvious analoga of our theorems in these settings. The dimension condition was already known for them. The existence of continuous linear right inverses for systems has been studied by Palamodov [19]. There, Phragmén-Lindelöf conditions arise also for varieties of codimension higher than 1. Although our methods can be applied to them, too, we do not investigate these problems here.

The contents of this paper form a part of the author's Habilitationsschrift.

1. Phragmén-Lindelöf conditions.

After fixing some notations, we recall the characterization of the surjective partial differential operators with constant coefficients on the space of all real analytic functions on a convex domain in $\mathbb{R}^N$, which was given by Hörmander [11]. There is a version of this theorem for the case of ultradifferentiable functions, due to Braun, Meise, and Vogt [6], [7], which will also be quoted here.

1.1. Notation. — The natural numbers are denoted by $\mathbb{N}=\{1,2,\ldots\}$, $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. The ϵ-neighborhood of $x \in \mathbb{K}^N$ is denoted by $U_\epsilon(x) := \{y \in \mathbb{K}^N \mid |y-x| < \epsilon\}$, where $|\cdot|$ is the euclidean norm. Sometimes, the elements of $\mathbb{K}^N$ are written as $z = (z', z_N)$ with $z' \in \mathbb{K}^{N-1}$.

By $A(\Omega)$, Ω a convex domain in $\mathbb{R}^N$, we denote the space of all real analytic, complex valued functions on Ω. For a polynomial $P \in \mathbb{C}[Z_1, \dots, Z_N]$, $P(z) = \sum_\alpha a_\alpha z^\alpha$, the partial differential operator $P(D)$ is defined by $P(D)(f) = \sum_\alpha i^{-|\alpha|} a_\alpha f^{(\alpha)}$.

Concerning notions from complex analytic geometry, we refer to the books of Whitney [21] and Narasimhan [17]. For a point θ in a variety V we denote by ${}_V\mathcal{O}_\theta$ the set of all germs in θ of holomorphic functions on V. In the case $V = \mathbb{C}^N$, we omit the subscript V. We write the elements of ${}_V\mathcal{O}_\theta$ as f_θ and the set germ of V in θ as V_θ. In Whitney [21], 3.8S, it is shown that a set germ V_θ is irreducible if and only if any sufficiently small analytic set W with $W_\theta = V_\theta$ is irreducible. So we may think of a germ V_θ as being given by one fixed representative V. The dimension of a real or complex analytic set germ V_θ is denoted by ${}_{\mathbb{K}}\dim_\theta V$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, respectively.

A function $\varphi: V \to [-\infty, \infty[$ on a (complex) analytic set V is plurisubharmonic if it is upper semicontinuous everywhere and plurisubharmonic in all regular points, *i.e.*, the compositions with all charts in all regular points are plurisubharmonic (see Hörmander [13]). The set of all plurisubharmonic functions on V is denoted by $\mathrm{PSH}(V)$.

1.2. THEOREM (Hörmander [11], 1.1–1.3). — *Let $P \in \mathbb{C}[Z_1, \dots, Z_N]$ be a polynomial of degree m, and denote by $V = P_m^{-1}(0)$ the variety of its principal part P_m. The differential operator $P(D): A(\mathbb{R}^N) \to A(\mathbb{R}^N)$ is surjective if and only if the following Phragmén-Lindelöf condition holds*

(HPL) *There is $A > 0$ such that every $\varphi \in \mathrm{PSH}(\mathbb{C}^N)$ with (α) and (β) also satisfies (γ) :*

- (α) *For all $\theta \in \mathbb{C}^N$: $\varphi(\theta) \le |\theta|$,*
- (β) *for all $\theta \in V \cap \mathbb{R}^N$: $\varphi(\theta) \le 0$,*
- (γ) *for all $\theta \in V$: $\varphi(\theta) \le A|\operatorname{Im}\theta|$.*

Remark. — For most of this paper, we need the case $\Omega = \mathbb{R}^N$ only. Therefore, we have stated Hörmander's result only for this case. We wish to point out, however, that he has given a characterization for the case of arbitrary convex sets.

1.3. DEFINITION (Andersson [1]). — *Let $P \in \mathbb{R}[Z_1, \dots, Z_N]$ be homogeneous, let ξ_0 be a real characteristic, and denote by q the lowest* *homogeneous part of* $\xi \mapsto P(\xi_0 + \xi)$. *Then* P *is called locally hyperbolic at* ξ_0 *if there are a vector* $\theta \in \mathbb{R}^N$ *with* $q(\theta) \neq 0$ *and a number* $\epsilon > 0$, *such that, for all* ξ *in a suitable real neighborhood of* ξ_0, *all roots of* $\tau \mapsto P(\xi + \tau\theta) = 0$ *with* $|\tau| < \epsilon$ *are real.*

1.4. THEOREM (Hörmander [11], 6.5). — *If* P_m *is locally hyperbolic at every real characteristic, then* $P(D): A(\mathbb{R}^N) \to A(\mathbb{R}^N)$ *is surjective. The converse holds if and only if* $N = 3$.

1.5. *Examples.* — Using 1.4, it is easy to see that the following operators $P(D): A(\mathbb{R}^N) \to A(\mathbb{R}^N)$ are surjective :

(1) every elliptic operator, since P_m does not have any real characteristics,

(2) every operator $P(D)$ whose tangential cone at infinity $V := P_m^{-1}(0)$ is regular outside the origin, provided that, for every real characteristic ξ, the following equality holds
$$_{\mathbb{R}}\dim_\xi V \cap \mathbb{R}^N = N - 1,$$

(3) $P(D) = \dfrac{\partial^4}{\partial x^2 \partial w^2} - \dfrac{\partial^4}{\partial y^4} - \dfrac{\partial^4}{\partial z^4}: A(\mathbb{R}^4) \to A(\mathbb{R}^4)$.

We recall the definition of a Roumieu class of ultradifferentiable functions given by Braun, Meise, and Taylor [5]. The most prominent examples are the Gevrey classes.

1.6. DEFINITION. — *A continuous function* $\omega: \mathbb{C}^N \to [0, \infty[$ *depending only on* $|z|$ *is called a weight function if it satisfies*

(α) $\omega(2t) = O(\omega(t))$, (β) $\displaystyle\int_0^\infty \frac{\omega(t)}{1+t^2}\,dt < \infty$,

(γ) $\log t = o(\omega(t))$, (δ) $\varphi: t \mapsto \omega(e^t)$ *is convex on* $\mathbb{R}$.

We define the Legendre transform – also called Young conjugate – of φ *by*
$$\varphi^*(y) = \sup_{x \geq 0} xy - \varphi(x).$$

1.7. DEFINITION. — *For a given convex domain* $\Omega \subset \mathbb{R}^N$ *fix a convex compact exhaustion* $K_1 \subset K_1 \subset \cdots \subset \Omega$. *Then define for* $n \in \mathbb{N}$
$$\mathcal{E}_{\{\omega\},n} = \Big\{ f \in C^\infty(\mathbb{R}^N) \;\Big|\; \sup_{\substack{x \in K_n \\ \alpha \in \mathbb{N}_0^N}} |f^{(\alpha)}(x)| \exp\Big(-\frac{1}{m}\varphi^*(|\alpha| m)\Big) < \infty \quad \text{for some } m \in \mathbb{N} \Big\}.$$

Endow $\mathcal{E}_{\{\omega\},n}$ with the inductive limit topology, and set

$$\mathcal{E}_{\{\omega\}}(\Omega) = \operatorname*{proj}_{\leftarrow n} \mathcal{E}_{\{\omega\},n}.$$

The elements of $\mathcal{E}_{\{\omega\}}(\Omega)$ are called ultradifferentiable functions of type ω. The classes $\mathcal{E}_{\{\omega\}}$ are called Roumieu classes. In Braun, Meise, and Taylor [5] it is shown that they contain sufficiently many test functions. That paper also contains a discussion of several theories of ultradifferentiable functions.

1.8. *Example.* — For $\omega(z) = |z|^{1/d}$, $d > 1$, the space $\mathcal{E}_{\{\omega\}}$ is the classical Gevrey class Γ^d of order d.

1.9. THEOREM ([4], 5.1.7). — *If $P(D): \mathcal{E}_{\{\omega\}}(\Omega) \to \mathcal{E}_{\{\omega\}}(\Omega)$ is surjective for some convex domain Ω, then it is surjective for $\Omega = \mathbb{R}^N$.*

1.10. THEOREM (Braun, Meise, Vogt [6], 3.9, [7], 3.3). — *Let P be a homogeneous polynomial in N variables with variety $V = P^{-1}(0)$. The differential operator $P(D): \mathcal{E}_{\{\omega\}}(\mathbb{R}^N) \to \mathcal{E}_{\{\omega\}}(\mathbb{R}^N)$ is surjective if and only if the following Phragmén-Lindelöf condition holds :*

(plh) *There is $k > 0$ such that, for every L, $\delta > 0$, there is $\epsilon_0 > 0$ such that, for every $\epsilon < \epsilon_0$ and every $\theta \in V$ with $|\theta| = 1$, every $\varphi \in \mathrm{PSH}(V \cap U_\epsilon(\theta))$ with (α) and (β) also satisfies (γ) :*

(α) *For all $z \in V \cap U_\epsilon(\theta)$: $\varphi(z) \leq \epsilon$,*

(β) *for all $z \in V \cap U_\epsilon(\theta)$: $\varphi(z) \leq L|\operatorname{Im} z|$,*

(γ) *$\varphi(\theta) \leq k|\operatorname{Im}\theta| + \delta\epsilon$.*

1.11. THEOREM (Braun, Meise, and Vogt [7], 3.5 and 3.6). — *Let P be a homogeneous polynomial.*

(1) *If $P(D): A(\mathbb{R}^N) \to A(\mathbb{R}^N)$ is surjective, then, for every weight function ω, the operator $P(D): \mathcal{E}_{\{\omega\}}(\mathbb{R}^N) \to \mathcal{E}_{\{\omega\}}(\mathbb{R}^N)$ is surjective, too.*

(2) *If $P(D): \mathcal{E}_{\{\omega\}}(\mathbb{R}^N) \to \mathcal{E}_{\{\omega\}}(\mathbb{R}^N)$ is surjective for some weight ω, then it is surjective for all weights.*

2. The dimension condition.

The dimension condition is due to Hörmander [11]. We give it here for the case of ultradifferentiable functions. The construction in the proof is similar, but estimates have to be more precise. This is accomplished by an application of the Tarski-Seidenberg theorem. For the theory of semi-algebraic sets, see the book of Bochnak, Coste, and Roy [2] or the appendix of Hörmander [12].

