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RANDOM WALKS ON THE AFFINE GROUP OF LOCAL FIELDS AND OF HOMOGENEOUS TREES

by

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1. Introduction.

The starting point for the work presented here is the study of probabilistic and potential theoretic properties of products of random affine transformations, that is, random matrices of the form

$$\begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ 0 & 1 \end{pmatrix}, \quad \text{where } \mathfrak{a} \neq 0.$$

Products of real affine transformations were, probably, one of the first examples of random walks on groups where non-commutativity critically influences asymptotic properties of the random walk and leads to essentially new phenomena, see Grenander [Gre]. Later on, the affine group over $\mathbb{R}$ always remained one of the first examples to be considered when addressing new problems connected with non-commutative random walks (see Molchanov [Mo], Grincevičjus [G1], [G2], Élie [E1], [E2], [E3] and others), the other typical example being that of free groups (see the survey by Woess [W2] for references).

Due to the structure theory of Lie groups, understanding the asymptotic behaviour of random walks on real Lie groups to a large extent

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amounts to considering random walks on solvable semi-direct products whose asymptotic properties are roughly the same as for random walks on the affine group, see Azencott [Az], Raugi [Rau], Guivarc'h [Gu].

Random walks on Lie groups over local fields have been studied much less (see, for example, Gérardin [Ge], Guimier [Gui]), although some of their asymptotic properties play an important role in questions connected with rigidity and arithmeticity (Raghunathan [Rag], Margulis [Ma]).

The present paper is devoted to a study of random walks on the affine group

$$\mathrm{A\scriptstyle FF}(\mathfrak{F}) = \left\{ \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ 0 & 1 \end{pmatrix} : \mathfrak{a}, \mathfrak{b} \in \mathfrak{F},\ \mathfrak{a} \neq 0 \right\}$$

over a non-archimedean local field $\mathfrak{F}$. This group is locally compact, totally disconnected, amenable and non-unimodular.

Recall a well known analogy with the case of the field of reals. In this situation the affine group $\mathrm{Aff}(\mathbb{R})$ as a subgroup of $GL(2, \mathbb{R})$ acts by isometries on the hyperbolic plane $\mathbb{H}^2$. The visibility boundary $\partial\mathbb{H}^2$ of the hyperbolic plane can be identified with the projective line $P^1\mathbb{R}$ (using, for example, the upper half-plane realization), and the induced action of $\mathrm{Aff}(\mathbb{R})$ on $\partial\mathbb{H}^2$ is isomorphic to its natural action on $P^1\mathbb{R}$.

In an analogous way, $GL(2, \mathfrak{F})$ acts on the corresponding *Bruhat-Tits building*, which turns out to be a *homogeneous tree* $\mathbb{T} = \mathbb{T}(\mathfrak{F})$, whose degree (number of edges meeting at each vertex) is one plus the order of the residual field, see Serre [S2]. The boundary $\partial\mathbb{T}$ of this tree (its *space of ends*) can be identified with the projective line $P^1\mathfrak{F}$ of the field $\mathfrak{F}$, so that the action of $\mathrm{Aff}(\mathfrak{F}) \subset GL(2, \mathfrak{F})$ on $\partial\mathbb{T}$ is isomorphic to its action on $P^1\mathfrak{F}$. In fact, one can give a realization of the action of $\mathrm{Aff}(\mathfrak{F})$ on $\mathbb{T}$ in a much more elementary way by identifying $\mathbb{T}$ with the tree of (ultrametric) balls in $\mathfrak{F}$, see § 4 below.

In the real case, simple matrix representation of the affine group suggests using coordinate language. However, taking a geometrical point of view and considering the action of $\mathrm{Aff}_0(\mathbb{R})$ (the component of the identity in $\mathrm{Aff}(\mathbb{R})$) on $\mathbb{H}^2$ makes asymptotic properties of the random walk much clearer. In the real case the difference between these two approaches («coordinate» and «geometric») bears mostly expository significance, because :

(1) $\mathrm{Aff}_0(\mathbb{R})$ acts on $\mathbb{H}^2$ simply transitively (so that elements of $\mathrm{Aff}_0(\mathbb{R})$ can be identified with points in $\mathbb{H}^2$) ; and

(2) $\mathrm{AFF}_0(\mathbb{R})$ is isomorphic with the group of orientation preserving isometries of $\mathbb{H}^2$ which fix a point $\omega \in \partial\mathbb{H}^2$.

In the case of local fields the situation is different. Let $\mathrm{AFF}(\mathbb{T})$ be the *affine group of the tree* $\mathbb{T}$, i.e., the group of *all* isometries of $\mathbb{T}$ which fix a given end (see § 2). Then the group $\mathrm{AFF}(\mathfrak{F})$ is a proper (and significantly smaller) subgroup of $\mathrm{AFF}(\mathbb{T})$. On the other hand, although $\mathrm{AFF}(\mathfrak{F})$ acts transitively on $\mathbb{T}$, all vertex stabilizers are uncountable.

From the geometrical point of view, it turns out that the crucial fact is that we have a closed *non-exceptional* subgroup Γ of the affine group $\mathrm{AFF}(\mathbb{T})$ of a homogeneous tree $\mathbb{T}$, and all the results we are interested in can be obtained under this assumption only (in fact, instead of homogeneous, one can consider here more general *bi-homogeneous* trees where the vertex degree takes two alternating values). « Non-exceptional » means (in complete analogy with the theory of Kleinian and hyperbolic groups, see e.g. Beardon [Be] and Gromov [Gro]) that the *limit set* of Γ in $\partial\mathbb{T}$ contains at least three points (and is in fact uncountable). A closed group $\Gamma \subset \mathrm{AFF}(\mathbb{T})$ is non-exceptional if and only if it is non-unimodular (Theorem 1). According to a theorem of Nebbia [Ne], any amenable group of isometries of T fixes an end. Thus, our results apply to *all amenable non-unimodular* groups of isometries of T.

Coming back to the topic of products of random transformations, we now consider a probability measure μ on $\mathrm{AFF}(\mathbb{T})$ whose support generates a non-exceptional subgroup Γ, and associate with it a sequence (X_n) of i.i.d. μ-distributed random isometries of $\mathbb{T}$. The *right random walk with law* μ is the sequence of random transformations $R_n = X_1 \cdots X_n$. A function g on Γ is *μ-harmonic* if $g(\alpha) = \int g(\alpha\gamma)\mu(\mathrm{d}\gamma)$ for every $\alpha \in \Gamma$.

In this context we obtain, under suitable conditions, the following main results (§ 3) :

- Convergence of the random walk to the boundary $\partial\mathbb{T}$ (Theorem 2), and hence, existence of a *harmonic measure* ν on $\partial\mathbb{T}$.

- The solution of the Dirichlet problem at infinity for μ-harmonic functions (Theorem 3).

- Law of large numbers (Theorems 4 and 5) and central limit theorem (Theorems 6–8), formulated with respect to two natural length functions on $\mathrm{AFF}(\mathbb{T})$.

- Identification of the Poisson boundary with the space $(\partial\mathbb{T}, \nu)$, that is, a description of the space of bounded μ-harmonic functions (Theorem 9).

The corresponding results for the affine group of a local field are then obtained as direct applications. Our point of view is that the tree furnishes a very useful geometrical visualization of our group and of the random walk. Hence, after all, local fields will appear only in a relatively short chapter (§ 4) where the interplay with the tree is explained and the results are «translated» (Corollaries 1–3).

Finally, in the Appendix we prove boundary convergence of the components of the random walk obtained by «splitting at the infimum» (Theorem 10). This is a main technical ingredient of our proof of the central limit theorem.

2. The tree and its affine group.

A. Geometry of the homogeneous tree.

The *homogeneous tree* $\mathbb{T} = \mathbb{T}_q$ is the unique connected graph without cycles in which every vertex has $q+1$ neighbours. Here, we always assume that $q \geq 2$. For every pair of vertices $x, y \in \mathbb{T}$ there is a unique *geodesic segment* $\overline{xy}$ of successive neighbours $x = x_0, x_1, \ldots, x_k = y$ in $\mathbb{T}$ with no backtracking; the *distance* between x and y is $d(x,y) = k$. A *geodesic ray* is an infinite sequence $x_0, x_1, x_2, \ldots$ of successive neighbours without repetitions. Two rays are *equivalent* if they differ only by finitely many vertices. An *end* is an equivalence class of rays. The set of all ends is denoted $\partial\mathbb{T}$, and we write $\widehat{\mathbb{T}} = \mathbb{T} \cup \partial\mathbb{T}$. If $\mathfrak{u} \in \partial\mathbb{T}$ and $x \in \mathbb{T}$ then there is a unique ray $\overline{x\mathfrak{u}}$ starting at x which represents $\mathfrak{u}$. We write $\mathfrak{u}_x(n)$ for the nth vertex on $\overline{x\mathfrak{u}}$ $(n \geq 0)$. Also, if $\mathfrak{u}, \mathfrak{v}$ are two different ends, then there is a unique (bi-infinite) *geodesic* $\overline{\mathfrak{u}\mathfrak{v}}$ connecting the two. (We shall usually use italic letters $o, x, y, \ldots$ for vertices of $\mathbb{T}$, and fraktur letters $\mathfrak{a}, \mathfrak{b}, \mathfrak{u}, \mathfrak{v}, \ldots$ — and in one case, a greek ω — for elements of $\partial\mathbb{T}$.)

We choose and fix, once and for all, a reference vertex $o \in \mathbb{T}$. For $x \in \mathbb{T}$, we write $|x| = d(o,x)$. If $\mathfrak{x}, \mathfrak{y} \in \widehat{\mathbb{T}}$, then the *confluent* $\mathfrak{x} \curlywedge \mathfrak{y}$ is the last common vertex on $\overline{o\mathfrak{x}}$ and $\overline{o\mathfrak{y}}$, unless $\mathfrak{x} = \mathfrak{y} \in \partial\mathbb{T}$, in which case we set $\mathfrak{x} \curlywedge \mathfrak{y} = \mathfrak{x}$. If $x, y \in \mathbb{T}$ then

$$|x \curlywedge y| = 2\big(|x| + |y| - d(x,y)\big)$$

coincides with what nowadays is often called the *Gromov product* and denoted $(x : y)$. We have the *ultrametric inequality*

$$|\mathfrak{x} \curlywedge \mathfrak{y}| \geq \min\big\{|\mathfrak{x} \curlywedge \mathfrak{z}|, |\mathfrak{z} \curlywedge \mathfrak{y}|\big\} \quad \text{for all } \mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \widehat{\mathbb{T}}.$$

Consequently,

$$\theta(\mathfrak{x}, \mathfrak{y}) = \begin{cases} q^{-|\mathfrak{x} \curlywedge \mathfrak{y}|} & \text{if } \mathfrak{x} \neq \mathfrak{y}, \\ 0 & \text{if } \mathfrak{x} = \mathfrak{y} \end{cases} \tag{2.1}$$

defines an ultrametric on $\widehat{\mathbb{T}}$, which thus becomes a totally disconnected, compact space with $\mathbb{T}$ open, dense and discrete. The following is a consequence of the ultrametric property.

LEMMA 1. — *A sequence (x_n) in $\mathbb{T}$ converges to an end if and only if* $|x_n \curlywedge x_{n+1}| \to \infty$.

B. Horocycles and the Busemann function.

We now choose and fix, once and for all, an end of $\mathbb{T}$ which we call ω and put $\partial^*\mathbb{T} = \partial\mathbb{T} \setminus \{\omega\}$. The *Busemann function* h (corresponding to the origin $o \in \mathbb{T}$ and the point at infinity ω) is defined on $\mathbb{T}$ by

$$h(x) = \lim_{n\to\infty} \big(|\omega_o(n)| - |\omega_x(n)|\big) = \lim_{n\to\infty} \big(n - |\omega_x(n)|\big) = d(x,c) - d(o,c),$$

where $x \in \mathbb{T}$ and $c = x \curlywedge \omega$. Note that the sequence $\big(n - |\omega_x(n)|\big)_n$ stabilizes, so that

$$\omega_x\big(n + h(x)\big) = \omega_o(n)$$

for all sufficiently large n. Clearly,

$$\big|h(x) - h(y)\big| \le d(x,y)$$

for any two points in $\mathbb{T}$; in particular, $|h(x)| \le |x|$. We think of $h(x)$ as the *height* of point x. For $m \in \mathbb{Z}$, the *horocycle* at level m is the (infinite) set

$$H_m = \big\{x \in \mathbb{T} \,:\, h(x) = m\big\}.$$

For our purpose, the best way to think of $\mathbb{T}$ is as an «infinite genealogical tree» with ω as the «mythical ancestor» (P. Cartier). The horocycles represent successive generations and are drawn as horizontal layers; each $x \in H_m$ has a unique «father» in H_{m-1} and q «sons» in H_{m+1}. We have a partial order $\preccurlyeq$ on $\mathbb{T} \cup \partial^*\mathbb{T}$ associated with the end ω :

$$\mathfrak{y} \preccurlyeq \mathfrak{x} \iff \mathfrak{y} \in \overline{\mathfrak{x}\omega}.$$

The elements of $\partial^*\mathbb{T}$ are maximal for this order. Every pair $\mathfrak{x}, \mathfrak{y} \in \mathbb{T} \cup \partial^*\mathbb{T}$ has a common ancestor

$$\mathfrak{x} \wedge \mathfrak{y} = \max\{\mathfrak{z} \in \mathbb{T} \cup \partial^*\mathbb{T} \,:\, \mathfrak{z} \preccurlyeq \mathfrak{x},\ \mathfrak{z} \preccurlyeq \mathfrak{y}\}.$$

Unless $\mathfrak{x} = \mathfrak{y} \in \partial^*\mathbb{T}$, this is a vertex of $\mathbb{T}$ and the lowest point on the geodesic $\overline{\mathfrak{x}\mathfrak{y}}$. For $x, y \in \mathbb{T}$,

$$d(x,y) = d(x, x \wedge y) + d(x \wedge y, y) = h(x) + h(y) - 2h(x \wedge y). \tag{2.2}$$

See Figure 1, which is drawn «upside down», so that points with positive heights are at the bottom of the picture.

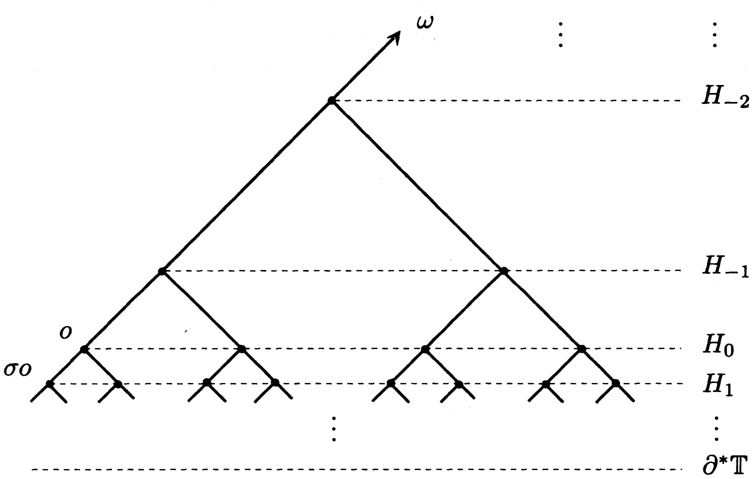

Figure 1

LEMMA 2. — *For any n points $x_1, \dots, x_n \in \mathbb{T}$ there is k, $1 \le k < n$, such that*

$$\bigwedge_{1 \le i \le n} x_i = x_k \wedge x_{k+1}.$$

In particular, $h\Big(\bigwedge_{1 \le i \le n} x_i\Big) = \min\big\{h(x_k \wedge x_{k+1}) \,:\, k = 1, \dots, n-1\big\}.$

Proof. — Using $\bigwedge_{1 \le i \le n} x_i = \bigwedge_{1 \le i < n} y_i$, where $y_i = x_k \wedge x_{k+1}$, and induction, we reduce the proof to checking the case $n = 3$, which is straightforward. $\square$

For $x \in \mathbb{T}$, we consider the «cone»

$$C_x = \{\mathfrak{y} \in \mathbb{T} \cup \partial^*\mathbb{T} : x \preccurlyeq \mathfrak{y}\}.$$

For $\mathfrak{u} \in \partial^*\mathbb{T}$, the family of all C_x which contain $\mathfrak{u}$ constitutes a neighbourhood base. We can consider the following ultrametric on $\mathbb{T} \cup \partial^*\mathbb{T}$:

$$\Theta(\mathfrak{x}, \mathfrak{y}) = \begin{cases} q^{-h(\mathfrak{x} \wedge \mathfrak{y})} & \text{if } \mathfrak{x} \neq \mathfrak{y}, \\ 0 & \text{if } \mathfrak{x} = \mathfrak{y}. \end{cases}$$

On the (noncompact) space $\mathbb{T} \cup \partial^*\mathbb{T} = \widehat{\mathbb{T}} \setminus \{\omega\}$, this induces the same topology as the one given in (2.1) by θ, and in addition $\mathfrak{u}_n \to \omega$ if and only if $\Theta(\mathfrak{u}_n, o) \to \infty$. If one thinks of the metric space $(\mathbb{T}, d)$ as a discrete analogue of the Poincaré plane with hyperbolic metric, then $\mathbb{T} \cup \partial\mathbb{T}$ with metric θ corresponds to the unit disc, while $\mathbb{T} \cup \partial^*\mathbb{T}$ with metric Θ and point at infinity ω corresponds to the upper half plane, in both cases with the Euclidean metric. The two ultrametrics θ and Θ are connected in a natural way with two «genealogical» partial orders, one having the vertex o and the other the end ω as the «ancestor».

