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STOKES PHENOMENON, MULTISUMMABILITY AND DIFFERENTIAL GALOIS GROUPS

by Michèle LODAY-RICHAUD

INTRODUCTION

This paper deals with the local study of linear differential systems of order one and dimension n

$$[A] \qquad \frac{\mathrm{d}X}{\mathrm{d}x} = AX$$

with coefficients meromorphic in a neighbourhood of a given point, say $x_0 = 0$, in $\mathbb{C}$. The variable x is the local coordinate around 0 in $\mathbb{C}$, the unknown X is a $n \times 1$ column vector and the matrix A of coefficients is a $n \times n$ matrix with meromorphic entries at $x = 0$.

A normal form $[A_0]$ being fixed, the *classifying set* $H^1(S^1; \Lambda(A_0))$ of Malgrange and Sibuya provides a non-Abelian cohomological description of the classes of formal transformations of $[A_0]$ up to the convergent ones. By transformations, we mean here linear changes of unknowns $Y = \widehat{F}X$, with formal meromorphic coefficients $\widehat{F} \in \mathrm{GL}(n, \mathbb{C}[[x]][1/x])$ acting inside systems with (convergent) meromorphic coefficients. In terms of connections, these classes correspond to the so-called *meromorphic marked pairs* (see [BV89]), and all results below could be rephrased in terms of connections. But we attempt here to be as constructive and algorithmic as possible; therefore, it will be more efficient in the following to speak in terms of systems.

In this paper, we first give a procedure to reduce any cocycle in $H^1(S^1; \Lambda(A_0))$ to a normalized unique form called a *Stokes cocycle*. This

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procedure is *natural* in the sense that it commutes with isomorphisms; and it endows the classifying set with a natural structure of a finite dimensional Lie group. Unlike the abstract approach given independently by Babbitt and Varadarajan in [BV89], the present approach is constructive and provides an explicit algorithm for reducing any cocycle to its Stokes form.

Other methods have been developed to describe meromorphic classes with a uniquely well-defined family of «invariants»: the method by Balser, Jurkat and Lutz (see [BJL79]) with a connection system (Stokes matrices), the method by Malgrange and Deligne using I-filtered systems (see [Mal83]), the geometric method of the wild π_1 by Ramis (see [MR91]), and the «bridge equation» by Écalle (see [Ec85]). The methods used by Balser, Jurkat, Lutz and also by Malgrange, Deligne are somewhat similar to ours. The main difference between the present work and both [BJL79] and [Mal83] is that we start with an arbitrary 1-cocycle while Balser, Jurkat and Lutz start with a fundamental solution and Malgrange and Deligne with a 0-cochain made of fundamental solutions. Of course, the Malgrange-Sibuya isomorphism gives a correspondence between 1-cocycles and fundamental solutions or 0-cochains; but, whereas this correspondence is constructive from fundamental solutions or 0-cochains to 1-cocycles, it is transcendental and nonconstructive the other way.

We then show in section III how our procedure applies to the *multisummability* of solutions of systems and to *differential Galois theory*.

Specifically, Ramis proved in [Ra85-1], [Ra85-3] that a multi-leveled transformation $\widehat{F}$ can be written essentially uniquely as a product of single-leveled $\widehat{F}_j$ (say k_j-summable $\widehat{F}_j$ for the different levels k_j of the system). His proof depends on the Ramis-Sibuya isomorphism theorem, a delicate analogue of the Malgrange-Sibuya theorem with Gevrey estimates. We give here a new proof of this result, which is mainly algebraic and much shorter; both proofs are nonconstructive. Furthermore, using a result of Martinet and Ramis (see [MR91], thm 14 i), we show that the natural sums associated to our Stokes cocycles coincide with those defined by Écalle using Borel-Laplace and acceleration integrals and with those defined by Ramis using asymptotics of Gevrey type. We also relate these sums to those defined by Malgrange and Ramis in [MalR92].

We also prove in section III that the Stokes automorphisms associated to our Stokes cocycles are *Galoisian*, i.e., that they belong to a faithful representation of the differential Galois group of the system. The proof we give of this result follows an idea of Deligne for using the *Tannakian theory*; we also include the discussion of an example of an usual Stokes matrix which is not Galoisian. Finally, we give a Tannakian proof of a theorem of Ramis (see [Ra85-2]) on the generation of the differential Galois group by the formal monodromy, the exponential torus and Stokes automorphisms.

CONTENTS

I. PRELIMINARIES

I.1. Formal and meromorphic classifications.

All over this paper, by *system* we mean a linear ordinary differential system of a given dimension $n \in \mathbb{N}^*$ and order 1 with complex meromorphic coefficients of a complex variable at the origin in $\mathbb{C}$. We denote

$$[A] \qquad \frac{\mathrm{d}X}{\mathrm{d}x} = AX,$$

where x is the complex variable, X is the unknown vector of size n and A is the $n \times n$ matrix of coefficients in $\mathrm{End}(n, \mathbb{C}\{x\}[1/x])$.

By *transformation* (or *meromorphic transformation*) of a system $[A]$ we mean a linear change of unknown $X \mapsto F^{-1}X$ (i.e. $Y = FX$) where F is an $n \times n$ invertible matrix with (convergent) meromorphic entries. We denote by $[{}^{F}\!A]$ the transformed system and $F \in G := \mathrm{GL}(n, \mathbb{C}\{x\}\,[1/x])$.

For F having formal meromorphic entries — we then usually denote it by $\widehat{F}$ — the transformed system $[{}^{\widehat{F}}\!A]$ may or may not have convergent meromorphic coefficients. And when it has, we call $\widehat{F}$ a *formal transformation* of $[A]$, thus restricting us to those changes of unknown which act inside the set of systems here considered. We denote this set by $\widehat{G}(A)$. An element $\widehat{F}$ of $\widehat{G}(A)$ in then an element of $\widehat{G} := \mathrm{GL}\big(n, \mathbb{C}[[x]]\,[1/x]\big)$ satisfying a relation ${}^{\widehat{F}}\!A = B$, for a convergent B (i.e. $B \in \mathrm{End}(n, \mathbb{C}\{x\}[1/x])$). This relation is equivalent to the linear differential system

$$[A, B] \qquad \frac{\mathrm{d}\widehat{F}}{\mathrm{d}x} = B\widehat{F} - \widehat{F}A,$$

One has $G \subset \widehat{G}(A) \subset \widehat{G}$.

Given $[A]$ and $[B]$ two systems, the relations «there exists a formal transformation or a transformation taking $[A]$ into $[B]$» are equivalence relations on the set of systems. They are respectively called *formal equivalence* and *meromorphic equivalence* and the associated classifications are called *formal classification of systems* and *meromorphic classification of systems*.

The formal classification initiated by Poincaré and Fabry has been solved by Balser, Jurkat and Lutz in [BJL79]. They proved there, that one can characterize each formal class by making the choice in this class of a particular (non unique) system

$$[A_0] \qquad \frac{\mathrm{d}X}{\mathrm{d}x} = A_0 X$$

called a *normal form* and satisfying the following properties: there exists a fundamental solution X_0 of $[A_0]$, i.e., a $n \times n$ matrix the columns of which are n $\mathbb{C}$-linearly independent solutions of $[A_0]$, of the form $X_0 = x^L \, \mathrm{e}^{Q(1/x)}$ where L is a constant matrix called the *matrix of exponents of formal monodromy* and $Q(1/x) = \mathrm{diag}(q_1(1/x), \ldots, q_n(1/x))$ is a diagonal matrix, the $q_j(1/x)$ being polynomials in a root $1/t = x^{-1/p}$, $p \in \mathbb{N}^*$, of the variable x without constant term. Such a solution will be called a *normal solution*. As usually x^L means $e^{L\ell n x}$.

Normal forms in a given formal class are generally not unique but they all belong to the same meromorphic class. Often one calls normal form any element in this class a fundamental solution of which takes the form $F(x)x^L \, \mathrm{e}^{Q(1/x)}$ with $F(x)$ convergent meromorphic. Without restricting our purpose we shall assume that our normal forms $[A_0]$ admit a fundamental solution $X_0 = x^L \, \mathrm{e}^{Q(1/x)}$.

We shall refer to the *unramified case* when the smallest possible p equals 1; otherwise we shall refer to the *ramified case*. The unramified case is essentially the case when Poincaré and Fabry achieved the formal classification. The ramified case appears as being technically much more difficult. Thus, for instance, in the ramified case, the matrices L and Q don't commute anymore and then $A_0 \neq L/x + \mathrm{d}(Q(1/x))/\mathrm{d}x$. Actually, the matrix A_0 is a polynomial in x and x^{-1}. It is identifiable by inspection of an arbitrary system $[A]$ in its formal class, since Q and L are identifiable. The identification includes the resolution of algebraic equations and offers at least as many choices as the Jordan form of a matrix. One of the normal forms $[A_0]$ will be provided by the solver DESIR2 made in the IMAG of Grenoble after a preliminary version DESIR. Such a program needs a large computer. Examples in dimension $n = 2$ or 3 can be found in [LR90-1] or in [LR91].

An *isotropy* of $[A_0]$ is a transformation $\widehat{F}$ which leaves $[A_0]$ invariant: ${}^{\widehat{F}}A_0 = A_0$. Thus, the isotropies are precisely the solutions of the system

$$[\mathrm{End}\, A_0] = [A_0, A_0] \qquad \frac{\mathrm{d}F}{\mathrm{d}x} = A_0 F - F A_0.$$

They are, a priori, formal transformations. Actually (see [BJL79]), the set of isotropies of a fixed normal form $[A_0]$ is a subgroup of $\mathrm{GL}(n, \mathbb{C}[1/x, x])$. We denote it by $G_0(A_0)$.

Two transformations $\widehat{F}_1$ and $\widehat{F}_2$ take $[A_0]$ into the same system ${}^{\widehat{F}_1}\!A_0 = {}^{\widehat{F}_2}\!A_0$ iff there exists $f_0 \in G_0(A_0)$ such that $\widehat{F}_1 = \widehat{F}_2 f_0$. Thus the *set of meromorphic classes of systems* can be identified to the left and right quotient $G \setminus \widehat{G}(A_0)/G_0(A_0)$ of $\widehat{G}(A_0)$ whereas the left quotient $G \setminus \widehat{G}(A_0)$ represents the *set of meromorphic classes of transformations* of $[A_0]$. The group $G_0(A_0)$ is easy to compute and is often trivial (cf. examples in [LR91] and [LR90-1]). Thus the structure of the set $G \setminus \widehat{G}(A_0)/G_0(A_0)$ of meromorphic classes of systems is easily deduced from the structure of $G\setminus\widehat{G}(A_0)$. But this one is not so easy to understand. The Malgrange-Sibuya isomorphism theorem which we recall below, describes this set $G\setminus\widehat{G}(A_0)$ as a non-Abelian cohomological set. The first aim of this paper is to endow this cohomological set with a natural and constructive structure of a unipotent Lie group.

I.2. The Malgrange-Sibuya isomorphism theorem.

Malgrange and Sibuya (see [Si76], [Mal79], [Si90]) have made a cohomological analysis of the set of meromorphic classes of transformations. In order to state their theorem we need a few notations and definitions.