2.1. *The dimension condition.* — Let $V \subset \mathbb{C}^N$ be a homogeneous algebraic set. We say that V satisfies the dimension condition if

$$_{\mathbb{R}}\mathrm{dim}_\theta\, W \cap \mathbb{R}^N = {}_{\mathbb{C}}\mathrm{dim}_\theta\, W$$

for all irreducible components W_θ of all germs V_θ with $\theta \in V \cap \mathbb{R}^N$, $\theta \neq 0$.

2.2. THEOREM. — *If for a homogeneous polynomial P, some weight ω, and some convex domain Ω, the operator $P(D)\colon \mathcal{E}_{\{\omega\}}(\Omega) \to \mathcal{E}_{\{\omega\}}(\Omega)$ is surjective, then the variety $V = P^{-1}(0)$ satisfies the dimension condition.*

The first step in the proof of this theorem is the following lemma.

2.3. LEMMA. — *Let $P \in \mathbb{C}[Z_1, \ldots, Z_N]$ be a homogeneous polynomial, let $V = P^{-1}(0)$, let $\theta \in V \cap \mathbb{R}^N \setminus \{0\}$, and let W_θ be an irreducible component of V_θ. Assume furthermore that θ is a regular point of $W \cap \mathbb{R}^N$ with $_{\mathbb{R}}\mathrm{dim}_\theta\, W \cap \mathbb{R}^N < N - 1$. Then there are a germ $h_\theta \in {}_V\mathcal{O}_\theta$ with $h_\theta | V \cap \mathbb{R}^N \equiv 0$ as well as $p, r_1, r_2, r_3 > 0$ and $0 < b < B$ such that $|h(z)| \leq B|\operatorname{Im} z|^p$ for all $z \in V \cap U_{r_2}(\theta)$ and such that for all r with $0 < r < r_1$ there is $z \in V \cap U_{r_3}(\theta)$ with $|\operatorname{Im} z| = r$ and $|h(z)| \geq b|\operatorname{Im} z|^p$.*

Proof. — We may assume that the germ W_θ is represented by a compact semi-algebraic set. We may also assume the existence of a polycylinder U and a semi-algebraic holomorphic function G on U such that the zero set of G consists of those irreducible components of V_θ that are distinct from W_θ. We let $X = W \cap \mathbb{R}^N$ and $k = {}_{\mathbb{R}}\mathrm{dim}_\theta\, X$. X is regular in θ. Therefore, by a suitable complex linear change of coordinates that maps $\mathbb{R}^N$ onto itself, we may assume

$$(1) \quad X_\theta = \left[\left\{(\xi, f_{k+1}(\xi), \ldots, f_N(\xi)) \middle| \xi \in \mathbb{R}^k, |\xi - (\theta_1, \ldots, \theta_k)| \text{ small}\right\}\right]_\theta,$$

where f_j, $k < j \leq N$, are holomorphic functions taking real values in real points. We claim that f_{N-1} is semi-algebraic. To see this, note that by Whitney [21], 3.8T, there is a representative Z, which may be chosen semi-algebraic, of the germ V_θ such that the irreducible components of the algebraic set Z correspond to the irreducible components of the set germ V_θ. By Whitney [21], 3.2B, the closures of the connected components of the set of regular points of Z are the irreducible components of Z. By Bochnak, Coste, and Roy [2], 2.4.5, connected components of semi-algebraic sets are again semi-algebraic. Thus the graph of f_{N-1}, which is the intersection of one of these components with some linear subspace, is semi-algebraic.

By Whitney [21], 3.3D, the change of coordinates that led to (1) can also be arranged in such a way that

$$W_\theta = \left[\left\{(z_1, \dots, z_{N-1}, g_N(z_1, \dots, z_{N-1})) \;\middle|\; |z' - \theta'| \text{ small}\right\}\right]_\theta$$

for a holomorphic multivalued function g_N. We let

$$h(z) = G(z)\,(f_{N-1}(z_1, \dots, z_k) - z_{N-1}), \quad z \in \mathbb{C}^N, |z - \theta| \le r_3,$$

where we choose the constant r_3 so small that all germs that have turned up so far have a representative that is defined for $z \in V$ with $|z - \theta| < 2r_3$. Then h vanishes identically on $V \cap \mathbb{R}^N \cap U_{r_3}(\theta)$, but not on $W \cap U_{r_3}(\theta)$. It is easy to see that the set $M = \{(r, y, z) \in \mathbb{R}^2 \times \mathbb{C}^N \mid z \in W, |z - \theta| \le r_3, |\operatorname{Im} z|^2 = r^{-2}, y = |h(z)|^2\}$ is semi-algebraic. By Hörmander [12], A.2.4, this implies that the function

$$g(r) = \sup\{|h(z)|^2 \mid z \in W, |\operatorname{Im} z| = r^{-1}\}$$

is semi-algebraic. By [12], A.2.5, we have for suitable a and A

$$g(r) = Ar^a(1 + o(1)), r \to \infty.$$

Since h does not vanish identically on W, the constant A is positive. To show that a is negative, note that W is compact and choose a sequence $(z_n)_n$ in W with $|z_n - \theta| \le r_3$, $|\operatorname{Im} z_n| = 1/n$, and $g(n) = |h(z_n)|^2$. A subsequence of $(z_n)_n$ converges to some $z \in W \cap \mathbb{R}^N$. Then $h(z) = 0$ implies $a < 0$. This yields the claim with $p = -a$. □

Proof of Theorem 2.2. — We may assume $|\theta| = 1$ and that θ is a regular point of the real analytic set $W \cap \mathbb{R}^N$. Let $g \in {}_V\mathcal{O}_\theta$ be a germ that vanishes identically on all irreducible components of V_θ except W_θ. Let h, b, B, p, r_1, r_2, and r_3 be as in Lemma 2.3. In view of 1.9, it suffices to disprove (hpl) of 1.10. For given $k > 0$ choose $\lambda < 1/4$ with

$$1 + \lambda \log\left(\sqrt[p]{\frac{b}{B}\frac{\lambda}{2k}}\right) > \frac{1}{2}.$$

Let $L = \exp(1/\lambda)$ and $\delta = 1/4$. Let a sufficiently small $\epsilon > 0$ be given. Choose $\xi \in V \cap U_{r_3/2}(\theta)$ with

$$(2) \qquad |\operatorname{Im}\xi| = \epsilon\lambda/k \quad \text{and} \quad |h(\xi)| \geq b|\operatorname{Im}\xi|^p.$$

Define, for sufficiently small $\mu > 0$,

$$\varphi(z) = \epsilon\left(1 + \frac{\lambda}{p}\log\left|\frac{h(z)}{B(2\epsilon)^p}\right|\right) + \mu\log|g(z)|,\ z \in V \cap U_\epsilon(\xi).$$

Then $|\operatorname{Im} z| \leq |\operatorname{Im}\xi| + \epsilon < 2\epsilon$ for $z \in V \cap U_\epsilon(\xi)$, thus $|h(z)| \leq B|\operatorname{Im} z|^p \leq B(2\epsilon)^p$ and $\varphi(z) \leq \epsilon$. Furthermore, for all z,

$$\varphi(z) \leq \epsilon\left(1 + \frac{\lambda}{p}\log\frac{B|\operatorname{Im} z|^p}{B(2\epsilon)^p}\right) \leq \epsilon\left(1 + \lambda\log\frac{|\operatorname{Im} z|}{2\epsilon}\right).$$

In particular, $\varphi(z) \leq 0$ provided $|\operatorname{Im} z| \leq 2\epsilon\exp(-1/\lambda)$. On the other hand, $L|\operatorname{Im} z| \geq 2\epsilon L\exp(-1/\lambda) = 2\epsilon \geq \varphi(z)$ for all z with $|\operatorname{Im} z| \geq 2\epsilon\exp(-1/\lambda)$. This shows that the inequalities (α) and (β) of (plh) are satisfied. However, the following two estimates show that (γ) of (plh) does not hold :

$$\begin{aligned}\varphi(\xi) &\geq \epsilon\left(1 + \frac{\lambda}{p}\log\frac{b|\operatorname{Im}\xi|^p}{B(2\epsilon)^p}\right) = \epsilon\left(1 + \frac{\lambda}{p}\log\left(\frac{b}{B}\left(\frac{\epsilon\lambda}{2k\epsilon}\right)^p\right)\right)\\ &= \epsilon\left(1 + \lambda\log\left(\sqrt[p]{\frac{b}{B}}\frac{\lambda}{2k}\right)\right) > \frac{\epsilon}{2},\end{aligned}$$

$$k|\operatorname{Im}\xi| + \delta\epsilon = \epsilon\lambda + \delta\epsilon < \epsilon/2.$$

□

2.4. *Examples.*

(1) Let $P \in \mathbb{C}[Z_1, \ldots, Z_N]$ be a homogeneous elliptic polynomial. If, for $M > N$, we consider P as a polynomial in M variables, then its variety $V = \{z \in \mathbb{C}^M \mid P(z) = 0\}$ does not satisfy the dimension condition. Thus, for every convex domain $\Omega \subset \mathbb{R}^M$, the operator $P(D)\colon \mathcal{E}_{\{\omega\}}(\Omega) \to \mathcal{E}_{\{\omega\}}(\Omega)$ is not surjective. For the case $\Omega = \mathbb{R}^M$, this has been shown by Braun, Meise, and Vogt [6], 3.3, using linear-topological invariants directly.

