LOU VAN DEN DRIES
CHRIS MILLER

Extending Tamm's theorem

Annales de l'institut Fourier, tome 44, nº 5 (1994), p. 1367-1395
<http://www.numdam.org/item?id=AIF_1994__44_5_1367_0>

Ann. Inst. Fourier, Grenoble
44, 5 (1994), 1367–1395

EXTENDING TAMM'S THEOREM

by L. van den DRIES & C. MILLER

Introduction.

The theorem of M. Tamm [T] referred to in the title of this paper can be stated as follows:

Given a finitely subanalytic function $f : U \to \mathbb{R}$ on an open set $U \subseteq \mathbb{R}^n$, there is a natural number N such that for all open $U' \subseteq U$, if $f \restriction U'$ is C^N, then $f \restriction U'$ is analytic.

(Here and throughout this paper, "analytic" means "real analytic".)

"Finitely subanalytic" [D2] is the same as "globally subanalytic" [KR], and is a better behaved notion than "subanalytic". We give several definitions of "finitely subanalytic" below. Here we just mention that bounded subanalytic sets in $\mathbb{R}^n$ as well as their complements are finitely subanalytic. (A map $f : A \to \mathbb{R}^n$ with $A \subseteq \mathbb{R}^m$ is finitely subanalytic if its graph is a finitely subanalytic subset of $\mathbb{R}^{m+n}$.)

In this paper we extend Tamm's theorem simultaneously in two ways:

(1) We allow U and f to depend on parameters, with an N independent of the parameters.

The first author is supported in part by the National Science Foundation under Grant No.DMS 92-02833. The second author is supported in part by a Department of Education National Needs Fellowship.
Key words: o-minimal structure – Polynomially bounded structure – Gateaux derivative – Real analytic – Quasianalytic – Finitely subanalytic – Tarski-Seidenberg property – Puiseux expansion.
A.M.S. Classification: 32B20 – 32C05 – 14P15 – 26E05 – 26E10 – 03C10 – 03C65.

(2) We allow f to be definable, not just in terms of addition, multiplication, and analytic functions on sets $[-1,1]^m$ for $m \in \mathbb{N}$ — this would give us just the finitely subanalytic functions — but also in terms of the power functions $x \mapsto x^r : (0,\infty) \to \mathbb{R}$, which are not subanalytic at 0 for irrational r.

In (2) above, "definable" is a certain technical notion arising from logic; we introduce it without referring explicitly to logical concepts.

DEFINITION. — *A structure $\mathcal{S}$ on $\mathbb{R}$ consists of a collection $\mathcal{S}_n$ of subsets of $\mathbb{R}^n$, for each $n \in \mathbb{N}$, such that*

(1) *$\mathcal{S}_n$ is a boolean algebra of subsets of $\mathbb{R}^n$, in particular $\mathbb{R}^n \in \mathcal{S}_n$;*

(2) *$\mathcal{S}_n$ contains the diagonals $\{(x_1,\dots,x_n) \in \mathbb{R}^n : x_i = x_j\}$ for $1 \le i < j \le n$;*

(3) *if $A \in \mathcal{S}_n$, then $A \times \mathbb{R}$ and $\mathbb{R} \times A$ belong to $\mathcal{S}_{n+1}$;*

(4) *if $A \in \mathcal{S}_{n+1}$, then $\pi(A) \in \mathcal{S}_n$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first n coordinates.*

We say that a set $A \subseteq \mathbb{R}^n$ *belongs to $\mathcal{S}$* if $A \in \mathcal{S}_n$, and that a map $f : A \to \mathbb{R}^k$ with $A \subseteq \mathbb{R}^n$ *belongs to $\mathcal{S}$* if its graph $\Gamma(f) := \{(x, f(x)) \in \mathbb{R}^{n+k} : x \in A\}$ belongs to $\mathcal{S}$. Instead of "A belongs to $\mathcal{S}$" we also say "$\mathcal{S}$ contains A"; (similarly with maps).

Given structures $\mathcal{S} = (\mathcal{S}_n)$ and $\mathcal{S}' = (\mathcal{S}'_n)$ on $\mathbb{R}$ we put $\mathcal{S} \subseteq \mathcal{S}'$ if $\mathcal{S}_n \subseteq \mathcal{S}'_n$ for all $n \in \mathbb{N}$; this defines a partial order on the set of all structures on $\mathbb{R}$. Given sets $A_i \subseteq \mathbb{R}^{m(i)}$ (i in some index set I), and functions $f_j : B_j \to \mathbb{R}$ with $B_j \subseteq \mathbb{R}^{n(j)}$ (j in some index set J), there is clearly a smallest structure on $\mathbb{R}$ containing all sets A_i and all functions f_j; we call this *the structure on $\mathbb{R}$ generated by the A_i's and the f_j's.* (A function $f : \mathbb{R}^0 = \{0\} \to \mathbb{R}$ is identified with the corresponding real constant $f(0)$.) A set $A \subseteq \mathbb{R}^n$ is said to be *definable in terms of the A_i's and the f_j's*, or to be *definable in* $(\mathbb{R}, (A_i)_{i\in I}, (f_j)_{j\in J})$, if A belongs to the structure on $\mathbb{R}$ generated by the A_i's and the f_j's; (similarly with maps). For example, by Tarski-Seidenberg, a set $X \subseteq \mathbb{R}^n$ is definable in $(\mathbb{R}, +, \cdot, (r)_{r\in\mathbb{R}})$ if and only if X is semialgebraic.

These notions all make sense with $\mathbb{R}$ replaced by any set. However, of special interest for analysis and topology are the "o-minimal" structures on $\mathbb{R}$, which are the simplest structures on $\mathbb{R}$ compatible with the ordering of the real line.

DEFINITION. — *A structure $\mathcal{S}$ on $\mathbb{R}$ is* o-minimal *("order-minimal") if*

(S1) $\{(x,y) : x < y\} \in \mathcal{S}_2$, *and* $\{a\} \in \mathcal{S}_1$ *for each* $a \in \mathbb{R}$;

(S2) *each set in* $\mathcal{S}_1$ *is a finite union of intervals* (a,b), $-\infty \leq a < b \leq +\infty$, *and points* $\{a\}$.

(We think of (S2) as a minimality requirement, since each structure on $\mathbb{R}$ satisfying (S1) must contain at least all finite unions of intervals and points.) If an o-minimal structure $\mathcal{S}$ is generated by sets $A_i \subseteq \mathbb{R}^{m(i)}$ (i in some index set I) and functions $f_j : B_j \to \mathbb{R}$ with $B_j \subseteq \mathbb{R}^{n(j)}$ (j in some index set J), then we also say that $(\mathbb{R}, (A_i)_{i\in I}, (f_j)_{j\in J})$ is o-minimal.

Each subset of $\mathbb{R}^n$ belonging to an o-minimal structure $\mathcal{S}$ on $\mathbb{R}$ has only finitely many connected components, and each component also belongs to $\mathcal{S}$. The class of semialgebraic sets is an o-minimal structure on $\mathbb{R}$, as is the larger class of finitely subanalytic sets: $B \subseteq \mathbb{R}^n$ is finitely subanalytic if and only if $B = f(A)$ for some bounded semianalytic set $A \subseteq \mathbb{R}^m$ and some semialgebraic map $f : \mathbb{R}^m \to \mathbb{R}^n$. (A map from a subset of $\mathbb{R}^m$ into $\mathbb{R}^n$ is semialgebraic if its graph is a semialgebraic subset of $\mathbb{R}^{m+n}$; unlike some authors, we do not require semialgebraic maps to be continuous.)

DEFINITION. — A structure on $(\mathbb{R}, +, \cdot)$ *is a structure on* $\mathbb{R}$ *containing the graphs of both addition and multiplication.*

Let $\mathcal{S}$ be a structure on $(\mathbb{R}, +, \cdot)$. Then the usual order relation $<$ necessarily belongs to $\mathcal{S}$; the set $\{(x,y) \in \mathbb{R}^2 : x \leq y\}$ is the projection of

$$\{(x,y,z) \in \mathbb{R}^3 : y = x + z^2\},$$

and $\{(x,y) \in \mathbb{R}^2 : x < y\} = \{(x,y) \in \mathbb{R}^2 : x \leq y\} - \{(x,y) \in \mathbb{R}^2 : x = y\}$. Given a set $X \in \mathcal{S}_n$, its closure and interior are also in $\mathcal{S}_n$. Given a function $f : U \to \mathbb{R}$ belonging to $\mathcal{S}$ with U open in $\mathbb{R}^n$, the set of points in U where f is differentiable belongs to $\mathcal{S}$, and if f is differentiable on U, then each partial derivative also belongs to $\mathcal{S}$. Throughout this paper, we use many such basic facts (familiar to logicians); proofs are left as exercises.

An o-minimal structure on $(\mathbb{R}, +, \cdot)$ shares many of the nice properties of the class of semialgebraic sets; the sets in such a structure can be triangulated by means of homeomorphisms in the structure, and Hardt's semialgebraic triviality theorem [H] extends to such o-minimal structures on $\mathbb{R}$. The theory of o-minimal structures is a wide-ranging generalization of semialgebraic and subanalytic geometry; one can view the subject as a realization of Grothendieck's idea of *topologie modérée*, (outlined in the unpublished notes *Esquisse d'un programme*, 1984). The first papers on o-minimality are [D1], [PS] and [KPS]; for an extensive and systematic account, see [D3].

We now return to the subject of this paper.

Notation. — Given a subfield K of $\mathbb{R}$, $\mathbb{R}_{\mathrm{an}}^K$ denotes the set $\mathbb{R}$ equipped with

(1) addition and multiplication (functions on $\mathbb{R}^2$),

(2) all analytic functions $f : [-1,1]^m \to \mathbb{R}$, for all $m \in \mathbb{N}$,

(3) the power functions $x \mapsto x^r : (0,\infty) \to \mathbb{R}$ for all $r \in K$.

Convention. — In this paper we say that $f : A \to B$ with $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$ is analytic if f is the restriction to A of an analytic map $g : U \to \mathbb{R}^n$ with U an open neighborhood of A in $\mathbb{R}^m$ and $g(A) \subseteq B$. We also say that such a map f is analytic at a point $a \in A$ if there is an open set $U \subseteq \mathbb{R}^n$ with $a \in U \subseteq A$ such that $f \restriction U$ is analytic; (note then that $a \in \mathrm{int}(A)$). We also work similarly with "analytic" replaced by "C^p", $1 \leq p \leq \infty$.

The sets definable in $\mathbb{R}_{\mathrm{an}}^K$ form an o-minimal structure on $(\mathbb{R},+,\cdot)$, and some basic properties of this structure are established in [M2].