Let $\bar{\omega} : \widetilde{\mathbb{C}} = S^1 \times [0, +\infty[\to \mathbb{C}$, $(\theta, \rho) \mapsto x = \rho\, e^{i\theta}$, be the real blow up of 0 in $\mathbb{C}$. One has $S^1 = \bar{\omega}^{-1}(0)$ and $\bar{\omega} : S^1 \times]0, +\infty[\to \mathbb{C}^*$ is a bijection. To each $\theta_0 \in S^1$ one considers the basis of open neigbourhoods in $\widetilde{\mathbb{C}}$ of the form:

$$\widetilde{U}(\theta_0, \varepsilon, \varepsilon') = \left\{(\theta, \rho) \in \widetilde{\mathbb{C}}; |\theta - \theta_0| < \tfrac{1}{2}\varepsilon,\ |\rho| < \varepsilon'\right\}$$

or their projections on $\mathbb{C}^*$ via $\bar{\omega}$:

$$U(\theta_0, \varepsilon, \varepsilon') = \left\{x \in \mathbb{C}^*; |\arg x - \theta_0| < \tfrac{1}{2}\varepsilon,\ |x| < \varepsilon'\right\}.$$

The intersection $S^1 \cap \widetilde{U}(\theta_0, \varepsilon, \varepsilon')$ is the arc $]\theta_0 - \frac{1}{2}\varepsilon, \theta_0 + \frac{1}{2}\varepsilon[$ in S^1. As generally the value of ε' does not matter, we shall simply speak of an *arc U on S^1* in place of a sector $U(\theta_0, \varepsilon, \varepsilon')$ or $\widetilde{U}(\theta_0, \varepsilon, \varepsilon')$ for a convenient ε'. We shall also denote $U(\theta_0, \varepsilon)$ or $]\theta_0 - \frac{1}{2}\varepsilon, \theta_0 + \frac{1}{2}\varepsilon[$.

The *sheaf* $\Lambda(A_0)$ *of flat isotropies* over S^1 is defined as follows: a germ f at $\theta_0 \in S^1$ is an invertible matrix $f \in \mathrm{GL}(n, \mathcal{O}(U))$ with holomorphic entries on an arc $U = U(\theta_0, \varepsilon, \varepsilon')$ for suitable $\varepsilon, \varepsilon'$ and satisfying the following conditions:

(i) *Flatness* (f is asymptotic to the identity on U at 0):

$$\lim_{\substack{x\to 0\\ x\in U}} f(x) = I \quad \text{and} \quad f \underset{U}{\sim} I;$$

(ii) *Isotropy of* $[A_0]$: ${}^fA_0 = A_0$.

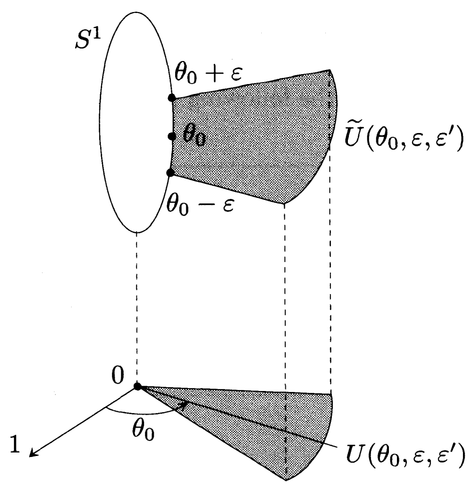

Figure I.1

We denote $f \in \Gamma(U; \Lambda(A_0))$.

The map $\exp_\mu$ in the Malgrange-Sibuya theorem below is defined as follows.

Let $\widehat{F} \in \widehat{G}(A_0)$. From the main asymptotic existence theorem, to a small open arc U on S^1 (actually a sector with vertex 0 in $\mathbb{C}^*$) there exists at least one *realization* F of $\widehat{F}$. This means $F \underset{U}{\sim} \widehat{F}$ and F satisfies the same system $[A_0, A]$ as $\widehat{F}$. To a finite covering $\mathcal{U} = \{U_j, j \in J\}$ of S^1 by open arcs of small size there can be then associated a family $\{F_j, j \in J\}$ of realizations of $\widehat{F}$ on the different U_j. The 1-cochain $(\dot{F}_{j,\ell} = F_j^{-1}F_\ell)$ is a 1-cocycle with values in $\Lambda(A_0)$. The different choices of a covering $\mathcal{U}$ and realizations F_j induce cohomologous 1-cocycles. On the other hand, the 1-cochain $(\dot{F}_{j,\ell})$ depends only on the (left) meromorphic class of $\widehat{F}$. This defines a map, denoted $\exp_{\mu_0}$ below, from $G \setminus \widehat{G}(A_0)$ to the non-Abelian cohomological set $H^1(S^1; \Lambda(A_0))$. It is easily seen that $\exp_{\mu_0}$is injective.

THEOREM I.2.1 (Malgrange-Sibuya). — *The map*

$$\exp_{\mu_0} : G \setminus \widehat{G}(A_0) \longrightarrow H^1(S^1; \Lambda(A_0))$$

is bijective.

Malgrange (see [Mal79], [Mal83]) and Sibuya (see [Si76], [Si90]) gave different proofs of this theorem. See also [MR82] for a variant of Malgrange's proof.

Remark I.2.2. — To another normal form $A_1 = {}^{\Phi}A_0$ (we recall that Φ is a meromorphic transformation) there correspond cochains which are conjugated via Φ. We get the following commutative diagram:

$$\begin{tikzcd}
G \setminus \widehat{G}(A_1) \arrow[r] \arrow[d] & G \setminus \widehat{G}(A_0) \arrow[d] \\
H^1(S^1\,;\,\Lambda(A_1)) \arrow[r] & H^1(S^1\,;\,\Lambda(A_0))
\end{tikzcd}$$

$$\begin{tikzcd}
\widehat{F} \arrow[r, mapsto] \arrow[d, mapsto] & \widehat{F}\Phi \arrow[d, mapsto] \\
\exp_{\mu_1}(\widehat{F}) \arrow[r, mapsto] & \exp_{\mu_0}(\widehat{F}\Phi) = \Phi^{-1}\exp_{\mu_1}(\widehat{F})\Phi
\end{tikzcd}$$

I.3. The Stokes phenomenon.

Given a formal solution $\widehat{F}X_0$ of a system $[A]$ in the formal class of $[A_0]$, (where $\widehat{F} \in \widehat{G}(A_0)$) there always exists, by the main asymptotic existence theorem, a realization FX_0 of $\widehat{F}X_0$ over a small sector U with vertex 0 in $\mathbb{C}^*$, and by the Cauchy-Lipschitz theorem, this solution can be indefinitely continued around 0 on the Riemann surface $\widetilde{\mathbb{C}}^*$ of $\log x$. But generally, this continuation can't be done by preserving asymptoticity of F to $\widehat{F}$: big jumps of exponential type appear. This is the *Stokes phenomenon.*

The unique case when no jump occurs is when $[A]$ is meromorphically equivalent to a normal form i.e. when $\widehat{F}$ is convergent. Thus, one can say that a system is a normal form when it has no Stokes phenomenon.

The Stokes phenomenon is strongly related to the matrix Q in the normal solution $X_0 = x^L\,\mathrm{e}^Q$. Notice that this matrix is well-defined up to a permutation on its diagonal terms. Jumps occur only when crossing the finitely many *Stokes directions* (Definition I.4.5.i). The Malgrange-Sibuya isomorphism theorem shows that these jumps characterize the meromorphic classes of transformations but in many somewhat intricate combinatoric ways. Our theorem II.2.1 below gives a unique rigid choice to characterize each class. Such other choices have been already made by Balser, Jurkat and Lutz (see [BJL79]) and by Malgrange and Deligne (see [Mal83]). But, whereas Balser, Jurkat and Lutz start with one fundamental solution asymptotic to $\widehat{F}X_0$ somewhere and Malgrange and Deligne with a 0-cochain, that is, a family of fundamental solutions asymptotic to $\widehat{F}X_0$, we start with an arbitrary 1-cocycle and we constructively change it into a rigid natural form called a *Stokes cocycle* (Definition II.1.8). Obviously, the correspondence between 0-cochains and 1-cocycles given by the Malgrange-Sibuya isomorphism theorem enables one to commute in between the different points of view. But, this commutation is transcendental and nonconstructive since the inverse map in the Malgrange-Sibuya theorem is transcendental and nonconstructive. Moreover, in our presentation, the phenomenon is described via *anti-Stokes* and not via *Stokes directions* (Definition I.4.5 ii).

In order to do this construction we need a precise analysis of the sheaf $\Lambda(A_0)$.

I.4. Some notations and definitions.

NOTATIONS I.4.1. — We set

$$\mathcal{Q}_{[A_0]} = \left\{q_1\Big(\frac{1}{x}\Big), \dots, q_n\Big(\frac{1}{x}\Big)\right\}$$

and

$$\mathcal{Q}_{[\mathrm{End}\,A_0]} = \left\{(q_j - q_\ell)\Big(\frac{1}{x}\Big);\, q_j \neq q_\ell \in \mathcal{Q}_{[A_0]}\right\}.$$

The set $\mathcal{Q}_{[A_0]}$ is the set of diagonal elements in a matrix Q associated to $[A_0]$. But, the set $\mathcal{Q}_{[\mathrm{End}\,A_0]}$ is the set of *non-zero* diagonal elements in a matrix Q associated to $[\mathrm{End}\,A_0]$.

For $q_j - q_\ell \in \mathcal{Q}_{[\mathrm{End}\,A_0]}$ we set

$$(q_j - q_\ell)\Big(\frac{1}{x}\Big) = \frac{a_{(j,\ell)}}{x^k} + \mathrm{o}\Big(\frac{1}{x^k}\Big), \qquad a_{(j,\ell)} \neq 0.$$

DEFINITION I.4.2. — The leading exponent k is called the *degree* $\deg(q_j - q_\ell)$ of $(q_j - q_\ell)$ and a *level* of $[A_0]$ (or of any system in the formal class of $[A_0]$).

Levels are rational numbers. The unramified case is the case when all levels are integers.

NOTATION I.4.3. — The set $\mathcal{K} = \{k_1 < \cdots < k_r\}$ denotes the set of all levels of $[A_0]$.

For elements $q_j \neq q_\ell$ in $\mathcal{Q}_{[A_0]}$ we define the two families of partial order relations $\underset{\tilde\theta}{\prec}$ and $\underset{\tilde\theta,\,\max}{\prec}$ as follows.

DEFINITION I.4.4.

$$q_j \underset{\tilde\theta}{\prec} q_\ell \iff \mathrm{e}^{(q_j-q_\ell)(1/x)} \text{ is flat at 0 in a neighbourhood of the direction } \tilde\theta$$
$$\iff \mathrm{Re}(a_{(j,\ell)}\,\mathrm{e}^{-ik\tilde\theta}) < 0;$$

$$q_j \underset{\tilde\theta,\,\max}{\prec} q_\ell \iff \mathrm{e}^{(q_j-q_\ell)(1/x)} \text{ is of maximal decay in the direction } \tilde\theta$$
$$\iff a_{(j,\ell)}\,\mathrm{e}^{-ik\tilde\theta} \text{ is a real negative number.}$$

In this latter case, we say that $\mathrm{e}^{q_j-q_\ell}$ is *led by* $\tilde\theta$.

In the unramified case these relations do not depend on the determination $\tilde\theta$ and thus we can write $\underset{\theta}{\prec}$ and $\underset{\theta,\,\max}{\prec}$.

DEFINITION I.4.5.

(i) θ is an *anti-Stokes direction* when for some, hence all, determination $\tilde\theta$ there is at least one pair (q_j, q_ℓ) in $\mathcal{Q}_{[A_0]}$ which satisfies $q_j \underset{\theta,\,\max}{\prec} q_\ell$.

(ii) θ is a *Stokes direction* when, for some, hence all, determination $\tilde\theta$ there is at least one pair $(q_j, q_\ell) \in \mathcal{Q}_{[A_0]}$ which satisfies neither $q_j \underset{\tilde\theta}{\prec} q_\ell$ nor $q_\ell \underset{\tilde\theta}{\prec} q_j$.