This class of examples contains the first non-surjective operator on Gevrey-classes, found by Cattabriga [9], 2.1, namely the Laplace operator in two variables considered as an operator in three variables. Note also that by Hörmander [11], 6.5 (see 1.5), every elliptic homogeneous operator is surjective.

(2) The operator

$$P(D) = \frac{\partial^4}{\partial x^2 \partial w^2} - \frac{\partial^4}{\partial y^2 \partial w^2} \pm \frac{\partial^4}{\partial z^4}$$

is not surjective.

2.5. COROLLARY. — *Assume that $P \in \mathbb{C}[Z_1, \dots, Z_N]$ is a homogeneous polynomial for which, for some weight ω and some convex domain $\Omega \subset \mathbb{R}^N$, the operator $P(D): \mathcal{E}_{\{\omega\}}(\Omega) \to \mathcal{E}_{\{\omega\}}(\Omega)$ is surjective. Then there is a decomposition $P = QR$, where $R \in \mathbb{R}[Z_1, \dots, Z_N]$ and Q is an elliptic polynomial or a complex constant.*

Proof. — Let S be an irreducible polynomial factor of P. Then S is again homogeneous. Set $W = S^{-1}(0)$, and assume $X := W \cap \mathbb{R}^N \neq 0$. We have to show that W coincides with its complex conjugate $\overline{W}$. Choose $\theta \in X$, $\theta \neq 0$. From Theorem 2.2 we know ${}_{\mathbb{R}}\dim_\theta X = N - 1$. We may assume that θ is a regular point of X. Thus there is a parametrization $\varphi: \{x \in \mathbb{R}^{N-1} \mid |x| < 1\} \to X$ of some real neighborhood of θ. We extend it to a complex neighborhood and call the extension φ, too. For $z \in C^{N-1}$ sufficiently small, we have $\varphi(\overline{z}) = \overline{\varphi(z)}$. Note that $\overline{W} = \overline{S}^{-1}(0)$ is algebraic, so $W \cap \overline{W}$ is an algebraic subvariety of the irreducible variety W having the same dimension. Thus $W \cap \overline{W} = W$.

3. The distance condition.

If P is a homogeneous polynomial such that $P(D): \mathcal{E}_{\{\omega\}}(\mathbb{R}^N) \to \mathcal{E}_{\{\omega\}}(\mathbb{R}^N)$ is surjective, we show that the distance of any point $\theta \in V$ to $V \cap \mathbb{R}^N$ is bounded above by $C|\operatorname{Im}\theta|$ for some C. The precise formulation is somewhat involved because we want to separate branches. The essence of the proof is the application of Theorem 1.10 to constant functions. That these can be chosen in a uniform way is shown using the Tarski-Seidenberg theorem.

3.1. LEMMA. — *Let $P \in \mathbb{C}[Z_1, \dots, Z_N]$ be a homogeneous polynomial whose variety $V = P^{-1}(0)$ has the following property : There is a sequence $(\theta_n)_n$ in $V \setminus \mathbb{R}^N$, satisfying $|\theta_n| = 1$ for all n, such that*

$$\inf_{n \in \mathbb{N}} \frac{\inf_{z \in V_n} |\operatorname{Im} z|}{|\operatorname{Im}\theta_n|} =: \eta > 0,$$

where V_n denotes an irreducible component of $V \cap U_{n|\operatorname{Im}\theta_n|}(\theta_n)$ containing θ_n.

Then there is no convex domain $\Omega \subset \mathbb{R}^N$ and no weight ω for which $P(D): A(\Omega) \to A(\Omega)$ or $P(D): \mathcal{E}_{\{\omega\}}(\Omega) \to \mathcal{E}_{\{\omega\}}(\Omega)$ are surjective.

Proof. — We have to show that (plh) is not satisfied. For fixed k, we choose $L > 2k/\eta$ and $\delta = 1/2$. Let ϵ_0 be given. Note that $n|\operatorname{Im}\theta_n| < 1$ for all n, since otherwise the origin were a real point in V_n. Thus there is $n > \eta L$ with $\epsilon := \eta L|\operatorname{Im}\theta_n| < \epsilon_0$. We let $\varphi(z) = \epsilon$ for all $z \in V_n$. There may be other irreducible components of $V \cap U_{n|\operatorname{Im}\theta_n|}(\theta_n)$ apart from V_n. We set $\varphi = 0$ there. Then it is clear that (α) holds, and (β) follows from the definition of η by

$$\varphi(z) \leq \eta L|\operatorname{Im}\theta_n| \leq L|\operatorname{Im} z|.$$

On the other hand, the following estimate shows that (γ) does not hold :

$$\varphi(\theta_n) = \eta L|\operatorname{Im}\theta_n| > k|\operatorname{Im}\theta_n| + \frac{\eta L}{2}|\operatorname{Im}\theta_n| = k|\operatorname{Im}\theta_n| + \delta\epsilon.$$

□

3.2. *The distance condition.* — We say that a homogeneous variety V in $\mathbb{C}^N$ satisfies the distance condition if, for each positively homogeneous semi-algebraic set $H \subset V$, there is a constant C with

$$\operatorname{dist}(z, (\mathbb{R}^N \cap H) \cup \partial H) \leq C|\operatorname{Im} z| \quad \text{for all } z \in H,$$

where ∂H denotes the boundary of H relative to V.

3.3. LEMMA. — $\Big|a/|a| - b/|b|\Big| \leq 2|a-b|/|a|$ *for* $a, b \in \mathbb{C}^N$ *with* $a, b \neq 0$.

Proof.

$$|a-b| = |a|\left|\frac{a}{|a|} - \frac{b}{|a|}\right| \geq |a|\left(\left|\frac{a}{|a|} - \frac{b}{|b|}\right| - |b|\left|\frac{1}{|b|} - \frac{1}{|a|}\right|\right)$$
$$= |a|\left|\frac{a}{|a|} - \frac{b}{|b|}\right| - \Big||a| - |b|\Big|.$$

□

3.4. PROPOSITION. — *If, for a convex domain $\Omega \subset \mathbb{R}^N$ and a homogeneous polynomial $P \in \mathbb{C}[Z_1, \ldots, Z_N]$, the operator $P(D): A(\Omega) \to A(\Omega)$ or, for some weight ω, the operator $P(D): \mathcal{E}_{\{\omega\}}(\Omega) \to \mathcal{E}_{\{\omega\}}(\Omega)$ is surjective, then the variety V of P satisfies the distance condition.*

Proof. — We argue by contradiction and assume the existence of a positively homogeneous semi-algebraic set $H \subset V$ and a sequence $(z_n)_n$ in H with

$$(3) \qquad 4n|\operatorname{Im} z_n| < \operatorname{dist}(z_n, (\mathbb{R}^N \cap H) \cup \partial H).$$

The set $(\mathbb{R}^N \cap H) \cup \partial H$ remains unchanged if H is replaced by its closure. Thus we can assume that H is closed. We abbreviate $\operatorname{dist}(\theta, (\mathbb{R}^N \cap H) \cup \partial H) =: d(\theta)$. Since H and the inequality (3) are positively homogeneous in z_n, we may assume $|z_n| = 1$ for all n. By Bochnak, Coste, and Roy [2], 2.2.8, the function d is semi-algebraic and continuous. Thus the set

$$M := \left\{(r, y, \theta) \in \mathbb{R}^2 \times H \;\middle|\; |\theta| = 1,\, d(\theta) = 1/r,\, y = |\operatorname{Im}\theta|^2\right\}$$

is semi-algebraic. This implies by Hörmander [12], A.2.4, that, for large R, the function

$$f\colon]R, \infty[\to \mathbb{R}, \quad r \mapsto \inf\left\{|\operatorname{Im}\theta|^2 \;\middle|\; \theta \in H,\, |\theta| = 1,\, d(\theta) = 1/r\right\} \tag{4}$$

is semi-algebraic. By [12], A.2.5, this implies the existence of $a, A \in \mathbb{R}$ with

$$f(r) = Ar^a(1 + o(1)), \quad r \to \infty.$$

Since H is closed, the infimum in (4) is really a minimum. This implies $A \neq 0$. Obviously, $f(r) \leq r^{-2}$, thus $a < 0$. We let $r_n = 1/d(z_n)$ and choose $\theta_n \in H$ with $|\theta_n| = 1$, $d(\theta_n) = 1/r_n$, and $f(r_n) = |\operatorname{Im}\theta_n|^2$. Then $|\operatorname{Im}\theta_n| \leq |\operatorname{Im} z_n|$, and (3) implies

$$4n|\operatorname{Im}\theta_n| < d(\theta_n). \tag{5}$$

To apply 3.1, we define V_n to be an irreducible component of $V \cap U_{n|\operatorname{Im}\theta_n|}(\theta_n)$ containing θ_n. Because of (5), V_n is disjoint to ∂H. It is an irreducible analytic set, thus connected, thus $\theta_n \in V_n$ implies $V_n \subset H$. If $z \in V_n$, then (5) implies $1/2 < |z| < 3/2$. Since H is positively homogeneous, $z/|z| \in H$ and thus because of (4) and (5)

$$|\operatorname{Im} z|^2 \geq |z|^2 f\left(\frac{1}{d(z/|z|)}\right) \geq \frac{1}{4} f\left(\frac{1}{d(z/|z|)}\right) \quad \text{for all } z \in V_n.$$