For $K = \mathbb{Q}$ the sets definable in $\mathbb{R}_{\mathrm{an}}^{\mathbb{Q}}$ are exactly the finitely subanalytic sets (see [DD], [D2]), and in fact the power functions x^q for $q \in \mathbb{Q}$ are superfluous here, since they are definable in terms of just multiplication.

We can now give a precise formulation of our extension of Tamm's theorem:

MAIN THEOREM. — *Let $f : A \to \mathbb{R}$ be definable in $\mathbb{R}_{\mathrm{an}}^K$, $A \subseteq \mathbb{R}^{m+n}$. Then there exists $N \in \mathbb{N}$ such that for all $x \in \mathbb{R}^m$ and all open sets $U \subseteq \mathbb{R}^n$ with $U \subseteq A_x := \{y \in \mathbb{R}^n : (x,y) \in A\}$, if $f(x,-)$ is C^N on U, then $f(x,-)$ is analytic on U.*

(We let $f(x,-)$ denote the function $y \mapsto f(x,y) : A_x \to \mathbb{R}$.)

COROLLARY. — *Let $A \subseteq \mathbb{R}^n$ be definable in $\mathbb{R}_{\mathrm{an}}^K$. Then* $\mathrm{Sing}(A)$, *the set of singular points of A, is definable in $\mathbb{R}_{\mathrm{an}}^K$.*

(See §5 for a definition of $\mathrm{Sing}(A)$).

We cannot follow here Tamm's original proof [T], nor the proof by Bierstone and Milman [BM], since these depend on properties of subanalytic sets not shared by all sets definable in $\mathbb{R}_{\text{an}}^K$ if $K \neq \mathbb{Q}$. Instead we adapt (and simplify in some places) the proof of Tamm's theorem given by Kurdyka [K]. One important tool used in [K] is Pawłucki's "Puiseux expansion with parameters for subanalytic functions" from [P]. Much of the technical work in this paper goes into establishing the Expansion Theorem of §4, which for $K = \mathbb{Q}$ is a somewhat stronger version of Pawłucki's result.

Here then is a brief outline of the contents of this paper. In §1 we review some basic properties of o-minimal structures needed for our purpose. In §2, we discuss Gateaux differentiability and its relation to analyticity and o-minimality. In §3, some results about $\mathbb{R}_{\text{an}}^K$ are given. The statement and proof of the aforementioned Expansion Theorem constitutes §4. Finally, in §5, we prove the Main Theorem and some corollaries.

1. o-minimal structures on $\mathbb{R}$.

Throughout this section, $\mathcal{S}$ denotes some fixed, but arbitrary, o-minimal structure on $\mathbb{R}$. "Definable" means "belonging to $\mathcal{S}$".

1.1. MONOTONICITY THEOREM. — *Let $f : \mathbb{R} \to \mathbb{R}$ be definable. Then there exist (extended) real numbers $-\infty = a_0 < a_1 < \dots < a_N < a_{N+1} = +\infty$ such that $f \upharpoonright (a_n, a_{n+1})$ is either constant, or strictly monotone and continuous, for $n = 0, \dots, N$.*

(See [D1] for a proof.)

Remarks.

(1) The statement holds with "differentiable" instead of "continuous" if $\mathcal{S}$ is an o-minimal structure on $(\mathbb{R}, +, \cdot)$; (see [D1]). Consequently, the ring of germs at $+\infty$ of all definable functions $f : \mathbb{R} \to \mathbb{R}$ is a Hardy field. The converse is also true: a structure $\mathcal{R}$ on $(\mathbb{R}, +, \cdot)$ containing all singletons $\{r\}$ for $r \in \mathbb{R}$ is o-minimal if every function $f : \mathbb{R} \to \mathbb{R}$ belonging to $\mathcal{R}$ is of constant sign (-1, 0 or 1) for all sufficiently large (depending on f) positive real arguments; (see [DMM]).

(2) For every presently-known o-minimal structure on $(\mathbb{R}, +, \cdot)$, the statement holds true with "analytic" in place of "continuous".

Cells and cell decomposition.

We define the *cells in* $\mathbb{R}^n$ as certain kinds of definable subsets of $\mathbb{R}^n$; the definition is by induction on n:

(1) The cells in $\mathbb{R}$ ($= \mathbb{R}^1$) are just the points $\{r\}$ and the open intervals (a, b), $-\infty \leq a < b \leq +\infty$;

(2) Let $C \subseteq \mathbb{R}^n$ be a cell and let $f, g : C \to \mathbb{R}$ be definable continuous functions such that $f < g$ on C, then $(f, g) := \{(x, r) \in C \times \mathbb{R} : f(x) < r < g(x)\}$ is a cell in $\mathbb{R}^{n+1}$; also, given definable continuous $f : C \to \mathbb{R}$ on a cell C in $\mathbb{R}^n$, the graph $\Gamma(f) \subseteq C \times \mathbb{R}$ and the sets $\{(x, r) \in C \times \mathbb{R} : r < f(x)\}$, $\{(x, r) \in C \times \mathbb{R} : f(x) < r\}$ and $C \times \mathbb{R}$ are cells in $\mathbb{R}^{n+1}$.

(We also consider $\mathbb{R}^0 = \{0\}$ as a cell in $\mathbb{R}^0$; so (2) even holds for $n = 0$.)

The *dimension* of a cell C in $\mathbb{R}^n$, denoted $\dim(C)$, is defined by induction on n:

(1) For $n = 1$, put $\dim(C) := 0$ if C is a singleton, and put $\dim(C) := 1$ if C is an open interval.

(2) Let C be a cell in $\mathbb{R}^{n+1}$. Then $\pi(C)$ is a cell in $\mathbb{R}^n$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}$ is the projection on the first n coordinates. Put $\dim(C) := \dim(\pi(C))$ if C is of the form $\Gamma(f)$ for some definable continuous $f : \pi(C) \to \mathbb{R}$, and put $\dim(C) := 1 + \dim(\pi(C))$ otherwise.

(We also put $\dim(\mathbb{R}^0) := 0$.)

Note. — Clearly, if C is a cell in $\mathbb{R}^n$ and C is open, then $\dim(C) = n$.

1.2. Given $i = (i_1, \ldots, i_m)$ with $1 \leq i_1 < \cdots < i_m \leq n$, define $\pi_i : \mathbb{R}^n \to \mathbb{R}^m$ by $\pi_i(x_1, \ldots, x_n) := (x_{i_1}, \cdots, x_{i_m})$. It is easy to check that if C is a cell in $\mathbb{R}^n$ of dimension m, then there is some $i = (i_1, \ldots, i_m)$ as above such that π_i maps C homeomorphically onto an open cell in $\mathbb{R}^m$. Note also that $\pi_i \restriction C$ is definable.

A *decomposition* of $\mathbb{R}^n$ is a special kind of partition of $\mathbb{R}^n$ into finitely many cells. Definition is by induction on n:

(1) A decomposition of $\mathbb{R}^1$ ($= \mathbb{R}$) is a collection of intervals and points of the form

$$\{(-\infty, a_1), (a_1, a_2), \ldots, (a_k, +\infty), \{a_1\}, \ldots, \{a_k\}\},$$

with $a_1 < \ldots < a_k$ real numbers. (For $k = 0$ this is just $\{(-\infty, \infty)\}$.)

(2) A decomposition of $\mathbb{R}^{n+1}$ is a finite partition of $\mathbb{R}^{n+1}$ into cells A such that the set of projections $\pi(A)$ is a decomposition of $\mathbb{R}^n$, where

$\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first n coordinates. (Note that different cells can have the same image under π.)

In a similar manner, one can define C^p cells and C^p decompositions, by requiring that the functions occurring in part (2) of the definition of cells be C^p, for p a positive integer or $p = \infty$; similarly for analytic cells and analytic decompositions. Each C^p cell in $\mathbb{R}^n$ is a connected C^p submanifold of $\mathbb{R}^n$, C^p diffeomorphic via some coordinate projection $\pi_i \restriction C$ to an open C^p cell in $\mathbb{R}^m$, for some $m \leq n$; similarly with "C^p" replaced by "analytic".

Note. — Cells and decompositions are always relative to some particular structure; (the structure $\mathcal{S}$ throughout this section).

The *projection* $\pi\mathcal{C}$ of a decomposition $\mathcal{C}$ of $\mathbb{R}^{m+n}$ onto $\mathbb{R}^m$ is the collection $\{\pi(C) : C \in \mathcal{C}\}$, where $\pi : \mathbb{R}^{m+n} \to \mathbb{R}$ is the projection map onto the first m coordinates. (Note that $\pi\mathcal{C}$ is then a decomposition of $\mathbb{R}^m$.) A decomposition of $\mathbb{R}^n$ is said to *partition* a set $A \subseteq \mathbb{R}^n$ if A is a union of cells in the decomposition.

THEOREM. — *The structure* $\mathcal{S}$ admits cell decomposition; *i.e.*,

(I_n) *given definable sets* $A_1, \dots, A_k \subseteq \mathbb{R}^n$, *there is a decomposition of* $\mathbb{R}^n$ *into cells partitioning* $A_1, \dots, A_k$,

(II_n) *for every definable function* $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}^n$, *there is a decomposition of* $\mathbb{R}^n$ *into cells partitioning* A *such that each restriction* $f \restriction C : C \to \mathbb{R}$ *is continuous for each cell* $C \subseteq A$ *in the decomposition.*

(See [PS] and [KPS].)

Remark. — If $\mathcal{S}$ is moreover a structure on $(\mathbb{R}, +, \cdot)$, then the statement holds with "C^N cells" and "C^N" in place of "cells" and "continuous", respectively, for every *fixed* positive integer N; i.e., $\mathcal{S}$ admits C^N cell decomposition. It is an open question at present as to whether or not every o-minimal structure on $(\mathbb{R}, +, \cdot)$ admits C^∞ cell decomposition, or even analytic cell decomposition.

Orders of growth of definable functions.

A structure $\mathcal{R}$ on $\mathbb{R}$ is *exponential* if the exponential function e^x belongs to $\mathcal{R}$; $\mathcal{R}$ is *polynomially bounded* if for every function $f : \mathbb{R} \to \mathbb{R}$ belonging to $\mathcal{R}$, there exists some $N \in \mathbb{N}$ such that ultimately $|f(x)| \leq x^N$. (*Ultimately* abbreviates "for all sufficiently large positive arguments".) If $\mathcal{R}$ is generated by sets $A_i \subseteq \mathbb{R}^{m(i)}$ (i in some index set I) and functions

$f_j : B_j \to \mathbb{R}$ with $B_j \subseteq \mathbb{R}^{n(j)}$ (j in some index set J), then we also say that $(\mathbb{R}, (A_i)_{i\in I}, (f_j)_{j\in J})$ is exponential if $\mathcal{R}$ is exponential; similarly for polynomially bounded.