Thus, with our convention, anti-Stokes directions are directions of maximal decay for the exponentials $\mathrm{e}^{q_j-q_\ell}, q_j - q_\ell \in \mathcal{Q}_{[\mathrm{End}\,A_0]}$ and Stokes directions are «oscillating» directions transitory from flat to exploding asymptoticity. Unfortunately, one can find the converse convention too.

NOTATION I.4.6. — $\mathbb{A} = \{\alpha_1, \ldots, \alpha_\nu\}$ $(\alpha_{j+\nu} = \alpha_j)$ denotes the set of anti-Stokes directions in a clockwise ordering and $\mathbb{S} = \{\sigma_1 < \cdots < \sigma_\mu\}$ $(\sigma_{j+\mu} = \sigma_j)$ the set of Stokes directions.

With the choice of a determination $\tilde{\theta}$ of θ entries of $Q(1/x)$ become actual functions near the direction θ in $\mathbb{C}$. With the choice of the function $\mathrm{e}^{Q(1/|x|\mathrm{e}^{i\tilde{\theta}})}$ as realization of the formal exponential $\mathrm{e}^{Q(1/x)}$ the normal solution X_0 becomes an actual solution $X_{0,\tilde{\theta}}$ near the direction θ in $\mathbb{C}$. By definition, a germ of isotropy $\varphi_\theta \in \Lambda_\theta(A_0)$ takes the fundamental solution $X_{0,\tilde{\theta}}$ into another fundamental solution say $X_{0,\tilde{\theta}}C_{0,\tilde{\theta}}$ with $C_{0,\tilde{\theta}} \in \mathrm{GL}(n,\mathbb{C})$. It is easily seen that an entry $c_{(j,\ell)}$ in $C_{0,\tilde{\theta}}$ equals zero unless $q_j \underset{\tilde{\theta}}{\prec} q_\ell$.

DEFINITION I.4.7. — The *representation* of a germ $\varphi_\theta \in \Lambda_\theta(A_0)$ relative to $X_{0,\tilde{\theta}}$ is the unique matrix

$$C_{X_{0,\tilde{\theta}}} = I + \sum_{(j,\ell)\,\mid\, q_j \underset{\tilde{\theta}}{\prec} q_\ell} c_{(j,\ell)} E_{(j,\ell)}$$

such that $\varphi_\theta(x) = X_{0,\tilde{\theta}}(x) C_{X_{0,\tilde{\theta}}} X_{0,\tilde{\theta}}(x)^{-1}$ near θ. The $c_{(j,\ell)}$ are complex numbers and the $E_{(j,\ell)}$ denote the elementary matrices filled with zeroes except the (j,ℓ) entry equal to 1.

CONSEQUENCE I.4.8. — *The sheaf $\Lambda(A_0)$ is a sheaf of non-Abelian unipotent groups.*

Proof. — For the q_j ordered monotonically with respect to $\underset{\tilde{\theta}}{\prec}$ the matrices $C_{X_{0,\tilde{\theta}}}$ are triangular with a diagonal equal to the identity. □

Remarks I.4.9.

1) In a change of determination $\tilde{\theta} \mapsto \tilde{\theta} + 2\pi$, representations of a germ φ_θ satisfy the relation

$$C_{X_{0,\tilde{\theta}+2\pi}} = \widehat{M}^{-1} C_{X_{0,\tilde{\theta}}} \widehat{M}$$

where $\widehat{M} \in \mathrm{GL}(n,\mathbb{C})$ is the so-called matrix of formal monodromy for X_0. This matrix $\widehat{M}$ is defined by $X_{0,\tilde{\theta}+2\pi}(x) = X_{0,\tilde{\theta}}(x)\widehat{M}$ independently of the choice of $\tilde{\theta}$ (see Definition III.3.2).

2) A determination $\tilde{\theta}$ of θ being fixed, the natural possible realizations of e^Q are given by $\mathrm{e}^{Q(1/|x|\,\mathrm{e}^{i\tilde{\theta}})}T$ where $T = \mathrm{diag}(t_1, \ldots, t_n)$ in $\mathrm{GL}(n,\mathbb{C})$

belongs to the exponential torus $\mathcal{T}$ of $[A_0]$ (see Definition III.3.3). The representation $C_{X_{0,\tilde\theta}}$ of φ_θ relative to $X_{0,\tilde\theta}$ and $T = I$ becomes then $T^{-1}C_{X_{0,\tilde\theta}}T$. We shall generally assume $T = I$.

3) A change of normal solution of the form $X_1(x) = X_0(x)\Gamma$ ($\Gamma \in \mathrm{GL}(n,\mathbb{C})$ and satisfying convenient commutation relations with Q and L) or of the form $X_1(x) = F_0(x)X_0(x)P$ (where $F_0 \in G_0(A_0)$ and P is a permutation) implies

$$C_{X_{0,\tilde\theta}} = \Gamma\, C_{X_{1,\tilde\theta}}\,\Gamma^{-1},$$

$$\varphi_\theta(x) = F_0(x)X_{0,\tilde\theta}(x)\,(P\,C_{X_{1,\tilde\theta}}P^{-1})\,X_{0,\tilde\theta}(x)^{-1}F_0(x)^{-1}.$$

Thus $C_{X_{0,\tilde\theta}}$ and $C_{X_{1,\tilde\theta}}$ are mainly conjugated via a permutation.

With these remarks one can check that the definitions given in terms of representations are intrinsic.

Notation I.4.10. — The set

$$\mathcal{K}(\varphi_\theta) = \Big\{\deg(q_j - q_\ell);\, c_{(j,\ell)} \neq 0 \text{ in some representation } I + \sum c_{(j,\ell)}E_{(j,\ell)} \text{ of } \varphi_\theta\Big\}$$

denotes the set of levels of the germ φ_θ.

Definition I.4.11. — A germ φ_θ is a *k-germ* when $\mathcal{K}(\varphi_\theta) = \{k\}$ or $\mathcal{K}(\varphi_\theta) = \emptyset$.

Definition I.4.12. — A germ φ_θ is a *Stokes germ* when it satisfies one of the two equivalent conditions:

(i) it is of maximal decay along θ;

(ii) for some, hence all, determination $\tilde\theta$, it has a representation $C_{X_{0,\tilde\theta}}$ of the form

$$C_{X_{0,\tilde\theta}} = I + \sum_{(j,\ell)\,\mid\, q_j \underset{\tilde\theta,\,\max}{\prec} q_\ell} c_{(j,\ell)}E_{(j,\ell)}.$$

In particular, if $\varphi_\theta \neq I$ then $\theta = \alpha \in \mathbb{A}$ is an anti-Stokes direction.

Remark I.4.13. — Definition I.4.12 (i) makes sense even when $[A_0]$ is not a normal form.

DEFINITION I.4.14. — For $\alpha \in \mathbb{A}$, the *Stokes group* $\mathrm{Sto}_\alpha(A_0)$ is the group of Stokes germs at α.

NOTATION I.4.15. — For $k \in \mathcal{K}$ and $\alpha \in \mathbb{A}$, we set:

$\Lambda^k(A_0) :=$ the subsheaf of $\Lambda(A_0)$ generated by k-germs;

$\Lambda^{\leq k}(A_0) :=$ the subsheaf of $\Lambda(A_0)$ generated by k'-germs for all $k' \leq k$;

$\Lambda^{<k}(A_0) :=$ the subsheaf of $\Lambda(A_0)$ generated by k'-germs for all $k' < k$;

$\mathrm{Sto}^k_\alpha(A_0) := \mathrm{Sto}_\alpha(A_0) \cap \Lambda^k_\alpha(A_0)$ the subgroup of $\mathrm{Sto}_\alpha(A_0)$ of Stokes k-germs at α;

$\mathrm{Sto}^{\leq k}_\alpha(A_0) := \mathrm{Sto}_\alpha(A_0) \cap \Lambda^{\leq k}_\alpha(A_0)$ the subgroup of $\mathrm{Sto}_\alpha(A_0)$ of Stokes germs of level $\leq k$ at α,

$\mathrm{Sto}^{<k}_\alpha(A_0) := \mathrm{Sto}_\alpha(A_0) \cap \Lambda^{<k}_\alpha(A_0)$ the subgroup of $\mathrm{Sto}_\alpha(A_0)$ of Stokes germs of level $< k$ at α;

$\mathbb{A}^k := \{\alpha \in \mathbb{A} \mid \mathrm{Sto}^k_\alpha(A_0) \neq \{I\}\}$ the set of anti-Stokes directions *bearing the level* k;

$\mathbb{A}^{\leq k} := \bigcup_{k' \leq k} \mathbb{A}^{k'}$ (resp. $\mathbb{A}^{<k} = \bigcup_{k'<k} \mathbb{A}^{k'}$);

$\mathcal{K}_\alpha := \{k \in \mathcal{K} \mid \mathrm{Sto}^k_\alpha(A_0) \neq \{I\}\}$ the set of levels *beared by* $\alpha \in \mathbb{A}$.

One can also define in a similar way $\Lambda^{\geq k}(A_0), \Lambda^{>k}(A_0), \ldots$ with similar properties. However, there is no real need to introduce them as long as $[A_0]$ is a normal form. The situation is quite different when $[A_0]$ is nolonger a normal form (cf. III.2.2).

I.5. Filtration of $\Lambda(A_0)$ and of the Stokes groups $\mathrm{Sto}_\alpha(A_0)$ by the levels.

It is easy to prove (see [LR91]) the following properties which are fundamental in the proof of the main theorem (Theorem II.2.1).

PROPOSITION I.5.1. — *For any level $k \in \mathcal{K}$ one has:*

(i) *the sheaves $\Lambda^k(A_0)$, $\Lambda^{\leq k}(A_0)$, $\Lambda^{<k}(A_0)$ are sheaves of subgroups of $\Lambda(A_0)$;*

(ii) *the sheaf* $\Lambda^k(A_0)$ *is normal in* $\Lambda^{\leq k}(A_0)$;

(iii) *the exact sequence of sheaves*

$$1 \to \Lambda^k(A_0) \xrightarrow{i} \Lambda^{\leq k}(A_0) \xrightarrow{p} \Lambda^{<k}(A_0) \to 1$$

splits. (Here i *denotes the canonical inclusion and* p *the truncation to terms of level* $< k$.)

COROLLARY I.5.2.

(i) *For any level* $k \in \mathcal{K}$ *the sheaf* $\Lambda^{\leq k}(A_0)$ *can be identified to a semi-direct product in the two following ways:*

$$\begin{aligned}\Lambda^{\leq k}(A_0) &\cong \Lambda^{<k}(A_0) \ltimes \Lambda^k(A_0)\\ &\cong \Lambda^k(A_0) \rtimes \Lambda^{<k}(A_0)\end{aligned}$$

i.e. any germ $f^{\leq k} \in \Lambda^{\leq k}(A_0)$ *can be uniquely factored in*

$$f^{\leq k} = f^{<k} g^k \quad \text{or} \quad f^{\leq k} = f^k f^{<k}$$

where $f^{<k} \in \Lambda^{<k}(A_0)$, f^k *and* $g^k \in \Lambda^k(A_0)$.

(ii) *A factorization algorithm can be: the factor* $f^{<k}$ *common to both factorizations is the truncation of* $f^{\leq k}$ *to terms of level* k *(in any representation* $I + \sum c_{(j,\ell)} E_{(j,\ell)}$ *of* $f^{\leq k}$ *keep only terms* (j, ℓ) *such that* $\deg(q_j - q_\ell) < k$). *Then* $g^k = (f^{<k})^{-1} f^{\leq k}$ *and* $f^k = f^{\leq k}(f^{<k})^{-1}$.