Lemma 3.3 implies $|z/|z| - \theta_n| \leq 2|z - \theta_n| < 2n|\operatorname{Im}\theta_n|$ and thus

$$\begin{aligned} d(z/|z|) &= \operatorname{dist}(z/|z|, (\mathbb{R}^N \cap H) \cup \partial H) \\ &\geq \operatorname{dist}(\theta_n, (\mathbb{R}^N \cap H) \cup \partial H) - |z/|z| - \theta_n| \\ &\geq \operatorname{dist}(\theta_n, (\mathbb{R}^N \cap H) \cup \partial H) - 2n|\operatorname{Im}\theta_n| \\ &\geq \frac{1}{2}\operatorname{dist}(\theta_n, (\mathbb{R}^N \cap H) \cup \partial H), \end{aligned}$$

where the last inequality follows from (5). Hence, for sufficiently large n and $z \in V_n$,

$$\begin{aligned} |\operatorname{Im} z|^2 \geq \frac{1}{4} f\left(\frac{1}{d(z/|z|)}\right) \geq \frac{A}{8} d\left(\frac{z}{|z|}\right)^{-a} &\geq \frac{A}{2^{3-a}} d(\theta_n)^{-a} \\ &\geq \frac{1}{2^{4-a}} f\left(\frac{1}{d(\theta_n)}\right) = \frac{|\operatorname{Im}\theta_n|^2}{2^{4-a}}. \end{aligned}$$

This verifies the hypotheses of Lemma 3.1, which yields the claim. □

Remark. — The semi-algebraic set H in Theorem 3.4 is meant to separate branches of V. Often, H will be the intersection of V with a half space. Note that finite intersections of half spaces are always semi-algebraic.

3.5. *Notation.* — We recall $z = (z', z_N) \in \mathbb{C}^{N-1} \times \mathbb{C}$. Let π be the projection $\pi(z) = z'$. Following Whitney [21], 2.10D, for a variety W and $\theta \in W$ for which θ is an isolated point of $W \cap \pi^{-1}(\theta')$, we define the branching locus Z' of the covering $\pi: W \to \mathbb{C}^{N-1}$ as the set of all $w' \in \mathbb{C}^{N-1}$ for which the cardinality of $\{z \in V \mid \pi(z) = w'\}$ is not maximal. For a C^1-path γ, we denote its derivative by $\dot{\gamma}$ to avoid confusion.

For $\theta \in W$, W a variety, the tangent cone $T_\theta W$ is defined as in Whitney [21], 7.1G :

$$T_\theta W = \{v \in \mathbb{C}^N \mid \forall \epsilon > 0 \, \exists q \in W, a \in \mathbb{C} : |q - \theta| < \epsilon, |a(q - \theta) - v| < \epsilon\}.$$

Note that tangent cones have their vertices in the origin.

3.6. THEOREM. — *Let $P \in \mathbb{C}[Z_1, \ldots, Z_N]$ be a homogeneous polynomial, let $V = P^{-1}(0)$, set $\pi: z = (z', z_N) \mapsto z'$, and let $\theta \in V \cap \mathbb{R}^N$ with $\pi(\theta) \neq 0$ be given. If there are a component W_θ of V_θ with representative W and a C^1-path $\gamma: [0, 1[\to W$ with*

(1) $\gamma(0) = \theta$, $\dot{\gamma}(0) \in \mathbb{R}^N$,

(2) *for all $t > 0$: $\gamma(t) \notin \mathbb{R}^N$ but $\pi \circ \gamma(t) \in \mathbb{R}^{N-1}$,*

(3) *θ is an isolated point of $W \cap \pi^{-1}(\theta')$ and $\pi(\dot{\gamma}(0))$ is transversal to the branching locus of the covering $\pi: W \to \mathbb{C}^{N-1}$,*

then there is no convex domain $\Omega \subset \mathbb{R}^N$ and no weight ω for which $P(D): A(\Omega) \to A(\Omega)$ or $P(D): \mathcal{E}_{\{\omega\}}(\Omega) \to \mathcal{E}_{\{\omega\}}(\Omega)$ are surjective.

Proof. — We may assume that V satisfies the dimension condition, since otherwise we apply Theorem 2.2. The dimension condition and hypothesis (2) cannot both hold in a regular point, thus $\pi: W \to \mathbb{C}^{N-1}$ is branched in θ'. We let $x := \dot{\gamma}(0)$. Note that hypothesis (3) implies $x' = \pi(x) \neq 0$. We have $\gamma(t) = \theta + tx + \alpha(t)$ with $\pi \circ \alpha(t) \in \mathbb{R}^{N-1}$ and $|\operatorname{Im} \gamma(t)| \leq |\alpha(t)| = o(t)$. Hypothesis (3) implies the existence of $\delta_1, \delta_2 > 0$ such that the translated truncated cone

$$C' = \theta' + \left\{ w' \in \mathbb{C}^{N-1} \,\middle|\, 0 < |w'| \leq \delta_1, \left| \frac{w'}{|w'|} - \frac{x'}{|x'|} \right| \leq \delta_2 \right\}$$

is disjoint to the branching locus Z' of π. There is $\delta_3 > 0$ such that

$$\operatorname{dist}(\pi \circ \gamma(t), \partial C') \geq \delta_3 t \quad \text{for small } t. \tag{6}$$

Let C be the connected component of $W \cap \pi^{-1}(C')$ that contains the path γ. C is semi-algebraic, since $\pi^{-1}(C')$ is obviously semi-algebraic and since by Bochnak, Coste, and Roy [2], 2.4.5, connected components of semi-algebraic sets are again semi-algebraic. As the covering π is unbranched over the contractible set C', its restriction $\pi: C \to C'$ is biholomorphic.

We claim $C \cap \mathbb{R}^N = \emptyset$. To see this, note first that the assumption that V satisfies the dimension condition implies that the hypervariety W can be written as the zero set of a function with real coefficients. Hence, the complex conjugate $\overline{W}$ of W is equal to W, and $\overline{C} \subset W$. Because of $\pi(\overline{C}) = C'$, the set $\overline{C}$ is a connected component of $\pi^{-1}(C')$. Since $\pi|C$ is biholomorphic, $\pi \circ \gamma = \overline{\pi} \circ \gamma$, but $\gamma \neq \overline{\gamma}$, the sets C and $\overline{C}$ are disjoint. In particular, $C \cap \mathbb{R}^N = C \cap \overline{C} = \emptyset$.

Since π is branched and P is homogeneous, $\mathbb{C}\theta' \subset Z'$. In particular, x and θ are linearly independent. Hence there is a complex linear form $A: \mathbb{C}^N \to \mathbb{C}$ with real coefficients satisfying $A(\theta) = 1$ and $A(x) = 0$. Set

$$H := \{z \in \mathbb{C}^N \mid z \in A(z)C\}.$$

This set is semi-algebraic and positively homogeneous. It suffices to disprove the distance condition for H. We have just shown $H \cap \mathbb{R}^N = \{0\}$, hence $\operatorname{dist}(\gamma(t), H \cap \mathbb{R}^N) \geq |\theta|/2$. We have to estimate $\operatorname{dist}(\gamma(t), \partial H)$ next. We assume for contradiction that, for sufficiently small t, there is $z \in \partial H$ with $|z - \gamma(t)| < \epsilon t$, where ϵ is chosen so small that

$$|A(z) - 1| = |A(z - \gamma(t)) + A(\gamma(t) - \theta)| \leq |A||z - \gamma(t)| + |A(\alpha(t))| < \frac{\delta_3}{8|\theta|} t.$$

Note that, if t is small enough, then $|z| < 2|\theta|$, $|A(z)| > 1/2$, and

$$\left| z' \left(1 - \frac{1}{A(z)} \right) \right| \leq |z'||A(z)|^{-1}|A(z) - 1| \leq 4|\theta| \frac{\delta_3}{8|\theta|} t = \frac{\delta_3}{2} t.$$

If $z \in \partial H$, then $z'/A(z) \in \partial C'$. Thus (6) leads to the following contradiction to $|z - \gamma(t)| < \epsilon t$

$$|z - \gamma(t)| \geq |z' - \pi \circ \gamma(t)| \geq \left| \frac{z'}{A(z)} - \pi \circ \gamma(t) \right| - \left| z' \left(1 - \frac{1}{A(z)} \right) \right| \geq \frac{\delta_3}{2} t.$$

□

3.7. *Example.* — Let $l \in \mathbb{N}$, $l \geq 2$, let $a, b = \pm 1$, but $(a, b, l) \neq (-1, -1, 2)$ (which was investigated in 1.5(3)). Then the operator

$$P(D) = \frac{\partial^{2l}}{\partial x^l \partial w^l} + a \frac{\partial^{2l}}{\partial y^{2l}} + b \frac{\partial^{2l}}{\partial z^{2l}} : \mathcal{E}_{\{\omega\}}(\mathbb{R}^4) \to \mathcal{E}_{\{\omega\}}(\mathbb{R}^4)$$

is not surjective.