C. Regular sequences in $\mathbb{T}$.

The results of this subsection are the geometrical ingredients for «law of large numbers» and «convergence to the boundary» which we shall study in the next section.

DEFINITION 1. — A sequence (x_n) of points in $\mathbb{T}$ is *regular* if there exist an end $\mathfrak{u} \in \partial\mathbb{T}$ and a real number $a \geq 0$ (the *rate of escape*) such that

$$\lim_{n\to\infty} \frac{1}{n} d\big(x_n, \mathfrak{u}_o(\lfloor an \rfloor)\big) = 0.$$

If $a = 0$, this means that $|x_n| = o(n)$, and we shall say that (x_n) is a *trivial regular sequence.*

Here, $\lfloor \cdot \rfloor$ denotes integer part. Note that $\mathfrak{u}$ is arbitrary when $a = 0$, but when $a > 0$, $x_n \to \mathfrak{u}$ and so $\mathfrak{u}$ is unique.

LEMMA 3. — *A sequence (x_n) in $\mathbb{T}$ is regular with rate of escape a if and only if*

(i) $\lim_{n\to\infty} \frac{1}{n} d(x_n, x_{n+1}) = 0$ *and*

(ii) $\lim_{n\to\infty} \frac{1}{n} |x_n| = a.$

Proof. — Clearly, if (x_n) is regular, then the conditions (i) and (ii) are satisfied. Conversely, assume (i) and (ii). If $a = 0$, then (x_n) is a trivial regular sequence. Suppose that $a > 0$. Then

$$|x_n \curlywedge x_{n+1}| = \tfrac{1}{2}\big(|x_n| + |x_{n+1}| - d(x_n, x_{n+1})\big) = na + o(n) \to \infty.$$

Thus, (x_n) converges to an end $\mathfrak{u} \in \partial\mathbb{T}$. Replacing $\wedge$ with $\curlywedge$ in Lemma 2 gives

$$|x_n \curlywedge x_m| \geq \min\big\{|x_i \curlywedge x_{i+1}| : n \leq i \leq m-1\big\}$$

for any $m \geq n$. Thus,

$$|x_n \curlywedge \mathfrak{u}| = \lim_{m\to\infty} |x_n \curlywedge x_m| \geq \inf\big\{|x_i \curlywedge x_{i+1}| : i \geq n\big\} = na + o(n).$$

Since $|x_n| = na + o(n) \geq |x_n \curlywedge \mathfrak{u}|$, this implies that

$$d(x_n, \mathfrak{u}_o(\lfloor an \rfloor)) = o(n). \qquad \square$$

We now reformulate condition (ii) in terms of the horospherical ordering of $\mathbb{T}$.

PROPOSITION 1. — *A sequence (x_n) in $\mathbb{T}$ is regular if and only if*

(i) $\displaystyle\lim_{n\to\infty} \frac{1}{n} d(x_n, x_{n+1}) = 0$ *and*

(ii′) *the limit* $\displaystyle a_h = \lim_{n\to\infty} \frac{1}{n} h(x_n)$ *exists.*

In this case the rate of escape of the sequence (x_n) equals $|a_h|$, and the following holds :

(1) *If $a_h < 0$, then (x_n) converges to the end ω.*

(2) *If $a_h = 0$, then (x_n) is a trivial regular sequence.*

(3) *If $a_h > 0$, then (x_n) converges to an end $\mathfrak{u} \in \partial^*\mathbb{T}$.*

Proof. — First suppose that (x_n) is a regular sequence with rate of escape a. If $a = 0$, then

$$\frac{1}{n}\big|h(x_n)\big| \leq \frac{1}{n}|x_n| \to 0.$$

If $a > 0$, then (x_n) converges to some $\mathfrak{u} \in \partial\mathbb{T}$, and

$$\big|h(x_n) - h\big(\mathfrak{u}_o(\lfloor an \rfloor)\big)\big| \leq d\big(x_n, \mathfrak{u}_o(\lfloor an \rfloor)\big) = o(n).$$

Now, if $\mathfrak{u} = \omega$ then $h(\mathfrak{u}_o(\lfloor an \rfloor)) = -\lfloor an \rfloor$, while for $\mathfrak{u} \in \partial^*\mathbb{T}$ we have $h(\mathfrak{u}_o(\lfloor an \rfloor)) = \lfloor an \rfloor - 2|\mathfrak{u} \curlywedge \omega|$, when n is sufficiently large. Thus, $n^{-1}h(x_n)$ converges and satisfies the above trichotomy.

Conversely, suppose that (i) and (ii′) hold. In view of Lemma 3 we only have to show that $n^{-1}|x_n| \to |a_h|$. Since

$$h(x_n \wedge x_{n+1}) = \tfrac{1}{2}\big(h(x_n) + h(x_{n+1}) - d(x_n, x_{n+1})\big) = na_h + o(n),$$

it is clear that $\liminf n^{-1}|x_n| \geq \lim n^{-1}|h(x_n)| = |a_h|$.

In proving that $\limsup n^{-1}|x_n| \leq |a_h|$, we may assume without loss of generality that $x_0 = o$. Let

$$z_n = \bigwedge_{0 \leq i \leq n} x_i.$$

Case 1. — Suppose $a_h \leq 0$. By Lemma 2,

$$h(z_n) = \min\big\{h(x_k \wedge x_{k+1}) \,:\, k = 1, \ldots, n-1\big\} = na_h + o(n).$$

Thus,

$$\begin{aligned}|x_n| &= d(x_0, x_n) = h(o) + h(x_n) - 2h(x_0 \wedge x_n) \\ &\leq h(x_n) - 2h(z_n) = n|a_h| + o(n).\end{aligned} \tag{2.3}$$

Case 2. — Suppose $a_h > 0$. Then $h(x_n \wedge x_{n+1}) \to \infty$, and by Lemma 2, the sequence $\big(h(z_n)\big)$ must stabilize. As $z_n \in \overline{o\omega}$, this means that (z_n) itself stabilizes, that is, there exists a vertex $z = \bigwedge_n x_n$ in $\mathbb{T}$. Consequently

$$|x_n| \leq h(x_n) - 2h(z) = na_h + o(n).$$

The above results on regular sequences are in direct analogy with the characterization of regular sequences in symmetric spaces in polar and horospheric coordinates, as introduced and studied by Kaimanovich [K2] motivated by the notion of Lyapunov regularity.

D. The affine group.

An *automorphism* of the tree $\mathbb{T}$ is a self-isometry of the metric space $(\mathbb{T}, d)$. Equipped with the topology of pointwise convergence, the group $\mathrm{Aut}(\mathbb{T})$ of all automorphisms of $\mathbb{T}$ is a locally compact Hausdorff group. A neighbourhood base at the identity is given by the family of all pointwise stabilizers of finite sets of vertices. These are open and compact, so that $\mathrm{Aut}(\mathbb{T})$ is totally disconnected. Every automorphism extends naturally to a homeomorphism (not necessarily an isometry!) of $\widehat{\mathbb{T}}$ onto itself, and the mapping $(\gamma, z) \mapsto \gamma z$ of $\mathrm{Aut}(\mathbb{T}) \times \widehat{\mathbb{T}}$ onto $\widehat{\mathbb{T}}$ is jointly continuous.

The *affine group* of the tree $\mathbb{T}$ is the group $\mathrm{AFF}(\mathbb{T})$ of all isometries $\gamma \in \mathrm{AUT}(\mathbb{T})$ which fix ω. (Changing the reference end ω means passing to a conjugate of this group. This justifies calling it «the» affine group.) The name is chosen because of the analogy with the Poincaré upper half plane : there, the group of all isometries which fix the point at infinity coincides with the affine group of the real line. Furthermore, as we shall see below, if q is a prime power, then the affine group of any local field having residual field of order q embeds naturally into $\mathrm{AFF}(\mathbb{T}_q)$. Note that $\mathrm{AFF}(\mathbb{T})$ is closed in $\mathrm{AUT}(\mathbb{T})$ and that it acts transitively on $\mathbb{T}$ and on $\partial^*\mathbb{T}$.

Since $\mathrm{AFF}(\mathbb{T})$ preserves ω, we have :

$$\gamma(\overline{x\,\omega}) = \overline{(\gamma x)\,\omega} \qquad \forall\,\gamma \in \mathrm{AFF}(\mathbb{T}),\ x \in \mathbb{T}.$$

This fundamental relation is used in many of the subsequent proofs without explicit mention. Hence, $\mathrm{AFF}(\mathbb{T})$ maps horocycles onto horocycles, and

$$h(\gamma x_1) - h(\gamma x_2) = h(x_1) - h(x_2) \qquad \forall\,\gamma \in \mathrm{AFF}(\mathbb{T}),\ x_1, x_2 \in \mathbb{T}.$$

In other words, there is a group homomorphism $\Phi : \mathrm{AFF}(\mathbb{T}) \to \mathbb{Z}$ such that

$$\gamma H_n = H_{n+\Phi(\gamma)} \qquad \forall\, n \in \mathbb{Z},$$

so that $\Phi(\gamma) = h(\gamma o)$. If $x_i = \omega_x(i)$ and $\Phi(\gamma) = m$ then

$$\gamma x_i = x_{i-m} \quad \forall i \geq d(\gamma x, x \curlywedge \gamma x).$$

We note the following useful relations for $x, y \in \mathbb{T}$, $\gamma \in \mathrm{AFF}(\mathbb{T})$ (the second follows from (2.2)) :

$$(2.4) \qquad h(\gamma x \wedge \gamma y) = \Phi(\gamma) + h(x \wedge y) \quad \text{and} \quad 2|o \wedge \gamma o| = |\gamma| - \Phi(\gamma).$$

In particular,

$$|\gamma| = |o \wedge \gamma^{-1} o| + |o \wedge \gamma o| \quad \text{and} \quad \Phi(\gamma) = |o \wedge \gamma^{-1} o| - |o \wedge \gamma o|.$$

By Nebbia [Ne] and Soardi and Woess [SW], the group $\mathrm{AFF}(\mathbb{T})$ is *amenable* and *non-unimodular* with modular function $\gamma \mapsto q^{\Phi(\gamma)}$, $\gamma \in \Gamma$.

The *horocyclic group* $\mathrm{HOR}(\mathbb{T})$ is the kernel of Φ : it consists of all $\beta \in \mathrm{AFF}(\mathbb{T})$ which preserve H_0 (and hence every other horocycle) as a set. If we write

$$\mathcal{O}_n = \{\beta \in \mathrm{AFF}(\mathbb{T}) \,:\, \beta \text{ fixes } \omega_o(n)\},$$

then each $\mathcal{O}_n$ is compact, $\mathcal{O}_n \subset \mathcal{O}_{n+1}$ and

$$\textsc{Hor}(\mathbb{T}) = \bigcup_{n=0}^{\infty} \mathcal{O}_n .$$

Indeed, if $\beta \in \textsc{Hor}(\mathbb{T})$, then β fixes $o \wedge \beta o \in \overline{o\omega}$. Note that an element $\beta \in \textsc{Hor}(\mathbb{T})$ is in $\mathcal{O}_n$ if and only if $|o \wedge \beta o| \le n$.

Every $\gamma \in \textsc{Aff}(\mathbb{T})$ with $\Phi(\gamma) \neq 0$ has a unique fixed point $\mathfrak{f}^\gamma \in \partial^*\mathbb{T}$, see for example Tits [Ti]. We choose and fix $\sigma \in \Phi^{-1}(\{1\})$ such that $\sigma^{-1}o = \omega_o(1)$. Hence, $\sigma^{-m}o = \omega_o(m)$, and $\{\sigma^m o : m \in \mathbb{Z}\}$ is the geodesic between ω and $\mathfrak{f}^\sigma$. Every $\gamma \in \textsc{Aff}(\mathbb{T})$ can be written uniquely as $\gamma = \beta\sigma^m$, where $m = \Phi(\gamma) \in \mathbb{Z}$, and $\beta = \beta(\gamma) = \gamma\sigma^{-m} \in \textsc{Hor}(\mathbb{T})$. The choice of σ determines a *semidirect product* decomposition

$$\textsc{Aff}(\mathbb{T}) = \mathbb{Z} \ltimes \textsc{Hor}(\mathbb{T}), \qquad \beta\sigma^m \equiv (m,\beta)\,;$$

the action of $\mathbb{Z}$ on $\textsc{Hor}(\mathbb{T})$ is given by group automorphisms $A^m\beta = \sigma^m\beta\sigma^{-m}$, so that

$$(m_1,\beta_1)\cdot(m_2,\beta_2) = \big(m_1+m_2, \beta_1(A^{m_1}\beta_2)\big).$$

We consider two «length functions» on $\textsc{Aff}(\mathbb{T})$: if $\gamma = (m,\beta) \in \textsc{Aff}(\mathbb{T})$ then

$$|\gamma| = |\gamma o| \quad \text{and} \quad \|\gamma\| = |m| + \tfrac{1}{2}|\beta|.$$

Note that zero length does in general *not* imply that $\gamma = \iota$, the identity of $\textsc{Aff}(\mathbb{T})$. It is obvious that $|\cdot|$ is subadditive (i.e. $|\gamma_1\gamma_2| \le |\gamma_1| + |\gamma_2|$) and symmetric (i.e. $|\gamma^{-1}| = |\gamma|$), while $\|\cdot\|$ is neither.

LEMMA 4.

(a) *If* $\beta_1, \beta_2 \in \textsc{Hor}(\mathbb{T})$, *then*

$$|\beta_1\beta_2| \le \max\big\{|\beta_1|, |\beta_2|\big\}.$$

(b) *If* $\beta \in \textsc{Hor}(\mathbb{T})$ *and* $m \ge 0$ *then* $|A^m\beta| \le |\beta|$. *Furthermore, if* $\gamma_1,\dots,\gamma_n \in \textsc{Aff}(\mathbb{T})$ *are such that* $\Phi(\gamma_1\cdots\gamma_k) \ge 0$ *for all* $k < n$ *then*

$$\big|\beta(\gamma_1\cdots\gamma_n)\big| \le \max\big\{|\beta(\gamma_1)|,\dots,|\beta(\gamma_n)|\big\}.$$

(c) *For all* $\gamma \in \textsc{Aff}(\mathbb{T})$,

$$\big|\beta(\gamma)\big| = 2d(\gamma^{-1}o, \overline{\mathfrak{f}^\sigma\,\omega}) \quad \textit{and} \quad |\gamma| = \max\big\{|\Phi(\gamma)|, \Phi(\gamma) + |\beta(\gamma)|\big\}.$$

(d) *For all* $\gamma \in \textsc{Aff}(\mathbb{T})$,

$$\tfrac{1}{2}|\gamma| \le \|\gamma\| \le 2|\gamma|.$$

Proof.

(a) Set $n = \frac{1}{2}\max\{|\beta_1|, |\beta_2|\}$. Then $\beta_1, \beta_2 \in \mathcal{O}_n$, so that $\beta = \beta_1\beta_2$ also belongs to $\mathcal{O}_n$, and hence $|\beta| = 2|o \wedge \beta o| \le 2n$.

(b) Set $x = o \wedge \beta o \in \overline{o\omega}$. Then $|\beta| = 2|x|$, and β fixes every vertex on $\overline{x\,\omega}$, in particular $\sigma^{-m}x$. Consequently, $A^m\beta x = x$, and $|A^m\beta| \le 2|x|$. For the second statement, observe that by the semidirect product decomposition,

$$\beta(\gamma_1\cdots\gamma_n) = \beta(\gamma_1)\big(A^{\Phi(\gamma_1)}\beta(\gamma_2)\big)\cdots\big(A^{\Phi(\gamma_1\cdots\gamma_{n-1})}\beta(\gamma_n)\big).$$

Now the first statement, together with (a), implies the result.