(iii) *The decomposition in semi-direct product and the algorithm can be extended to all levels:*

$$\Lambda(A_0) \cong \mathop{\Large\ltimes}_{k \in \mathcal{K}} \Lambda^k(A_0)$$

the semi-direct product being taken in an ascending or a descending order of levels k.

In order to get semi-direct products in an arbitrary order of levels one can extend Proposition I.5.1 and Corollary I.5.2 in:

PROPOSITION I.5.3. — *Let* k, $k' \in \mathcal{K}$, $k' < k$ *be levels of* $[A_0]$. *Then one has:*

(i) *the sheaf* $\Lambda^{\geq k'}(A_0) \cap \Lambda^{\leq k}(A_0)$ *is normal in* $\Lambda^{\leq k}(A_0)$;

(ii) *the exact sequence of sheaves*

$$1 \to \Lambda^{\geq k'}(A_0) \cap \Lambda^{\leq k}(A_0) \xrightarrow{i} \Lambda^{\leq k}(A_0) \xrightarrow{p} \Lambda^{<k'}(A_0) \to 1$$

splits. (Here i *still denotes the canonical inclusion and* p *the truncature to terms of levels* $< k'$.)

Corollary I.5.4. — *Let $\mathcal{K} = \{k_1 < k_2 < \cdots < k_r\}$ be the set of levels.*

(i) *The filtration*

$$\Lambda^{k_r}(A_0) \equiv \Lambda^{\geq k_r}(A_0) \subset \Lambda^{\geq k_{r-1}}(A_0) \subset \cdots \subset \Lambda^{\geq k_1}(A_0) = \Lambda(A_0)$$

is normal.

(ii) *The decomposition in iterated semi-direct products*

$$\Lambda(A_0) \cong \bigtimes_{k \in \mathcal{K}} \Lambda^k(A_0)$$

and the algorithm in Corollary I.5.2 *can be extended to an arbitrary order of levels $k \in \mathcal{K}$.*

Proposition I.5.5. — *The previous results can be restricted to the Stokes groups; in particular, for $\alpha \in \mathbb{A}$, one has*

$$\mathrm{Sto}_\alpha(A_0) \cong \bigtimes_{k \in \mathcal{K}_\alpha} \mathrm{Sto}_\alpha^k(A_0)$$

the semi-direct product being taken in an arbitrary order.

I.6. Leading directions of k-germs and rules of commutations (unramified case).

This section contains technical rules for the explicitation of the algorithm in Section II.3.4. It can be skipped by readers interested only in the main theorem.

We restrict ourselves to the unramified case and we fix a level $k \in \mathcal{K}$ (here $k \in \mathbb{N}$). For any $\theta \in S^1$, we set

$$\mathbb{A}_\theta^k := \mathbb{A}^k \cap U(\theta, \pi/k) = \{\alpha_1(\theta) < \alpha_2(\theta) < \cdots < \alpha_{s(\theta)}(\theta)\},$$

where (cf. I.2) $U(\theta, \pi/k)$ denotes an open arc on S^1 bisected by θ with opening π/k. The set $\mathbb{A}_\theta^k$ is piecewise constant with respect to θ and discontinuities occur only at Stokes directions bearing the level k. For $\alpha \in \mathbb{A}_\theta^k$, the Stokes germs in $\mathrm{Sto}_\alpha^k(A_0)$ can be analytically continued up to θ. Thus the Stokes groups $\mathrm{Sto}_\alpha^k(A_0)$, $\alpha \in \mathbb{A}_\theta^k$, can be regarded as subgroups of $\Lambda_\theta(A_0)$. Moreover one has:

PROPOSITION I.6.1. — *The Stokes groups* $\mathrm{Sto}^k_\alpha(A_0)$*, for* $\alpha \in \mathbb{A}^k_\theta$*, generate the group* $\Lambda^k_\theta(A_0)$ *of all k-germs at* θ*.*

We set $\Lambda^k_\theta(A_0) =: \bigvee_{\alpha \in \mathbb{A}^k_\theta} \mathrm{Sto}^k_\alpha(A_0)$.

Let $\varphi_\theta \in \Lambda^k_\theta(A_0)$ be a k-germ at θ with representation

$$C_{X_{o,\tilde\theta}} = I + \sum_{(j,\ell)\,|\,q_j \underset{\theta}{\prec} q_\ell} c_{(j,\ell)} E_{(j,\ell)}.$$

DEFINITION I.6.2. — The set

$$\mathbb{A}^k(\varphi_\theta) := \big\{\alpha \in \mathbb{A}^k_\theta ; q_j \underset{\alpha,\,\max}{\prec} q_\ell \text{ and } c_{(j,\ell)} \neq 0\big\}$$

is the set of *leading directions* of the k-germ φ_θ.

Obviously φ_θ is an element of

$$\bigvee_{\alpha \in \mathbb{A}^k(\varphi_\theta)} \mathrm{Sto}^k_\alpha(A_0),$$

φ_θ is a Stokes k-germ iff $\sharp\mathbb{A}^k(\varphi_\theta) = 1$ and when φ_θ is an element of

$$\bigvee_{j \in J} \mathrm{Sto}^k_{\alpha_j(\theta)}(A_0)$$

then one has

$$\mathbb{A}^k(\varphi_\theta) \subset \big[\alpha_{\min_J(j)}(\theta), \alpha_{\max_J(j)}(\theta)\big].$$

Besides the filtration by levels on the group of all germs $\Lambda_\theta(A_0)$, one has a normal filtration by leading directions on the groups of k-germs $\Lambda^k_\theta(A_0)$. Precisely, one has:

PROPOSITION I.6.3. — *Let* $\mathbb{A}^k_\theta = \big\{\alpha_1(\theta) < \cdots < \alpha_{s(\theta)}(\theta)\big\}$.

(i) *The group* $\bigvee_{j=2}^{s} \mathrm{Sto}^k_{\alpha_j(\theta)}(A_0)$ *is a normal subgroup of* $\Lambda^k_\theta(A_0)$.

(ii) *The exact sequence of groups*

$$1 \to \bigvee_{j=2}^{s} \mathrm{Sto}^k_{\alpha_j(\theta)}(A_0) \xrightarrow{\ i\ } \Lambda^k_\theta(A_0) = \bigvee_{j=1}^{s} \mathrm{Sto}^k_{\alpha_j(\theta)}(A_0) \xrightarrow{\ p\ } \mathrm{Sto}^k_{\alpha_1(\theta)}(A_0) \to 1$$

splits. (Here i means the canonical injection and p the truncation to entries led by $\alpha_1(\theta)$ *in any representation.)*

COROLLARY I.6.4.

(i) *The group*

$$\Lambda_\theta^k(A_0) = \bigvee_{j=1}^{s} \mathrm{Sto}_{\alpha_j(\theta)}^k(A_0)$$

can be identified to a semi-direct product

$$\Lambda_\theta^k(A_0) \cong \mathrm{Sto}_{\alpha_1(\theta)}^k(A_0) \ltimes \bigvee_{j=2}^{s} \mathrm{Sto}_{\alpha_j(\theta)}^k(A_0).$$

(ii) *As a consequence any k-germ φ_θ can be uniquely factored in the form*

$$\varphi_\theta = \varphi_{\alpha_1(\theta)}\psi_\theta$$

where $\varphi_{\alpha_1(\theta)} \in \mathrm{Sto}_{\alpha_1(\theta)}^k(A_0)$ is a Stokes k-germ led by $\alpha_1(\theta)$ and ψ_θ is a k-germ led by anti-Stokes directions in

$$\mathbb{A}_\theta^k \setminus \{\alpha_1(\theta)\} = \{\alpha_2(\theta) < \cdots < \alpha_s(\theta)\}.$$

A factorization algorithm can be the following: $\varphi_{\alpha_1(\theta)}$ is deduced from φ_θ by keeping the terms led by $\alpha_1(\theta)$ in a representation of φ_θ; then ψ_θ is given by $\psi_\theta = \varphi_{\alpha_1(\theta)}^{-1}\varphi_\theta$.

(iii) *The previous decomposition and the algorithm can be extended to all anti-Stokes directions in $\mathbb{A}_\theta^k$:*

$$\Lambda_\theta^k(A_0) \cong \mathrm{Sto}_{\alpha_1(\theta)}^k(A_0) \ltimes \Big(\mathrm{Sto}_{\alpha_2(\theta)}^k(A_0) \ltimes \big(\cdots \ltimes \mathrm{Sto}_{\alpha_s(\theta)}^k(A_0)\big)\cdots\Big).$$

COROLLARY I.6.5. — *The filtration*

$$\{I\} \subset \mathrm{Sto}_{\alpha_s(\theta)}^k(A_0) \subset \bigvee_{j=s-1}^{s} \mathrm{Sto}_{\alpha_j(\theta)}^k(A_0)$$
$$\subset \cdots \subset \bigvee_{j=2}^{s} \mathrm{Sto}_{\alpha_j(\theta)}^k(A_0) \subset \bigvee_{j=1}^{s} \mathrm{Sto}_{\alpha_j(\theta)}^k(A_0) = \Lambda_\theta^k(A_0)$$

is normal.

Given $\varphi, \psi \in \Lambda_\theta(A_0)$ two germs at θ we denote by

$$[\varphi, \psi] = \varphi^{-1}\psi^{-1}\varphi\psi$$

the commutator of φ with ψ. Then we have $\varphi\psi = \psi\varphi[\varphi,\psi] = [\varphi^{-1},\psi^{-1}]\psi\varphi$.

The following proposition gives the commutation rules of Stokes k-germs. Notice that, in this commutation, new leading directions may appear but they all appear in between the previous ones.

Proposition I.6.6. — *Let φ_θ and φ'_θ be germs at θ. Let us assume that $\varphi_\theta \in \mathrm{Sto}^k_\alpha(A_0)$, $\varphi'_\theta \in \mathrm{Sto}^{k'}_{\alpha'}(A_0)$, $k' \leq k$ and $\alpha \neq \alpha'$, say $\alpha < \alpha'$, in $U(\theta, \pi/k')$ (cf. I.2).*

(i) *If $k' < k$ then $[\varphi_\theta, \varphi'_\theta]$ is an element of $\mathrm{Sto}^k_\alpha(A_0)$.*

(ii) *If $k' = k$ then $[\varphi_\theta, \varphi'_\theta]$ is an element of $\bigvee_j \mathrm{Sto}^k_{\alpha_j(\theta)}(A_0)$ for all j satisfying $\alpha_j(\theta) \in \mathbb{A}^k_\theta$ and $\alpha < \alpha_j(\theta) < \alpha'$ in $\mathbb{A}^k_\theta$.*

II. THE MAIN THEOREM

From now on, $[A_0]$ is assumed to be a normal form.

II.1. Basic topics in Čech cohomology for $H^1(S^1; \Lambda(A_0))$.

We orient S^1 *clockwise.* This orientation is chosen because we want our computations to be compatible with most of the classical computations made around infinity with a counterclockwise orientation. We use the terminology *left-right* with the following meaning: we go to the left when moving counterclockwise and to the right when moving clockwise.

The non-Abelian cohomological set $H^1(S^1; \Lambda(A_0))$ is defined as the inductive limit, over coverings $\mathcal{U}$ filtered with inclusion, of Čech cohomological sets $H^1(\mathcal{U}; \Lambda(A_0))$ (see [Fr56]). Without loss of generality we restrict the limit to the particular coverings which we call cyclic coverings.