Proof. — Set $\theta = (0,0,0,1)$ in (x,y,z,w)-space. The hypotheses imply that at least one of the numbers $-a$ and $-b$ admits a non-real lth root λ. Without restriction we assume $\lambda^l = -a$, $\lambda \notin \mathbb{R}$, and we set, for sufficiently small $t > 0$,

$$\gamma(t) = (\lambda t^2, t, 0, 1).$$

We verify hypotheses (1) to (3) of Theorem 3.6 for the covering

$$\pi\colon (x,y,z,w) \mapsto (y,z,w).$$

The first two of them are clearly satisfied. To see (3), note that the branching locus is equal to $Z' = \{(y,z,w) \mid ay^{2l} + bz^{2l} = 0\}$. This is transversal to $\pi(\dot\gamma(0)) = (1,0,0)$. □

Let P be again a homogeneous polynomial. If its variety has only regular points outside the origin, then the dimension condition is neccessary and sufficient by Hörmander [11], 6.5, (1.4 in this paper). On the other hand, examples like 1.5(3) show that the mere presence of an irreducible singularity does not exclude surjectivity. Thus the singularities have to be investigated. Theorem 3.8 is a result in this direction. Notations are like in 3.5.

3.8. THEOREM. — *Let $P \in \mathbb{C}[Z_1,\dots,Z_N]$ be a homogeneous polynomial, and let $V = P^{-1}(0)$ be its variety. If, in some point $\theta \in V \cap \mathbb{R}^N$, $\theta \neq 0$, there is an irreducible component W_θ of V_θ with the following properties (1), (2), and (3), then there is no convex domain $\Omega \subset \mathbb{R}^N$ and no weight ω for which $P(D)\colon A(\Omega) \to A(\Omega)$ or $P(D)\colon \mathcal{E}_{\{\omega\}}(\Omega) \to \mathcal{E}_{\{\omega\}}(\Omega)$ are surjective :*

(1) $\theta' \neq 0$,

(2) *the Nth standard basis vector e_N is not in $T_\theta W$,*

(3) *there is a two dimensional vector space $X \subset \mathbb{R}^N$ with $e_N \in X$ and $\theta \notin X$ such that the curve $[W \cap (\theta + (X \oplus iX))]_\theta$ has an irreducible component K_θ with*

(a) *K is singular in θ,*

(b) ${}_{\mathbb{R}}\dim_\theta X \cap T_\theta K = 1$,

(c) $\pi(T_\theta K) \cap T_{\theta'} Z' = \{0\}$.

Proof. — We may assume that V satisfies the dimension condition; otherwise, the claim follows from 2.2. X is spanned by e_N and some other vector x. The expansion of K_θ into its Puiseux series has the form

$$K_\theta = [\{\theta + t^q x + f(t)e_N \mid |t| < \delta_1\}]_\theta \, . \tag{7}$$

Since e_N is not tangential to W, let alone to K, the vanishing order of f in the origin is at least q. Furthermore, q is strictly larger than 1, since otherwise K would be regular in θ. It is easily seen from (7) that

$$T_\theta K = \mathbb{C}T, \quad T = x + \frac{f^{(q)}(0)}{q!} e_N.$$

Hypothesis (3)(b) implies $f^{(q)}(0) \in \mathbb{R}$. If f has non-real Taylor coefficients, then define $\sigma = 1$. If all Taylor coefficients of f are real, then let σ be a primitive $2q$th root of unity. Then σ is not real because q is greater than 1. Some of the Taylor coefficients of $t \mapsto f(\sigma t)$ are not real, then, since otherwise f would be a Taylor series in t^q, and K would be regular. So in both cases we have for sufficiently small $\delta_2 > 0$ and sufficiently large C_1 :

$$\text{for } 0 < t < \delta_2 : \operatorname{Im} f(\sigma t) \neq 0 \text{ and } |\operatorname{Im} f(\sigma t)| \leq C_1 t^{q+1}.$$

Define

$$\gamma(t) = \theta + \sigma^q t x + f(\sigma \sqrt[q]{t}) e_N, \quad 0 < t < \delta_2.$$

This is a C^1-path in K, which is easily seen to satisfy hypotheses (1) and (2) of Theorem 3.6. Hypothesis (2) implies that θ is an isolated point of $W \cap \pi^{-1}(\theta')$, and (3)(c) that $\pi(\dot\gamma(0)) = T'$ is transversal to the branching locus Z'. □

3.9. *Examples.* — We wish to study the operator

$$P(D) = \frac{\partial^n}{\partial x^l \, \partial w^{n-l}} + a \frac{\partial^n}{\partial y^m \partial w^{n-m}} + b \frac{\partial^n}{\partial z^n} : \mathcal{E}_{\{\omega\}}(\mathbb{R}^4) \to \mathcal{E}_{\{\omega\}}(\mathbb{R}^4), \tag{8}$$

with $a, b = \pm 1$, $1 \leq l \leq m \leq n$.

(1) If m is not a multiple of l, then $P(D)$ is not surjective.

Proof. — Choose $\theta := (0,0,0,1)$ in (x,y,z,w)-space. The tangent cone there is the hyperplane $\{x = 0\}$, thus we may use the map $\pi\colon (x,y,z,w) \mapsto (y,z,w)$ as covering in Theorem 3.8. Let $X = \{z = w = 0\}$, let λ, μ be relatively prime satisfying $\lambda/\mu = l/m$. Then $\lambda \neq 1$, since m is not a multiple of l. Choose $c, d \in \mathbb{C} \setminus \{0\}$ with $c^l + a d^m = 0$. Then

$$K = V \cap (\theta + (X \oplus iX)) = \{(ct^\mu, dt^\lambda, 0, 1) \mid t \in \mathbb{C}\}.$$

K_θ is irreducible and singular in θ, and $T_\theta K = \{(0, y, 0, 0) \mid y \in \mathbb{C}\}$. Near θ, the covering π is branched only where $x = 0$, thus $Z' = \{(y,z,w) \mid$

$ay^m w^{n-m} + bz^n = 0\}$. If $m < n$, then $T_{\theta'}Z' = \{(0, z, w) \mid z, w \in \mathbb{C}\}$, and if $m = n$, then $T_{\theta'}Z' = \{(y, z, w) \mid ay^n + bz^n = 0, w \in \mathbb{C}\}$. In both cases, the hypotheses of 3.8 are satisfied. □

(2) If $n \geq l + 2$ and n is not a multiple of $n - l$, then $P(D)$ is not surjective.

Proof. — Choose $\theta := (1, 0, 0, 0)$. Since $T_\theta V = \{w = 0\}$, the map $\pi\colon (x, y, z, w) \mapsto (x, y, z)$ can serve as covering. To calculate the branching locus Z', let (x, y, z, w) with $P(x, y, z, w) = (\partial P/\partial w)(x, y, z, w) = 0$ be given. Then $(n - l)x^l w^{n-l-1} + a(n - m)y^m w^{n-m-1} = 0$. We solve for w and insert into $P(x, y, z, w) = 0$ to get

$$y^{m\frac{n-l}{m-l}} x^{-l\frac{n-m}{m-l}} a^{\frac{n-l}{m-l}} \left(\left(\frac{m-n}{n-l} \right)^{\frac{n-l}{m-l}} + \left(\frac{m-n}{n-l} \right)^{\frac{n-m}{m-l}} \right) + bz^n = 0.$$

If $m < n$, then $T_{\theta'}Z' = \{z = 0\}$, otherwise $T_{\theta'}Z' = \{cy^n + z^n = 0\}$ for some $c \neq 0$. In both cases, if we define $X := \{x = y = 0\}$, then $\pi(X) \cap T_{\theta'}Z' = \{0\}$. This shows that hypothesis (3)(c) of Theorem 3.8 is satisfied for

$$K = V \cap (\theta + (X \oplus iX)) = \{(1, 0, z, w) \mid w^{n-l} + bz^n = 0\}.$$

Since n is not a multiple of $n - l$, this curve has a singular irreducible component satisfying the hypotheses of 3.8. □

4. Carry over to the tangent cone.

We show that if $P(D)\colon \mathcal{E}_{\{\omega\}}(\mathbb{R}^N) \to \mathcal{E}_{\{\omega\}}(\mathbb{R}^N)$, P a homogeneous polynomial, is surjective, then, for all $\theta \in V = P^{-1}(0) \cap \mathbb{R}^N$, $\theta \neq 0$, also the tangent cone in θ satisfies the Phragmén-Lindelöf condition (plh). This is a useful necessary condition, because the tangent cones are, after a suitable change of coordinates, of the form $\mathbb{C} \times V'$, where V' is a cone in $\mathbb{C}^{N-1}$. Thus the new problem has fewer variables (see 5.2).