(c) Let $\gamma = (m, \beta) = \beta\sigma^m$. For the first identity,

$$|\beta| = |\beta^{-1}| = d(\gamma^{-1}o, \sigma^{-m}o) = 2d(\gamma^{-1}o, \gamma^{-1}o \wedge \mathfrak{f}^\sigma) = 2d(\gamma^{-1}o, \overline{\mathfrak{f}^\sigma\,\omega}).$$

To see the second identity, set $h = h(o \wedge \beta^{-1}o)$. Then $|\beta| = -2h$ and $-h = d(\gamma^{-1}o, \overline{\mathfrak{f}^\sigma\,\omega})$. Thus, $o \wedge \beta^{-1}o = \sigma^h o$ lies on the geodesic segment connecting $\beta^{-1}o$ and $\sigma^m o$, and

$$\begin{aligned}|\gamma| &= d(\beta^{-1}o, \sigma^h o) + d(\sigma^h o, \sigma^m o) = -h + |m - h|\\ &= \max\{-m, m - 2h\} = \max\big\{|m|, m + |\beta|\big\}.\end{aligned}$$

(d) Since $\|\gamma\| = |m| - h$ in the notation of (c), this is a direct corollary of the identity $|\gamma| = -h + |m - h|$ obtained in the proof of (b). □

E. Exceptional and non-exceptional subgroups of $\text{Aff}(\mathbb{T})$**.**

In the following, we consider closed subgroups Γ of $\text{Aff}(\mathbb{T})$, and denote by Γ_x the stabilizer of vertex $x \in \mathbb{T}$ in Γ.

We say that Γ is *exceptional,* if $\Gamma \subset \text{Hor}(\mathbb{T})$ or if Γ fixes an element of $\partial^*\mathbb{T}$. (This is an analogue of a one-dimensional subgroup of the affine group over $\mathbb{R}$.)

THEOREM 1. — *Γ is exceptional if and only if it is unimodular.*

Proof. — First, suppose that Γ is exceptional.

If $\Gamma \subset \text{Hor}(\mathbb{T})$ then it is the union of an increasing sequence of compact groups and hence unimodular. If Γ fixes $\mathfrak{u} \in \partial^*\mathbb{T}$ and is not contained in $\text{Hor}(\mathbb{T})$, then the restriction Φ_Γ of the homomorphism $\Phi : \text{Aff}(\mathbb{T}) \to \mathbb{Z}$ onto Γ is nontrivial. On the other hand, Γ preserves the infinite geodesic $\overline{\mathfrak{u}\omega}$ as a set and acts on it by translations. Thus,

$\ker \Phi_\Gamma = \Gamma_x$ for any point x on $\overline{\mathfrak{u}\omega}$. As Γ_x is compact, and $\Phi_\Gamma(\Gamma) \cong \mathbb{Z}$, the group Γ must be unimodular.

Second, suppose that Γ is non-exceptional. We shall prove existence of $\gamma \in \Gamma$ and $x \in \mathbb{T}$ such that

$$\sharp(\Gamma_x \gamma x) \neq \sharp(\Gamma_{\gamma x} x)$$

(with $\sharp(\cdot)$ denoting cardinality). By Schlichting [Sch] and Trofimov [Tr], this is equivalent with non-unimodularity of Γ.

As $\Gamma \setminus \mathrm{HOR}(\mathbb{T}) \neq \emptyset$, the homomorphism Φ_Γ is non-trivial and so $\Phi(\Gamma) = k_0\mathbb{Z}$, where

$$k_0 = \min\{\Phi(\gamma) > 0 : \gamma \in \Gamma\}.$$

Choose $\gamma \in \Gamma$ with $\Phi(\gamma) = k_0$. It acts as a translation on the geodesic $\overline{\omega\, \mathfrak{f}^\gamma}$. By assumption, Γ does not fix $\mathfrak{u} = \mathfrak{f}^\gamma$. Hence there is $\alpha \in \Gamma$ such that $\alpha\mathfrak{u} \neq \mathfrak{u}$. Now $\Phi(\alpha) = \ell \cdot k_0$ for some $\ell \in \mathbb{Z}$. Let $\beta = \gamma^{-\ell}\alpha$, and set $\mathfrak{v} = \beta\mathfrak{u}$. Then $\Phi(\beta) = 0$ and $\mathfrak{v} \in \partial^*\mathbb{T} \setminus \{\mathfrak{u}\}$. Let $x = \mathfrak{u} \wedge \mathfrak{v}$. Then $\gamma x = \mathfrak{u}_x(k_0)$ and $\beta\gamma x = \mathfrak{v}_x(k_0)$ are different, and $\beta \in \Gamma_x$; see Figure 2.

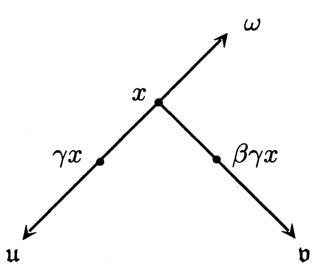

Figure 2

Hence $\Gamma_x\gamma x$ contains γx and $\beta\gamma x$, while $\Gamma_{\gamma x} x = \{x\}$ due to the fact that Γ fixes ω. □

In particular, we see from Theorem 1 that $\mathrm{AFF}(\mathbb{T})$ contains no discrete non-exceptional subgroups.

The *limit set* $\partial\Gamma$ of Γ is the set of accumulation points of an orbit Γx in $\partial\mathbb{T}$. It does not depend on the choice of $x \in \mathbb{T}$.

PROPOSITION 2. — *If Γ is non-exceptional then $\omega \in \partial\Gamma$, and $\partial\Gamma$ is uncountable. Let $\partial^*\Gamma = \partial\Gamma \setminus \{\omega\}$. Then for each $\mathfrak{u} \in \partial^*\Gamma$, the orbit $\Gamma\mathfrak{u}$ is dense in $\partial\Gamma$.*

Proof. — There are $\gamma_1, \gamma_2 \in \Gamma$ and distinct $\mathfrak{u}_1, \mathfrak{u}_2 \in \partial^*\mathbb{T}$ such that $\Phi(\gamma_i) = k_0 > 0$ and γ_i acts as a translation on $\overline{\omega\,\mathfrak{u}_i}$, $i = 1, 2$. Indeed, with γ and α as in the second part of the proof of Theorem 1, take $\gamma_1 = \gamma$ and $\gamma_2 = \alpha\gamma\alpha^{-1}$. Again, let $x = \mathfrak{u}_1 \wedge \mathfrak{u}_2$. For every sequence (k_n) of positive integers,

$$\gamma_1^{k_1}\gamma_2^{k_1}\gamma_1^{k_2}\gamma_2^{k_2}\cdots\gamma_1^{k_n}\gamma_2^{k_n}o$$

converges to an end in $\partial^*\mathbb{T}$ as $n \to \infty$, and different sequences give rise to different limits. Hence, $\partial^*\Gamma$ is uncountable. (This type of argument bears some similarity with Klein's « ping-pong lemma », see e.g. de la Harpe [dH].)

Let $\mathfrak{v} \in \partial^*\Gamma$, so that $\gamma_n o \to \mathfrak{v}$ for some sequence (γ_n) in Γ. Then $\gamma_n^{-1}o \to \omega$. By Cartwright and Soardi [CS], Lemma 2.2, $\gamma_n^{-1}\mathfrak{u} \to \omega$ for every $\mathfrak{u} \in \partial^*\mathbb{T} \setminus \{\mathfrak{v}\}$, so that $\omega \in \partial\Gamma$ and ω is an accumulation point of $\Gamma\mathfrak{u}$ for every $\mathfrak{u} \in \partial^*\Gamma$. Also by [CS], $\gamma_n\mathfrak{u} \to \mathfrak{v}$ for every $\mathfrak{u} \in \partial^*\mathbb{T}$. Given $\mathfrak{u}$, choose $\alpha \in \Gamma$ with $\alpha\mathfrak{u} \neq \mathfrak{u}$. Then at least one of $(\gamma_n\mathfrak{u})_n$ and $(\gamma_n\alpha\mathfrak{u})_n$ has an infinite subsequence of elements different from $\mathfrak{v}$. This shows that $\mathfrak{v}$ is a limit point of $\Gamma\mathfrak{u} \setminus \{\mathfrak{v}\}$. □

We see from Theorem 1 and Proposition 2 that closed subgroups Γ of $\mathrm{A{\scriptsize FF}}(\mathbb{T})$ can be classified in terms of the cardinality of their limit sets as follows :

(i) $\sharp\,\partial\Gamma = 0$ if and only if Γ is compact;

(ii) $\sharp\,\partial\Gamma = 1$ if and only if Γ is noncompact and contained in $\mathrm{H{\scriptsize OR}}(\mathbb{T})$;

(iii) $\sharp\,\partial\Gamma = 2$ if and only if Γ is a compact extension of the infinite cyclic group;

(iv) $\sharp\,\partial\Gamma = \infty$ if and only if Γ is non-exceptional.

We can consider $\partial\Gamma$ as the boundary in a compactification $\widehat{\Gamma}$ of Γ : a sequence (γ_n) in Γ tends to $\mathfrak{u} \in \partial\Gamma$, if $\gamma_n x \to \mathfrak{u}$ for some (and hence every) $x \in \mathbb{T}$.

3. Random walks on $\mathrm{A{\scriptsize FF}}(\mathbb{T})$.

A. Right and left random walk.

Let μ be a Borel probability measure on $\mathrm{A{\scriptsize FF}}(\mathbb{T})$. *We shall always assume that the closed subgroup $\Gamma = \Gamma(\mu)$ of $\mathrm{A{\scriptsize FF}}(\mathbb{T})$ generated by the support of μ is non-exceptional.* In other words, $\operatorname{supp}\mu$ neither fixes a horocycle nor an element of $\partial^*\mathbb{T}$.

Let $(X_n)_{n\geq 1}$ be a sequence of independent Γ-valued random variables with common distribution μ. The *right random walk on* Γ (or on $\mathrm{AFF}(\mathbb{T})$) *with law* μ is the sequence of random variables

$$R_0 = \iota, \quad R_n = X_1 \cdots X_n \qquad (n \geq 1),$$

and the *left* random walk is

$$L_0 = \iota, \quad L_n = X_n \cdots X_1 \qquad (n \geq 1).$$

For their increments we shall use the notation

$$R_{k,n} = R_k^{-1} R_n \quad \text{and} \quad L_{n,k} = L_n L_k^{-1} \qquad (k \leq n).$$

Both R_n and L_n have distribution $\mu^{(n)}$, the n-th convolution power of μ. As Γ is non-unimodular by Theorem 1, both (R_n) and (L_n) are *transient,* that is, with probability one they leave every compact set after finite time, see Guivarc'h, Keane and Roynette [GKR], Thm 51.

Note that for $x \in \mathbb{T}$, $(L_n x)$ is a Markov chain, while in general $(R_n x)$ is not. On the other hand, the right random walk has independent distance increments $d(R_{n-1}x, R_n x) = d(x, X_n x)$, which makes it in some sense better adapted to the tree structure than the left walk. We shall denote by $G(\,\cdot\,,\cdot\,)$ the *Green function* of the left random walk on $\mathbb{T}$

$$G(x,y) = \sum_{n=0}^{\infty} \Pr[L_n x = y],$$

and by $\check{G}(\,\cdot\,,\cdot\,)$ the Green function of the left random walk with law $\check{\mu}$ (the image of μ under the reflection $\gamma \mapsto \gamma^{-1}$). Thus,

$$\check{G}(x,y) = \sum_{n=0}^{\infty} \Pr[R_n y = x]. \tag{3.1}$$

The (absolute) moment of order $r > 0$ of the measure μ is

$$m_r(\mu) = \mathbb{E}\big(|X_1|^r\big) = \int_\Gamma d(o, \gamma o)^r \mu(\mathrm{d}\gamma).$$

Let $\Phi(\mu)$ be the image of μ on $\mathbb{Z}$, that is, $\Phi(\mu)(k) = \mu(\Phi^{-1}\{k\})$. Then

$$\Phi(R_n) = \Phi(L_n) = \Phi(X_1) + \cdots + \Phi(X_n)$$

is a sum of i.i.d. random variables $\Phi(X_i)$ with law $\Phi(\mu)$. We denote by $m_r\big(\Phi(\mu)\big)$ the (absolute) moment of order r of $\Phi(\mu)$. If $m_1\big(\Phi(\mu)\big)$ is finite, then the mean displacement (drift) is

$$\overline{\Phi(\mu)} = \sum_{k\in\mathbb{Z}} k\,\Phi(\mu)(k) = \int_\Gamma \Phi(\gamma)\,\mu(\mathrm{d}\gamma).$$

B. Convergence to ends.

Transience implies that $d(o, R_n o)$ tends to infinity almost surely. We first address the problem of convergence in the end topology.

THEOREM 2.

(a) *If $m_1(\Phi(\mu)) < \infty$ and $\overline{\Phi(\mu)} < 0$, then $R_n o \to \omega$ almost surely.*

(b) *If $m_1(\mu) < \infty$ and $\overline{\Phi(\mu)} > 0$ then $(R_n o)$ converges almost surely to a random element of $\partial^*\mathbb{T}$.*

(c) *If $m_1(\mu) < \infty$, $\overline{\Phi(\mu)} = 0$ and*

$$\mathbb{E}\big(|o\wedge X_1^{-1}o|\; q^{|o\wedge X_1 o|}\big) < \infty$$

then $R_n o \to \omega$ almost surely. This holds, in particular, under the simpler condition

$$\mathbb{E}\big(q^{|X_1|}\big) = \int_\Gamma q^{d(o,\gamma o)}\,\mu(\mathrm{d}\gamma) < \infty.$$

Proof.

(a) The assumptions yield $\Phi(R_n) \to -\infty$. But then $R_n o \to \omega$.

(b) We have by the classical law of large numbers

$$\lim_{n\to\infty} \frac{1}{n}\Phi(R_n) = \overline{\Phi(\mu)} \quad \text{almost surely.}$$

Integrability of $|X_1|$ yields

$$\lim_{n\to\infty} \frac{1}{n} d(R_n o, R_{n+1} o) = \lim_{n\to\infty} \frac{1}{n}|X_{n+1}| = 0 \quad \text{almost surely.}$$

Setting $x_n = R_n o$, properties (i) and (ii′) of Proposition 1 are satisfied with $a_h = \overline{\Phi(\mu)}$. Therefore, $R_n o$ converges almost surely to a random end in $\partial^*\mathbb{T}$.

(c) We have to show that $|o\wedge R_n o| \to \infty$ almost surely. Since the random walk $\big(\Phi(R_n)\big)$ on $\mathbb{Z}$ has mean zero, it is recurrent, see Spitzer [Sp], § 8. Thus, with probability one $R_n o \in H_0$ infinitely often. On the other hand, $|R_n| \to \infty$ by transience. Hence $|o \wedge R_n o|$ is unbounded.

Suppose that $|o \wedge R_n o| > |o \wedge R_{n+1} o|$, so that $o \wedge R_n o \preccurlyeq o \wedge R_{n+1} o$ and $R_n o \wedge R_{n+1} o = o \wedge R_n o$. Then

$$\begin{aligned}|o \wedge X_{n+1} o| &= d(R_n o, R_n o \wedge R_{n+1} o) = d(R_n o, o \wedge R_n o),\\ d(X_{n+1} o, o \wedge X_{n+1} o) &= d(R_{n+1} o, R_n o \wedge R_{n+1} o)\\ &\geq |o \wedge R_n o| - |o \wedge R_{n+1} o|,\end{aligned}$$

see Figure 3.