Definition II.1.1. — A covering $\mathcal{U} = \{U_j; j \in J\}$ of S^1 is a *cyclic covering* when:

(i) the set J is finite and cyclic $J = \mathbb{Z}/\nu\mathbb{Z}$;

(ii) the U_j and, except when $\# J = 2$, the $U_j \cap U_{j+1}$ are connected arcs on S^1 (cf. I.2);

(iii) the bisecting directions of the U_j are in ascending order with respect to the clockwise orientation of S^1;

(iv) the U_j are not encased: the arcs $U_j \setminus U_\ell$ are connected arcs (when U_ℓ is included in U_j then U_ℓ and U_j coincide at one end).

Definition II.1.2. — The *nerve* of a cyclic covering $\mathcal{U} = \{U_j; j \in J\}$ is the family $\dot{\mathcal{U}} = \{\dot{U}_j; j \in J\}$ of connected arcs defined by:

- $\dot{U}_j = U_j \cap U_{j+1}$ when $\sharp J > 2$,
- $\dot{U}_1$ and $\dot{U}_2$ the two connected components of $U_1 \cap U_2$ when $\sharp J = 2$.

There is a one-to-one correspondence between cyclic coverings and their nerve: when $\dot{\mathcal{U}} = \{\dot{U}_j; j \in J\}$ is a nerve then it is the nerve of a unique cyclic covering $\mathcal{U} = \{U_j; j \in J\}$ and U_j is the connected clockwise hull arc from $\dot{U}_{j-1}$ to $\dot{U}_j$. Coverings of thickness 2 often used by many authors (Malgrange, Ramis, Sibuya,...) are cyclic.

By definition, a covering $\mathcal{V}$ refines a covering $\mathcal{U}$ when each open V_ℓ in $\mathcal{V}$ is included in at least one open U_j in $\mathcal{U}$.

Proposition II.1.3. — *Let $\dot{\mathcal{U}} = \{\dot{U}_j; j \in J\}$ and $\dot{\mathcal{V}} = \{\dot{V}_\ell; \ell \in L\}$ be nerves of cyclic coverings $\mathcal{U}$ and $\mathcal{V}$. Then $\mathcal{V}$ refines $\mathcal{U}$ when each $\dot{U}_j$ contains at least one $\dot{V}_\ell$.*

In particular $\mathcal{V}$ refines $\mathcal{U}$ when $\dot{\mathcal{V}}$ results from $\dot{\mathcal{U}}$ by the narrowing of one arc in $\dot{\mathcal{U}}$ or by the addition of a new arc.

Proposition II.1.4. — *Let $\mathcal{U} = \{U_j; j \in J\}$ be a cyclic covering. One can identify the set of 1-cocycles on $\mathcal{U}$ to the set $\prod_{j\in J} \Gamma(\dot{U}_j; \Lambda(A_0))$ of partial 1-cochains on $\mathcal{U}$ without any condition.*

In the following, cocycle or cochain means 1-cocycle or 1-cochain given in this partial form.

As the sheaf $\Lambda(A_0)$ is piecewise constant with finitely many discontinuities, the inductive limit in the definition of $H^1(S^1; \Lambda(A_0))$ is stationary.

Definition II.1.5. — A covering $\mathcal{U}$ beyond which the inductive limit is stationary is said to be *adequate* to describe $H^1(S^1; \Lambda(A_0))$, briefly, adequate to $\Lambda(A_0)$. Similarly, one defines coverings adequate to $\Lambda^k(A_0)$, $\Lambda^{\leq k}(A_0)$ or $\Lambda^{<k}(A_0)$.

Definition II.1.6. — Let $\alpha \in \mathbb{A}$ be an anti-Stokes direction and $k \in \mathcal{K}_\alpha$ be a level beared by α (notation I.4.15). An arc $U(\alpha, \pi/k)$ (cf. I.2) bisected by α with opening π/k is called a *Stokes arc of level k* at α.

Proposition II.1.7. — *Let $\mathcal{U} = \{U_j \in J\}$ be a cyclic covering. If each Stokes arc of level k, of level $\leq k$ or of any level contains at least one arc $\dot{U}_j$ from the nerve $\dot{\mathcal{U}}$ of $\mathcal{U}$, then $\mathcal{U}$ is adequate to $\Lambda^k(A_0)$, to $\Lambda^{\leq k}(A_0)$ or to $\Lambda(A_0)$.*

DEFINITION II.1.8. — A cocycle $\dot{\varphi} = (\dot{\varphi}_j)_{j\in J}$ on a cyclic covering $\mathcal{U} = \{U_j; j \in J\}$ is said to be a *Stokes cocycle* when the components $\dot{\varphi}_j$, if non trivial, represent Stokes germs $\dot{\varphi}_{j,\alpha_j} \in \mathrm{Sto}_{\alpha_j}(A_0)$ at anti-Stokes directions α_j in a cyclic ascending ordering.

If we add or remove trivial components to a Stokes cocycle we get a Stokes cocycle and we preserve its cohomology class. In this way, we can also reduce the set J to be the set $\mathbb{A}$ of anti-Stokes directions.

From now on, for Stokes cocycles, we assume the set $J = \mathbb{A}$ (possibly a subset of $\mathbb{A}$ in special cases when some components are a priori known to be trivial).

The map h. — To any finite family $\dot{\varphi} = (\dot{\varphi}_{\theta_j})_{j\in J}$ of germs $\dot{\varphi}_{\theta_j}$ in $\Lambda_{\theta_j}(A_0)$, one can associate a cohomology class in the following way: let $\dot{\varphi}_j$ be the function representing the germ $\dot{\varphi}_{\theta_j}$ on its maximal arc of definition Ω_j around θ_j (later we shall keep the same notation $\dot{\varphi}_j = \dot{\varphi}_{\theta_j}$); when a cyclic covering $\mathcal{U} = \{U_j; j \in J\}$ satisfies the conditions $\dot{U}_j \subset \Omega_j$ for all $j \in J$ one can define the 1-cocycle $(\dot{\varphi}_{j|\dot{U}_j})_{j\in J}$ on $\mathcal{U}$ and to different $\mathcal{U}$ correspond cohomologous 1-cocycles. Thus the following definition makes sense.

DEFINITION II.1.9. — The map

$$h : \prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A_0) \longrightarrow H^1(S^1; \Lambda(A_0))$$

is the map which canonically takes a family $\dot{\varphi} = (\dot{\varphi}_\alpha)_{\alpha\in\mathbb{A}}$ of Stokes germs into the cohomology class of the cocycle induced by $\dot{\varphi} = (\dot{\varphi}_\alpha)_{\alpha\in\mathbb{A}}$ over any cyclic covering $\mathcal{U}$ the nerve $\dot{\mathcal{U}} = \{\dot{U}_\alpha; \alpha \in \mathbb{A}\}$ of which satisfies $\dot{U}_\alpha \subset \Omega_\alpha$ for all $\alpha \in \mathbb{A}$ (Ω_α is the natural arc of definition of $\dot{\varphi}_\alpha$).

Since germs in the family $\dot{\varphi}$ are Stokes germs, then the induced cocycles are Stokes cocycles.

II.2. The main theorem.

THEOREM II.2.1. — *The map*

$$h : \prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A_0) \to H^1(S^1; \Lambda(A_0))$$

is bijective and natural.

Natural means here that h commutes to isomorphisms and constructions (cf. Section III.3.3) over systems or connections they represent.

As we shall see in Section III.1, this isomorphism endows the classifying set $H^1(S^1; \Lambda(A_0))$ with a natural structure of a finite dimensional linear variety and, although it is transcendental, it provides a nice description of systems by the arbitrary family of their Stokes matrices.

The map h is easily seen to be injective. To prove the theorem we then have to prove that to each cohomology class belongs a Stokes cocycle. Our proof separates the unramified and the ramified case. In the unramified case, the proof proceeds by descending induction on levels. The ramified case is deduced from the unramified one by blowing up and descent. This proof is constructive and provides an algorithm for putting any cocycle in its cohomologous Stokes form.

II.3. Proof in the unramified case.

The description of the cohomology at the different levels requires the coverings to be adequate to the sheaves $\Lambda^k(A_0)$, $\Lambda^{\leq k}(A_0)$ and $\Lambda^{<k}(A_0)$. In order to make this description as simple as possible we look for coverings with as few 0-cochains as possible. But in order to carry out inductions, we need to compare the cohomological sets at different levels and then we need comparable coverings. These last two conditions mainly tend in opposite directions. The cyclic coverings $\mathcal{U}^k$, $\mathcal{U}^{\leq k}$ and $\mathcal{U}^{<k}$ below offer a good compromise. We define them by their nerve.

II.3.1. Adequate coverings.

Let k be an element of $\mathcal{K}$.

The cyclic covering $\mathcal{U}^k = \{U^k_\alpha; \alpha \in \mathbb{A}^k\}$. — The family

$$\left(\dot{U}^k_\alpha := U(\alpha, \pi/k)\right)_{\alpha \in \mathbb{A}^k}$$

of Stokes arcs bisected by α with opening π/k when the level k is fixed and α runs through the set $\mathbb{A}^k$ of anti-Stokes directions bearing the level k is the nerve of a cyclic covering. We define

$$\dot{\mathcal{U}}^k := \left\{\dot{U}^k_\alpha; \alpha \in \mathbb{A}^k\right\}$$

as being the *nerve* of $\mathcal{U}^k$.

In general, when extended to several levels the previous family is no longer a nerve. Neither is a nerve the family of Stokes arcs of level $k' \leq k$ after the selection of the smallest arc in each anti-Stokes direction:

$$\dot{V}_{\alpha}^{\leq k} = U(\alpha, \pi/K) \text{ for } \alpha \in \mathbb{A}^{\leq k} \text{ and } K = \max\{k'; k' \in \mathcal{K}_{\alpha} \cap [0, k]\}.$$

The integer K is the *k-maximal level* at $\alpha \in \mathbb{A}^{\leq k}$ and we denote by $\{K_1 < \cdots < K_s = k\}$ the set of all k-maximum levels.

Let us now construct the nerve $\dot{\mathcal{U}}^{\leq k}$ of the required covering $\mathcal{U}^{\leq k}$ by descending induction on k-maximum levels.

The cyclic covering $\mathcal{U}^{\leq k} = \{U_{\alpha}^{\leq k}; \alpha \in \mathbb{A}^{\leq k}\}$. — For all $\alpha \in \mathbb{A}^{K_s}$ (recall that in this case α has $K_s = k$ as k-maximum level) we set

$$\dot{U}_{\alpha}^{\leq k} := \dot{U}_{\alpha}^{k} = U(\alpha, \pi/K_s).$$

This family is a nerve, the nerve $\dot{\mathcal{U}}^{k}$ of level $k = K_s$.

Now, let us assume that the $\dot{U}_{\alpha}^{\leq k}$ are defined for all $\alpha \in \mathbb{A}^{\leq k}$ with k-maximum level greater than K_i in such a way that their complete family be a nerve and let $\alpha \in \mathbb{A}^{\leq k}$ be an anti-Stokes direction with k-maximum level K_i. This direction is located in between α^-, the nearest on the left, and α^+, the nearest on the right, anti-Stokes directions with k-maximum level $> K_i$. If $\dot{U}_{\alpha^-}^{\leq k} =]O^-, E^-[$ and $\dot{U}_{\alpha^+}^{\leq k} =]O^+, E^+[$ (S^1 is clockwise oriented) then we set

$$\dot{U}_{\alpha}^{\leq k} := U(\alpha, \pi/K_i) \cap\,]O^-, E^+[\,.$$

This complete family of $\dot{U}_{\alpha}^{\leq k}$ for all α with k-maximum level $\geq K_i$ is a nerve. This achieves the induction and defines the nerve $\dot{\mathcal{U}}^{\leq k}$ of the required covering $\mathcal{U}^{\leq k}$.

Notice that when α has a k-maximum level equal to k then $\dot{U}_{\alpha}^{\leq k}$ is the Stokes arc $U(\alpha, \pi/k)$ and then no 0-cochain with level k or $\geq k$ can exist on the covering $\mathcal{U}^{\leq k}$.