The proof uses techniques from Hörmander [11] and Meise, Taylor, and Vogt [16], who investigated carrying over to and from the tangent cone at infinity, *i.e.*, the variety of the principal part of the operator.

4.1. THEOREM. — *Let $P \in \mathbb{C}[Z_1, \ldots, Z_N]$ be a homogeneous polynomial with variety $V = P^{-1}(0)$. If V satisfies the Phragmén-Lindelöf* *condition (plh), then so does the tangent cone $T_\Xi V$ for every $\Xi \in V \cap \mathbb{R}^N$, $\Xi \neq 0$.*

Proof. — We may assume $|\Xi| = 1$. We choose coordinates with the following property : except for the elements of an analytic subset, all points of V admit a neighborhood and holomorphic functions $\alpha_1, \dots, \alpha_m$ and $\beta_1, \dots, \beta_{m'}$ on this neighborhood such that

$$V = \{(z', \alpha_j(z')) + \Xi \mid z' \in U_{\delta_1}(0), j = 1, \dots, m\},$$
$$T_\Xi V = \{(z', \beta_j(z')) \mid z' \in U_{\delta_1}(0), j = 1, \dots, m'\}.$$

We may choose the coordinates in such a way that $(0, \dots, 0, 1) \notin T_\Xi V$. Then there is $C > 0$ with

$$|\alpha_j(z')| \leq C|z'|, \; |\beta_j(z')| \leq C|z'|.$$

Let D be the discriminant of the β_j, *i.e.*, $D(z') = b \prod_{i \neq j} (\beta_i(z') - \beta_j(z'))$, where $b \neq 0$ is chosen so small that $|D(z')| \leq 1$ for all z' with $|z'| \leq 1$. Let $d := \deg D$. By a homogeneity argument there is, for sufficiently small $\lambda > 0$, a constant $c(\lambda) > 0$ with

$$(9) \qquad |\beta_i(z') - \beta_j(z')| > c(\lambda)|z'| \quad \text{provided } i \neq j \text{ and } |D(z')| > \lambda|z'|^d.$$

On the other hand, $\Xi + T_\Xi V$ approaches V faster than linearly, thus there is $\eta > 0$ such that, for $|z'| < \delta_1$,

(10)
$$\max_{j=1}^{m'} \min_{i=1}^{m} |\beta_j(z') - \alpha_i(z')| \leq |z'|^{1+\eta} \text{ and } \max_{i=1}^{m} \min_{j=1}^{m'} |\beta_j(z') - \alpha_i(z')| \leq |z'|^{1+\eta}.$$

The entities k, L, δ, etc. appearing in (plh) will carry a prime if they refer to V and will be plain if referring to $T_\Xi V$. We set $k := 2k'$ and let L and δ be given. Then $L' := L/2$ and $\delta' := \delta/3$. There exists ϵ_0' as in (plh). Let $\epsilon_0 := 1/2$. Fix $\epsilon < \epsilon_0$ and $\theta \in T_\Xi V$ with $|\theta| = 1$. Let $\varphi \in \mathrm{PSH}(T_\Xi V \cap U_\epsilon(\theta))$ with (α) and (β) be given. It suffices to prove (γ) only for θ with $D(\theta') \neq 0$. Choose $\lambda < 1$ so small that

$$\frac{-1}{\log \lambda} \leq \frac{-\delta}{3 \log |D(\theta'/2)|}.$$

Choose $r > 0$ so small that

$$r < \delta_1, \; 8r^\eta < c(\lambda), \text{ and } (2L + 1)r^\eta \leq \epsilon\delta/3.$$

Set $\epsilon' := r\epsilon/2$. By (10), there is $z = \Xi + (r\theta', \alpha_i(r\theta'))$ in V satisfying $|\alpha_i(r\theta') - r\theta_N| < r^{1+\eta}$, and for w with $\Xi + w \in V \cap U_{\epsilon'}(z)$ there is j with

$|\beta_j(w') - w_N| < |w'|^{1+\eta} \leq 2r^{1+\eta}$. Then

$$\begin{aligned}\left|\left(\frac{w'}{r}, \frac{\beta_j(w')}{r}\right) - \theta\right| &\leq \frac{1}{r}\left(|\Xi + (w', \beta_j(w')) - z| + |z - \Xi - r\theta|\right)\\ &\leq \frac{1}{r}\left(|\Xi + w - z| + |\beta_j(w') - w_N| + |\alpha_i(r\theta') - r\theta_N|\right)\\ &< \frac{1}{r}(\epsilon' + 3r^{1+\eta}) \leq \epsilon.\end{aligned}$$

Thus we can define a function value $\psi(\Xi + w)$ as follows

$$\psi(\Xi + w) = \max\left(0, \max\left\{r\varphi\left(\frac{w'}{r}, \frac{\beta_j(w')}{r}\right) \Big| |\beta_j(w') - w_N| < 2r^{1+\eta}\right\} - \frac{\epsilon r}{\log\lambda}\log\left|\frac{D(w')}{(2r)^d}\right|\right).$$

We show that ψ is plurisubharmonic. Consider first the case $|D(w')| \leq \lambda|w'|^d$. Then we have for j with $|\beta_j(w') - w_N| < 2r^{1+\eta}$

$$r\varphi\left(\frac{w'}{r}, \frac{\beta_j(w')}{r}\right) - \frac{\epsilon r}{\log\lambda}\log\left|\frac{D(w')}{(2r)^d}\right| \leq r\epsilon - \frac{\epsilon r}{\log\lambda}\log\frac{\lambda|w'|^d}{(2r)^d} = 0.$$

In the other case $|D(w')| > \lambda|w'|^d$, choose j with $|\beta_j(w') - w_N| < 2r^{1+\eta}$ and let $l \neq j$. Then (9) implies

$$\begin{aligned}|\beta_l(w') - w_N| &\geq |\beta_l(w') - \beta_j(w')| - |\beta_j(w') - w_N| > c(\lambda)|w'| - 2r^{1+\eta}\\ &\geq c(\lambda)r/2 - 2r^{1+\eta} = r(c(\lambda)/2 - 2r^\eta) \geq 2r^{1+\eta}.\end{aligned}$$

Thus branches are separated and ψ is plurisubharmonic, being the maximum of finitely many plurisubharmonic functions. The estimates $(\alpha)'$ and $(\beta)'$ will be derived from the following

$$\psi(\Xi + w) \leq r \max_{|\beta_j(w') - w_N| < 2r^{1+\eta}} \varphi\left(\frac{w'}{r}, \frac{\beta_j(w')}{r}\right) \leq r\epsilon = 2\epsilon',$$

$$\begin{aligned}\psi(\Xi + w) &\leq L \max_{|\beta_j(w') - w_N| < 2r^{1+\eta}} |\operatorname{Im}(w', \beta_j(w'))| - \frac{\epsilon r}{\log\lambda}\log\frac{|w'|^d}{(2r)^d}\\ &\leq L|\operatorname{Im} w| + L|\beta_j(w') - w_N| \leq L|\operatorname{Im} w| + 2Lr^{1+\eta}.\end{aligned}$$

Now (plh) is applied to $\psi/2 - Lr^{1+\eta}$. This gives

$$\psi(z) \leq 2k'|\operatorname{Im} z| + 2\delta'\epsilon' + 2Lr^{1+\eta}$$

and

$$\begin{aligned}\varphi(\theta) &\leq \frac{1}{r}\psi(z) + \frac{\epsilon r}{\log\lambda}\log\left|\frac{D(r\theta')}{(2r)^d}\right|\\ &\leq \frac{1}{r}\left(2k'(|\operatorname{Im} r\theta| + |\alpha_i(r\theta') - r\theta_N|) + 2\delta'\epsilon' + 2Lr^{1+\eta}\right) + \frac{\epsilon}{\log\lambda}\log\left|\frac{D(\theta')}{2^d}\right|\\ &\leq 2k'|\operatorname{Im}\theta| + r^\eta + \epsilon\delta' + 2Lr^\eta + \epsilon\delta/3\\ &\leq k|\operatorname{Im}\theta| + \epsilon\delta.\end{aligned}$$

□

Theorem 4.1 about carrying over and the dimension condition 2.2 imply the following statement.

4.2. COROLLARY. — *$P \in \mathbb{C}[Z_1, \dots, Z_N]$ be homogeneous, let $V = P^{-1}(0)$. If there is $\Xi \in V \cap \mathbb{R}^N$ such that the tangent cone $T_\Xi V$ contains $\theta \neq 0$ and an irreducible component W_θ of $[T_\Xi V]_\theta$ satisfying ${}_{\mathbb{R}}\dim_\theta W \cap \mathbb{R}^N < N-1$, then there is no convex domain $\Omega \subset \mathbb{R}^N$ and no weight ω for which $P(D): A(\Omega) \to A(\Omega)$ or $P(D): \mathcal{E}_{\{\omega\}}(\Omega) \to \mathcal{E}_{\{\omega\}}(\Omega)$ are surjective.*

4.3. *Example.* — Corollary 4.2 shows that the following operators are not surjective. The point Ξ in which the tangent cone is investigated is $x = y = z = 0$, $w = 1$ in both cases :

$$(1)\ \ P(D) = \frac{\partial^n}{\partial x^2 \partial w^{n-2}} + \frac{\partial^n}{\partial y^2 \partial w^{n-2}} \pm \frac{\partial^n}{\partial z^n},\ n > 2,$$

$$(2)\ \ P(D) = \frac{\partial^n}{\partial x^l \partial w^{n-l}} \pm \frac{\partial^n}{\partial y^l \partial w^{n-l}} \pm \frac{\partial^n}{\partial z^n},\ 3 \leq l < n.$$

5. Applications.

The following result is an immediate corollary to Theorem 3.8 and Hörmander [11], 6.5 (see 1.4). It generalizes [3], 12, to the case of $\mathcal{E}_{\{\omega\}}(\mathbb{R}^3)$, thus showing that, for a homogeneous partial differential operator in three variables, surjectivity on $A(\mathbb{R}^3)$ and on $\mathcal{E}_{\{\omega\}}(\mathbb{R}^3)$, ω any weight function, are equivalent. In three variables, we look upon the homogeneous variety V as a curve in complex projective space $\mathbb{P}^2$. The real projective space $\mathbb{RP}^2$ is embedded canonically into $\mathbb{P}^2$.