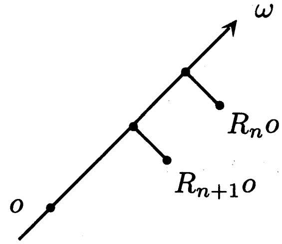

Figure 3

Fix a positive integer ℓ. Then for any point $y \in \mathbb{T}$ such that $|o \wedge y| > \ell$ we have :

$$\begin{aligned}&\Pr\big[R_n o = y, |o \wedge R_{n+1} o| = \ell\big]\\ &\quad \leq \Pr\big[R_n o = y\big] \Pr\big[\,|o \wedge X_1 o| = d(u, o \wedge y),\\ &\qquad\qquad\qquad\qquad d(X_1 o, o \wedge X_1 o) \geq |o \wedge y| - \ell\big].\end{aligned}$$

For positive integers j, m let

$$A_{m,j} = \big\{y \in \mathbb{T} \,:\, d(y, o \wedge y) = j,\ |o \wedge y| = \ell + m\big\},$$

so that $A_{m,j}$ is contained in the ball of radius j centered at $\sigma^{-\ell-m} o$. Then

$$\begin{aligned}&\sum_{n=0}^{\infty} \Pr\big[|o \wedge R_{n+1} o| = \ell < |o \wedge R_n o|\big]\\ &\leq \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=1}^{\infty} \sum_{y \in A_{m,j}} \Pr[R_n o = y] \Pr\big[|o \wedge X_1 o| = j\\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{and } d(X_1 o, o \wedge X_1 o) \geq m\big]\\ &= \sum_{j=0}^{\infty} \sum_{m=1}^{\infty} \sum_{y \in A_{m,j}} \check{G}(y, o) \Pr\big[|o \wedge X_1 o| = j \text{ and } d(X_1 o, o \wedge X_1 o) \geq m\big]\\ &\leq \check{G}(o, o) \sum_{j=0}^{\infty} \sum_{m=1}^{\infty} (\sharp\, A_{m,j}) \Pr\big[|o \wedge X_1 o| = j \text{ and } d(X_1 o, o \wedge X_1 o) \geq m\big]\\ &\leq \text{const} \sum_{m=1}^{\infty} \sum_{j=0}^{\infty} q^j \Pr\big[|o \wedge X_1 o| = j \text{ and } d(X_1 o, o \wedge X_1 o) \geq m\big]\\ &= \text{const} \cdot \mathbb{E}\big(|o \wedge X_1 o|\, q^{d(X_1 o, o \wedge X_1 o)}\big) < \infty.\end{aligned}$$

(We have used the inequality $\check{G}(x, y) \leq \check{G}(y, y)$ for the Green function (3.1), well known in the theory of Markov chains.) Thus, given any $\ell > 0$, by the Borel-Cantelli Lemma, with probability one the events

$$\big[|o \wedge R_{n+1}o| = \ell < |o \wedge R_n o|\big]$$

may occur only for finitely many n. Since $|o \wedge R_n o|$ is almost surely unbounded, this means that $|o \wedge R_n o| \to \infty$ almost surely. $\square$

We remark that for (a) and (b), it is not needed that $\operatorname{supp}\mu$ generates a non-exceptional subgroup of $\mathrm{A{\scriptstyle FF}}(\mathbb{T})$.

C. Solution of the Dirichlet problem.

When $(R_n o)$ converges in the end topology (equivalently, (R_n) converges in the compactification $\widehat{\Gamma}$ of Γ, see § 2.E), let ν be the distribution of its limit $R_\infty \in \partial\Gamma \subset \partial\mathbb{T}$; see Theorem 2. This is the *harmonic measure* of the random walk. Since $X_1^{-1}R_\infty$ (being the limit of $X_2 \cdots X_n o$) has the same distribution as R_∞, the measure ν is *μ-stationary*, that is, $\mu * \nu = \nu$, where

$$\int_{\partial\Gamma} f(\mathrm{u})\, \mu * \nu(\mathrm{du}) = \int_\Gamma \int_{\partial\Gamma} f(\gamma \mathrm{u})\, \nu(\mathrm{du})\mu(\mathrm{d}\gamma)$$

for $f \in C(\partial\Gamma)$. Thus, for such f, the function

$$g(\gamma) = \int_{\partial\mathbb{T}} f(\gamma\mathrm{u})\, \nu(\mathrm{du})\,, \quad \gamma \in \Gamma \tag{3.2}$$

is *bounded* and *μ-harmonic,* that is,

$$\int_\Gamma g(\alpha\gamma)\, \mu(\mathrm{d}\gamma) = g(\alpha) \quad \text{for all } \alpha \in \Gamma\,. \tag{3.3}$$

The solution of the Dirichlet problem for μ-harmonic functions on Γ relative to the compactification $\partial\Gamma$ is statement (3) below.

THEOREM 3. — *If $m_1(\mu) < \infty$ and $\overline{\Phi(\mu)} > 0$ then*

(1) *ν is a continuous measure (it carries no point mass).*

If in addition $\operatorname{supp}\mu$ *generates Γ as a closed semigroup, then*

(2) *ν is supported by the whole of $\partial\Gamma$, and*

(3) *for every continuous function f on $\partial\Gamma$, (3.2) gives its unique continuous extension to $\widehat{\Gamma}$ which is harmonic on Γ.*

Proof.

(1) (Compare with [CS].) We have $\nu(\{\omega\}) = 0$, as $R_\infty \in \partial^*\Gamma$ almost surely. Let $M = \max\{\nu(\{\mathfrak{u}\}) : \mathfrak{u} \in \partial\mathbb{T}\}$. Suppose that $M > 0$. Then

$$S = \{\mathfrak{u} \in \partial\mathbb{T} : \nu(\{\mathfrak{u}\}) = M\}$$

is finite and contained in $\partial^*\mathbb{T}$. Furthermore, as $\mu * \nu = \nu$, we have $\gamma S = S$ for every $\gamma \in \operatorname{supp}\mu$. But then Γ must fix S, in contradiction with Proposition 2. Hence ν is continuous.

(2) (Compare with Woess [W1], Lemma 3.7.) For verifying that $\operatorname{supp}\nu = \partial\Gamma$, it is enough to show that $\nu(C_x) > 0$ for every x with $C_x \cap \partial^*\Gamma \neq \emptyset$. For such x there must be $\alpha \in \Gamma$ with $\alpha x \in C_x \setminus \{x\}$. But then $\Phi(\alpha) > 0$, and α must fix some end $\mathfrak{u} \in C_x$. Once more, α acts as a translation on $\overline{\mathfrak{u}\,\omega}$. Consequently, $\alpha^{-n}C_x = C_{\alpha^{-n}x}$ is an increasing sequence of sets tending to $\partial^*\mathbb{T}$. As $\nu(\partial^*\mathbb{T}) = 1$, there is n with $\nu(\alpha^{-n}C_x) > 0$. Without loss of generality, suppose that $n = 1$. Set $y = \alpha^{-1}x$. Then $\alpha\Gamma_y = \{\gamma \in \Gamma : \gamma y = x\}$ is nonempty and compact-open in Γ. Therefore $\mu^{(k)}(\alpha\Gamma_y) > 0$ for some $k \geq 1$. We get

$$\nu(C_x) = \mu^{(k)} * \nu(C_x) \geq \int_{\alpha\Gamma_y} \nu(\gamma^{-1}C_x)\,\mu^{(k)}(\mathrm{d}\gamma) = \nu(C_y)\,\mu^{(k)}(\alpha\Gamma_y) > 0.$$

(3) (Compare with [W1], Thm 6.2.) If $f \in C(\partial\Gamma)$ then it is clear that its harmonic extension, as defined in (3.2), is continuous on Γ. Uniqueness of the extension follows from the maximum principle. It remains to prove that for every $\mathfrak{u} \in \partial\Gamma$, $\gamma_n \to \mathfrak{u}$ implies $g(\gamma_n) \to f(\mathfrak{u})$, or, equivalently, that $\gamma_n\,\nu = \delta_{\gamma_n} * \nu \to \delta_{\mathfrak{u}}$ weakly.

Let $\gamma_n \to \mathfrak{u}$ for a sequence (γ_n) in Γ, that is $\gamma_n o \to \mathfrak{u}$. By compactness, we may assume without loss of generality that $\gamma_n^{-1}o$ converges to some $\mathfrak{v} \in \partial\Gamma$. But then, once more by [CS], Lemma 2.2, $\gamma_n\mathfrak{z} \to \mathfrak{u}$ for every $\mathfrak{z} \in \partial\mathbb{T} \setminus \{\mathfrak{v}\}$. As $\nu(\{\mathfrak{v}\}) = 0$, we get that $\gamma_n\,\nu \to \delta_{\mathfrak{u}}$. □

Observe that solvability of the Dirichlet problem requires the existence of a solution ν of $\mu * \nu = \nu$ which is supported by the whole of $\partial\Gamma$. But in this case, [W1], Cor. 3.6 and Thm 3.3 show that $(R_n o)$ must converge with continuous limiting distribution. (In [W1], Γ is assumed to act transitively, but the proofs do not change when Γ is non-exceptional.) On the other hand, when $\overline{\Phi(\mu)} < 0$ then $R_n o \to \omega$ almost surely; when $\overline{\Phi(\mu)} = 0$ then transience of (R_n) and recurrence of $(\Phi(R_n))$ imply that $(R_n o)$ accumulates at ω with probability one. Hence, if $m_1(\Phi(\mu)) < \infty$ and $\overline{\Phi(\mu)} \le 0$ then the Dirichlet problem does not admit solution.

D. Law of large numbers.

As $d(o,x) \ge |h(x)|$ for $x \in \mathbb{T}$ and $\Phi(\gamma) = h(\gamma o)$ for $\gamma \in \mathrm{AFF}(\mathbb{T})$, we have

$$m_1(\mu) \ge m_1(\Phi(\mu)).$$

If $m_1(\mu)$ is finite, then by Kingman's subadditive ergodic theorem [Ki],

$$\frac{1}{n} d(o, R_n o) = \frac{1}{n}|R_n| \quad \text{and} \quad \frac{1}{n} d(o, L_n o) = \frac{1}{n}|L_n|$$

converge almost surely and in L^1 with constant limits, see Guivarc'h [Gu] and Derriennic [De]. We now determine this constant. Note that it must be at least $|\overline{\Phi(\mu)}|$ by the inequality $|\Phi(\gamma)| \le |\gamma|$.

THEOREM 4. — *If $m_1(\mu) < \infty$ then*

$$\lim_{n\to\infty} \frac{1}{n}|R_n| = \lim_{n\to\infty} \frac{1}{n}|L_n| = |\overline{\Phi(\mu)}| \quad \textit{almost surely and in } L^1.$$

Proof. — As in the proof of Theorem 2 (b), setting $x_n = R_n o$, the conditions of Proposition 1 are satisfied with $a_h = \overline{\Phi(\mu)}$. Observing that (L_n^{-1}) is the right random walk with law $\check{\mu}$, the result for (L_n) follows immediately. □

The interpretation of the fact that the limit is not any larger than $|\overline{\Phi(\mu)}|$ is that, asymptotically, the contribution coming from horizontal moves of $R_n o$ is negligible compared with the vertical drift. (« Horizontal » and « vertical » refer to viewing the tree as in Fig. 1.)

Next, recall that $\|\cdot\|$ is not symmetric. It seems to be difficult to derive a law of large numbers for $\|R_n\|$. However, this can be achieved for $\|R_n^{-1}\|$ (or equivalently, replacing μ with $\check{\mu}$, for $\|L_n\|$).

THEOREM 5. — *If $m_1(\mu) < \infty$ then*

$$\lim_{n\to\infty} \frac{1}{n}\|R_n^{-1}\| = \begin{cases} 2\overline{\Phi(\mu)} & \textit{if } \overline{\Phi(\mu)} > 0, \\ |\,\overline{\Phi(\mu)}\,| & \textit{if } \overline{\Phi(\mu)} \le 0 \end{cases}$$

almost surely and in L^1.

Proof. — We show that

$$\lim_{n\to\infty} \frac{1}{n} |\beta(R_n^{-1})| = 2\max\{0, \overline{\Phi(\mu)}\} \quad \text{almost surely.}$$

Case 1. — If $\overline{\Phi(\mu)} > 0$ then by Theorem 2, $R_n o \to R_\infty$ almost surely, and $R_\infty \in \partial^*\mathbb{T}$. Also, $\Pr[R_\infty = \mathfrak{f}^\sigma] = 0$ by Theorem 3 (1). Consequently, with probability one,

$$R_n o \wedge \mathfrak{f}^\sigma = R_\infty \wedge \mathfrak{f}^\sigma \in \mathbb{T}$$

for all sufficiently large n. Hence, by Lemma 4 (c)

$$\begin{aligned}(3.4)\qquad |\beta(R_n^{-1})| &= 2d(R_n o, \overline{\mathfrak{f}^\sigma\,\omega}) = 2d(R_n o, R_\infty \wedge \mathfrak{f}^\sigma) \\ &= 2\big(|R_n| - |R_\infty \wedge \mathfrak{f}^\sigma|\big).\end{aligned}$$

Divided by n, this tends to $2\overline{\Phi(\mu)}$ almost surely by Theorem 4.

Case 2. — If $\overline{\Phi(\mu)} = 0$ then by Lemma 4 (d)

$$\frac{1}{n} |\beta(R_n^{-1})| \le \frac{2}{n} \|R_n^{-1}\| \le \frac{4}{n} |R_n| \to 0 \quad \text{almost surely.}$$

Case 3. — If $\overline{\Phi(\mu)} < 0$ then (once more using Lemma 4 (c)) with probability one

$$\frac{1}{n} d\big(R_n o, \overline{\mathfrak{f}^\sigma\,\omega}\big) = \frac{1}{n} d(R_n o, \overline{o\omega})$$

for all but finitely many n. By Proposition 1 the sequence $(R_n o)$ is regular, hence the last term tends to zero almost surely. □

E. Central limit theorem.

In the case of nonzero drift, the CLT is an easy consequence of Theorem 2.

THEOREM 6. — *If $m_1(\mu) < \infty$, $\overline{\Phi(\mu)} \ne 0$ and $0 < \operatorname{Var}\Phi(\mu) < \infty$ then*

$$\frac{|R_n| - n\,|\,\overline{\Phi(\mu)}\,|}{\sqrt{n \operatorname{Var}\Phi(\mu)}} \longrightarrow N(0,1)$$

in law.

Proof. — First suppose that $\overline{\Phi(\mu)} > 0$. As $R_n o \to R_\infty \in \partial^* \mathbb{T}$,

$$o \wedge R_n o \longrightarrow o \wedge R_\infty \quad \text{almost surely,}$$

and the latter is a (random) vertex on $\overline{o\omega}$. Hence, with probability one we have

$$d(o, R_n o) = \Phi(R_n) + 2|o \wedge R_\infty|$$

for all but finitely many n. Now $|o \wedge R_\infty|/\sqrt{n} \to 0$ almost surely, while $\Phi(R_n)$ is a sum of n i.i.d. integer valued random variables with mean $\overline{\Phi(\mu)}$ and variance $\operatorname{Var}\Phi(\mu)$. The statement now follows from the classical central limit theorem.

Second, suppose that $\overline{\Phi(\mu)} < 0$. We have

$$d(o, R_n o) = d(o, R_n^{-1} o) = d(o, \check{L}_n o) \overset{\text{in law}}{=\!=\!=} d(o, \check{R}_n o),$$

where $(\check{L}_n)$ and $(\check{R}_n)$ are the left and right random walks with law $\check{\mu}$. As $\overline{\Phi(\check{\mu})} = -\overline{\Phi(\mu)}$, the result now follows from the case $\overline{\Phi(\mu)} > 0$. □

The driftfree case $(\overline{\Phi(\mu)} = 0)$ is much more complicated. Even finiteness of $m_2(\mu)$ does not seem to be sufficient for a CLT; we need a slightly more restrictive condition. The method extrapolates that of Grincevičjus [G1], [G2], but both context and technical details are quite different in several points. The proof relies on Proposition 3, which considers convergence in law of the process in $\Gamma \times \Gamma$ obtained by «splitting at the infimum» of $\Phi(R_n)$; Proposition 3 is stated and proved in the Appendix.

Suppose that $\overline{\Phi(\mu)} = 0$ and $\operatorname{Var}\Phi(\mu) < \infty$. Note that by the assumption of non-exceptionality, $\operatorname{Var}\Phi(\mu)$ and $\overline{\Phi(\mu)}$ cannot be equal to zero simultaneously. Set

$$\begin{cases} \overline{M}_n = \max\{\Phi(R_k) \,:\, k = 0, \dots, n\} \\ \underline{M}_n = \min\{\Phi(R_k) \,:\, k = 0, \dots, n\}. \end{cases} \tag{3.5}$$

By duality (see e.g. Feller [Fe]),

$$\Phi(R_n) - \underline{M}_n \overset{\text{in law}}{=\!=\!=} \overline{M}_n,$$

and it is known that

$$\frac{1}{\sqrt{n \operatorname{Var}\Phi(\mu)}} \big(\Phi(R_n), \overline{M}_n\big) \to (U, V) \quad \text{in law,}$$

where (U, V) is an $\mathbb{R}^2$-valued random variable whose distribution has density

$$f(u,v) = \begin{cases} \sqrt{2/\pi}\,(2v-u)\,\mathrm{e}^{-(2v-u)^2/2} & \text{if } v \geq \max\{0,u\}, \\ 0 & \text{otherwise,} \end{cases} \tag{3.6}$$

see for example Billingsley [Bi], (11.2) and (11.11). Thus, U has standard Gaussian distribution $N(0,1)$ and V has distribution $N^+(0,1)$ (the law of $|U|$). Below, we shall also use the random variables $2V-U$ and $|U|+V$, which have densities

$$f_{2V-U}(t) = \sqrt{2/\pi}\,t^2\,\mathrm{e}^{-t^2/2}, \quad t \geq 0,$$

$$f_{|U|+V}(t) = \sqrt{2/\pi}\,\left(\tfrac{1}{3}\,\mathrm{e}^{-t^2/8} + \mathrm{e}^{-t^2/2} - \tfrac{4}{3}t^{-2t^2}\right), \quad t \geq 0,$$

respectively.