The cyclic covering $\mathcal{U}^{<k} = \{U_{\alpha}^{<k}; \alpha \in \mathbb{A}^{<k}\}$. — We set $\mathcal{U}^{<k} := \mathcal{U}^{\leq k'}$ where $k' = \max\{k'' \in \mathcal{K}; k'' < k\}$.

The following proposition is easily deduced from the definitions and Proposition II.1.7.

PROPOSITION II.3.1. — *For all* $k \in \mathcal{K}$,

(i) *the coverings* $\mathcal{U}^k, \mathcal{U}^{\leq k}, \mathcal{U}^{<k}$ *do not depend on* $[A_0]$ *itself or* X_0 *but only on the set* $\mathcal{Q}_{[\operatorname{End} A_0]}$ (*cf.* I.4);

(ii) *the covering* $\mathcal{U}^{\leq k}$ *refines* $\mathcal{U}^k$ *and* $\mathcal{U}^{<k}$;

(iii) *the coverings* $\mathcal{U}^k$, $\mathcal{U}^{\leq k}$, $\mathcal{U}^{<k}$ *are adequate to* $\Lambda^k(A_0)$, $\Lambda^{\leq k}(A_0)$, $\Lambda^{<k}(A_0)$ *respectively*;

(iv) *on the covering* $\mathcal{U}^k$ *there exists no* 0*-cochain in* $\Lambda^k(A_0)$. *On the covering* $\mathcal{U}^{\leq k}$ *there exists no 0-cochain in* $\Lambda^{\leq k}(A_0)$ *with a level equal to* k: 0*-cochains all belong to* $\Lambda^{<k}(A_0)$.

We need to compare cochains when both coverings and sheaves are different. Thus, to be precise, we need to introduce some notations.

For simplicity's sake we denote the product $\prod_{\alpha\in\mathbb{A}^k} \Gamma(\dot{U}^k_\alpha; \Lambda^k(A_0))$ by $\Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A_0))$ and so on.

Inclusions. — The maps we use between sets of indices are only the canonical inclusions

$$\mathbb{A}^k \hookrightarrow \mathbb{A}^{\leq k} \quad \text{and} \quad \mathbb{A}^{<k} \hookrightarrow \mathbb{A}^{\leq k}.$$

To compare cocycles we use the following injective maps (with $k\in \mathcal{K}$):

$$\sigma^k : \begin{cases} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A_0)) \to \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A_0)), \\ \dot{f} = (\dot{f}_\alpha)_{\alpha\in\mathbb{A}^k} \mapsto \sigma^k(\dot{f}) = (\dot{F}_\alpha)_{\alpha\in\mathbb{A}^{\leq k}} \end{cases}$$

where

$$\dot{F}_\alpha = \begin{cases} \dot{f}_\alpha & \text{restricted to } \dot{U}^{\leq k}_\alpha \text{ and seen as being in } \Lambda^{\leq k}(A_0) \text{ when } \alpha \in \mathbb{A}^k, \\ I & \text{(the identity) when } \alpha \notin \mathbb{A}^k; \end{cases}$$

and

$$\sigma^{<k} : \Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A_0)) \longrightarrow \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A_0))$$

defined in a similar way.

By means of the maps σ^k and $\sigma^{<k}$ we can multiply 1-cocycles in $\Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A_0))$ with 1-cocycles in $\Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A_0))$ by making them become 1-cochains in $\Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A_0))$.

While we have $\Lambda(A_0) = \Lambda^{\leq k_r}(A_0)$ (recall that $k_r \in \mathcal{K}$ is the largest level) we also denote by $\mathcal{U}$ the covering $\mathcal{U}^{\leq k_r}$.

We introduce also the injective map

$$\tau^k : \begin{cases} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A_0)) \to \Gamma(\dot{\mathcal{U}}; \Lambda(A_0)), \\ \dot{f} = (\dot{f}_\alpha)_{\alpha\in\mathbb{A}^k} \mapsto \tau^k(\dot{f}) = (\dot{F}_\alpha)_{\alpha\in\mathbb{A}} \end{cases}$$

where

$$\dot{F}_\alpha = \begin{cases} \dot{f}_\alpha \text{ restricted to } \dot{U}_\alpha \text{ and seen as being in } \Lambda(A_0) \text{ when } \alpha \in \mathbb{A}^k, \\ \mathrm{I} \text{ (the identity) when } \alpha \notin \mathbb{A}^k. \end{cases}$$

The maps τ^k are deduced from the previous ones by composition.

II.3.2. The case of a unique level.

By «unique level» we mean both the case where the sheaf $\Lambda(A_0)$ itself has only one level (see [Mal83]) and the case where we restrict ourselves to a given level k: we replace the sheaf $\Lambda(A_0)$ by $\Lambda^k(A_0)$ and the groups $\mathrm{Sto}_\alpha(A_0)$ by $\mathrm{Sto}_\alpha^k(A_0)$.

LEMMA II.3.2. — *Let $k \in \mathcal{K}$. The canonical injective map i^k and the canonical surjective map s^k*

$$\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha^k(A_0) \xrightarrow[\sim]{i^k} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A_0)) \xrightarrow[\sim]{s^k} H^1(S^1; \Lambda^k(A_0))$$

are bijective.

Proof. — The map i^k is the canonical extension of germs to their natural arc of definition. It is a group isomorphism. The map s^k is the quotient map. It is surjective while $\mathcal{U}^k$ is adequate to $\Lambda^k(A_0)$ (cf. Definition II.1.5 and Proposition II.3.1 (iii)) and it is injective while, on $\mathcal{U}^k$, there exists no 0-cochain in $\Lambda^k(A_0)$ (Proposition II.3.1 (iv)). □

This lemma proves Theorem II.2.1 in the case when the sheaf $\Lambda(A_0)$ itself has only one level.

II.3.3. The case of several levels.

LEMMA II.3.3. — *Let $k \in \mathcal{K}$.*

(i) *The product-map of cocycles*

$$\mathfrak{S}^{\leq k} : \Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A_0)) \times \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A_0)) \longrightarrow \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A_0))$$

defined by

$$\mathfrak{S}^{\leq k}(\dot{f},\dot{g}) = (\dot{F}_\alpha \dot{G}_\alpha)_{\alpha\in\mathbb{A}^{\leq k}}$$

where we denote

$$(\dot{F}_\alpha)_{\alpha\in\mathbb{A}^{\leq k}} = \sigma^{<k}(\dot{f}) \quad \text{and} \quad (\dot{G}_\alpha)_{\alpha\in\mathbb{A}^{\leq k}} = \sigma^{k}(\dot{g})$$

is injective.

(ii) *If the cocycles $\mathfrak{S}^{\leq k}(\dot{f},\dot{g})$ and $\mathfrak{S}^{\leq k}(\dot{f}',\dot{g}')$ are cohomologous in $\Gamma(\dot{\mathcal{U}}^{\leq k}\,;\,\Lambda^{\leq k}(A_0))$ then $\dot{f}$ and $\dot{f}'$ are cohomologous in $\Gamma(\dot{\mathcal{U}}^{<k}\,;\,\Lambda^{<k}(A_0))$.*

(iii) *Any cocycle in $\Gamma(\dot{\mathcal{U}}^{\leq k};\Lambda^{\leq k}(A_0))$ is cohomologous to a cocycle in the range of $\mathfrak{S}^{\leq k}$.*

Proof.

(i) The proof is obvious as injectivity for germs implies injectivity for sections (Proposition I.5.1).

(ii) Let us denote by α^+ the nearest anti-Stokes direction in $\mathbb{A}^{\leq k}$ on the right of α. The cocycles $\mathfrak{S}^{\leq k}(\dot{f},\dot{g})$ and $\mathfrak{S}^{\leq k}(\dot{f}',\dot{g}')$ are cohomologous when there exists a 0-cochain $c = (c_\alpha)_{\alpha\in\mathbb{A}^{\leq k}} \in \Gamma(\mathcal{U}^{\leq k};\,\Lambda^{\leq k}(A_0))$ such that the relations

$$\dot{F}_\alpha \dot{G}_\alpha = c_\alpha^{-1}(\dot{F}'_\alpha \dot{G}'_\alpha)c_{\alpha^+}, \qquad \alpha \in \mathbb{A}^{\leq k}$$

hold. Actually, we know (Proposition II.3.1 (iv)) that c is with values in $\Lambda^{<k}(A_0)$ (i.e. $c \in \Gamma(\mathcal{U}^{\leq k};\,\Lambda^{<k}(A_0))$). The sheaf $\Lambda^{k}(A_0)$ being normal in $\Lambda^{\leq k}(A_0)$ we can give these relations the form

$$\dot{F}_\alpha \dot{G}_\alpha = (c_\alpha^{-1}\dot{F}'_\alpha c_{\alpha^+})\dot{G}''_\alpha, \qquad \alpha \in \mathbb{A}^{\leq k}$$

where $\dot{G}''_\alpha = c_{\alpha^+}^{-1}\dot{G}'_\alpha c_{\alpha^+}$ is an element of $\Gamma(\dot{\mathcal{U}}_\alpha^{\leq k};\Lambda^{k}(A_0))$. And by the identification of $\Lambda^{\leq k}(A_0)$ to the semi-direct product $\Lambda^{<k}(A_0) \ltimes \Lambda^{k}(A_0)$ we get for all $\alpha \in \mathbb{A}^{\leq k}$

$$\dot{F}_\alpha = c_\alpha^{-1}\dot{F}'_\alpha c_{\alpha^+} \quad \text{and} \quad \dot{G}_\alpha = \dot{G}''_\alpha.$$

The former relation means that $(\dot{F}_\alpha)$ and $(\dot{F}'_\alpha)$ are cohomologous as cocycles with values in $\Lambda^{<k}(A_0)$ on $\mathcal{U}^{\leq k}$; they also are cohomologous on $\mathcal{U}^{<k}$ while $\mathcal{U}^{<k}$ is already adequate to $\Lambda^{<k}(A_0)$.