5.1. THEOREM. — *Let $P \in \mathbb{C}[Z_1, Z_2, Z_3]$ be a homogeneous polynomial, and let $V = P^{-1}(0)$. The following are equivalent :*

(1) *$P(D): A(\mathbb{R}^3) \to A(\mathbb{R}^3)$ is surjective,*

(2) *$P(D): \mathcal{E}_{\{\omega\}}(\mathbb{R}^3) \to \mathcal{E}_{\{\omega\}}(\mathbb{R}^3)$ is surjective for every (some) weight ω,*

(3) *for every $\theta \in V \cap \mathbb{RP}^2$ and every irreducible component W_θ of V_θ :*

$${}_{\mathbb{R}}\dim_\theta W \cap \mathbb{RP}^2 = 1 \quad \text{and} \quad W_\theta \text{ is regular in } \theta.$$

Theorem 4.1 about carrying over of Phragmén-Lindelöf conditions to tangent cones gives us operators in $N-1$ variables acting on $\mathbb{R}^N$. For these, results of Meise, Taylor, and Vogt show that surjectivity already implies the existence of a continuous linear right inverse.

5.2. THEOREM. — *Let ω be a weight function, let $P \in \mathbb{C}[Z_1, \dots, Z_N]$ be a homogeneous polynomial, and let $M \in \mathbb{N}$ (recall $0 \notin \mathbb{N}$). If $\mathbb{R}^N$ is regarded as a subspace of $\mathbb{R}^{N+M}$, then the operator $P(D)$ acts on both spaces. Equivalent are*

(1) *$P(D)\colon \mathcal{E}_{\{\omega\}}(\mathbb{R}^N) \to \mathcal{E}_{\{\omega\}}(\mathbb{R}^N)$ admits a continuous linear right inverse,*

(2) *$P(D)\colon A(\mathbb{R}^N) \to A(\mathbb{R}^N)$ is surjective, and P has no elliptic factor,*

(3) *$P(D)\colon \mathcal{E}_{\{\omega\}}(\mathbb{R}^{N+M}) \to \mathcal{E}_{\{\omega\}}(\mathbb{R}^{N+M})$ is surjective,*

(4) *$P(D)\colon A(\mathbb{R}^{N+M}) \to A(\mathbb{R}^{N+M})$ is surjective.*

Proof. — The equivalence of (1) and (2) has been shown by Meise, Taylor, and Vogt [16], 3.14, [15], 4.5.

To show that (1) implies (3) and (4), note that by Braun, Meise, and Taylor [5], 8.1, the operator in (3) is the tensor product of the operator in (1) with the identity map of $\mathcal{E}_{\{\omega\}}(\mathbb{R}^M)$. Thus the operator in $N+M$ variables admits a right inverse, namely the tensor product of the right inverse for N variables with the identity map in M variables. Now the arguments that show the equivalence of (1) and (2) yield that (4) is a consequence of the existence of a right inverse for the operator in (3).

(4) implies (3) because of Theorem 1.11.

To show that (3) implies (1), we verify the Phragmén-Lindelöf condition HPL($\mathbb{R}^N$, loc) of Meise, Taylor, and Vogt [16], 3.2, at zero. By Meise, Taylor, and Vogt [16], 3.3, [15], 4.5, HPL($\mathbb{R}^N$, loc) at zero is equivalent to the existence of a right inverse for $P(D)\colon \mathcal{E}_{\{\omega\}}(\mathbb{R}^N) \to \mathcal{E}_{\{\omega\}}(\mathbb{R}^N)$. To write down this condition, set $V = \{z \in \mathbb{C}^N \mid P(z) = 0\}$. Then HPL($\mathbb{R}^N$, loc) means that there are bounded open sets $U_1 \subset U_2 \subset U_3 \subset \mathbb{C}^N$ with $0 \in U_1$, such that, for each $K > 0$, there are $Q > 0$ and $\eta > 0$ such that each $\varphi \in \mathrm{PSH}(V \cap U_3)$ satisfying (α') and (β') also satisfy (γ'), where

(α') $\varphi(z) \le K|\operatorname{Im} z| + \eta$, $z \in V \cap U_3$,

(β') $\varphi(x) \le 0$, $x \in V \cap \mathbb{R}^N \cap U_2$,

(γ') $\varphi(z) \le Q|\operatorname{Im} z|$, $z \in V \cap U_1$.

We show that (plh) for the operator in $N+M$ variables implies HPL($\mathbb{R}^N$, loc) for the operator in N variables. Let $K > 0$ be given, let

k be as in (plh), set $Q = k(K+1)$, $U_1 = U_1(0) \subset \mathbb{C}^N$ and $U_3 = U_2 = U_3(0)$. Let $\varphi \in \mathrm{PSH}(V \cap U_2)$ satisfy (α') and (β') of $\mathrm{HPL}(\mathbb{R}^N, \mathrm{loc})$. Let $\eta > 0$ be arbitrary, set $\delta = \eta/(K+2)$. Since φ is upper semicontinuous, $\{z \in U_2 \mid \varphi(z) < \delta\}$ is an open neighborhood of $V \cap U_{5/2}(0) \cap \mathbb{R}^N$. Let L be so large that, for $z \in U_2(0)$ outside this set, $L|\operatorname{Im} z| > K+1$. Then $\varphi(z) - \delta < L|\operatorname{Im} z|$ for all $z \in V \cap U_2(0)$. For δ and L there is ϵ_0 as in (plh). Set $\epsilon = \epsilon_0/2$. Fix $w \in V \cap U_1$. Denote by e_{N+M} the $N+M$th standard basis vector and set

$$\theta = \epsilon w + \sqrt{1-\epsilon^2}\, e_{N+M}.$$

Then $P(\theta) = 0$. Denote by π the projection $\pi\colon \mathbb{R}^{N+M} \to \mathbb{R}^N$ which forgets the last M components. For $z \in U_\epsilon(\theta) \subset \mathbb{R}^{N+M}$ we have $|\pi(z)/\epsilon| < 1$. Thus

$$\psi(z) := \frac{\epsilon}{K+1}\left(\varphi \circ \pi\left(\frac{1}{\epsilon} z\right) - \delta\right), \quad z \in \mathbb{C}^{N+M}, P(z) = 0, |z - \theta| < \epsilon,$$

is a plurisubharmonic function satisfying (α) and (β). Thus it also satisfies (γ), hence

$$\varphi(w) = \frac{K+1}{\epsilon}\psi(\theta) + \delta \le \frac{K+1}{\epsilon}(k|\operatorname{Im}\theta| + \delta\epsilon) + \delta = (K+1)k|\operatorname{Im} w| + \eta.$$

□

Remark. — There is a way to prove the equivalence of (3) and (4) without recurrence to continuous linear right inverses. The main tool there is a Sibony-Wong inequality for homogeneous varieties. But this is also one main ingredient of the proof of Meise, Taylor, and Vogt [16], 3.14.

5.3. *Example.* — The following partial differential operator is surjective :

$$P(D) = \frac{\partial^3}{\partial x^2 \partial w} - \frac{\partial^3}{\partial y^2 \partial w} \pm \frac{\partial^3}{\partial z^3}\colon A(\mathbb{R}^4) \to A(\mathbb{R}^4).$$

Remember that by 1.11 surjectivity for $A(\mathbb{R}^4)$ implies surjectivity for $\mathcal{E}_{\{\omega\}}(\mathbb{R}^4)$.

Proof. — It is no restriction to assume that "$\pm$" stands for "$+$". We claim that for each $\theta \in V := P^{-1}(0)$ with $|\theta| = 1$ there are a neighborhood U_θ of θ and some $A_\theta > 0$ with $\varphi(\zeta) \le A_\theta|\operatorname{Im}\zeta|$ for all $\zeta \in U_\theta$ provided $\varphi \in \mathrm{PSH}(\mathbb{C}^N)$ satisfies (α) and (β) of (HPL) (see 1.2). Once this claim is established, (HPL) is easily proved by a compactness argument since because of the homogeneity of all data it suffices to consider $\theta \in V$ with $|\theta| = 1$.