1.3. THEOREM (Growth Dichotomy). — *Let $\mathcal{R}$ be an o-minimal structure on $(\mathbb{R}, +, \cdot)$. Then either $\mathcal{R}$ is exponential, or $\mathcal{R}$ is polynomially bounded. If $\mathcal{R}$ is polynomially bounded, then for every $f : \mathbb{R} \to \mathbb{R}$ belonging to $\mathcal{R}$, either f is ultimately identically equal to 0, or there exist nonzero $c \in \mathbb{R}$ and a real power function x^r belonging to $\mathcal{R}$ such that $f(x) = cx^r + o(x^r)$ as $x \to +\infty$.*

(See [M1] for the proof.)

The first known example of an exponential o-minimal structure on $(\mathbb{R}, +, \cdot)$ is due to Wilkie [W], who established that the structure on $\mathbb{R}$ generated by addition, multiplication, all real constants, and exponentiation is o-minimal. The structure on $\mathbb{R}$ generated by addition, multiplication, exponentiation and all analytic functions $f : [-1, 1]^m \to \mathbb{R}$ for all $m \in \mathbb{N}$, is o-minimal and admits analytic cell decomposition; (see [DM] and [DMM]).

Polynomially bounded o-minimal structures on $(\mathbb{R}, +, \cdot)$.

We will be particularly concerned in this paper with the polynomially bounded case. *For the remainder of this section, we assume that $\mathcal{S}$ is a polynomially bounded o-minimal structure on $(\mathbb{R}, +, \cdot)$.*

The following variant of a result from [M2] is crucial to later developments:

1.4. THEOREM (Piecewise Uniform Asymptotics). — *Let $f : A \times \mathbb{R} \to \mathbb{R}$ be definable, $A \subseteq \mathbb{R}^m$. Then there exist $r_1, \ldots, r_\ell \in \mathbb{R}$ such that for all $x \in A$, either $t \mapsto f(x,t) : \mathbb{R} \to \mathbb{R}$ vanishes identically for all sufficiently small (depending on x) positive t, or $f(x,t) = ct^{r_i} + o(t^{r_i})$ as $t \to 0^+$ for some $i \in \{1, \ldots, \ell\}$ and $c = c(x) \in \mathbb{R}$, $c \neq 0$.*

Remark. — A "definable" version of the Łojasiewicz inequality follows from this fact; (see [M2]).

Let U be an open subset of $\mathbb{R}^n$, $a \in U$, and let $f : U \to \mathbb{R}$ be given. If f is C^N at a and all partial derivatives of f of order less than or equal to N vanish at a, then f is said to be *N-flat* at a. If f is N-flat at a for all $N \in \mathbb{N}$ then f is said to be *flat* at a.

1.5. THEOREM (Uniform Bounds on Orders of Vanishing). — *Let $f : A \to \mathbb{R}$ be definable, $A \subseteq \mathbb{R}^{m+n}$. Then there exists $N \in \mathbb{N}$ such that for all $(x, y) \in A$, if $y \in \operatorname{int}(A_x)$ and $f(x, -)$ is N-flat at y, then $f(x, z) = 0$ for all $z \in A_x$ sufficiently close to y.*

(See [M3] for the proof.)

In the special case that $m = 0$ and A is open, we have that for all $y \in A$, if f is flat at y, then f vanishes identically in a neighborhood of y. It follows easily then that the set of all definable C^∞ functions $f : U \to \mathbb{R}$, for a fixed connected definable open set $U \subseteq \mathbb{R}^n$, is an integral domain; we denote it by $C^\infty_{\mathrm{df}}(U)$. Furthermore, $C^\infty_{\mathrm{df}}(U)$ is a *quasianalytic class*; i.e., if $f \in C^\infty_{\mathrm{df}}(U)$ and f is flat at some $x_0 \in U$, then $f = 0$.

The descending chain condition on zero sets.

Given $f : A \to \mathbb{R}^n$, $A \subseteq \mathbb{R}^m$, put $Z(f) := \{a \in A : f(a) = 0\}$. Note that if f is definable, then so is $Z(f)$.

1.6. PROPOSITION. — *Assume that $\mathcal{S}$ admits C^∞ cell decomposition. Then given a family $(f_i : A \to \mathbb{R})_{i \in \mathbb{N}}$ of definable C^∞ functions, $A \subseteq \mathbb{R}^n$, there exists $M \in \mathbb{N}$ such that*

$$\bigcap_{i \in \mathbb{N}} Z(f_i) = \bigcap_{i \leq M} Z(f_i).$$

Proof. — To avoid trivialities, let us suppose that $\emptyset \neq Z(f_0) \neq A$. By taking a C^∞ decomposition of $\mathbb{R}^n$ partitioning A, we may assume that A is a C^∞ cell; in particular, A is connected. We proceed now by induction on $\dim(A)$ and n.

The result is trivial if $\dim(A) = 0$. So suppose that $\dim(A) = d > 0$, and that the result holds for all lower values of d and n.

If A is nonopen, then A is C^∞ diffeomorphic via some coordinate projection $\pi = \pi_i \restriction A$ to an open cell $\pi(A) \subseteq \mathbb{R}^m$ with $m < n$; (see 1.2). By the inductive assumption, we have

$$\bigcap_{i \in \mathbb{N}} Z(f_i \circ \pi^{-1}) = \bigcap_{i \leq M} Z(f_i \circ \pi^{-1})$$

for some $M \in \mathbb{N}$; thus,

$$\bigcap_{i \in \mathbb{N}} Z(f_i) = \bigcap_{i \leq M} Z(f_i),$$

as desired.

Now suppose that A is open. Take a partition $\mathcal{P}$ of $Z(f_0)$ into finitely many C^∞ cells B; note that $\dim(B) < d$, since otherwise f_0 would vanish on a nonempty open subset of A, hence $f_0 = 0$ (by quasianalyticity). By the inductive assumption, for each $B \in \mathcal{P}$ there exists $M(B) \in \mathbb{N}$ such that

$$\bigcap_{i\in\mathbb{N}} Z(f_i \restriction B) = \bigcap_{i\leq M(B)} Z(f_i \restriction B).$$

Hence,

$$\bigcap_{i\in\mathbb{N}} Z(f_i) = \bigcap_{i\leq M} Z(f_i),$$

where $M := \max\{M(B) : B \in \mathcal{P}\}$. □

Remark. — The assumption that $\mathcal{S}$ is polynomially bounded and admits C^∞ cell decomposition may be removed if one assumes that A is a definable analytic submanifold of $\mathbb{R}^n$ and that each f_i is analytic; (see Tougeron [To]).

2. Gateaux differentiability, analyticity and o-minimality.

In this section, we give a characterization of analyticity (at a point) for real functions that is a slight variant of a result of Bochnak and Siciak [BS].

First, we reformulate a result of Abhyankar and Moh on power series:

2.1. PROPOSITION. — *Let $F(X_1,\dots,X_n) \in \mathbb{R}[\![X_1,\dots,X_n]\!]$ and suppose that for all $x \in \mathbb{R}^n$ the series $F(x_1T,\dots,x_nT) \in \mathbb{R}[\![T]\!]$ is convergent. Then $F(X_1,\dots,X_n)$ is convergent.*

Proof. — We proceed by induction on n; the case $n = 1$ is trivial. Assume the result for n. Let $F(X_1,\dots,X_{n+1}) \in \mathbb{R}[\![X_1,\dots,X_{n+1}]\!]$, and suppose that for all $x_1,\dots,x_{n+1} \in \mathbb{R}$, the series $F(x_1T,\dots,x_{n+1}T) \in \mathbb{R}[\![T]\!]$ is convergent. Let $r \in \mathbb{R}$, and $x \in \mathbb{R}^n$. Then the series $F(x_1T,\dots,x_nT, rx_nT)$ is convergent. By the inductive assumption, the series $F(X_1,\dots,X_n, rX_n)$ is convergent. It follows then from [AM] that $F(X_1,\dots,X_n)$ is convergent. □

DEFINITION. — *Let $f : U \to \mathbb{R}$ be a function, U open in $\mathbb{R}^n$, $x \in U$. Let k be a positive integer and suppose that for each $y \in \mathbb{R}^n$, the (partial) function $t \mapsto f(x + ty)$ is k-times differentiable at $t = 0$. If the map*

$$y \mapsto \frac{d^k f(x+ty)}{dt^k}(0) : \mathbb{R}^n \to \mathbb{R}$$

is given by a homogeneous polynomial in y of degree k, then f is k-times Gateaux differentiable *at x, or G^k at x. If f is G^k at x for all $k > 0$, then f is G^∞ at x.*

For f and x as in the preceding definition, if f is C^k at x, then f is G^k at x. The converse fails; indeed, f can be G^∞ at a point x, and yet not even be continuous at x. (For example, consider the characteristic function of $\{(x, x^2) : x > 0\}$, which is G^∞ at $(0, 0)$.)

Notation. — For $x \in \mathbb{R}^n$, $||x||$ denotes the usual euclidean norm of x.

2.2. PROPOSITION. — *Let $U \subseteq \mathbb{R}^n$ be open, let $x \in U$. Then $f : U \to \mathbb{R}$ is analytic at x if and only if f is G^∞ at x and there exists $\varepsilon > 0$ such that for all $y \in \mathbb{R}^n$ with $||y|| \leq 1$, the function $t \mapsto f(x + ty)$ is defined and analytic on $(-\varepsilon, \varepsilon)$.*

Proof. — The forward implication is clear. For the other direction, it suffices to show the result for U a neighborhood of 0, with $x = 0$ and $f(0) = 0$.

Since f is G^∞ at 0, for all $k > 0$ the function $\delta_k : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\delta_k(y) := \frac{d^k f(ty)}{dt^k}(0)$$

is given by a homogeneous real polynomial $\delta_k(Y_1, \ldots, Y_n)$ of degree k. Put

$$F(Y_1, \ldots, Y_n) := \sum_{k=1}^{\infty} (1/k!)\delta_k(Y_1, \ldots, Y_n) \in \mathbb{R}[\![Y_1, \ldots, Y_n]\!];$$

(the "Taylor series" of f at 0).