(iii) Let $(\dot{h}_\alpha)_{\alpha\in\mathbb{A}^{\leq k}} \in \Gamma(\dot{\mathcal{U}}^{\leq k};\,\Lambda^{\leq k}(A_0))$. Using again the identification of $\Lambda^{\leq k}$ to the semi-direct product $\Lambda^{<k}(A_0) \ltimes \Lambda^{k}(A_0)$ we can write

$$\dot{h}_\alpha = \dot{F}_\alpha \dot{G}_\alpha$$

where

$$\dot{F}_\alpha \in \Gamma(\dot{\mathcal{U}}^{\leq k};\, \Lambda^{<k}(A_0)), \quad \dot{G}_\alpha \in \Gamma(\dot{\mathcal{U}}^{\leq k};\, \Lambda^{k}(A_0)).$$

The covering $\mathcal{U}^{<k}$ is adequate to $\Lambda^{<k}(A_0)$ and $\mathcal{U}^{\leq k}$ refines $\mathcal{U}^{<k}$ by Proposition II.3.1. Then the cocycle $(\dot{F}_\alpha)_{\alpha\in\mathbb{A}^{\leq k}}$ is cohomologous to a cocycle $(\dot{F}'_\alpha)_{\alpha\in\mathbb{A}^{\leq k}}$ which comes from an $(\dot{f}'_\alpha)$ in $\Gamma(\dot{\mathcal{U}}^{<k};\, \Lambda^{<k}(A_0))$: there is a 0-cochain $(c_\alpha)_{\alpha\in\mathbb{A}^{\leq k}} \in \Gamma(\mathcal{U}^{\leq k};\, \Lambda^{<k}(A_0))$ with values in $\Lambda^{<k}(A_0)$ such that

$$c_\alpha^{-1}\dot{F}_\alpha c_{\alpha+} = \dot{F}'_\alpha \quad \text{and} \quad \dot{F}'_\alpha = \sigma^{<k}(\dot{f}'_\alpha).$$

Then $c_\alpha^{-1}\dot{h}_\alpha c_{\alpha+} = \dot{F}'_\alpha\dot{G}'_\alpha$ where, from the normality of $\Lambda^k(A_0)$ in $\Lambda^{\leq k}(A_0)$, the cocycle $(\dot{G}'_\alpha = c_{\alpha+}^{-1}\dot{G}_\alpha c_{\alpha+})_{\alpha\in\mathbb{A}^{\leq k}}$ is with values in $\Lambda^k(A_0)$. But the covering $\mathcal{U}^k$ is adequate to $\Lambda^k(A_0)$ and $\mathcal{U}^{\leq k}$ refines $\mathcal{U}^k$. Then the cocycle $(\dot{G}'_\alpha)_{\alpha\in\mathbb{A}^{\leq k}}$ is cohomologous to a cocycle $\sigma^k(\dot{g}'_\alpha)$ which comes from a $(\dot{g}'_\alpha)$ in $\Gamma(\dot{\mathcal{U}}^k; \Lambda^k)$. And they necessarily are cohomologous through a 0-cochain with values in $\Lambda^{<k}(A_0)$ (Proposition II.3.1 (iv)). Then this 0-cochain is trivial: $(\dot{G}'_\alpha) = \sigma^k(\dot{g}'_\alpha)$ and the cocycle $(c_\alpha^{-1}\dot{h}_\alpha c_{\alpha+})_{\alpha\in\mathbb{A}^{\leq k}}$ answers the question. □

From Lemmas II.3.2 and II.3.3 we deduce:

PROPOSITION II.3.4.

(i) *The product map of single-leveled cocycles*

$$\tau : \begin{cases} \displaystyle\prod_{k\in\mathcal{K}} \Gamma(\dot{\mathcal{U}}^k;\, \Lambda^k(A_0)) \longrightarrow & \Gamma(\dot{\mathcal{U}};\, \Lambda(A_0)), \\ \qquad\qquad (\dot{f}^k)_{k\in\mathcal{K}} \longmapsto & \displaystyle\prod_{k\in\mathcal{K}} \tau^k(\dot{f}^k) \end{cases}$$

following an ascending ordering of levels is injective.

(ii) *It induces on the cohomology a bijective and natural map*

$$\begin{aligned} \mathcal{T} : \prod_{k\in\mathcal{K}} \Gamma(\dot{\mathcal{U}}^k;\, \Lambda^k(A_0)) \cong \prod_{k\in\mathcal{K}} H^1(S^1;\, \Lambda^k(A_0)) \\ \longrightarrow H^1(\mathcal{U};\, \Lambda(A_0)) \cong H^1(S^1;\, \Lambda(A_0)). \end{aligned}$$

Proof. — It is an immediate consequence of Lemmas II.3.2 and II.3.3. Naturality must be understood in the same sense as in the main Theorem II.2.1 and is obvious. □

Remark II.3.5. — As it was done in Corollary I.5.4, we can extend Proposition II.3.4 to an arbitrary order of levels.

Proof of the main theorem II.2.1. — Let

$$\mathcal{I} : \prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A_0) \longrightarrow \prod_{k\in\mathcal{K}} \Gamma(\dot{\mathcal{U}}^k;\ \Lambda^k(A_0))$$

be the bijection composed as follows:

$$\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A_0) \to \prod_{\alpha\in\mathbb{A}}\prod_{k\in\mathcal{K}} \mathrm{Sto}_\alpha^k(A_0) \equiv \prod_{k\in\mathcal{K}}\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha^k(A_0)$$
$$\longrightarrow \prod_{k\in\mathcal{K}} \Gamma(\dot{\mathcal{U}}^k;\ \Lambda^k(A_0)),$$

where $i_\alpha : \mathrm{Sto}_\alpha(A_0) \to \prod\limits_{k\in\mathcal{K}} \mathrm{Sto}_\alpha^k(A_0)$ gives the factors in the factorization of Stokes germs in ascending levels (Corollary I.5.2, Proposition I.5.5) and $i^k : \prod\limits_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha^k(A_0) \to \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A_0))$ is the canonical map (Lemma II.3.2).

The bijection $\mathcal{T}\circ\mathcal{I} : \prod\limits_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A_0) \to H^1(S^1;\ \Lambda(A_0))$ is the map h in Theorem II.2.1. Naturality is obvious. □

In order for this proof to provide an algorithm for reducing any cocycle $\dot{\varphi}$ into its cohomologous Stokes form $h^{-1}(\dot{\varphi})$, one must detail the cohomological relation $(c_\alpha^{-1}\dot{h}_\alpha c_{\alpha^+})$ in Lemma II.3.3 (iii). This is done by using Section I.6.

II.3.4. An algorithm for the reduction of 1-cocycles to their cohomologous Stokes form.

We assume that $\sharp\mathbb{A} > 2$, leaving to the reader the quite simple case when $\sharp\mathbb{A} = 2$.

- Input: a cocycle $\dot{g} = (\dot{g}_j)$ over a cyclic covering $\mathcal{V}$ with nerve $\dot{\mathcal{V}} = \{\dot{V}_j; j\in J\}$.
- Output: the Stokes cocycle $(\dot{f}_\alpha)_{\alpha\in\mathbb{A}}$ cohomologous to $\dot{g}$.
- Algorithm:

1) Choose a first element α_1 among the set of cyclically ordered anti-Stokes directions

$$\mathbb{A} = \{\alpha_1 < \alpha_2 < \cdots < \alpha_s\}.$$

As previously, $U(\alpha,\pi/k)$ denotes an arc on S^1 bisected by α with opening π/k (Section I.2) and $\mathcal{K}=\{k_1<\cdots<k_r\}$ denotes the set of levels (Notation I.4.3).

2) Forget the j's such that $\dot{g}_j=I$.

3) Factorize the remaining $\dot{g}_j$'s by levels k_i, following the algorithm in Corollary I.5.2

$$\dot{g}_j=\dot{g}_j^{k_1}\dot{g}_j^{k_2}\ldots\dot{g}_j^{k_r},\quad \text{where } \dot{g}_j^{k_i}\in\Gamma\big(\dot{V}_j;\,\Lambda^{k_i}(A_0)\big).$$

4) For all (j,k), list the set of possible leading directions of level k for sections over $\dot{V}_j$

$$\mathbb{A}^k_{\dot{V}_j}:=\big\{\alpha\in\mathbb{A}^k;\dot{V}_j\subset U(\alpha,\pi/k)\big\}=\big\{\alpha^k_{j,1}<\cdots<\alpha^k_{j,s(j,k)}\big\}$$

and factorize the $\dot{g}_j^k$ by leading directions following the algorithm in Corollary I.6.4:

$$\dot{g}_j^k=\dot{g}^k_{j,\alpha^k_{j,1}}\cdots\dot{g}^k_{j,\alpha^k_{j,s(j,k)}}.$$

5) List the Stokes k-germs $\dot{g}^k_{j,\alpha}$ into a cyclic list preserving the lexicographic order on (j,k,α).

6) Using the commutation rules in Proposition I.6.6 change this list into a list of Stokes k-germs ordered according to their leading anti-Stokes directions α. (As commutators are not trivial, the previous germs $\dot{g}^k_{j,\alpha}$ must of course be changed.)

One can proceed by induction on anti-Stokes directions as follows:

first step: using the commutation rules in proposition I.6.6 collect together germs led by α_1.

j-th step: suppose the list is now arranged such that, according to the clockwise orientation, it contains all germs led by α_1, all germs led by $\alpha_2,\ldots$, all germs led by α_{j-1} and then germs led by the others anti-Stokes directions.

Using the commutation rules in Proposition I.6.6 collect all germs led by α_j and write them to the right of germs led by α_{j-1}. New germs led by $\alpha_1,\ldots,\alpha_{j-1}$ may appear out of order. Using again the commutation rules in Proposition I.6.6 rearrange them from α_{j-1} to α_1.

7) In the final list, for all $\alpha\in\mathbb{A}$, denote by $\dot{f}_\alpha$ the product (according to the final order) of all germs led by α. The cocycle $(\dot{f}_\alpha)_{\alpha\in\mathbb{A}}$ is the required Stokes cocycle cohomologous to the given cocycle $\dot{g}$.

Remark II.3.6. — A normal solution being given (for instance using the solver DESIR and the forthcoming DESIR2), one can compute germs using their representations, i.e., Stokes matrices, as it is done in [BJL79]. But one must notice that, as one turns all around S^1, one cannot avoid a jump of determination. This jump is controlled by the formal monodromy $\widehat{M}$ (cf. Remarks I.4.9) and implies the closure relation in [BJL79]. We avoid this problem when considering germs and cocycles instead of Stokes matrices.

II.4. Proof in the ramified case.

We denote by $\mathbb{C}_x$ and $\mathbb{C}_t$ copies of the complex line $\mathbb{C}$ whose coordinates are x and t and by S^1_x, S^1_t the corresponding real blow-up of 0.

Let $p \in \mathbb{N}$, $p \neq 0$. The map $\mathbb{C}_t \to \mathbb{C}_x$ such that $t \mapsto x = t^p$ defines a p-foiled ramified covering of $\mathbb{C}_x$ and takes a system $[A] : \mathrm{d}X/\mathrm{d}x = AX$ into the system $[A^{*p}] : \mathrm{d}Y/\mathrm{d}t = A^{*p}Y$ where $A^{*p}(t) = p\,t^{p-1}A(t^p)$. Obviously, when the matrix A is meromorphic with respect to x, the matrix A^{*p} is meromorphic with respect to t. The system $[A^{*p}]$ is called the *lifting* of $[A]$ by the ramification $x = t^p$.

We consider also the following liftings:

- $[A_0^{*p}]$ is the lifting of the normal form $[A_0]$. It is a normal form and has $X_0^{*p}(t) = X_0(t^p)$ as a normal solution when $X_0(x)$ is a normal solution of $[A_0]$.

- $\mathcal{U}^{*p} = \{U^{*p}_{j,i}; j \in J,\ 0 \le i \le p-1\}$ is the lifting of a cyclic covering $\mathcal{U} = \{U_j; j \in J\}$ of S^1. The $U^{*p}_{j,i}$, for $i = 0, \dots, p-1$, denote the p liftings of U_j. The covering $\mathcal{U}^{*p}$ is cyclic. Its nerve $\dot{\mathcal{U}}^{*p} = \{\dot{U}^{*p}_{j,i}; j \in J,\ 0 \le i \le p-1\}$ is the lifting of the nerve $\dot{\mathcal{U}} = \{\dot{U}_j; j \in J\}$ of $\mathcal{U}$. It is adequate to $\Lambda(A_0^{*p})$ (cf. figure II.1).

- $c^{*p} = (c^{*p}_{j,i})_{j\in J,\ 0\le i\le p-1}$ is the lifting of a 0-cochain $c = (c_j)$ in $\Gamma(\mathcal{U};\ \Lambda(A_0))$ defined by $c^{*p}_{j,i}(t) = c_j(t^p)$. It is a 0-cochain in $\Gamma(\mathcal{U}^{*p};\ \Lambda(A_0^{*p}))$.