For $\theta \in V \setminus \mathbb{R}^N$, $|\theta| = 1$, choose $A_\theta = 3/|\operatorname{Im}\theta|$ and $U_\theta = U_{|\operatorname{Im}\theta|/2}(\theta)$. To handle the case that V is regular in $\theta \in V \cap \mathbb{R}^N$, $|\theta| = 1$, set $D = \{t \in \mathbb{C} \mid |t| < 1, \operatorname{Im} t > 0\}$ and recall that a standard estimate of the harmonic measure of the half disk (see Nevanlinna [18], 38) implies that for all functions ρ, subharmonic in a neighborhood of D, we have

$$\rho(it) \leq \frac{4}{\pi} t \quad \text{for all } t \in [0,1[, \tag{11}$$

provided

$$\rho(t) \leq 1 \text{ for all } t \in D \text{ and } \rho(t) \leq 0 \text{ for all } t \in]-1,1[.$$

This is applied to a suitable parametrization for V near θ. For the investigation of the singular points, introduce for a moment the variables $a = x + y$ and $b = x - y$. In these coordinates, P has the form $abw + z^3$. So the singularities of V are where z and at least two of the variables a, b, and w vanish. From this we see that all singularities are isomorphic. To actually investigate them, it is better to use x and y again. It is enough to consider, in (x,y,z,w)-space, $\theta := (0,0,0,1)$ only. We set $U_\theta = U_{1/4}(\theta)$ and $A_\theta = 24$. Start with z, w real, *i.e.*, fix $z \in [-1/2, 1/2]$ and $w \in [1/2, 3/2]$. Consider the case $z \leq 0$ first. For given y with $|y| < 1/2$ and $y \notin \mathbb{R}$, define the subharmonic function

$$\psi\colon \{t \in \mathbb{C} \mid |t| < 1/2, \operatorname{Im} t > 0\} \to \mathbb{R}$$

$$t \mapsto \max_{\pm} \varphi\left(\pm\sqrt{\left(\operatorname{Re} y + t\frac{\operatorname{Im} y}{|\operatorname{Im} y|}\right)^2 - \frac{z^3}{w}},\ \operatorname{Re} y + t\frac{\operatorname{Im} y}{|\operatorname{Im} y|},\ z,\ w\right).$$

Because of $z \leq 0$, the radicand is positive and thus $\psi(t) \leq 0$ for $t \in]-1/2, 1/2[$, and $\psi(t) \leq 5$ for all t with $|t| < 1/2$. By (11) this implies $\psi(t) \leq (40/\pi)|\operatorname{Im} t|$ provided $\operatorname{Re} t = 0$, in particular $\varphi(x,y,z,w) \leq (40/\pi)|\operatorname{Im} y|$ for both values of x for which $(x,y,z,w) \in V$. In the case $z > 0$, the argument is somewhat more interesting. Set $r = z^{3/2}w^{-1/2}$. Define

$$M = (]-2,2[+]-1,1[i) \setminus (]-2,-r] \cup [r,2[).$$

M is simply connected, and $y^2 - z^3/w \neq 0$ for all $y \in M$, thus over M there are two distinct branches of $\sqrt{y^2 - z^3/w}$. Denote one of them by $g(y)$. Then $\operatorname{Im} g(y) \neq 0$ for all $y \in M$, so it has the same sign everywhere. In particular, $|\operatorname{Im} g(y)|$ is harmonic on M. We show first

$$|\operatorname{Im} y| \leq 4|\operatorname{Im} g(y)| \quad \text{for all } y \in \partial M. \tag{12}$$

If $y \in \partial M$ and $\operatorname{Im} y \neq 0$, then $|g(y)|^2 = |y^2 - r^2| \geq 1 - r^2 > 3/4$, thus at least one of $|\operatorname{Re} g(y)|$ and $|\operatorname{Im} g(y)|$ is greater than $\sqrt{3/8}$. If this is $|\operatorname{Im} g(y)|$

then (12) is clear. In the other case, (12) follows from $\operatorname{Im} g(y)^2 = \operatorname{Im} y^2$ and thus

$$|\operatorname{Im} g(y)| = \frac{|\operatorname{Im} y||\operatorname{Re} y|}{|\operatorname{Re} g(y)|} \leq 4|\operatorname{Im} y|.$$

We want to estimate φ on the vertical parts of ∂M next. To this end, define

$$\psi: D \to \mathbb{R},\ t \mapsto \varphi(g(t+2), t+2, z, w).$$

The hypotheses imply $\psi(t) \leq 0$ for $t \in]-1,1[$ and $\psi(t) \leq 4$ for all t with $|t| < 1$. Thus it follows from (11) that

$$\psi(it) \leq \frac{16}{\pi} t \quad \text{for all } t \in [0,1[. \tag{13}$$

On the horizontal parts of ∂M, *i.e.*, for $y = \pm i + t$, $-2 < t < 2$, we have $\varphi(g(y), y, z, w) \leq \sqrt{5} = \sqrt{5}|\operatorname{Im} y|$. Thus

$$\varphi(g(y), y, z, w) \leq 6|\operatorname{Im} y| \leq 24|\operatorname{Im} g(y)| \quad \text{for all } y \in \partial M. \tag{14}$$

Since $|\operatorname{Im} g(y)|$ is harmonic, (14) holds on all of M. Repeating the argument for $-g$, we conclude

$$\begin{aligned} &\varphi(x,y,z,w) \leq 24|\operatorname{Im} x| \\ &\text{if } (x,y,z,w) \in V, z \in]-1/2, 1/2[, w \in]1/2, 3/2[,\ y \in]-2,2[+]-1,1[i. \end{aligned} \tag{15}$$

If now z, w with $|z| < 1/4$ and $|w-1| < 1/4$ are given arbitrarily, consider $\psi: D \to \mathbb{R}$

$$t \mapsto \max_{\pm} \varphi\left(\pm\sqrt{y^2 - \frac{(\operatorname{Re} z + \frac{t}{4}\frac{\operatorname{Im} z}{|\operatorname{Im}(z,w)|})^3}{\operatorname{Re} w + \frac{t}{4}\frac{\operatorname{Im} w}{|\operatorname{Im}(z,w)|}}},\ y,\ \operatorname{Re} z + \frac{t}{4}\frac{\operatorname{Im} z}{|\operatorname{Im}(z,w)|},\right.$$
$$\left.\operatorname{Re} w + \frac{t}{4}\frac{\operatorname{Im} w}{|\operatorname{Im}(z,w)|}\right).$$

Then, for $t \in]-1,1[$, (15) implies $\psi(t) \leq 24|\operatorname{Im} x|$, where x denotes the first entry above. Now (11) implies $\psi(i|\operatorname{Im}(z,w)|) \leq (32/\pi)|\operatorname{Im}(z,w)| + 24|\operatorname{Im} x|$. This completes the proof. □

The homogeneous partial differential operators in three variables are dealt with in 5.1. In four variables, a series of non trivial operators is given by

$$P(D) = A\frac{\partial^n}{\partial x^l \partial w^{n-l}} + B\frac{\partial^n}{\partial y^m \partial w^{n-m}} + C\frac{\partial^n}{\partial z^n},\ A, B, C \in \mathbb{C},\ l, m, n \in \mathbb{N}_0,$$
$$l, m \leq n.$$

By 2.5, we can restrict our attention to the case of real A, B, and C. If one of them is zero, then the situation is governed by 5.1 and 5.2. The same argument applies to the cases $l = 0$, $m = 0$, or $l = m = n$. Since we can scale the problem, we may assume $A = 1$, $B, C = \pm 1$. The assumption $l \leq m$ is without restriction.

5.4. PROPOSITION. — *The operator*

$$P(D) = \frac{\partial^n}{\partial x^l \partial w^{n-l}} + a\frac{\partial^n}{\partial y^m \partial w^{n-m}} + b\frac{\partial^n}{\partial z^n}, \quad a, b = \pm 1, \quad l, m, n \in \mathbb{N},$$
$$l \leq m \leq n, \; l \neq n,$$

is surjective on $A(\mathbb{R}^4)$ if and only if one of the following cases is fulfilled :

(1) $l = 1$, $n = 2$,

(2) $l = m = 2$, $n = 3$, $a = -1$,

(3) $l = 2$, $m = n = 4$, $a = b = -1$.

These are also exactly the cases for which $P(D)$ is surjective on $\Gamma^d(\mathbb{R}^4)$ for some (every) $d > 1$.

Proof. — Case (1) satisfies the hypothesis of 1.5(2) if $m = 2$. If $m = 1$, then, after introducing the coordinates $\xi = x + y$, $\eta = x - y$, the polynomial is independent of either ξ or η, so the result follows from 5.1 and 5.2. Case (3) is treated in 1.5(3) and case (2) in 5.3.

If $l = 1$ and $n \geq 3$, then $n - l \geq 2$ and n is not a multiple of $n - l$; so the claim follows from 3.9(2). If m is not a multiple of l, then the claim follows from 3.9(1).

For the remainder of the proof, m is a multiple of l and $l \geq 2$.

If $m = l$, then 4.3 or case (2) applies, or $l = m = 2$ and $n > 3$ and $a = -1$. If $(l, m, n, a) = (2, 2, 4, -1)$, then 2.4(2) applies. If $(l, m, a) = (2, 2, -1)$ and $n > 4$, then 3.9(2) can be used.

If $m = 2l$, then either n is not a multiple of $n - l$, and 3.9(2) applies, or $n = m = 2l$. This is either the operator of case (3) or it has been investigated in 3.7. If m is strictly greater than $2l$, then n is not a multiple of $n - l \geq m - l$. □

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Manuscrit reçu le 28 janvier 1994,
révisé le 21 juin 1994.

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