Let $y \in \mathbb{R}^n$, $||y|| \leq 1$. Then, for the formal series F, we have

$$F(y_1T, \ldots, y_nT) = \sum_{k=1}^{\infty} (1/k!)\delta_k(y_1T, \ldots, y_nT) = \sum_{k=1}^{\infty} (1/k!)\delta_k(y)T^k \in \mathbb{R}[\![T]\!].$$

Now there exists $\varepsilon > 0$ such that $f(ty)$ is defined and

$$f(ty) = \sum_{k=1}^{\infty} (1/k!)\delta_k(y)t^k$$

for all $|t| < \varepsilon$. Thus, $F(y_1T, \dots, y_nT)$ is convergent. By the previous proposition, $F(Y_1, \dots, Y_n)$ is convergent, say on some open neighborhood $V \subseteq (-\varepsilon, \varepsilon)^n$ of $0 \in \mathbb{R}^n$. Let F also denote the analytic function on V thus obtained. Then for every line $L \subseteq \mathbb{R}^n$ through the origin, we have $f \restriction (V \cap L) = F \restriction (V \cap L)$. Hence, $f \restriction V = F \restriction V$, and f is analytic at 0. □

We will need the following fact; (the proof is left to the reader).

2.3. Let $n \in \mathbb{N}$. Then for all $k \in \mathbb{N}$ there exist points $p(k,1), \dots, p(k, \mu(k)) \in \mathbb{R}^n$ and linear functions $a_1, \dots, a_{\mu(k)} : \mathbb{R}^{\mu(k)} \to \mathbb{R}$ such that for all $x \in \mathbb{R}^{\mu(k)}$,

$$P_k(x, Y) := \sum_{j=1}^{\mu(k)} a_j(x) M_j(Y) \in \mathbb{R}[Y]$$

is the unique homogeneous real polynomial $P(Y)$ of degree k with $P(p(k,i)) = x_i$ for $i = 1, \dots, \mu(k)$, where $\mu(k)$ is the dimension of the vector space of homogeneous polynomials in $Y := (Y_1, \dots, Y_n)$ of degree k over $\mathbb{R}$ and $M_1(Y), \dots, M_{\mu(k)}(Y)$ are the monomials of degree k in Y.

2.4. Lemma. — *Let $\mathcal{S}$ be a structure on $(\mathbb{R}, +, \cdot)$, and let $f : A \to \mathbb{R}$ belong to $\mathcal{S}$, $A \subseteq \mathbb{R}^{m+n}$, such that A_x is open in $\mathbb{R}^n$ for all $x \in \mathbb{R}^m$. Then for all $k > 0$ there exists $w_k : A \times \mathbb{R}^n \to \mathbb{R}$ belonging to $\mathcal{S}$ such that for all $(x, y) \in A$, $f(x, -)$ is G^k at y if and only if $w_k(x, y, z) = 0$ for all $z \in \mathbb{R}^n$.*

Proof. — For positive integers k define $\phi_k : A \times \mathbb{R}^n \to \mathbb{R}$ as follows: if $(x, y) \in A$ and $t \mapsto f(x, y+tz)$ is k-times differentiable at 0 for all $z \in \mathbb{R}^n$, then put

$$\phi_k(x, y, z) := \frac{d^k f(x, y + tz)}{dt^k}(0);$$

otherwise, put $\phi_k(x, y, z) := 1$. Note that ϕ_k belongs to $\mathcal{S}$.

For each $k > 0$, choose points $p(k,1), \dots, p(k, \mu(k)) \in \mathbb{R}^n$ as in 2.3, and define $v_k : A \times \mathbb{R}^n \to \mathbb{R}$ by

$v_k(x, y, z) := P_k(\phi_k(x, y, p(k,1)), \dots, \phi_k(x, y, p(k, \mu(k))), z)$, ($P_k$ as in 2.3).

Define $w_k : A \times \mathbb{R}^n \to \mathbb{R}$ by $w_k := v_k - \phi_k$. Then w_k belongs to $\mathcal{S}$, and for all $(x, y) \in A$, $f(x, -)$ is G^k at y if and only if $w_k(x, y, z) = 0$ for all $z \in \mathbb{R}^n$. □

2.5. Proposition. — *Keep all assumptions and notàtion as in the preceding lemma and its proof. Assume in addition that*

(1) $\mathcal{S}$ *is o-minimal, polynomially bounded and admits* C^∞ *cell decomposition;*

(2) $A \times \mathbb{R}^n$ *is a union of sets* $B_1, \ldots, B_\ell$, *each belonging to* $\mathcal{S}$, *such that* $\phi_k \restriction B_i$ *is* C^∞ *for all* $k \in \mathbb{N}$ *and* $i \in \{1, \ldots, \ell\}$.

Then there exists $N \in \mathbb{N}$ *such that for all* $(x, y) \in A$, *if* $f(x, -)$ *is* G^N *at* y, *then* $f(x, -)$ *is* G^∞ *at* y.

Proof. — Examining the proof above, we see that then $w_k \restriction B_i$ is C^∞ for all $k \geq 1$ and $i \in \{1, \ldots, \ell\}$. By 1.6, there exists $N \in \mathbb{N}$ such that

$$\bigcap_{k=1}^{\infty} Z(w_k) = \bigcap_{k=1}^{N} Z(w_k).$$

Hence, for all $(x, y) \in A$, $f(x, -)$ is G^∞ at y if and only if $w_i(x, y, z) = 0$ for all $i \leq N$ and $z \in \mathbb{R}^n$; i.e., if and only if $f(x, -)$ is G^N at y. □

3. Some results on $\mathbb{R}_{\text{an}}^K$.

Throughout the remainder of this paper, K denotes some fixed subfield of $\mathbb{R}$; *"definable" means "definable in* $\mathbb{R}_{\text{an}}^K$*" unless stated otherwise.*

We will state here some facts (established in [M2]) about $\mathbb{R}_{\text{an}}^K$, and prove a lemma on definable functions that we will need in the next section.

Definition. — *Let* $G(X_1, \ldots, X_m)$ *be a real power series converging on some open neighborhood* U *of* $[-1, 1]^m$ *to an analytic function* $g : U \to \mathbb{R}$. *Then* $\tilde{g} : \mathbb{R}^m \to \mathbb{R}$ *given by*

$$\tilde{g}(x) := \begin{cases} g(x), & \text{if } x \in [-1,1]^m \\ 0, & \text{otherwise} \end{cases}$$

is a restricted analytic function. *(For* $m = 0$, $\tilde{g}$ *is just the corresponding real constant.)*

Note that $\tilde{g}$ is finitely subanalytic, hence definable.

Definition. — *The* $\mathbb{R}_{\text{an}}^K$-functions on $\mathbb{R}^n$ *are defined inductively:*

(1) *The projection functions* $x \mapsto x_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, \ldots, n)$ *are* $\mathbb{R}_{\text{an}}^K$*-functions on* $\mathbb{R}^n$.

(2) *If* $f : \mathbb{R}^n \to \mathbb{R}$ *is an* $\mathbb{R}_{\text{an}}^K$*-function, then* $-f$ *is an* $\mathbb{R}_{\text{an}}^K$*-function on* $\mathbb{R}^n$.

(3) *If* $f, g : \mathbb{R}^n \to \mathbb{R}$ *are* $\mathbb{R}_{\text{an}}^K$*-functions, then both* $f + g$ *and* fg *are* $\mathbb{R}_{\text{an}}^K$*-functions on* $\mathbb{R}^n$*.*

(4) *If* $f : \mathbb{R}^n \to \mathbb{R}$ *is an* $\mathbb{R}_{\text{an}}^K$*-function on* $\mathbb{R}^n$*, then for each* $r \in K$*, the function*

$$x \mapsto \begin{cases} f(x)^r, & \text{if } f(x) > 0 \\ 0, & \text{otherwise} \end{cases}$$

is an $\mathbb{R}_{\text{an}}^K$*-function on* $\mathbb{R}^n$*.*

(5) *If* $f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ *are* $\mathbb{R}_{\text{an}}^K$*-functions on* $\mathbb{R}^n$ *and* $\tilde{g} : \mathbb{R}^m \to \mathbb{R}$ *is a restricted analytic function, then the composition* $\tilde{g}(f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}$ *is an* $\mathbb{R}_{\text{an}}^K$*-function on* $\mathbb{R}^n$*.*

Note that $\mathbb{R}_{\text{an}}^K$-functions are definable.

3.1. FACTS.

(1) $\mathbb{R}_{\text{an}}^K$ *is o-minimal, polynomially bounded, and admits analytic cell decomposition.*

(2) *Every definable set in* $\mathbb{R}^n$ *is a finite union of (definable) sets of the form*

$$\{x \in \mathbb{R}^n : f(x) = 0, g_1(x) < 0, \ldots, g_\ell(x) < 0\},$$

where $f, g_1, \ldots, g_\ell$ *are* $\mathbb{R}_{\text{an}}^K$*-functions on* $\mathbb{R}^n$*.*

(3) *Given a definable function* $f : A \to \mathbb{R}$*,* $A \subseteq \mathbb{R}^n$*, there are* $\mathbb{R}_{\text{an}}^K$*-functions* $f_1, \ldots, f_\ell$ *on* $\mathbb{R}^n$ *such that for all* $x \in \mathbb{R}^n$ *there exists* $i \in \{1, \ldots, \ell\}$ *with* $f(x) = f_i(x)$*; (i.e.,* f *is given piecewise by* $\mathbb{R}_{\text{an}}^K$*-functions).*

(4) *For every definable function* $f : (0, \varepsilon) \to \mathbb{R}$ *with* $f(t) \neq 0$ *for all* $t \in (0, \varepsilon)$*, there exist a convergent real power series* $F(Y_1, \ldots, Y_d)$ *with* $F(0) \neq 0$ *and* $r_0, r_1, \ldots, r_d \in K$ *with* $r_1, \ldots, r_d > 0$ *such that* $f(t) = t^{r_0} F(t^{r_1}, \ldots, t^{r_d})$ *for all sufficiently small positive* t*.*

(These facts were previously known for the case $K = \mathbb{Q}$; see [DD], [D2], [DM] and [DMM].)

Remark. — Item (2) above expresses a kind of Tarski-Seidenberg property for $\mathbb{R}_{\text{an}}^K$, and presents definable sets in a form similar to semialgebraic sets.

3.2. LEMMA. — *Let* $f : A \times (0, 1) \to \mathbb{R}$ *be definable,* $A \subseteq \mathbb{R}^m$ *(*$m > 0$*). Then* A *is a disjoint union of definable sets* $A_1, \ldots, A_M$*, and there exist definable analytic functions* $h_i : A_i \to (0, 1]$*, (*$i = 1, \ldots, M$*) and* *(not necessarily distinct)* $\mathbb{R}_{\mathrm{an}}^{K}$*-functions* $f_1, \ldots, f_M : \mathbb{R}^{m+1} \to \mathbb{R}$ *such that* $f \upharpoonright (0, h_i)$ *is analytic and* $f \upharpoonright (0, h_i) = f_i \upharpoonright (0, h_i)$ *for* $i = 1, \ldots, M$.