THEOREM 7. — *If $m_{2+\varepsilon}(\mu) < \infty$ for some $\varepsilon > 0$ and $\overline{\Phi(\mu)} = 0$, then*

$$\frac{|R_n|}{\sqrt{n\mathrm{Var}\,\Phi(\mu)}} \longrightarrow 2V-U \ \textit{in law.}$$

Proof. — Let

$$T(n) = \max\{k \in \{0,\dots n\} \,:\, \Phi(R_k) = \underline{M}_n\}. \tag{3.7}$$

We then have, using (2.2) and (2.4),

$$\begin{aligned} |R_n| &= d\big(R_{T(n)}^{-1}o\,,\, R_{T(n),n}o\big) \\ &= h\big(R_{T(n)}^{-1}o\big) + h\big(R_{T(n),n}o\big) - 2h\big(R_{T(n)}^{-1}o \wedge R_{T(n),n}o\big) \\ &= \Phi(R_n) - 2\underline{M}_n - 2h\big(R_{T(n)}^{-1}o \wedge R_{T(n),n}o\big). \end{aligned}$$

We show in the Appendix (Proposition 3) that

$$h\big(R_{T(n)}^{-1}o \wedge R_{T(n),n}o\big)$$

converges in law to a finite random variable. Hence

$$\frac{1}{\sqrt{n}}h\big(R_{T(n)}^{-1}o \wedge R_{T(n),n}o\big) \to 0 \quad \text{in probability.}$$

Consequently, as $n \to \infty$, $|R_n|/\sqrt{n}$ behaves in law like $\big(\Phi(R_n) - 2\underline{M}_n\big)/\sqrt{n}$. Now (3.6) yields the result. □

Remark. — Under the more general assumptions $m_2(\mu) < \infty$, $\overline{\Phi(\mu)} = 0$ and $\operatorname{Var}\Phi(\mu) > 0$, one can get the estimates

$$\begin{aligned}\sqrt{\frac{2}{\pi}} \int_0^t u^2 e^{-u^2/2}\,du &\leq \liminf_{n\to\infty} \Pr\left[\frac{|R_n o|}{\sqrt{n\operatorname{Var}\Phi(\mu)}} \leq t\right] \\ &\leq \limsup_{n\to\infty} \Pr\left[\frac{|R_n o|}{\sqrt{n\operatorname{Var}\Phi(\mu)}} \leq t\right] \\ &\leq \sqrt{\frac{2}{\pi}} \int_0^t \mathrm{e}^{-u^2/2}\,du.\end{aligned}$$

Indeed, the lim sup-estimate follows from the inequality $|R_n| \geq |\Phi(R_n)|$. On the other hand, using (2.2) and the inequality (2.3), we have

$$\begin{aligned}|R_n| &= h(R_n o) - 2h(o \wedge R_n o) \\ &\leq \Phi(R_n) - 2\min\{h(R_{k-1}o \wedge R_k o) \,:\, k = 1,\dots,n\} \\ &= \Phi(R_n) - 2\min\{\Phi(R_k) + h(o \wedge X_k^{-1}o) \,:\, k = 1,\dots,n\} \\ &\leq \Phi(R_n) - 2\underline{M}_n + E_n,\end{aligned}$$

where

$$E_n = 2\max\{|X_k| \,:\, k = 1,\dots,n\}.$$

Finiteness of $m_2(\mu)$ is sufficient for having $E_n/\sqrt{n} \to 0$ almost surely, so that (3.6) implies the lim inf-estimate.

Next, we study the central limit theorem for $\|R_n\|$.

THEOREM 8.

(a) *If* $m_1(\mu) < \infty$, $\overline{\Phi(\mu)} > 0$ *and* $0 < \operatorname{Var}\Phi(\mu) < \infty$ *then* $|\beta(R_n)|$ *converges almost surely to* $2|R_\infty \wedge o|$, *and*

$$\frac{\|R_n\| - n\,\overline{\Phi(\mu)}}{\sqrt{n\operatorname{Var}\Phi(\mu)}} \longrightarrow N(0,1) \quad \text{in law.}$$

(b) *If* $m_1(\mu) < \infty$, $\overline{\Phi(\mu)} < 0$ *and* $0 < \operatorname{Var}\Phi(\mu) < \infty$ *then both*

$$\frac{|\beta(R_n)| + 2n\,\overline{\Phi(\mu)}}{2\sqrt{n\operatorname{Var}\Phi(\mu)}},\quad \frac{\|R_n\| + 2n\,\overline{\Phi(\mu)}}{2\sqrt{n\operatorname{Var}\Phi(\mu)}} \longrightarrow N(0,1) \quad \text{in law.}$$

(c) *If* $m_{2+\varepsilon}(\mu) < \infty$ *for some* $\varepsilon > 0$ *and* $\overline{\Phi(\mu)} = 0$ *then*

$$\frac{|\beta(R_n)|}{2\sqrt{n\operatorname{Var}\Phi(\mu)}} \longrightarrow V \ \text{ and } \ \frac{\|R_n\|}{\sqrt{n\operatorname{Var}\Phi(\mu)}} \longrightarrow |U| + V \ \text{ in law.}$$

Proof.

(a) The proposed limit of $|\beta(R_n)|$ is an immediate consequence of Theorem 2 (b), and the result follows from the classical CLT, applied to $\Phi(R_n)$.

(b) We have

$$|\beta(R_n)| \overset{\text{in law}}{=\!=} |\beta(L_n)| = |\beta(\check{R}_n^{-1})| = 2d(\check{R}_n o, \overline{\mathfrak{f}^\sigma\,\omega}) \tag{3.8}$$

by Lemma 4 (c), and the reflected random walk has $\Phi(\check{\mu}) = -\overline{\Phi(\mu)} > 0$. In particular, from (3.4) in the proof of Theorem 5 we get that with probability one

$$|\beta(\check{R}_n^{-1})| = 2|\check{R}_n| - 2|\check{R}_\infty \wedge \mathfrak{f}^\sigma|$$

for all but finitely many n, and $|\check{R}_\infty \wedge \mathfrak{f}^\sigma|$ is almost surely finite by Theorem 3 (1). This and Theorem 6 yield the result.

(c) Once more, this relies on Proposition 3, proved in the Appendix. By (3.8), and as $\overline{\Phi(\mu)} = 0$, in the proof we may replace R_n with R_n^{-1}. With the same technique as in the proof of Theorem 7,

$$\begin{aligned}|\beta(R_n^{-1})| &= |\sigma^{-\Phi(R_n)}R_n| = d\big(R_{T(n)}^{-1}\sigma^{\Phi(R_n)}o,\ R_{T(n),n}o\big)\\ &= 2\big(\Phi(R_n) - \underline{M}_n\big) - 2h\big(R_{T(n)}^{-1}\sigma^{\Phi(R_n)}o \wedge R_{T(n),n}o\big).\end{aligned}$$

In view of (3.6), the proof will be completed by showing that

$$\frac{1}{\sqrt{n}}h\big(R_{T(n)}^{-1}\sigma^{\Phi(R_n)}o \wedge R_{T(n),n}o\big) \longrightarrow 0 \quad \text{in probability.}$$

Observe that

$$h\big(R_{T(n),n}o\big) = h\big(R_{T(n)}^{-1}\sigma^{\Phi(R_n)}o\big) = \Phi(R_n) - \underline{M}_n$$

and

$$h\big(R_{T(n)}^{-1}o\big) = -\underline{M}_n.$$

Since $R_{T(n)}^{-1}o$ and $R_{T(n)}^{-1}\sigma^{\Phi(R_n)}o$ are comparable with respect to $\preccurlyeq$,

$$h\big(R_{T(n)}^{-1}\sigma^{\Phi(R_n)}o \wedge R_{T(n),n}o\big) = h\big(R_{T(n)}^{-1}o \wedge R_{T(n),n}o\big)$$

unless $R_{T(n)}^{-1}o \preccurlyeq R_{T(n),n}o$. But in this case

$$h\big(R_{T(n)}^{-1}o \wedge R_{T(n),n}o\big) = -\underline{M}_n.$$

By Proposition 3, the probability of the latter event tends to zero (as $-\underline{M}_n \to \infty$ almost surely when $n \to \infty$). Applying once more Proposition 3 yields the desired result. □

F. The Poisson boundary.

We now return to the study of harmonic functions (3.3) which we have already considered in the context of the Dirichlet problem. Another question is whether *every* bounded harmonic function can be presented as a Poisson integral (3.2) for some $f \in L^\infty(\partial\mathbb{T}, \nu)$. In this case we say that $(\partial\mathbb{T}, \nu)$ is the *Poisson boundary* [Fu]. The Poisson boundary is unique up to measure-theoretical isomorphism and can be introduced in various equivalent ways (see [K4] and the references therein for precise definitions). In particular, triviality of the Poisson boundary is equivalent with the absence of non-constant bounded harmonic functions (*Liouville property*). Kaimanovich [K5] has recently extended from discrete [K1] to topological groups a useful geometric criterion for identifying the Poisson boundary. In accordance with the present approach, we formulate it in terms of compactifications.

Let Γ be a second countable Hausdorff topological group with a left-invariant Haar measure λ. We say that an increasing sequence $\mathcal{A}$ of measurable sets $A_1 \subset A_2 \subset \cdots$ which exhaust Γ is a *C-gauge* if

$$\lambda(A_k) \leq \mathrm{e}^{Ck}.$$

For a gauge $\mathcal{A}$ let $|\gamma|_{\mathcal{A}} = \min\{k : \gamma \in A_k\}$ be the corresponding *gauge function* (note that no conditions on the value of the gauge function at the identity of the group are imposed). Let μ be a probability measure on Γ which is *irreducible* ($\operatorname{supp}\mu$ generates Γ as a closed semigroup), *spread out* (some convolution power $\mu^{(k)}$ is non-singular with respect to λ) and has finite first moment $\int |\gamma|_{\mathcal{A}}\, \mu(d\gamma)$ with respect to some gauge $\mathcal{A}$ which is *subadditive* (i.e., $|\gamma_1\gamma_2|_{\mathcal{A}} \leq |\gamma_1|_{\mathcal{A}}|\gamma_2|_{\mathcal{A}}$). Let $\widehat{\Gamma}$ be a compactification of Γ to which the left action of Γ on itself extends continuously. Suppose that the right random walk (R_n) with law μ converges almost surely to a random variable $R_\infty \in \partial\Gamma = \widehat{\Gamma} \setminus \Gamma$ with distribution ν. Then the space $(\partial\Gamma, \nu)$ is the Poisson boundary of the random walk, provided the following holds : for ν-almost every point $\mathrm{u} \in \partial\Gamma$ there are $C > 0$ and a sequence of C-gauges $\mathcal{A}^{(n)}(\mathrm{u})$ such that

$$\lim_{n\to\infty} \frac{1}{n} |R_n|_{\mathcal{A}^{(n)}(R_\infty)} = 0 \quad \text{almost surely.} \tag{3.9}$$

Remarks.

1) The irreducibility hypothesis can be omitted, but then the definition of the Poisson boundary becomes slightly more delicate; see [K4] for a discussion.

2) The criterion is of interest in the transient case only; the Poisson boundary is always trivial when (R_n) is recurrent.

3) If the group Γ is compactly generated, and J is a compact neighbourhood of the identity, then the family $\mathcal{A}_J = \{J^k\}$ is a subadditive gauge for an appropriate choice of C [Gu]. Any left translation of the gauge $\mathcal{A}$ is again a gauge with the same C. Thus, in this case (3.9) follows from the following simpler condition : there is a sequence of maps $\Pi_n : \partial\Gamma \to \Gamma$ such that

$$\lim_{n\to\infty} \frac{1}{n} \left|R_n^{-1}\Pi_n(R_\infty)\right|_{\mathcal{A}_J} = 0 \quad \text{almost surely.} \tag{3.10}$$

THEOREM 9. — *Let μ be a probability measure on a closed subgroup Γ of* AFF$(\mathbb{T})$ *which is irreducible, spread out and has finite first moment $m_1(\mu)$ in the group* AFF$(\mathbb{T})$.

(a) *If $\overline{\Phi(\mu)} \le 0$, then the Poisson boundary of the pair (Γ, μ) is trivial.*

(b) *If $\overline{\Phi(\mu)} > 0$, then the Poisson boundary is the space $(\partial\mathbb{T}^*, \nu)$, where ν is the distribution of R_∞.*

Proof. — Fix a left-invariant Haar measure λ on Γ. For a point $x \in \mathbb{T}$ put $\Gamma^x = \{\gamma \in \Gamma : \gamma o = x\}$. In particular, Γ^o $(= \Gamma_o)$ is a compact neighbourhood of the identity in Γ, so that $\lambda(\Gamma^o) < \infty$. If a set Γ^x is non-empty, then $\Gamma^x = \gamma\Gamma^o$ for every $\gamma \in \Gamma^x$, so that $\lambda(\Gamma^x) = \lambda(\Gamma^o)$. Thus, for any point $x \in \mathbb{T}$ the family $\mathcal{A}^x$ of sets $\{\gamma \in \Gamma : d(x, \gamma o) \le k\}$, $k \ge 0$, is a gauge in Γ. Moreover, the gauge $\mathcal{A}^o$ is subadditive. The condition $m_1(\mu) < \infty$ means that the first moment of the measure μ with respect to the gauge $\mathcal{A}^o$ is finite.

If $\overline{\Phi(\mu)} = 0$, then we may take the one-point compactification $\widehat{\Gamma} = \Gamma \cup \{\infty\}$, and by transience, $R_n \to R_\infty = \infty$. By Theorem 4

$$\frac{1}{n}|R_n|_{\mathcal{A}} = \frac{1}{n} d(o, R_n o) \longrightarrow 0 \quad \text{almost surely}$$

for the gauge $\mathcal{A} = \mathcal{A}^o$, so that the Poisson boundary is trivial.

In order to make the proof more transparent in the case $\overline{\Phi(\mu)} \ne 0$ we first consider the situation when the group Γ acts transitively on the tree. Then Γ is generated by the compact neighbourhood of the identity $J = \{\gamma \in \Gamma : d(o, \gamma o) \le 1\}$, and $|\gamma|_{\mathcal{A}_J} = |\gamma| = d(o, \gamma o)$. We show that condition (3.10) is applicable to the right random walk.

Let $a = |\overline{\Phi(\mu)}|$. For every $x \in \mathbb{T}$, we may fix $\gamma_x \in \Gamma$ with $\gamma_x o = x$. If $\mathfrak{u} \in \partial\mathbb{T}$ and $n \in \mathbb{N}$, then we define

$$\Pi_n(\mathfrak{u}) = \gamma_x, \quad \text{where} \quad x = \mathfrak{u}_o(\lfloor an \rfloor).$$

Under our assumptions, we know from Theorem 4 that

$$\frac{1}{n} d(o, R_n o) \longrightarrow a \text{ and } \frac{1}{n} d(R_n o, R_{n+1} o) \longrightarrow 0 \text{ almost surely.}$$

Thus, Lemma 3 yields

$$\frac{1}{n} \left| R_n^{-1} \Pi_n(R_\infty) \right| = \frac{1}{n} d\big(R_n o, (R_\infty)_o(\lfloor an \rfloor)\big) \longrightarrow 0 \text{ almost surely,}$$

and condition (3.10) of Kaimanovich's geometric criterion is satisfied. Thus, the Poisson boundary is $(\partial\mathbb{T}, \nu)$. If $\overline{\Phi(\mu)} > 0$, then $R_\infty \in \partial^*\mathbb{T}$ almost surely and $\nu(\{\omega\}) = 0$, that is, $(\partial\mathbb{T}, \nu) \cong (\partial^*\mathbb{T}, \nu)$. If $\overline{\Phi(\mu)} < 0$, then $\nu = \delta_\omega$, and the boundary is trivial.

In Γ is not transitive, applying condition (3.10) becomes more complicated. However, replacing balls around the points $\Pi_n(\mathfrak{u})$ with the corresponding gauge sets in Γ allows one to use condition (3.9) instead. For $\mathfrak{u} \in \partial\mathbb{T}$ and $n \in \mathbb{N}$ we define

$$\mathcal{A}^{(n)}(\mathfrak{u}) = \mathcal{A}^x, \quad \text{where } x = \mathfrak{u}_o(\lfloor an \rfloor).$$

All the rest is as in the transitive case.

Remark. — The description of the Poisson boundary of $\mathrm{AFF}(\mathbb{T})$ obtained in Theorem 9 is completely analogous to the description of the Poisson boundary of the real affine group. The Poisson boundary for a measure μ with a finite first moment on $\mathrm{AFF}_0(\mathbb{R})$ is trivial if its drift $\overline{\Phi(\mu)}$ is non-negative and can be identified with $\mathbb{R}$ if $\overline{\Phi(\mu)} < 0$. (Here the map $\Phi : \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \mapsto \log a$ is a homomorphism from $\mathrm{AFF}_0(\mathbb{R})$ to the additive group $\mathbb{R}$). The original proofs in the real case are based on other ideas. If $\overline{\Phi(\mu)} \ge 0$, then for any $\gamma \in \mathrm{HOR}(\mathbb{R}) = \left\{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\right\}$ and almost every path (R_n) of the corresponding right random walk the sequence $(R_n^{-1}\gamma R_n)$ has a limit point in $\mathrm{AFF}_0(\mathbb{R})$, so that the horocyclic group $\mathrm{HOR}(\mathbb{R})$ is contained in the group of μ-periods, and the Poisson boundary is trivial. In the contracting case $\overline{\Phi(\mu)} < 0$, one uses the fact that $\mathrm{HOR}(\mathbb{R})$ acts on $\mathbb{R}$ simply transitively and that there are no $\mathrm{HOR}(\mathbb{R})$-invariant harmonic functions to deduce that $\mathbb{R}$ with the corresponding harmonic measure is the whole Poisson boundary [Az], [Rau].

For an arbitrary non-exceptional subgroup Γ of $\mathrm{AFF}(\mathbb{T})$ we had to use more sophisticated methods, relying on entropy (although the above ideas can be still applied to the affine group $\mathrm{AFF}(\mathfrak{F})$ of a local field $\mathfrak{F}$). These methods bear a general character and are also applicable to the real affine group and to the (discrete) affine group of the dyadic-rational line $\mathrm{AFF}(\mathbb{Z}[\frac{1}{2}])$. Note that $\mathrm{AFF}(\mathbb{Z}[\frac{1}{2}])$ is contained both in $\mathrm{AFF}(\mathbb{Q}_2)$ and in $\mathrm{AFF}_0(\mathbb{R})$, although neither of these imbeddings is discrete. However, $\mathrm{AFF}(\mathbb{Z}[\frac{1}{2}])$ is a lattice in the product $\mathrm{AFF}(\mathbb{Q}_2) \times \mathrm{AFF}_0(\mathbb{R})$, and its Poisson boundary is either $\mathbb{R}$ or $\mathbb{Q}_2$ according to the sign of the drift [K3]. Apparently, this phenomenon has a more general nature.

As a next step, one may ask for a representation of *positive* harmonic functions (instead of bounded ones) in terms of boundary integrals. This is done via the construction of the *Martin compactification,* by techniques which are usually very different from those used for identifying the Poisson boundary. (However, see [W1] for a case where partial knowledge of the Martin boundary is used to obtain full knowledge of the Poisson boundary.) For more details and for sufficient conditions on μ which guarantee that $\widehat{\mathbb{T}}$ is the Martin compactification of the random walk, see Woess [W3].

4. Application to the affine group over a local field.

A. Local fields and ends of trees.

For the necessary background on local fields, see for example Cassels [Ca] (where, however, the valuation is an exponential of ours) or Serre [S2].

Let $\mathfrak{F}$ be a field with a *discrete valuation* v, that is, v is a homomorphism of the multiplicative group $\mathfrak{F}^*$ onto $\mathbb{Z}$ such that :

$$v(\mathfrak{u} + \mathfrak{v}) \geq \inf\{v(\mathfrak{u}), v(\mathfrak{v})\}, \qquad \mathfrak{u}, \mathfrak{v} \in \mathfrak{F}.$$

We also set $v(\mathbf{0}) = \infty$ (here $\mathbf{0}$ and $\mathbf{1}$ are the neutral elements of the additive group $\mathfrak{F}$ and the multiplicative group $\mathfrak{F}^*$, respectively). For the sake of simplicity we shall assume that $\mathfrak{F}$ is commutative. However, with obvious modifications our results carry over to the non-commutative case. Let $\mathfrak{O} = \{\mathfrak{u} \in \mathfrak{F} : v(\mathfrak{u}) \geq 0\}$ be the *valuation ring* (or, the ring of *integers*) of the field $\mathfrak{F}$, and $\mathfrak{P} = \{\mathfrak{u} \in \mathfrak{F} : v(\mathfrak{u}) \geq 1\}$ its *maximal ideal.* Then $\mathfrak{F}$ is called a *(non-archimedean) local field*, if

(I) the cardinality q of the *residual field* $\mathfrak{O}/\mathfrak{P}$ is finite (it must be a prime power), and

(II) $\mathfrak{F}$ is complete when equipped with the metric :

$$\Theta(\mathfrak{u}, \mathfrak{v}) = |\mathfrak{u} - \mathfrak{v}| = q^{-v(\mathfrak{u}-\mathfrak{v})}. \tag{4.1}$$

Assuming (II), (I) is equivalent to the local compactness of $\mathfrak{F}$.

We choose a *uniformizer* $\mathfrak{p} \in \mathfrak{F}^*$ such that $v(\mathfrak{p}) = 1$. Then $\mathfrak{P} = \mathfrak{p}\mathfrak{O}$. Let $\mathfrak{S} \subset \mathfrak{O}$ be a set of representatives of the residual field such that $\mathbf{0} \in \mathfrak{S}$. Then every $\mathfrak{u} \in \mathfrak{F}^*$ can be uniquely written as

$$\mathfrak{u} = \sum_{i=n}^{\infty} \mathfrak{s}_i \mathfrak{p}^i, \qquad \mathfrak{s}_i \in \mathfrak{S},\ \mathfrak{s}_n \neq \mathbf{0}, \tag{4.2}$$

where $n = v(\mathfrak{u}) \in \mathbb{Z}$. In the particular case when $\mathfrak{F} = \mathbb{Q}_p$ is the *field of p-adic numbers*, the valuation ring is the ring $\mathfrak{O} = \mathbb{Z}_p$ of p-adic integers, and the residual field is $\mathbb{F}_p$. We can take $\mathfrak{p} = p$ and $\mathfrak{S} = \{0, 1, \dots, p-1\}$. Then (4.2) is the standard representation of p-adic numbers.

We now describe how to associate with $\mathfrak{F}$ the tree $\mathbb{T} = \mathbb{T}_q$ in such a way that $\mathfrak{F} \equiv \partial^*\mathbb{T} = \partial\mathbb{T} \setminus \{\omega\}$, where $\partial\mathbb{T}$ is the space of ends of $\mathbb{T}$ and ω a fixed element of $\partial\mathbb{T}$. This construction is certainly known to specialists; compare for example with Figà-Talamanca [Fi] (for compact ultrametric spaces) and Choucroun [Ch].

For a point $\mathfrak{u} \in \mathfrak{F}$ and a number $k \in \mathbb{Z}$, let :

$$U_k(\mathfrak{u}) = \{\mathfrak{v} \in \mathfrak{F} : \Theta(\mathfrak{v}, \mathfrak{u}) \leq q^{-k}\} = \{\mathfrak{v} \in \mathfrak{F} : v(\mathfrak{v} - \mathfrak{u}) \geq k\}.$$

By the ultrametric property, any two balls $U_k(\mathfrak{u})$, $U_\ell(\mathfrak{v})$ either are disjoint or one of the two is contained in the other. In particular, $U_k(\mathfrak{u}) = U_k(\mathfrak{v})$ for every $\mathfrak{v} \in U_k(\mathfrak{u})$. Thus, for a fixed k, the family

$$H_k = \{U_k(\mathfrak{u}) : \mathfrak{u} \in \mathfrak{F}\}$$

forms a partition of $\mathfrak{F}$. In view of (4.2), the balls $U_k(\mathfrak{u}) \in H_k$ can be identified with the sums of the form $\sum_{i=n}^{k-1} \mathfrak{s}_i \mathfrak{p}^i$. In particular, $H_0 \cong \mathfrak{F}/\mathfrak{O}$. The vertex set of our tree is

$$\mathbb{T} = \{U_k(\mathfrak{u}) : \mathfrak{u} \in \mathfrak{F},\ k \in \mathbb{Z}\}.$$

The *father* of a vertex $x = U_k(\mathfrak{u})$ is $y = U_{k-1}(\mathfrak{u})$. This defines the tree structure in $\mathbb{T}$, and H_k is the k-th horocycle. By (4.2), $\mathbb{T}$ is homogeneous of degree $q+1$. For every vertex $x = U_k(\mathfrak{u})$ and representative $\mathfrak{s} \in \mathfrak{S}$, we may label the edge between x and its *son* $v = U_{k+1}(\mathfrak{u} + \mathfrak{p}^k\mathfrak{s})$ with $\mathfrak{s}$.

Given $\mathfrak{u} \in \mathfrak{F}$, the sequence of balls $U_k(\mathfrak{u})$, $k \in \mathbb{Z}$, constitutes the vertices on the geodesic path in $\mathbb{T}$ from ω to an element of $\partial^*\mathbb{T}$; as $\{\mathfrak{u}\} = \bigcap_{k\in\mathbb{Z}} U_k(\mathfrak{u})$ is uniquely determined by this path, we may identify $\mathfrak{u}$ with that end. Also, the series expansion (4.2) of $\mathfrak{u}$ may be recovered by reading the labels of the edges along that geodesic. The end ω corresponds to increasing sequences of balls.

We may consider the zero element $\mathbf{0} \in \mathfrak{F}$ as the «leftmost» end in $\partial^*\mathbb{T}$ when viewing $\mathbb{T}$ as in Figure 1, and we choose $o = U_0(\mathbf{0})$ as the reference point in $\mathbb{T}$. Identifying vertices with balls $U_k(\cdot)$, we have

$$\mathfrak{u} \wedge \mathfrak{v} = U_{v(\mathfrak{v}-\mathfrak{u})}(\mathfrak{u}) = U_{v(\mathfrak{v}-\mathfrak{u})}(\mathfrak{v}),$$

whenever $\mathfrak{u}, \mathfrak{v} \in \mathfrak{F}$, $\mathfrak{u} \neq \mathfrak{v}$. The metric (4.1) coincides with the restriction to $\partial^*\mathbb{T}$ of the metric Θ defined in §2.B. We can now recover the valuation of $\mathfrak{u} \in \mathfrak{F}$ in terms of the tree structure :

$$v(\mathfrak{u}) = h(\mathfrak{u} \wedge \mathbf{0}), \quad \text{when } \mathfrak{u} \neq \mathbf{0}.$$

B. The affine group over $\mathfrak{F}$ and its action on $\mathbb{T}$.

The *affine group* over the field $\mathfrak{F}$ is

$$\mathrm{AFF}(\mathfrak{F}) = \left\{\mathfrak{u} \mapsto \mathfrak{a}\mathfrak{u} + \mathfrak{b} \ : \ \mathfrak{a} \in \mathfrak{F}^*,\ \mathfrak{b} \in \mathfrak{F}\right\} \cong \left\{\begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \ : \ \mathfrak{a}, \mathfrak{b} \in \mathfrak{F}, \mathfrak{a} \neq \mathbf{0}\right\},$$

the group generated by the canonical actions on the field $\mathfrak{F}$ of its additive and multiplicative groups. If $\gamma = \left(\begin{smallmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathbf{0} & \mathbf{1} \end{smallmatrix}\right) \in \mathrm{AFF}(\mathfrak{F})$ then we write $\mathfrak{a}(\gamma) = \mathfrak{a}$ and $\mathfrak{b}(\gamma) = \mathfrak{b}$.

The action of $\mathrm{AFF}(\mathfrak{F})$ naturally extends from the field $\mathfrak{F}$ to $\widehat{\mathbb{T}}$. Indeed, if $x = U_k(\mathfrak{u}) \in \mathbb{T}$ and $\mathfrak{a}, \mathfrak{b} \in \mathfrak{F}$, $\mathfrak{a} \neq \mathbf{0}$, then we set :

$$\mathfrak{a}x + \mathfrak{b} = \left\{\mathfrak{a}\mathfrak{v} + \mathfrak{b} \ : \ \mathfrak{v} \in U_k(\mathfrak{u})\right\} = U_{v(\mathfrak{a})+k}(\mathfrak{a}\mathfrak{u} + \mathfrak{b}). \tag{4.3}$$

It is easy to see that (4.3) defines an automorphism of $\mathbb{T}$ which fixes ω and extends the mapping $\mathfrak{u} \mapsto \mathfrak{a}\mathfrak{u} + \mathfrak{b}$ from $\partial^*\mathbb{T} \equiv \mathfrak{F}$ to $\widehat{\mathbb{T}}$. In this way we have realized $\mathrm{AFF}(\mathfrak{F})$ as a closed subgroup of $\mathrm{AFF}(\mathbb{T})$ which acts transitively on $\mathbb{T}$.

Remark. — There is a well known construction of a canonical action of the group $\mathrm{GL}(n,\mathfrak{F})$ on the corresponding *Bruhat-Tits building.* In the case $n = 2$ this building is the tree $\mathbb{T}_q$ whose space of ends can be identified with the projective line of $\mathfrak{F}$. When restricted to the affine group $\mathrm{AFF}(\mathfrak{F}) \subset \mathrm{GL}(2,\mathfrak{F})$, this action coincides with the action of $\mathrm{AFF}(\mathfrak{F})$ on the tree $\mathbb{T}(\mathfrak{F})$ constructed above [Ch].

We have a canonical semidirect product decomposition

$$\mathrm{AFF}(\mathfrak{F}) = \mathfrak{F}^* \leftthreetimes \mathfrak{F}, \quad \gamma \equiv \big(\mathfrak{a}(\gamma), \mathfrak{b}(\gamma)\big),$$

where the action of the multiplicative group $\mathfrak{F}^* = \mathfrak{F} \setminus \{\mathbf{0}\}$ on the additive group $\mathfrak{F}$ is given by multiplication. This decomposition arises from the homomorphism $\mathrm{AFF}(\mathfrak{F}) \to \mathfrak{F}^*$, $\gamma \mapsto \mathfrak{a}(\gamma)$.

The homomorphism $\pi : \mathrm{AFF}(\mathfrak{F}) \to \mathbb{Z}$, $\pi(\gamma) = v\big(\mathfrak{a}(\gamma)\big)$, together with the choice

$$\sigma = \begin{pmatrix} \mathfrak{p} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

for the «vertical shift» (compare with § 2.D), gives rise to a *different* (non-canonical) semidirect product decomposition $\mathrm{AFF}(\mathfrak{F}) = \mathbb{Z}^* \leftthreetimes \mathrm{HOR}(\mathfrak{F})$, with

$$\mathrm{HOR}(\mathfrak{F}) = \ker \pi = \mathrm{HOR}(\mathbb{T}) \cap \mathrm{AFF}(\mathfrak{F}) = \left\{ \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} : \mathfrak{a}, \mathfrak{b} \in \mathfrak{F},\ v(\mathfrak{a}) = 0 \right\},$$

and the action of $\mathbb{Z}$ on $\mathrm{HOR}(\mathfrak{F})$ is given by

$$A^m\beta = \sigma^m \beta \sigma^{-m} = \begin{pmatrix} \mathfrak{a} & \mathfrak{p}^m\mathfrak{b} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \quad \text{where} \quad \beta = \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \in \mathrm{HOR}(\mathfrak{F}).$$

We now give a list of useful relations.

LEMMA 5. — *Let* $\gamma = \left(\begin{smallmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathbf{0} & \mathbf{1} \end{smallmatrix}\right) \in \mathrm{AFF}(\mathfrak{F}) \leq \mathrm{AFF}(\mathbb{T})$. *Then* :

$$\Phi(\gamma) = v(\mathfrak{a}), \quad \beta(\gamma) = \begin{pmatrix} \mathfrak{a}\,\mathfrak{p}^{-v(\mathfrak{a})} & \mathfrak{b} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \tag{1}$$

$$\begin{cases} o \wedge \gamma o = o \wedge \mathfrak{a} \wedge \mathfrak{b} = U_{\min\{0, v(\mathfrak{a}), v(\mathfrak{b})\}}(\mathbf{0}), \\ |o \wedge \gamma o| = \max\{0, -v(\mathfrak{a}), -v(\mathfrak{b})\}\,; \end{cases} \tag{2}$$

$$\begin{aligned} |\gamma| &= \max\{0, -v(\mathfrak{a}), -v(\mathfrak{b})\} + \max\{0, v(\mathfrak{a}), v(\mathfrak{a}) - v(\mathfrak{b})\} \\ &= v(\mathfrak{a}) + 2\max\{0, -v(\mathfrak{a}), -v(\mathfrak{b})\}\,; \end{aligned} \tag{3}$$

(4) $$\|\gamma\| = |v(\mathfrak{a})| + \max\{0, -v(\mathfrak{b})\}.$$

Proof.