(Here, $(0, h_i) := \{(x, t) : x \in A_i \text{ and } 0 < t < h_i(x)\}$ for $i = 1, \ldots, M$.)

Proof. — By 3.1(3), there exist $\mathbb{R}_{\mathrm{an}}^{K}$-functions $f_1, \ldots, f_\ell : \mathbb{R}^{m+1} \to \mathbb{R}$ such that for all $(x, t) \in A \times (0, 1)$, there is an $i \in \{1, \ldots, \ell\}$ with $f(x, t) = f_i(x, t)$. Then for $i = 1, \ldots, \ell$ the sets

$$B_i := \{(x, t) \in A \times (0, 1) : f(x, t) = f_i(x, t)\}$$

are definable, and $A \times (0, 1) = B_1 \cup \ldots \cup B_\ell$. By 3.1(1), there exists a decomposition $\mathcal{C}$ of $\mathbb{R}^{m+1}$ into (definable) analytic cells partitioning $B_1, \ldots, B_\ell$ such that $f \upharpoonright C$ is analytic and is the restriction to C of an $\mathbb{R}_{\mathrm{an}}^{K}$- function, for each cell $C \in \mathcal{C}$ with $C \subseteq A \times (0, 1)$. Let $\pi : \mathbb{R}^{m+1} \to \mathbb{R}^m$ be the projection onto the first m coordinates. Then $\pi\mathcal{C}$ is a decomposition of $\mathbb{R}^m$ partitioning A, say that A is the disjoint union of analytic cells $A_1, \ldots, A_M \in \pi\mathcal{C}$. It suffices to consider the case that $M = 1$. By 3.1(1), we have

$$A \times (0, 1) = \bigcup_{i=1}^{k-1} \Gamma(g_i) \cup \bigcup_{i=1}^{k} (g_{i-1}, g_i)$$

where $g_0 < \cdots < g_k : A \to \mathbb{R}$ are analytic and definable, with $g_0 = 0$ and $g_k = 1$. Now put $h := g_1$. □

4. The Expansion Theorem.

The goal of this section is to prove a "parametric" version of 3.1(4).

It will be convenient to introduce some working definitions and notation.

Given tuples of distinct variables $X := (X_1, \ldots, X_m)$ and $Y := (Y_1, \ldots, Y_d)$, we let $M(X; Y)$ denote the ring of all power series $F \in \mathbb{R}[\![X, Y]\!]$ that converge on an open neighborhood of $[-1, 1]^m \times [-\varepsilon, \varepsilon]^d$, for some $\varepsilon > 0$ that depends on F. For $d = 0$, we just write $M(X)$. *From now on, we assume that we have chosen such an* ε *for each* $F \in M(X_1, \ldots, X_m; Y_1, \ldots, Y_d)$. Given $(x, y) \in \mathbb{R}^{m+d}$, we let $F(x, y)$ be the value given by the power series if $(x, y) \in [-1, 1]^m \times [-\varepsilon, \varepsilon]^d$, and put $F(x, y) := 0$ otherwise. The resulting function F is finitely subanalytic, hence definable.

Notation. — Given a property $P(t)$ of positive real numbers t, we say that $P(t)$ holds *at* 0^+ if there exists $\varepsilon > 0$ such that $P(t)$ holds for all $t \in (0, \varepsilon)$. When the property $P(x, t)$ also depends on a parameter x ranging over a set $A \subseteq \mathbb{R}^p$, then we allow $\varepsilon = \varepsilon(x) > 0$ also to depend on this parameter.

Let $f : A \times \mathbb{R} \to \mathbb{R}$ be definable, $A \subseteq \mathbb{R}^p$. We wish to expand $f(x,t)$ at 0^+ in a power series in t with exponents from K, uniformly in the parameter $x \in A$.

DEFINITION. — *The function* f has a uniform expansion on A *if there exist*

(1) $F \in M(X_1, \ldots, X_m; Y_1, \ldots, Y_d)$ *for some* $m, d \in \mathbb{N}$,

(2) $r_0, r_1, \ldots, r_d \in K$ *with* $r_1, \ldots, r_d > 0$,

(3) *definable analytic maps* $a : A \to (0, \infty)$, $b = (b_1, \ldots, b_m) : A \to [-1, 1]^m$ *and* $c : A \to [1, \infty)$, *such that for each* $x \in A$, $F(b(x), 0) \neq 0$ *and* $f(x, t) = a(x)t^{r_0} F(b(x), (c(x)t)^{r_1}, \ldots, (c(x)t)^{r_d})$ *at* 0^+. *(In particular,* $f(x,t) \neq 0$ *at* 0^+.*)*

Remarks.

(1) Suppose that f has a uniform expansion on A. Then there is a sequence $(f_n : A \to \mathbb{R})_{n \geq 0}$ of definable analytic functions and an unbounded strictly increasing sequence $(\alpha_n)_{n \geq 0}$ of real numbers such that for all $x \in A$, $f_0(x) \neq 0$ and there exists $\varepsilon(x) > 0$ such that $f(x,t) = \sum_{n \geq 0} f_n(x) t^{\alpha_n}$ for $t \in (0, \varepsilon(x))$, where the convergence is absolute and uniform on each subinterval $(0, \delta] \subseteq (0, \varepsilon(x))$. Consequently, $f(x, -)$ is analytic on $(0, \varepsilon(x))$; (see §4 of [M2]).

(2) For the case $K = \mathbb{Q}$, if f has a uniform expansion on A, then there exist a rational number q, a positive integer k, a power series $F \in M(X; Y_1)$, and finitely subanalytic, analytic maps $a : A \to (0, \infty)$, $c : A \to [1, \infty)$ and $b = (b_1, \ldots, b_m) : A \to [-1, 1]^m$ such that for each $x \in A$ we have $F(b(x), 0) \neq 0$ and $f(x,t) = a(x)t^q F(b(x), (c(x)t)^{1/k})$ at 0^+. Thus, there exists a sequence $(f_n : A \to \mathbb{R})_{n \geq 0}$ of finitely subanalytic, analytic functions such that for all $x \in A$, $f_0(x) \neq 0$ and there exists $\varepsilon(x) > 0$ such that $f(x,t) = t^q \sum_{n \geq 0} f_n(x) t^{n/k}$ for $t \in (0, \varepsilon(x))$.

DEFINITION. — *Let* $f : A \times \mathbb{R} \to \mathbb{R}$ *be definable,* $A \subseteq \mathbb{R}^p$. *For definable* $B \subseteq A$, f has a uniform expansion on B *if* $f \restriction B \times \mathbb{R}$ *has a* *uniform expansion on* B*;* f has a piecewise uniform expansion on A *if* A *is a union of definable sets* $A_1, \ldots A_\ell$ *such that for* $i = 1, \ldots, \ell$*, either* f *has a uniform expansion on* A_i*, or* $f(x,t)$ *vanishes identically at* 0^+ *for all* $x \in A_i$*.*

(Note that we can take $A_1, \ldots, A_\ell$ to be disjoint.)

We can now state the main result of this section:

EXPANSION THEOREM. — *Let* $f : A \times \mathbb{R} \to \mathbb{R}$ *be definable,* $A \subseteq \mathbb{R}^p$*. Then* f *has a piecewise uniform expansion on* A*.*

We have some work to do before we begin the proof.

Given a tuple of variables $X := (X_1, \ldots, X_m)$ and $\nu \in \mathbb{N}^m$, we let X^ν denote the monomial $X_1^{\nu_1} \cdots X_m^{\nu_m}$. We note here a fact from analysis that we will need:

(*) *Let* $F \in M(X)$*,* $X := (X_1, \ldots, X_m)$*, and let* $Y := (Y_1, \ldots, Y_m)$ *be a tuple of new variables. Then there exists* $\varepsilon > 0$ *such that the power series*

$$G(X,Y) := \sum \frac{1}{\nu!} \frac{\partial^{|\nu|} F}{\partial X^\nu} Y^\nu \ (\nu \in \mathbb{N}^m)$$

converges on a neighborhood of $[-1,1]^m \times [-\varepsilon, \varepsilon]^m$ *(i.e.,* $G \in M(X;Y)$*), and such that for all* $(u,v) \in [-1,1]^m \times [-\varepsilon,\varepsilon]^m$*, with* $|u+v| \leq 1$*, we have* $F(u+v) = G(u,v)$*.*

We are thus justified in denoting the power series G by $F(X+Y)$.

N.B. — The following reductions will be used throughout this section, often without mention.

(1) Let $f : A \times \mathbb{R} \to \mathbb{R}$ be definable with $A \subseteq \mathbb{R}^p$. Then the set

$$\{x \in A : f(x,t) = 0 \text{ at } 0^+\}$$

is definable. Thus, in order to show that f has a piecewise uniform expansion on A, we may remove this set from A and assume (by 1.1) that $f(x,t) \neq 0$ at 0^+; i.e., for all $x \in A$ there exists $\varepsilon(x) > 0$ such that $f(x,t) \neq 0$ for all $t \in (0, \varepsilon(x))$.

(2) Suppose $r_0, r_1, \ldots, r_d$, F and a, b, c are as in the definition of "uniform expansion on A", except that instead of requiring a, b and c to be definable and analytic, we only assume that they are definable. It follows then from 3.1(1) that f has a piecewise uniform expansion on A. (We do not actually need this observation, but it will relieve us in the coming pages

of doing the easy, but quite frequently occurring, verifications that certain definable functions are analytic.)

LEMMA. — *Let $r_1, \dots, r_d \in K \cap (0, \infty)$, $d \geq 1$. Let $0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots$ be the elements of the monoid $r_1\mathbb{N} + \ldots + r_d\mathbb{N}$ in increasing order. Then given $N \in \mathbb{N}$, there exist $s_1, \dots, s_e \in K \cap (0, \infty)$ such that $\{\alpha_n - \alpha_N : n \geq N\} \subseteq s_1\mathbb{N} + \ldots + s_e\mathbb{N}$.*

(See [M2], e.g., for a proof.)

MAIN LEMMA. — *Let $r_1, \dots, r_d \in K \cap (0, \infty)$, $F \in M(X_1, \dots, X_m; Y_1, \dots, Y_d)$, and let $b = (b_1, \dots, b_m) : A \to [-1, 1]^m$ and $c_1, \dots, c_d : A \to [1, \infty)$ be definable maps, $A \subseteq \mathbb{R}^p$. Then the definable function $f : A \times \mathbb{R} \to \mathbb{R}$ given by*

$$f(x, t) := F(b(x), (c_1(x)t)^{r_1}, \dots, (c_d(x)t)^{r_d})$$

has a piecewise uniform expansion on A.

Proof. — First, we do the case that $c_1 = \ldots = c_d$.

Note that f has a piecewise uniform expansion on the definable set

$$\{x \in A : F(b(x), 0) \neq 0\},$$

so we may reduce to the case that $F(b(x), 0) = 0$ and $f(x, t) \neq 0$ at 0^+ for each $x \in A$. Write

$$F(X, Y) = \sum F_\nu(X) Y^\nu, \quad F_\nu(X) \in M(X),$$

where the sum is taken over $\nu \in \mathbb{N}^d$. Let $0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots$ be the elements of the monoid $r_1\mathbb{N} + \ldots + r_d\mathbb{N}$ in increasing order. For all $n \in \mathbb{N}$ put

$$G_n(X) := \sum F_\nu(X) \in M(X),$$

where the (finite) sum is taken over all $\nu \in \mathbb{N}^d$ such that $r_1\nu_1 + \cdots + r_d\nu_d = \alpha_n$. Then for each $x \in A$, we have

$$F(b(x), (c(x)t)^{r_1}, \dots, (c(x)t)^{r_d}) = \sum_{n \geq 0} G_n(b(x))(c(x)t)^{\alpha_n} \text{ at } 0^+.$$

Note that $G_0(b(x)) = F(b(x), 0) = 0$ for all $x \in A$, and that $x \mapsto G_n(b(x)) : A \to \mathbb{R}$ is definable for all $n \in \mathbb{N}$. Since f is definable, and we assume that $f(x, t) \neq 0$ at 0^+, there exists by 1.4 some $N \in \mathbb{N}$ such that for all $x \in A$, there is an $n \leq N$ with $G_n(b(x)) \neq 0$. Now for each $N > 0$, the set

$$\{x \in A : G_0(b(x)) = \ldots = G_{N-1}(b(x)) = 0, G_N(b(x)) \neq 0\}$$

is definable. Partitioning A suitably, we may thus reduce to the case that there exists $N > 0$ such that for all $x \in A$, we have $G_0(b(x)) = \ldots = G_{N-1}(b(x)) = 0$ and $G_N(b(x)) \neq 0$. Then for each $x \in A$ we have

$$f(x,t) = (c(x)t)^{\alpha_N}\Big(G_N(b(x)) + \sum_{n>N} G_n(b(x))(c(x)t)^{\alpha_n-\alpha_N}\Big) \text{ at } 0^+.$$

Let $s = (s_1, \ldots, s_e) \in \mathbb{N}^e$ be as in the previous lemma. For each $n > N$, choose $\mu(n) \in \mathbb{N}^e$ such that $s_1\mu(n)_1 + \cdots + s_e\mu(n)_e = \alpha_n - \alpha_N$. Put

$$H(X,Z) := G_N(X) + \sum_{n>N} G_n(X)Z^{\mu(n)} \in M(X;Z),$$

where $Z := (Z_1, \ldots, Z_e)$ is a tuple of new variables. Put $a(x) := c(x)^{\alpha_N}$ for all $x \in A$. Note that $H(b(x),0) = G_N(b(x)) \neq 0$ and $a(x) > 0$ for all $x \in A$. Then for each $x \in A$ we have

$$f(x,t) = a(x)t^{\alpha_N} H(b(x), (c(x)t)^{s_1}, \ldots, (c(x)t)^{s_e}) \text{ at } 0^+,$$

as desired.

Next, let $c_1, \ldots, c_d : A \to [1,\infty)$ be definable functions. For each $i = 1, \ldots, d$, the set

$$A_i := \{x \in A : c_i(x) \geq c_j(x) \text{ for } j = 1, \ldots, d\}$$

is definable; thus, we may reduce to the case that, say, $A = A_1$. Define $b_{m+j} : A \to [-1,1]$ for $j = 1, \ldots, d$ by $b_{m+j}(x) := (c_j(x)/c_1(x))^{r_j}$. Put

$$G(X, X_{m+1}, \ldots, X_{m+d}, Y) := F(X, X_{m+1}Y_1, \ldots, X_{m+d}Y_d).$$

Then $G \in M(X, X_{m+1}, \ldots, X_{m+d}; Y)$ and

$$F(b(x), (c_1(x)t)^{r_1}, \ldots, (c_d(x)t)^{r_d}) = G(b'(x), (c_1(x)t)^{r_1}, \ldots, (c_1(x)t)^{r_d}) \text{ at } 0^+$$

for each $x \in A$, where $b' := (b_1, \ldots, b_{m+d})$. By the previous case, we are done. □

Proof of the Expansion Theorem.

To show that a definable function $f : A \times \mathbb{R} \to \mathbb{R}$ with $A \subseteq \mathbb{R}^p$ has a piecewise uniform expansion on A, we may (by 3.2) reduce to the case that f is the restriction to $A \times \mathbb{R}$ of an $\mathbb{R}_{\mathrm{an}}^K$-function on $\mathbb{R}^{p+1}$. We now proceed by "induction on complexity of $\mathbb{R}_{\mathrm{an}}^K$-functions" to show that each $\mathbb{R}_{\mathrm{an}}^K$-function f on $\mathbb{R}^{p+1}$ has a piecewise uniform expansion on the definable set $A \subseteq \mathbb{R}^p$. As usual, we will assume that $f(x,t) \neq 0$ at 0^+ for each $x \in A$. *Throughout the proof, the parameter x will range over A.*

Case. f is a projection function $\mathbb{R}^{p+1} \to \mathbb{R}$.

(Trivial.)

Case. $f = -g$, where g is an $\mathbb{R}^K_{\text{an}}$-function having a piecewise uniform expansion on A.

(Easy; details omitted.)

Case. $f = g + h$, where g and h are $\mathbb{R}^K_{\text{an}}$-functions having piecewise uniform expansions on A.

Now f has a piecewise uniform expansion on the set

$$\{x \in A : g(x,t) = 0 \text{ at } 0^+\} \cup \{x \in A : h(x,t) = 0 \text{ at } 0^+\},$$

so we may reduce to the case that g and h have uniform expansions on A; say that

$$g(x,t) = a(x)t^{r_0} G(b(x), (c(x)t)^{r_1}, \dots, (c(x)t)^{r_d}) \text{ at } 0^+$$

and

$$h(x,t) = a'(x)t^{s_0} H(b'(x), (c'(x)t)^{s_1}, \dots, (c'(x)t)^{s_e}) \text{ at } 0^+$$

of the required form. We may assume that $a = a'$. (To see this, note that the sets $A_1 := \{x \in A : a(x) \leq a'(x)\}$ and $A_2 := \{x \in A : a(x) > a'(x)\}$ are definable, so we may assume that either $A = A_1$ or $A = A_2$. If $A = A_1$, then

$$g(x,t) = a'(x)t^{r_0} G'(b(x), (a(x)/a'(x)), (c(x)t)^{r_1}, \dots, (c(x)t)^{r_d}) \text{ at } 0^+,$$

where $G'(X, X_{m+1}, Y) := X_{m+1}G(X,Y)$. We use the same trick in case $A = A_2$.) Put $b'' := (b, b') : A \to [-1,1]^{m+n}$, for appropriate m and n. Suppose without loss of generality that $s_0 \leq r_0$.

Subcase. $s_0 = r_0$.

Introducing new variables as needed, put

$$F(X,Y) := G(X_1, \dots, X_m, Y_1, \dots, Y_d) \\ + H(X_{m+1}, \dots, X_{m+n}, Y_{d+1}, \dots, Y_{d+e}).$$

Then at 0^+ we have

$$f(x,t) = a(x)t^{s_0} F(b''(x), (c(x)t)^{r_1}, \dots, (c(x)t)^{r_d}, (c'(x)t)^{s_1}, \dots, (c'(x)t)^{s_e}).$$

Apply the Main Lemma.

Subcase. $s_0 < r_0$.

Put $c''(x) := 1$ for $x \in A$, and put $F(X, Y)$ equal to

$$Y_{d+e+1}G(X_1, \dots, X_m, Y_1, \dots, Y_d) + H(X_{m+1}, \dots, X_{m+n}, Y_{d+1}, \dots, Y_{d+e}).$$

Then at 0^+, $f(x,t)$ is equal to

$$a(x)t^{s_0}F(b''(x), (c(x)t)^{r_1}, \dots, (c(x)t)^{r_d}, (c'(x)t)^{s_1}, \dots, (c'(x)t)^{s_e}, (c''(x)t)^{r_0 - s_0}).$$

Apply the Main Lemma.

Case. $f = gh$, where g and h are $\mathbb{R}^K_{\mathrm{an}}$-functions having piecewise uniform expansions on A.

This case is similar to, but easier than, the previous case, and we omit the details.

Case. $f = h^s$, where $s \in K$, and h is an $\mathbb{R}^K_{\mathrm{an}}$-function having a piecewise uniform expansion on A. (Recall that we put $h(x,t)^s := 0$ for $h(x,t) \le 0$; see §3.)

For $s = 0$, the result is trivial, so suppose that $s \neq 0$. Since $f(x,t) \neq 0$ at 0^+, we have $h(x,t) > 0$ at 0^+. We may assume that h has a uniform expansion on A; say that

$$h(x,t) = a(x)t^{r_0}H(b(x), (c(x)t)^{r_1}, \dots, (c(x)t)^{r_d}) \text{ at } 0^+$$

of the required form. Then

$$f(x,t) = a(x)^s t^{sr_0}(H(b(x), (c(x)t)^{r_1}, \dots, (c(x)t)^{r_d}))^s \text{ at } 0^+.$$

Thus, it suffices to show that the definable function $u : A \times \mathbb{R} \to \mathbb{R}$ given by

$$u(x,t) := (H(b(x), (c(x)t)^{r_1}, \dots, (c(x)t)^{r_d}))^s$$

has a piecewise uniform expansion on A.

Write

$$H(X,Y) = \sum H_\nu(X)Y^\nu, H_\nu(X) \in M(X),$$

where the sum is taken over all $\nu \in \mathbb{N}^d$. Note that $H_0(b(x)) = H(b(x), 0) > 0$, and that the sets $A_1 := \{x \in A : H_0(b(x)) \ge 1\}$ and $A_2 := \{x \in A : H_0(b(x)) < 1\}$ are definable. So we may assume that either $A = A_1$ or $A = A_2$.

Subcase. $H_0(b(x)) \geq 1$ for all x.