(1) By (4.3), $\gamma o = U_{v(\mathfrak{a})}(\mathfrak{b}) \in H_{v(\mathfrak{a})}$. Hence $\Phi(\gamma) = v(\mathfrak{a})$, and

$$\beta(\gamma) = \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathfrak{p}^{-v(\mathfrak{a})} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}.$$

(2) First of all, observe that for $x = U_k(\mathfrak{u}) \in \mathbb{T}$,

$$\mathbf{0} \wedge x = U_{\min\{k, v(\mathfrak{u})\}}(\mathbf{0})$$

and consequently

$$o \wedge x = U_{\min\{0, k, v(\mathfrak{u})\}}(\mathbf{0}).$$

Therefore

$$o \wedge \gamma o = o \wedge U_{v(\mathfrak{a})}(\mathfrak{b}) = U_{\min\{0, v(\mathfrak{a}), v(\mathfrak{b})\}}(\mathbf{0}) = o \wedge \mathfrak{a} \wedge \mathfrak{b}.$$

In particular,

$$|o \wedge \gamma o| = -h(o \wedge \gamma o) = -\min\{0, v(\mathfrak{a}), v(\mathfrak{b})\}.$$

(3) As $\gamma^{-1} = \left(\begin{smallmatrix} \mathfrak{a}^{-1} & -\mathfrak{a}^{-1}\mathfrak{b} \\ \mathbf{0} & \mathbf{1} \end{smallmatrix}\right)$, this follows from (2) together with the formula $|\gamma| = |o \wedge \gamma o| + |o \wedge \gamma^{-1} o|$, see (2.4).

(4) This is obtained from (1) and (3). □

In concluding, we remark that everything which has been said in § 2.E about (non-) exceptionality and the limit set carries over to closed subgroups of $\mathrm{AFF}(\mathfrak{F})$ (in particular, $\partial^*\Gamma \subset \mathfrak{F}$).

C. Application to random walks.

All results of § 3 apply to random walks on $\mathrm{AFF}(\mathfrak{F})$, or, in other words, to products of random affine matrices over $\mathfrak{F}$. We supply the necessary «translations» from tree to local field setting.

Let μ be a Borel probability measure on $\mathrm{AFF}(\mathfrak{F})$. The requirement that $\operatorname{supp}\mu$ generates a non-exceptional group now becomes the following :

(i) There is $\gamma = \left(\begin{smallmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathbf{0} & \mathbf{1} \end{smallmatrix}\right) \in \operatorname{supp}\mu$ with $v(\mathfrak{a}) \neq 0$, and

(ii) There are $\gamma_i = \left(\begin{smallmatrix} \mathfrak{a}_i & \mathfrak{b}_i \\ \mathbf{0} & \mathbf{1} \end{smallmatrix}\right) \in \operatorname{supp}\mu$, $i = 1, 2$, such that $(\mathbf{1} - \mathfrak{a}_1)\,\mathfrak{b}_2 \neq (\mathbf{1} - \mathfrak{a}_2)\,\mathfrak{b}_1$.

For our sequence (X_n) of i.i.d. random variables in $\mathrm{AFF}(\mathfrak{F})$ with law μ, write $A_n = \mathfrak{a}(X_n)$ and $B_n = \mathfrak{b}(X_n)$. Then

$$R_n = \begin{pmatrix} A_1 & B_1 \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \cdots \begin{pmatrix} A_n & B_n \\ \mathbf{0} & \mathbf{1} \end{pmatrix},$$

$$L_n = \begin{pmatrix} A_n & B_n \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \cdots \begin{pmatrix} A_1 & B_1 \\ \mathbf{0} & \mathbf{1} \end{pmatrix}.$$

Consequently,

$$\mathfrak{a}(R_n) = \mathfrak{a}(L_n) = A_1 \cdots A_n,$$
$$\mathfrak{b}(R_n) = \sum_{i=1}^{n} A_1 \cdots A_{i-1} B_i\,, \quad \text{and} \quad \mathfrak{b}(L_n) = \sum_{i=1}^{n} B_i A_{i+1} \cdots A_n.$$

We have by Lemmas 4 and 8

$$m_1(\mu) < \infty \iff \mathbb{E}\big(|v(A_1)|\big) < \infty \quad \text{and} \quad \mathbb{E}\big(\max\{0, -v(B_1)\}\big) < \infty.$$

Also, $\overline{\Phi(\mu)} = \mathbb{E}\big(v(A_1)\big)$ and $\operatorname{Var}\Phi(\mu) = \operatorname{Var}\big(v(A_1)\big)$.

From convergence to ends (Theorems 1 and 2 (1)) we obtain the following. (The case $m_1(\mu) < \infty$, $\overline{\Phi(\mu)} < 0$ is of minor interest here.)

COROLLARY 1.

(a) *If $m_1(\mu) < \infty$ and $\overline{\Phi(\mu)} > 0$ then*

$$\mathfrak{a}(R_n) \to \mathbf{0} \quad \textit{and} \quad \mathfrak{b}(R_n) \to R_\infty \in \mathfrak{F} \quad \textit{almost surely,}$$

and $\Pr[R_\infty = \mathfrak{u}] = 0$ *for every* $\mathfrak{u} \in \mathfrak{F}$.

(b) *If $m_1(\mu) < \infty$, $\overline{\Phi(\mu)} = 0$ and*

$$\mathbb{E}\Big(\max\big\{0, v(A_1), v(A_1) - v(B_1)\big\} \cdot q^{-\min\{0, v(A_1), v(B_1)\}}\Big) < \infty$$

then

$$\min\big\{v\big(\mathfrak{a}(R_n)\big), v\big(\mathfrak{b}(R_n)\big)\big\} \to -\infty \quad \textit{almost surely.}$$

This holds in particular when

$$\mathbb{E}\Big(q^{|v(A_1)| - \min\{0, v(B_1)\}}\Big) < \infty \quad \textit{and} \quad \overline{\Phi(\mu)} = 0.$$

Besides the obvious rewriting of $n^{-1}|R_n|$ in the terms of Lemma 5, the law of large numbers (in the version of Theorem 5) implies the following.

COROLLARY 2. — *Assume that* $m_1(\mu) < \infty$.

(a) *If* $\overline{\Phi(\mu)} \geq 0$ *then*

$$\liminf_{n\to\infty} \frac{1}{n} v\big(\mathfrak{b}(L_n)\big) \geq 0 \quad \textit{almost surely.}$$

(b) *If* $\overline{\Phi(\mu)} < 0$ *then*

$$\lim_{n\to\infty} \frac{1}{n} v\big(\mathfrak{b}(L_n)\big) = \overline{\Phi(\mu)} \quad \textit{almost surely.}$$

Finally the following is obtained from the central limit theorem in the version of Theorem 8 (b), (c) (this time the case $\overline{\Phi(\mu)} > 0$ is of minor interest). Recall that V is a real random variable with distribution $N^+(0,1)$.

COROLLARY 3.

(a) *If* $m_1(\mu) < \infty$ *and* $\overline{\Phi(\mu)} < 0$ *and* $0 < \operatorname{Var}\Phi(\mu) < \infty$ *then*

$$\frac{v(\mathfrak{b}(R_n)) - n\,\overline{\Phi(\mu)}}{\sqrt{n\operatorname{Var}\Phi(\mu)}} \longrightarrow N(0,1) \quad \textit{in law.}$$

(b) *If* $m_{2+\varepsilon}(\mu) < \infty$ *for some* $\varepsilon > 0$, $\overline{\Phi(\mu)} = 0$ *and* $0 < \operatorname{Var}\Phi(\mu) < \infty$ *then*

$$\frac{v(\mathfrak{b}(R_n))}{\sqrt{n\operatorname{Var}\Phi(\mu)}} \longrightarrow -V \quad \textit{in law.}$$

We skip the solution of the Dirichlet problem and the Poisson boundary, whose rewriting in terms of $\mathfrak{F}$ is obvious.

Appendix. Splitting at the infimum in the driftfree case.

In this section, we shall complete the proof of Theorems 6 and 7 by proving the following. (Recall the definitions (3.5) and (3.7) of $\underline{M}_n$ and $T(n)$, respectively.)

PROPOSITION 3. — *If* $\mathbb{E}\big(d(o, X_1 o)^{2+\varepsilon}\big) < \infty$ *for some* $\varepsilon > 0$, $\overline{\Phi(\mu)} = 0$ *and* $\operatorname{Var}\Phi(\mu) > 0$ *then*

$$h\big(R_{T(n)}^{-1} o \wedge R_{T(n),n} o\big)$$

converges in law to a finite random variable.

This will be a corollary of a stronger result, obtained at the end of this appendix. We proceed in several steps, always assuming that $|\Phi(\mu)| < \infty$, $\overline{\Phi(\mu)} = 0$ and that $\operatorname{supp}\mu$ generates a non-exceptional subgroup of $\mathrm{A{\scriptsize FF}}(\mathbb{T})$.

A. Ladder indices and induced random walks.

For the right random walk (R_n) with law μ we define recursively the following Markov stopping times (*non-strictly descending ladder indices* of $(\Phi(R_n))$) :

$$\mathbf{s}_0 = 0, \quad \mathbf{s}_k = \min\{n > \mathbf{s}_{k-1} \,:\, \Phi(R_n) \le \Phi(R_{\mathbf{s}_{k-1}})\}.$$

By recurrence of $(\Phi(R_n))$, all $\mathbf{s}_n$ are almost surely finite. Denote by η the distribution of $R_{\mathbf{s}_1}$. Then the increments $R_{\mathbf{s}_k}^{-1}R_{\mathbf{s}_{k+1}}$ are i.i.d. with distribution η, so that $(R_{\mathbf{s}_k})$ is the right random walk with law η.

PROPOSITION 4.

(a) *The support of η generates a non-exceptional subgroup of* $\mathrm{A{\scriptsize FF}}(\mathbb{T})$.

(b) *If* $\operatorname{Var}\Phi(\mu) = \mathbb{E}(\Phi(X_1)^2) < \infty$ *and* $\mathbb{E}(|\beta(X_1)|^{2+\varepsilon}) < \infty$ *for some* $\varepsilon > 0$ *then*

$$m_1(\eta) = \mathbb{E}(|R_{\mathbf{s}_1}|) < \infty.$$

Proof.

(a) We show that the group $\Gamma(\eta)$ generated by $\operatorname{supp}\eta$ coincides with $\Gamma(\mu)$. Clearly, $\Gamma(\eta) \subset \Gamma(\mu)$, and $\operatorname{supp}\mu \cap \{\gamma \,:\, \Phi(\gamma) \le 0\} \subset \operatorname{supp}\eta$. Since $\Gamma(\mu)$ is non-exceptional and $\overline{\Phi(\mu)} = 0$, there is $\alpha \in \operatorname{supp}\mu$ with $\Phi(\alpha) < 0$, and hence $\alpha \in \operatorname{supp}\eta$. For an arbitrary element $\gamma \in \operatorname{supp}\mu$ with $\Phi(\gamma) > 0$ there exists a minimal $k \ge 1$ such that $\Phi(\gamma) + k\Phi(\alpha) < 0$. Then $\gamma\alpha^k \in \operatorname{supp}\eta$, so that $\gamma \in \Gamma(\eta)$.

(b) (Compare with [G2] and [E3], § V.C.) For proving finiteness of $\mathbb{E}(|R_{\mathbf{s}_1}|)$, in view of Lemma 4(d) we have show that

$$\mathbb{E}(|\Phi(R_{\mathbf{s}_1})|) < \infty \quad \text{and} \quad \mathbb{E}(|\beta(R_{\mathbf{s}_1})|) < \infty.$$

By Feller [Fe], § XIII.7 and XVIII.5 or Spitzer [Sp], Thm 18.1 and Prop. 20.1,

$$\mathbb{E}(|\Phi(R_{\mathbf{s}_1})|) < \infty \iff \operatorname{Var}\Phi(\mu) < \infty,$$

and in this case

$$\Pr[\mathbf{s}_1 \ge n] \le \frac{\text{const}}{\sqrt{n}}.$$

Next, we have $\Phi(R_{\mathbf{s}_{n-1},i}) > 0$ for all i between $\mathbf{s}_{n-1}$ and $\mathbf{s}_n$, so that by Lemma 4 (b),

$$\big|\beta(R_{\mathbf{s}_{n-1},\mathbf{s}_n})\big| \le \max\big\{|\beta(X_i)| \,:\, i = \mathbf{s}_{n-1}+1,\dots,\mathbf{s}_n\big\} = W_n.$$

The W_n are i.i.d. $\mathbb{N}$-valued random values. We shall show that there is a constant $M < \infty$ such that

$$\limsup_{n\to\infty} \frac{1}{n} W_n \le M \quad \text{almost surely.}$$

By the Borel-Cantelli Lemma, this implies integrability of W_n and consequently also of $|\beta(R_{\mathbf{s}_1})|$.

Now $\mathbf{s}_1$ is not integrable, but $\mathbf{s}_1^{1/(2+\varepsilon)}$ is.

$$\sum_{n=1}^{\infty} \Pr\big[\mathbf{s}_1^{1/(2+\varepsilon)} \ge n\big] \le \text{const} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^{2+\varepsilon}}} < \infty.$$

Write

$$\frac{1}{n} W_n = \left\{ \frac{1}{n} \sum_{k=1}^{n} (\mathbf{s}_k - \mathbf{s}_{k-1})^{1/(2+\varepsilon)} \right\} \times \frac{W_n}{\sum_{k=1}^{n} (\mathbf{s}_k - \mathbf{s}_{k-1})^{1/(2+\varepsilon)}}.$$

As $n \to \infty$, the first factor on the right hand side tends to $\mathbb{E}(\mathbf{s}_1^{1/(2+\varepsilon)})$ almost surely by the law of large numbers. Taking the second factor to the power $2+\varepsilon$, we get

$$\begin{aligned}
\left\{ \frac{W_n}{\sum_{k=1}^{n} (\mathbf{s}_k - \mathbf{s}_{k-1})^{1/(2+\varepsilon)}} \right\}^{2+\varepsilon} &\le \frac{1}{\mathbf{s}_n} W_n^{2+\varepsilon} \\
&= \frac{\max\big\{|\beta(X_j)|^{2+\varepsilon} \,:\, j = \mathbf{s}_{n-1}+1,\dots,\mathbf{s}_n\big\}}{\mathbf{s}_n} \\
&\le \frac{1}{\mathbf{s}_n} \sum_{j=1}^{\mathbf{s}_n} \big|\beta(X_j)\big|^{2+\varepsilon}.
\end{aligned}$$

Once more by the law of large numbers, the last term tends to $\mathbb{E}(|\beta(X_1)|^{2+\varepsilon})$ almost surely. Setting

$$M = \mathbb{E}\big(\mathbf{s}_1^{1/(2+\varepsilon)}\big)\, \mathbb{E}\big(|\beta(X_1)|^{2+\varepsilon}\big)^{1/(2+\varepsilon)},$$

we obtain $\limsup n^{-1} W_n \le M < \infty$ almost surely. □

B. Convergence in law.

In the sequel we shall always assume that the hypotheses of Proposition 3 are satisfied. Let

$$\mathbf{t}_0 = 0, \qquad \mathbf{t}_k = \min\bigl\{n > \mathbf{t}_{k-1} \,:\, \Phi(R_n) < \Phi(R_{\mathbf{t}_{k-1}})\bigr\}$$

be the *(strictly) descending ladder indices* of $\bigl(\Phi(R_n)\bigr)$, and for $k \geq 1$ let

$$\mathbf{r}_k = \max\bigl\{\mathbf{s}_m \,:\, \mathbf{s}_m < \mathbf{t}_{k+1}\bigr\}.$$

Thus, $0 = \mathbf{t}_0 \leq \mathbf{r}_0 < \mathbf{t}_1 \leq \mathbf{r}_2 < \mathbf{t}_2 \cdots$, and $\mathbf{t}_k$ is the moment when the sequence $\Phi(R_n)$ attains its k-th record minimal value, whereas $\mathbf{r}_k$ is the last moment when the k-th record value is reproduced before attaining a new record value $\Phi(R_{\mathbf{t}_{k+1}})$. Also, observe that both $(\mathbf{t}_k)$ and $(\mathbf{r}_k)$ are subsequences of $(\mathbf{s}_k)$. Define

$$V_k = R_{\mathbf{t}_k}^{-1} R_{\mathbf{r}_k}, \quad k \geq 0, \quad \text{and} \quad W_k = R_{\mathbf{r}_{k-1}}^{-1} R_{\mathbf{t}_k}, \quad k \geq 1.$$

Then we can decompose

$$R_{\mathbf{t}_k} = V_0 W_1 V_1 \cdots V_{k-1} W_k, \qquad R_{\mathbf{r}_k} = V_0 W_1 V_1 \cdots V_{k-1} W_k V_k,$$

and (V_k) and (W_k) are two independent sequences of i.i.d. random variables in $\mathrm{A{\scriptsize FF}}(\mathbb{T})$. We denote by η_0 and η_- their respective common distributions. They can be obtained as follows. Let $\bar{\eta}$ be the restriction of η onto $\mathrm{H{\scriptsize OR}}(\mathbb{T})$, and let $\bar{m} \geq 0$ be the total mass of $\bar{\eta}$. Then

$$\eta_- = \frac{1}{1-\bar{m}}(\eta - \bar{\eta}) \quad \text{and} \quad \eta_0 = \frac{1}{1-\bar{m}}\Bigl(\delta_\iota + \sum_{n=1}^{\infty} \bar{\eta}^{(n)}\Bigr).$$

In particular, $(R_{\mathbf{t}_k})$ is the right random walk with law $\eta_0 {*} \eta_-$ (recall that « $*$ » denotes convolution). Finiteness of the first moment of η (Proposition 4) implies finiteness of the first moments of η_- and η_0.