At 0^+ we have

$$H(b(x),(c(x)t)^{r_1},\ldots,(c(x)t)^{r_d}) \\ = H_0(b(x))(1+H'(b'(x),(c(x)t)^{r_1},\ldots,(c(x)t)^{r_d})),$$

where $b' := (b_1,\ldots,b_m,1/(H_0(b)))$ and

$$H'(X,X_{m+1},Y) := X_{m+1}\sum_{\nu\neq 0} H_\nu(X)Y^\nu.$$

Then

$$u(x,t) = H_0(b(x))^s F(b'(x),(c(x)t)^{r_1},\ldots,(c(x)t)^{r_d}) \text{ at } 0^+,$$

where

$$F(X,X_{m+1},Y) := \sum_{k\geq 0}\binom{s}{k}(H'(X,X_{m+1},Y))^k \in M(X,X_{m+1};Y).$$

Subcase. $0 < H_0(b(x)) < 1$ for all x.

For $i = 1,\ldots,d$, put $c_i(x) := c(x)(H_0(b(x)))^{-1/r_i}$. Note that $c_i(x) > 1$ and $(c_i(x)t)^{r_i} = (c(x)t)^{r_i}(1/H_0(b(x)))$. Then at 0^+ we have

$$u(x,t)=H_0(b(x))^s F(b(x),(c(x)t)^{r_1},\ldots,(c(x)t)^{r_d},(c_1(x)t)^{r_1},\ldots,(c_d(x)t)^{r_d}),$$

where

$$F(X,Y_1,\ldots,Y_{2d}) := \sum_{k\geq 0}\binom{s}{k}(H^*(X,Y_1,\ldots,Y_{2d}))^k,$$

and

$$H^*(X,Y_1,\ldots,Y_{2d}) := \sum_{\nu_1\neq 0} H_\nu(X)Y_{d+1}Y_1^{\nu_1-1}Y_2^{\nu_2}\cdots Y_d^{\nu_d} \\ + \sum_{\substack{\nu_1=0\\ \nu_2\neq 0}} H_\nu(X)Y_{d+2}Y_2^{\nu_2-1}Y_3^{\nu_3}\cdots Y_d^{\nu_d} + \\ \cdots + \sum_{\substack{\nu_1=\ldots=\nu_{d-1}=0\\ \nu_d\neq 0}} H_\nu(X)Y_{2d}Y_d^{\nu_d-1}.$$

(Note that $F, H^* \in M(X;Y_1,\ldots,Y_{2d})$ and $H^*(X,0) = 0$.)

Apply the Main Lemma.

(We alert the reader here that we will use again in the next case the construction of the series H^*.)

Case. $f = \tilde{g}(h_1, \ldots, h_\ell)$, where $\tilde{g}$ is a restricted analytic function and $h_1, \ldots, h_\ell$ are $\mathbb{R}_{\text{an}}^K$-functions each having piecewise uniform expansions on A.

For each $i = 1, \ldots, \ell$, the set $A_i := \{x \in A : h_i(x,t) = 0 \text{ at } 0^+\}$ is definable, so we may assume by the monotonicity theorem that $h_i(x,t) \neq 0$ at 0^+ for $i = 1, \ldots, \ell$, and that each h_i has a uniform expansion on A. Since $f(x,t) \neq 0$ at 0^+, by the definition of "restricted analytic function" we must have $|h_i(x,t)| \leq 1$ at 0^+ for $i = 1, \ldots, \ell$.

For simplicity, we do the case $\ell = 1$; the case $\ell > 1$ is similar, but notationally cumbersome. Put $h := h_1$. Then

$$h(x,t) = a(x)t^{r_0} H(b(x), (c(x)t)^{r_1}, \ldots, (c(x)t)^{r_d}) \text{ at } 0^+$$

of the required form. Note that we must have $r_0 \geq 0$, since $|h(x,t)| \leq 1$ at 0^+. The sets $A_1 := \{x \in A : a(x) \leq 1\}$ and $A_2 := \{x \in A : a(x) > 1\}$ are definable, so we may as well assume that either $A = A_1$ or $A = A_2$. We must also consider separately the cases $r_0 = 0$ and $r_0 > 0$. Thus, there are four subcases to treat. We will show that in each subcase, h can be represented at 0^+ in the form

$$h(x,t) = B'(x) + H'(B(x), (C_1(x)t)^{s_1}, \ldots, (C_e(x)t)^{s_e}),$$

where the maps $B : A \to [-1,1]^n$ for some $n \in \mathbb{N}$, $B' : A \to [-1,1]$, and $C_1, \ldots, C_e : A \to [1, \infty)$ are definable, $s_1, \ldots, s_e \in K \cap (0, \infty)$, and $H'(X,Y) \in M(X;Y)$ for $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_e)$, with $H'(X,0) = 0$. It follows then from Fact $(*)$ that

$$f(x,t) = F(B(x), B'(x), (C_1(x)t)^{s_1}, \ldots, (C_e(x)t)^{s_e}) \text{ at } 0^+,$$

where G is the Taylor series at 0 of $\tilde{g}$ and

$$F := G(H'(X,Y) + X_{n+1}) \in M(X, X_{n+1}; Y).$$

The Main Lemma then applies, finishing each subcase, and thus finishing the proof as well.

Subcase. $r_0 > 0$ and $a(x) \leq 1$ for all x.

Put $B := (b,a) : A \to [-1,1]^{m+1}$, $c' := 1 : A \to \mathbb{R}$, and $B' := 0 : A \to \mathbb{R}$. Then

$$h(x,t) = B'(x) + H'(B(x), (c(x)t)^{r_1}, \ldots, (c(x)t)^{r_d}, (c'(x)t)^{r_0}) \text{ at } 0^+,$$

where

$$H'(X, X_{m+1}, Y, Y_{d+1}) := X_{m+1} Y_{d+1} H(X,Y) \in M(X, X_{m+1}; Y, Y_{d+1}).$$

Subcase. $r_0 > 0$ and $a(x) > 1$ for all x.

Put $B' := 0 : A \to \mathbb{R}$ and $c'(x) := a(x)^{1/r_0}$; note that $c'(x) > 1$ and $(c'(x)t)^{r_0} = a(x)t^{r_0}$. Then

$$h(x,t) = B'(x) + H'(b(x), (c(x)t)^{r_1}, \dots, (c(x)t)^{r_d}, (c'(x)t)^{r_0}) \text{ at } 0^+,$$

where

$$H'(X, Y, Y_{d+1}) := Y_{d+1}H(X,Y) \in M(X; Y, Y_{d+1}).$$

Subcase. $r_0 = 0$ and $a(x) > 1$ for all x.

Write

$$H(X,Y) = \sum H_\nu(X)Y^\nu, H_\nu(X) \in M(X),$$

where the sum is taken over $\nu \in \mathbb{N}^d$. Note that we must have $|a(x)H_0(b(x))| \le 1$ for $x \in A$. Put $B'(x) := a(x)H_0(b(x))$ and put $c_i(x) := a(x)^{1/r_i}c(x)$ for $i = 1, \dots, d$; then $c_i(x) > 1$ and $(c_i(x)t)^{r_i} = a(x)(c(x)t)^{r_i}$. Let $H' := H^*$, where H^* is constructed from H as in the previous case. Hence, at 0^+ we have

$$h(x,t) = B'(x) + H'(b(x), (c(x)t)^{r_1}, \dots, (c(x)t)^{r_d}, (c_1(x)t)^{r_1}, \dots, (c_d(x)t)^{r_d}).$$

Subcase. $r_0 = 0$ and $a(x) \le 1$ for all x.

With H_0 as in the previous subcase, let B' also be as in the previous subcase and put

$$H' := X_{m+1}(H(X,Y) - H_0(X))$$

and

$$B := (b, a) : A \to [-1, 1]^{m+1}.$$

$\square$

5. Proof of the Main Theorem.

Let $f : A \to \mathbb{R}$ be definable, $A \subseteq \mathbb{R}^{m+n}$. Replacing A by

$$\{(x,y) \in A : y \in \operatorname{int}(A_x)\},$$

and f by its restriction to this definable set, we may assume that $A_x \subseteq \mathbb{R}^n$ is open for all $x \in \mathbb{R}^m$. We must show that there exists $N > 0$ such that for all $(x,y) \in A$, if $f(x,-)$ is C^N at y, then $f(x,-)$ is analytic at y.

Consider the definable function $F : A \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ given by

$$F(x, y, z, t) := \begin{cases} f(x, y + tz), & \text{if } y + tz \in A_x \\ 0, & \text{otherwise.} \end{cases}$$

For notational convenience, let the variable v range over $\mathbb{R}^{m+n+n}$.

CLAIM. — *There exists $N > 0$ such that for all $v \in A \times \mathbb{R}^n$, if $F(v, -)$ is C^N at 0, then $F(v, -)$ is analytic at 0.*

Proof of Claim. — First, suppose that $D \subseteq A \times \mathbb{R}^n$ is definable, and that F has a uniform expansion on D. Arguing as in the proof of the Main Lemma, we write

$$F(v, t) = a(v)t^{r_0} \sum_{n \geq 0} G_n(b(v))(c(v)t)^{\alpha_n} \text{ at } 0^+,$$

with v ranging over D. Note that if there exist $v \in D$ and a positive integer $p > r_0$ such that $F(v, -)$ is C^p at 0, then $r_0 \in \mathbb{N}$, (since $a(v)F(b(v), 0) \neq 0$). So we may as well assume that $r_0 \in \mathbb{N}$. Put

$$G(X, Y_1) := \sum G_n(X) Y_1^{\alpha_n} \in M(X; Y_1),$$

where the sum is taken over all $n \in \mathbb{N}$ with $\alpha_n \in \mathbb{N}$. Then the function $g : D \times \mathbb{R} \to \mathbb{R}$ given by $g(v, t) := a(v)t^{r_0} G(b(v), c(v)t)$ is definable. Furthermore, there exists an $\varepsilon > 0$ (depending only on F) such that g is analytic on the set

$$\{(v, t) \in D \times \mathbb{R} : |t| < \varepsilon / c(v)\};$$

in particular, $g(v, -)$ is analytic at 0 for all $v \in D$. Note also that for each $k \in \mathbb{N}$, the function

$$v \mapsto \frac{d^k g(v, t)}{dt^k}(0) : D \to \mathbb{R}$$

is analytic. Put $h := F \upharpoonright (D \times \mathbb{R}) - g$. Then h is definable, and we have

$$h(v, t) = \sum a(v) G_n(b(v)) c(v)^{\alpha_n} t^{\alpha_n + r_0} \text{ at } 0^+,$$

where the sum is taken over all $n \in \mathbb{N}$ with $\alpha_n \notin \mathbb{N}$. Thus,

$$h(v, t) = \sum_{k=0}^{\infty} h_k(v) t^{\beta_k} \text{ at } 0^+,$$

where (β_k) is a strictly increasing sequence of nonintegral real numbers (since $r_0 \in \mathbb{N}$) and each $h_k : D \to \mathbb{R}$ is definable. By 1.5, there is an $N \in \mathbb{N}$ such that for all $v \in D$, if $h(v, t) = O(t^N)$ as $t \to 0^+$, then $h(v, t) = 0$ at 0^+. Thus, there exists $N > 0$ such that for all $v \in D$, if $F(v, -)$ is C^N at 0, then

$F(v,-) = g(v,-)$ on some interval (depending on v) about 0; thus, $F(v,-)$ is analytic at 0. (*Note:* For each $k \in \mathbb{N}$ the function

$$v \mapsto \frac{d^k F(v,t)}{dt^k}(0) : B \to \mathbb{R}$$

is definable and analytic, where $B := \{v \in D : F(v,-) \text{ is } C^N \text{ at } 0\}$.)