Fix $n \geq 0$ and consider

$$\begin{aligned} K(n) &= \max\{k \,:\, \mathbf{t}_k \leq n\}, \\ S(n) &= \mathbf{t}_{K(n)}, \\ T(n) &= \max\{\mathbf{s}_k \,:\, \mathbf{s}_k \leq n\}. \end{aligned}$$

Thus, $S(n)$ is the first time when $\Phi(R_k)$ attains its minimal value up to time n, and $T(n)$ is the last moment (again up to time n) when this minimal value is attained. Note that $T(n)$ coincides with $\mathbf{r}_{K(n)}$ only if $\mathbf{r}_{K(n)} \leq n$. Put

$$\overline{V}(n) = R_{S(n)}^{-1} R_{T(n)}.$$

Let $\Lambda = \{0,1,2,\ldots\} \times \bigcup_{i=0}^{\infty} (\mathrm{A{\scriptstyle FF}}(\mathbb{T}))^i$. Denote by P_n and Q_n the distributions of the Λ-valued random variables

$$\big(K(n); V_0, W_1, \ldots, V_{K(n)-1}, W_{K(n)}\big),$$

$$\big(K(n); V_0, W_1, \ldots, V_{K(n)-1}, W_{K(n)}, \overline{V}(n)\big),$$

respectively. By restricting to the events $[K(n) = k,\ S(n) = s]$ and $[K(n) = k,\ T(n) = t]$, we obtain the decompositions

$$P_n = \sum_{k,s} P_n^{k,s}, \quad Q_n = \sum_{k,t} Q_n^{k,t},$$

respectively. Let also $P_n^k = \sum_s P_n^{k,s}$ and $Q_n^k = \sum_t Q_n^{k,t}$.

Next, denote by Π^k the distribution of the random variable $\big(k; V_0, W_1, \ldots, V_{k-1}, W_k\big)$, and by Π_n^k its restriction onto the set $[K(n) = k]$ (so that $\Pi^k = \sum_n \Pi_n^k$). Then it is clear that $P_n^k = \Pi_n^k$ for any k.

The following can be directly checked using the definition of the measures $P_n^{k,s}$ and $Q_n^{k,t}$.

LEMMA 6.

(1) *For any given k, s the measures $P_n^{k,s}$ and P_n^k are invariant with respect to the action of the product of the symmetric groups $\mathcal{S}_k \times \mathcal{S}_k$ by permutations of the coordinates $(V_0, \ldots, V_{k-1})$ and $(W_1, \ldots, W_k)$.*

(2) *For any k, t the measures $Q_n^{k,s}$ and Q_n^k are invariant with respect to the action of $\mathcal{S}_{k+1} \times \mathcal{S}_k$ by permutations of the coordinates $(V_0, \ldots, V_{k-1}, \overline{V}(n))$ and $(W_1, \ldots, W_k)$.*

Denote by $\check{\eta}_0$ and by $\check{\eta}_-$ the reflected measures of η_0 and η_-, respectively. Then the convolutions $\check{\eta}_0 * \check{\eta}_-$ and $\check{\eta}_- * \check{\eta}_0$ both have a finite first moment, and their projection $\Phi(\check{\eta}_0 * \check{\eta}_-) = \Phi(\check{\eta}_- * \check{\eta}_0)$ onto $\mathbb{Z}$ has strictly positive drift. Put :

$$Y_k = W_k^{-1} V_{k-1}^{-1} \quad \text{and} \quad Z_k = V_{k-1}^{-1} W_k^{-1}.$$

Then the measures $\check{\eta}_- * \check{\eta}_0$ and $\check{\eta}_0 * \check{\eta}_-$ are the common distributions of the sequences of i.i.d. random variables (Y_k) and (Z_k), respectively.

The harmonic measures ν_1 and ν_2 of the right random walks with laws $\check{\eta}_- * \check{\eta}_0$ and $\check{\eta}_0 * \check{\eta}_-$, respectively, satisfy the relations $\check{\eta}_- * \nu_2 = \nu_1$ and $\check{\eta}_0 * \nu_1 = \nu_2$. Also, ν_1 coincides with the harmonic measure of the random walk with law $\check{\eta}$ (because $R_{\mathbf{t}_k}^{-1} = Y_k^{-1} \cdots Y_1^{-1}$). Since the support η generates a non-exceptional subgroup of $\mathrm{AFF}(\mathbb{T})$ (Proposition 4 (a)), the measures ν_1 and ν_2 are continuous (Theorem 3).

PROPOSITION 5.

(a) *As $n \to \infty$, the distribution of $R_{S(n)}^{-1} o$ converges weakly in $\widehat{\mathbb{T}}$ (that is, $R_{S(n)}^{-1} o$ converges in law) to ν_1. Furthermore,*

$$\lim_{m,n\to\infty} \theta\Big(R_{S(n)}^{-1} o, Y_{K(n)} \cdots Y_{(K(n)-m+1)\vee 1} o\Big) \to 0$$

in probability.

(b) *The distribution of $R_{T(n)}^{-1} o$ converges weakly to ν_2, and*

$$\lim_{m,n\to\infty} \theta\Big(R_{T(n)}^{-1} o,\ \bar{V}(n)^{-1} W_{K(n)}^{-1} V_{K(n)-1} \cdots V_{(K(n)-m+1)\vee 1} W_{(K(n)-m+1)\vee 1} o\Big) \to 0$$

in probability.

Proof. — Since $K(n) \to \infty$ almost surely, we can simplify notations by assuming that $K(n) \geq m$.

(a) By definition, $R_{S(n)}^{-1} = Y_{K(n)} \cdots Y_1$. By Lemma 6, for any fixed value of $K(n)$ the distribution of $Y_{K(n)} \cdots Y_1$ coincides with the distribution of $Y_1 \cdots Y_{K(n)}$. Since $Y_1 \cdots Y_k o$ converges almost surely to a random variable R_∞^1 with distribution ν_1 (Theorem 2), the distribution of $R_{S(n)}^{-1} o$ also converges to ν_1. The second part now follows from the fact that $\theta(Y_1 \cdots Y_m o, R_\infty^1) \to 0$ almost surely as m tends to infinity.

(b) By Lemma 6, $R_{T(n)}^{-1} o$ has the same distribution as

$$Z_1 \cdots Z_{K(n)} \bar{V}_{K(n)}^{-1} o,$$

and

$$d\big(Z_1 \cdots Z_{K(n)} \bar{V}_{K(n)}^{-1} o, Z_1 \cdots Z_{K(n)} o\big) = \big|\bar{V}_{K(n)}^{-1}\big|$$

has the same distribution as $|V_0|$. Since $|Z_1 \cdots Z_{K(n)}| \to \infty$ in probability, this implies that

$$\theta\big(Z_1 \cdots Z_{K(n)} \bar{V}_{K(n)}^{-1} o, Z_1 \cdots Z_{K(n)} o\big) \to 0$$

in probability. All the rest is analogous to part (a). □

C. Asymptotic independence.

Our next step is to show that splitting R_n at $T(n)$ gives rise to two pieces which are asymptotically independent.

Once again suppose that n is fixed. Put $\widehat{X}_k^n = X_{n+1-k}^{-1}$ for $k = 1, 2, \ldots, n$. Then

$$\widehat{R}_k^n = \widehat{X}_1^n \widehat{X}_2^n \cdots \widehat{X}_k^n = R_n^{-1} R_{n-k} \overset{\text{in law}}{=\!=\!=} \check{R}_k, \quad k \le n,$$

where $(\check{R}_k)$ stands for the right random walk with law $\check{\mu}$. For the objects associated with the (initial n-segment of the) random walk $(\widehat{R}_k) = (\widehat{R}_k^n)$ we shall use the same notations as used above for the random walk (R_k), adding a hat «$\hat{\ }$». Note that Proposition 4 applies to the reflected measure $\check{\mu}$ as well as to μ.

Clearly, $\widehat{S}(n) = n - T(n)$ is the first time when the sequence $\Phi(\widehat{R}_k) = \Phi(R_{n-k}) - \Phi(R_n)$ attains its minimal value on the segment $[0, n]$, and

$$R_{n,T(n)} = R_{T(n)}^{-1} R_n = \widehat{R}_{n-T(n)} = \widehat{R}_{\widehat{S}(n)}.$$

Moreover, for a fixed value of $T(n)$ the sequences of increments $(X_1, X_2, \ldots, X_{T(n)})$ and $(\widehat{X}_1, \widehat{X}_2, \ldots, \widehat{X}_{n-T(n)}) = (X_n^{-1}, X_2^{-1}, \ldots X_{T(n)+1}^{-1})$ are conditionally independent. Thus, the distribution of the $\Lambda \times \Lambda$-valued random variable

$$\Big(\big(K(n); V_0, W_1, \ldots, W_{K(n)}, \bar{V}(n)\big); \big(\widehat{K}(n); \widehat{V}_0, \widehat{W}_1, \ldots, \widehat{V}_{\widehat{K}(n)-1}, \widehat{W}_{\widehat{K}(n)}\big) \Big)$$

coincides with

$$\sum_{k,\hat{k},t,s\,:\,t+s=n} Q_n^{k,t} \otimes \widehat{P}_n^{\hat{k},s}. \tag{A.1}$$

PROPOSITION 6. — *For any fixed $m \ge 1$ the Λ-valued random variables*

$$\big(K(n) \vee m; V_0, W_1, \ldots, V_{(K(n)\vee m)-1}, W_{K(n)\vee m}\big)$$

and

$$\big(\widehat{K}(n) \vee m; \widehat{V}_0, \widehat{W}_1, \ldots, \widehat{V}_{(\widehat{K}(n)\vee m)-1}), \widehat{W}_{\widehat{K}(n)\vee m}\big)$$

are asymptotically independent, and their joint distribution π_n^m converges weakly to the product of the measures Π^m and $\widehat{\Pi}^m$.

Proof. — Let $(\pi_n^m)_1$ and $(\pi_n^m)_2$ be the marginals of the measure (π_n^m). Since $K(n) \to \infty$ in probability, $(\pi_n^m)_1$ converges weakly to Π^m.

Analogically, $(\pi_n^m)_2 \to \widehat{\Pi}^m$. Now we have to show asymptotic independence of the marginals. Fix two measurable sets $A, \widehat{A} \subset \{m\} \times \big(\mathrm{Aff}(\mathbb{T})\big)^{2m} \subset \Lambda$, then

$$(\pi_n^m)_1(A) = \sum_{k\le n} q_k \quad \text{and} \quad (\pi_n^m)_2(\widehat{A}) = \sum_{\hat{k}\le n} \hat{q}_{\hat{k}},$$

where

$$q_k = \Pr\big[\mathbf{t}_m = k,\ \big(m; V_0, W_1, \dots, V_{m-1}, W_m\big) \in A\big],$$

$$\hat{q}_{\hat{k}} = \Pr\big[\hat{\mathbf{t}}_m = \hat{k},\ \big(m; \widehat{V}_0, \widehat{W}_1, \dots, \widehat{V}_{m-1}, \widehat{W}_m\big) \in \widehat{A}\big].$$

On the other hand, the decomposition (A.1) implies that

$$\pi_n^m(A \times \widehat{A}) = \sum_{k+\hat{k}\le n} q_k \hat{q}_{\hat{k}},$$

so that

$$\begin{aligned}(\pi_n^m)_1(A)\cdot(\pi_n^m)_2(\widehat{A}) - \pi_n^m(A\times\widehat{A})\\ = \sum_{\substack{k\le n,\,\hat{k}\le n\\ k+\hat{k}>n}} q_k\hat{q}_{\hat{k}} \le \Pr\big[\mathbf{t}_m \ge \tfrac12 n\big] + \Pr\big[\hat{\mathbf{t}}_m \ge \tfrac12 n\big].\end{aligned}$$

As m is fixed, the latter tends to zero when $n \to \infty$. □

D. Conclusion.

Proposition 3 is a consequence of our final result.

THEOREM 10. — *As $n \to \infty$, the distribution π_n of $\big(R_{T(n)}^{-1} o, R_{n,T(n)} o\big)$ converges weakly in $\widehat{\mathbb{T}} \times \widehat{\mathbb{T}}$ to the measure $\nu_2 \otimes \hat{\nu}_1$.*

Proof. — For $x \in \mathbb{T}$, set $D_x = \{\mathfrak{z} \in \widehat{\mathbb{T}} : x \in \overline{o\mathfrak{z}}\}$. It is sufficient to prove that

$$\pi_n(D_x \times D_y) \longrightarrow \nu_2(D_x) \times \hat{\nu}_1(D_y)$$

for any two points $x, y \in T$. Fix a number $\epsilon > 0$. Then by Proposition 5 there exists a number m such that

$$\Pr\big([R_{T(n)}^{-1} o \in D_x] \,\triangle\, B\big) < \epsilon \quad \text{and} \quad \Pr\big([R_{n,T(n)} o \in D_y] \,\triangle\, \widehat{B}\big) < \epsilon,$$

where $\triangle$ denotes symmetric difference and

$$B = \big[\overline{V}(n)^{-1} W_{K(n)}^{-1} V_{K(n)-1}^{-1} \cdots V_{K(n)-m+1}^{-1} W_{K(n)-m+1}^{-1} o \in D_x\big],$$

$$\widehat{B} = \big[\widehat{W}_{\widehat{K}(n)}^{-1} \widehat{V}_{\widehat{K}(n)-1}^{-1} \cdots \widehat{W}_{\widehat{K}(n)-m+1}^{-1} \widehat{V}_{\widehat{K}(n)-m}^{-1} o \in D_y\big].$$

As $K(n), \widehat{K}(n) \to \infty$ in probability, we can always assume, up to an error of ϵ, that $K(n), \widehat{K}(n) \geq m$. Thus,

$$\left|\pi_n(D_x \times D_y) - \Pr(B \cap \widehat{B})\right| \leq 3\epsilon.$$

From Lemma 6 and decomposition (A.1) we get

$$\Pr\bigl(B \cap \widehat{B}\bigr) = \Pr\bigl[Z_1 \cdots Z_m \in D_x, \widehat{Y}_1 \cdots \widehat{Y}_m \in D_y\bigr],$$

and by Proposition 6,

$$\begin{aligned}\Pr\bigl[Z_1 \cdots Z_m \in D_x, \widehat{Y}_1 \cdots \widehat{Y}_m \in D_y\bigr] \\ \longrightarrow \Pr\bigl[Z_1 \cdots Z_m \in D_x\bigr] \times \Pr\bigl[\widehat{Y}_1 \cdots \widehat{Y}_m \in D_y\bigr]\end{aligned}$$

as n tends to infinity. Now, for $m \to \infty$

$$\Pr\bigl[Z_1 \cdots Z_m \in D_x\bigr] \to \nu_2(D_x) \quad \text{and} \quad \Pr\bigl[\widehat{Y}_1 \cdots \widehat{Y}_m \in D_y\bigr] \to \hat{\nu}_1(D_y).$$

Gathering all these relations we obtain that

$$\pi_n(D_x \times D_y) \longrightarrow \nu_2(D_x) \times \hat{\nu}_1(D_y). \qquad \square$$

By Theorem 3 the measures ν_2 and $\hat{\nu}_1$ on $\partial\mathbb{T}$ are continuous, so that

$$\nu_2 \otimes \hat{\nu}_1\bigl(\{(\mathfrak{u}, \mathfrak{u}) \,:\, \mathfrak{u} \in \partial\mathbb{T}\}\bigr) = 0.$$

Consequently, the function $(\mathfrak{x}, \mathfrak{y}) \mapsto h(\mathfrak{x} \wedge \mathfrak{y})$ is finite and continuous at $\nu_2 \times \hat{\nu}_1$-almost every point $(\mathfrak{x}, \mathfrak{y}) \in \widehat{\mathbb{T}} \times \widehat{\mathbb{T}}$. Thus, Proposition 3 follows from Theorem 10 by a well known property of weak convergence (see for example [Bi], Thm 5.1).

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