Next, suppose that $D \subseteq A \times \mathbb{R}^n$ is definable and that $F(v,t) = 0$ at 0^+ for each $v \in D$. Note that for any positive integer p and $v \in D$, if $F(v,-)$ is C^p at 0, then $F(v,-)$ is p-flat at 0. By 1.5, there exists $N > 0$ such that for all $v \in D$, if $F(v,-)$ is C^N at 0, then $F(v,-)$ vanishes identically on some interval about 0, hence is analytic at 0. (*Note:* For each $k \in \mathbb{N}$ the function

$$v \mapsto \frac{d^k F(v,t)}{dt^k}(0) : B \to \mathbb{R}$$

is identically zero, where $B := \{v \in D : F(v,-) \text{ is } C^N \text{ at } 0\}$.)

The claim now follows easily from the Expansion Theorem applied to F.

Proof of Main Theorem from Claim. — Let $N > 0$ be as in the claim. Put

$$A' := \{(x,y) \in A : F(x,y,z,-) \text{ is } C^N \text{ at } 0 \in \mathbb{R} \text{ for all } z \in \mathbb{R}^n\}.$$

Note that if $(x,y) \in A - A'$, then $f(x,-)$ is not C^N at y, hence not analytic at y. So, we may replace A by its definable subset A' and assume that for all $v \in A \times \mathbb{R}^n$, the function $F(v,-)$ is C^N at $0 \in \mathbb{R}$, and hence analytic at 0 by the claim. The arguments in the proof of the claim then establish that there exist definable sets $B_1, \ldots, B_\ell$ with $A \times \mathbb{R}^n = B_1 \cup \cdots \cup B_\ell$ such that

$$v \mapsto \frac{d^k F(v,t)}{dt^k}(0) : B_i \to \mathbb{R}$$

is analytic for all $k \in \mathbb{N}$ and $i \in \{1, \ldots, \ell\}$. Increasing (if necessary) N as in the claim and applying 2.5 and 3.1(1), we may assert that for all $(x,y) \in A$, if $f(x,-)$ is C^N at y, then $f(x,-)$ is G^∞ at y.

Now let $(x,y) \in A$ and suppose that $f(x,-)$ is C^N on some open euclidean ball U centered at y of radius $\varepsilon > 0$. Then $f(x,-)$ is G^∞ at y. Furthermore, $F(x,y',z,-)$ is C^N, and thus analytic, at $t = 0$ for all $(y',z) \in U \times \mathbb{R}^n$. Thus, $t \mapsto f(x, y+tz)$ is defined and analytic on $(-\varepsilon, \varepsilon)$ for all $z \in \mathbb{R}^n$ with $||z|| \leq 1$. Hence, by 2.2, $f(x,-)$ is analytic at y. □

Note. — The result clearly holds for definable maps $F : A \to \mathbb{R}^p$, $A \subseteq \mathbb{R}^{m+n}$.

COROLLARY. — *With assumptions as in the Main Theorem, the set*

$$\{(x,y) \in A : f(x,-) \ \text{ is analytic at } y\}$$

is definable.

Proof. — The set $\{(x,y) \in A : f(x,-) \ \text{ is } C^M \text{ at } y\}$ is definable for any fixed positive integer M. □

DEFINITION. — *Let $X \subseteq \mathbb{R}^p$. Then $x \in X$ is a* smooth point *of X of dimension k if $X \cap U$ is an analytic submanifold of $\mathbb{R}^p$ of dimension k for some open neighborhood U of x. The* singular set *of X, denoted* $\operatorname{Sing}(X)$, *is the complement in X of the smooth points of highest dimension.*

COROLLARY. — *Let $X \subseteq \mathbb{R}^p$ be definable. Then for each $k \in \mathbb{N}$, the set of smooth points of X of dimension k is definable; in particular,* $\operatorname{Sing}(X)$ *is a closed definable subset of X.*

Proof. — Let $k \in \{0, \ldots, p\}$. Given $\varepsilon > 0$ and $x \in \mathbb{R}^p$, put

$$B(x,\varepsilon) := \{y \in \mathbb{R}^p : |x-y| < \varepsilon\}.$$

Note that $x \in X$ is a smooth point of dimension k if and only if for some $\varepsilon > 0$ and some $i = (i_1, \ldots, i_k)$ with $1 \le i_1 < \cdots < i_k \le p$, the coordinate projection π_i (as in 1.2) maps $X \cap B(x,\varepsilon)$ bijectively onto an open subset C of $\mathbb{R}^k$, and the inverse of $\pi_i \restriction A \cap B(x,\varepsilon)$ (as a map $C \to \mathbb{R}^p$) is analytic at $\pi_i(x)$. Now use the previous corollary. □

COROLLARY. — *Let $A \subseteq \mathbb{R}^{m+n}$ be definable. Then $\{(x,y) \in \mathbb{R}^{m+n} : y \in \operatorname{Sing}(A_x)\}$ is definable.*

The proof is similar to that of the preceding corollary.

Note. — Suppose that $\mathcal{S}$ is a structure on $(\mathbb{R}, +, \cdot)$ such that all sets in $\mathcal{S}$ are definable in $\mathbb{R}_{\text{an}}^K$. Then the above corollaries (suitably rephrased) hold with the notion of definable in $\mathbb{R}_{\text{an}}^K$ replaced by the notion of belonging to $\mathcal{S}$.

In closing, we point out that the results of this section *never* hold in o-minimal structures $\mathcal{S}$ on $(\mathbb{R}, +, \cdot)$ which are *not* polynomially bounded. By 1.3, the exponential function belongs to every such $\mathcal{S}$, and thus

$$t \mapsto \begin{cases} e^{-1/t}, & t > 0 \\ 0, & t \le 0 \end{cases}$$

belongs to $\mathcal{S}$, which is C^∞ on $\mathbb{R}$ but not analytic at $t = 0$. Also, the function $F : \mathbb{R}^2 \to \mathbb{R}$ given by

$$F(x,y) := \begin{cases} |y|^{1/x} \cdot \exp\left(-1/(x^2+y^2)\right), & \text{if } x > 0 \text{ and } y \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

belongs to $\mathcal{S}$. Note that $F(x,-)$ is C^∞ at $y = 0$ iff $x \in (-\infty, 0] \cup \{1/(2n) : n \geq 1\}$, which has infinitely many connected components; also, F is C^n at $(0,0)$ for every $n > 0$, but not C^∞ at $(0,0)$.

BIBLIOGRAPHY

[AM] S. ABHYANKAR and T. MOH, Reduction theorem for divergent power series, J. Reine Angew. Math., 241 (1970), 27–33.

[BM] E. BIERSTONE and P. MILMAN, Semianalytic and subanalytic sets, Inst. Hautes Études Sci. Publ. Math., 67 (1988), 5–42.

[BS] J. BOCHNAK and J. SICIAK, Analytic functions in topological vector spaces, Studia Math., 39 (1971), 77–112.

[DD] J. DENEF and L. van den DRIES, p-adic and real subanalytic sets, Ann. of Math., 128 (1988), 79–138.

[D1] L. van den DRIES, Remarks on Tarski's problem concerning $(\mathbb{R}, +, \cdot, \exp)$, Logic Colloquium 1982, eds. G. Lolli, G. Longo and A. Marcja, North Holland, Amsterdam (1984), 97–121.

[D2] L. van den DRIES, A generalization of the Tarski-Seidenberg theorem and some nondefinability results, Bull. Amer. Math. Soc. (N.S.), 15 (1986), 189–193.

[D3] L. van den DRIES, Tame topology and o-minimal structures, (monograph in preparation).

[DMM] L. van den DRIES, A. MACINTYRE and D. MARKER, The elementary theory of restricted analytic fields with exponentiation, Ann. of Math., 140 (1994), 183–205.

[DM] L. van den DRIES and C. MILLER, On the real exponential field with restricted analytic functions, Israel J. Math., 85 (1994), 19–56.

[H] R. HARDT, Semi-algebraic local triviality in semi-algebraic mappings, Amer. J. Math., 102 (1980), 291–302.

[KPS] J. KNIGHT, A. PILLAY and C. STEINHORN, Definable sets in ordered structures. II, Trans. Amer. Math. Soc., 295 (1986), 593–605.

[KR] T. KAYAL and G. RABY, Ensembles sous-analytiques: quelques propriétés globales, C. R. Acad. Sci. Paris, Sér. I Math., 308 (1989), 521–523.

[K] K. KURDYKA, Points réguliers d'un sous-analytique, Ann. Inst. Fourier (Grenoble), 38-1 (1988), 133–156.

[M1] C. MILLER, Exponentiation is hard to avoid, Proc. Amer. Math. Soc., 122 (1994), 257–259.

[M2] C. MILLER, Expansions of the real field with power functions, Ann. Pure Appl. Logic, 68 (1994), 79–94.

[M3] C. MILLER, Infinite differentiability in polynomially bounded o-minimal structures, Proc. Amer. Math. Soc., (to appear).

[P] W. PAWŁUCKI, Le théorème de Puiseux pour une application sous-analytique, Bull. Polish Acad. Sci. Math., 32 (1984), 556–560.

[PS] A. PILLAY and C. STEINHORN, Definable sets in ordered structures. I, Trans. Amer. Math. Soc., 295 (1986), 565–592.

[T] M. TAMM, Subanalytic sets in the calculus of variations, Acta Math., 146 (1981), 167–199.

[To] J.-Cl. TOUGERON, Algèbres analytiques topologiquement noethériennes. Théorie de Khovanskii, Ann. Inst. Fourier (Grenoble), 41-4 (1991), 823–840.

[W] A. WILKIE, Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function, Jour. Amer. Math. Soc., (to appear).

Manuscrit reçu le 9 mars 1994.

L. van den DRIES & C. MILLER,
University of Illinois
at Urbana-Champaign
Department of Mathematics
Urbana, IL 61801 (USA).
vddries@math.uiuc.edu
miller@math.uiuc.edu