- $\dot{\varphi}^{*p} = (\dot{\varphi}^{*p}_{j,i})_{j\in J,\ 0\le i\le p-1}$ is the lifting of a cocycle $\dot{\varphi} = (\dot{\varphi}_j)$ in $\Gamma(\dot{\mathcal{U}};\ \Lambda(A_0))$ defined by $\varphi^{*p}_{j,i}(t) = \varphi_j(t^p)$. It is a cocycle in $\Gamma(\dot{\mathcal{U}}^{*p};\ \Lambda(A_0^{*p}))$.

- $\mathbb{A}^{*p} = \{\alpha^{*p}_{1,0} < \alpha^{*p}_{2,0} < \cdots < \alpha^{*p}_{r,0} < \alpha^{*p}_{1,1} < \cdots < \alpha^{*p}_{r,p-1}\}$ is the lifting of the set $\mathbb{A} = \{\alpha_1 < \cdots < \alpha_r\}$ of anti-Stokes directions of $[A_0]$. The $\alpha^{*p}_{j,i}$, for $i = 0, 1, \dots, p-1$, denote the p liftings of α_j. As the set $\mathcal{Q}_{[A_0]}$ is invariant under the substitution $x \mapsto xe^{2i\pi}$, the set $\mathbb{A}^{*p}$ is the set of anti-Stokes directions of $[A_0^{*p}]$.

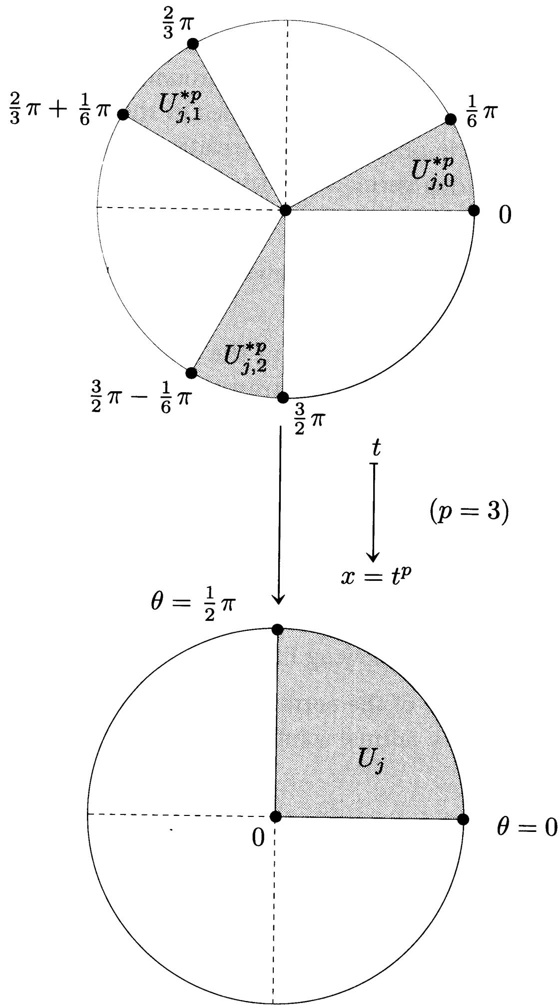

Figure II.1

These liftings are $\mathbb{Z}/p\mathbb{Z}$-invariant, that is, invariant under the rotation of angle $2\pi/p$.

Proof of the main theorem II.2.1. — Let $[A_0]$ belong to the p-ramified case. Let $\dot{g} = (\dot{g}_j)_{j\in J} \in \Gamma(\dot{\mathcal{U}};\, \Lambda(A_0))$ be a cocycle on a cyclic covering $\mathcal{U} = \{U_j;\, j \in J\}$ of S^1_x.

The system $[A_0^{*p}]$ belongs to the unramified case and the lifting $\dot{g}^{*p}$

of $\dot{g}$ is a $\mathbb{Z}/p\mathbb{Z}$-invariant cocycle in $\Gamma(\dot{\mathcal{U}}^{*p}; \Lambda(A_0^{*p}))$. The theorem in the unramified case shows that $\dot{g}^{*p}$ is cohomologous to a Stokes cocycle in

$$\prod_{\substack{j\in J\\ 0\le i\le p-1}} \mathrm{Sto}_{\alpha_{j,i}^{*p}}(A_0^{*p}).$$

The algorithm applied in a $\mathbb{Z}/p\mathbb{Z}$-symmetric way shows that this Stokes cocycle is $\mathbb{Z}/p\mathbb{Z}$-invariant and then it is the lifting $\dot{f}^{*p}$ of a Stokes cocycle $\dot{f}$ in $\prod\limits_{j\in J} \mathrm{Sto}_{\alpha_j}(A_0)$. Moreover the 0-cochain achieving the reduction of $\dot{g}^{*p}$ to $\dot{f}^{*p}$ is $\mathbb{Z}/p\mathbb{Z}$-invariant and then it «descends» in an 0-cochain achieving the reduction of $\dot{g}$ to $\dot{f}$: the cocycle $\dot{g}$ and the Stokes cocycle $\dot{f}$ belong to the same cohomology class in $H^1(S^1; \Lambda(A_0))$. This ends the proof of the surjectivity of h.

Injectivity of h is obvious. In particular, notice that $\mathbb{Z}/p\mathbb{Z}$-invariant cohomologous cocycles are cohomologous through a $\mathbb{Z}/p\mathbb{Z}$-invariant 0-cochain: a relation of cohomology that would not be $\mathbb{Z}/p\mathbb{Z}$-invariant can be symmetrized. □

Remark II.4.1. — When $[A_0]$ belongs to the ramified case the lifting X_0^{*p} of a normal solution X_0 is not $\mathbb{Z}/p\mathbb{Z}$-invariant. Then, while lifted cocycles are $\mathbb{Z}/p\mathbb{Z}$-invariant, their representations by usual Stokes matrices are not. This is the reason why Stokes matrices are much more difficult to handle than cocycles.

Let us summarize the above results.

Proposition II.4.2. — *The following diagram commutes*:

$$\begin{tikzcd}
G_t \setminus \widehat{G}(A_0^{*p}) \arrow[r, "\exp_{\mu_t}", "\sim"'] & H^1(S_t^1\,;\,\Lambda(A_0^{*p})) \arrow[r, "h_t^{-1}", "\sim"'] & \prod\limits_{\alpha^{*p}\in\mathbb{A}^{*p}} \mathrm{Sto}_{\alpha^{*p}}(A_0^{*p}) \arrow[r, "\mathrm{rest}"] & \prod\limits_{\alpha^{*p}\in\mathbb{A}_0^{*p}} \mathrm{Sto}_{\alpha^{*p}}(A_0^{*p}) \\
G_x \setminus \widehat{G}(A_0) \arrow[u, "\rho_p"'] \arrow[r, "\exp_{\mu_x}", "\sim"'] & H^1(S_x^1\,;\,\Lambda(A_0)) \arrow[u, "R_p"'] \arrow[r, "h_x^{-1}", "\sim"'] & \prod\limits_{\alpha\in\mathbb{A}} \mathrm{Sto}_{\alpha}(A_0) \arrow[u, "R'_p"'] \arrow[ur, "r_p"', "\sim"] &
\end{tikzcd}$$

The notation $\mathbb{A}_0^{*p}$ *means the restriction of* $\mathbb{A}^{*p}$ *to the first sheat*: $\mathbb{A}_0^{*p} = \{\alpha_{j,0}^{*p}; j\in J\}$; *the map* rest *denotes the projection on the restricted family*; *vertical arrows are injective liftings*; *the map* $r_p := \mathrm{rest}\circ R'_p$ *and horizontal arrows except* rest *are bijective.*

III. APPLICATIONS

As previously, $[A_0]$ denotes a normal form and $X_0 = x^L \mathrm{e}^{Q(1/x)}$ a normal solution.

III.1. H^1 as a linear affine variety.

The Stokes groups $\mathrm{Sto}_\alpha(A_0)$ are unipotent Lie groups with finite dimension

$$\dim_{\mathbb{C}} \mathrm{Sto}_\alpha(A_0) = \sum_{q_j \underset{\tilde{\alpha},\,\max}{\prec} q_\ell} \deg(q_j - q_\ell)$$

the summation being taken over couples (j, ℓ), $1 \leq j, \ell \leq n$ such that $\mathrm{e}^{q_j - q_\ell}$ is led by a given determination $\widetilde{\alpha}$ of α. (cf. Definition I.4.4).

The natural isomorphism h endows the classifying set $H^1(S^1; \Lambda(A_0))$ with the product structure of a unipotent Lie group with finite dimension

$$N = \sum_{1 \leq j, \ell \leq n} \deg(q_j - q_\ell).$$

This number N is known to be the irregularity of $[\mathrm{End}\, A_0]$ (see [Mal74], [De77] and [BV89], Prop. 2.6.3, Thm 3.4.1).

The Lie algebra $\mathrm{sto}_\alpha(A_0)$ of $\mathrm{Sto}_\alpha(A_0)$ is nilpotent and the exponential map induces an homeomorphism

$$\exp : \mathrm{sto}_\alpha(A_0) \longrightarrow \mathrm{Sto}_\alpha(A_0).$$

We denote by $\ln = \exp^{-1}$ the inverse map. Then the tangent linear structure is defined all over $\mathrm{Sto}_\alpha(A_0)$ by the following laws:

- addition $\dot{F}_\alpha \top \dot{G}_\alpha = \exp(\ln \dot{F}_\alpha + \ln \dot{G}_\alpha)$;
- multiplication by a scalar $\lambda \in \mathbb{C}$

$$\lambda * \dot{F}_\alpha = \exp(\lambda \ln \dot{F}_\alpha).$$

The natural isomorphism h endows the classifying set $H^1(S^1; \Lambda(A_0))$ with the tangent product structure of a linear affine variety of dimension N.

It can be convenient especially for numerical calculations (see [LR91]) to use another linear structure induced from $\mathrm{sto}_\alpha(A_0)$ by the homeomorphism

$$\begin{array}{ccc} \mathrm{sto}_\alpha(A_0) & \xrightarrow{I+\cdot} & \mathrm{Sto}_\alpha(A_0), \\ \dot{f}_\alpha & \longmapsto & I+\dot{f}_\alpha\,. \end{array}$$

The laws are:

- the addition $(I+f_\alpha)\perp(I+g_\alpha) = I+(\dot{f}_\alpha+\dot{g}_\alpha)$;
- the multiplication by a scalar $\lambda\in\mathbb{C}$
$$\lambda\cdot(I+\dot{f}_\alpha) = I+\lambda\dot{f}_\alpha.$$

Except in dimension 2 this structure differs from the previous one and the product endows $H^1(S^1;\Lambda(A_0))$ with a new structure of a linear affine variety of dimension N. The relation between the two structure is summarized in

Proposition III.1.1. — *The map*
$$\big((I+\cdot)\circ\ln\big)_{\alpha\in\mathbb{A}} : \prod_{\alpha\in\mathbb{A}}\mathrm{Sto}_\alpha(A_0) \longrightarrow \prod_{\alpha\in\mathbb{A}}\mathrm{Sto}_\alpha(A_0)$$
is polynomial and a homomorphism of the two previous linear structures. More precisely, the diagram

$$\begin{tikzcd} \prod_{\alpha\in\mathbb{A}}(\mathrm{Sto}_\alpha(A_0),\top,*) \arrow[rr, "((I+\cdot)\circ\ln)_{\alpha\in\mathbb{A}}"] & & \prod_{\alpha\in\mathbb{A}}(\mathrm{Sto}_\alpha(A_0),\perp,\cdot) \\ & \prod_{\alpha\in\mathbb{A}}(\mathrm{sto}_\alpha(A_0),+,\cdot) \arrow[ul, "\exp"] \arrow[ur, "I+\cdot"'] & \end{tikzcd}$$

commutes.

III.2. Summability.

Let $\widehat{F}\in\widehat{G}(A_0)$ be a transformation of $[A_0]$, i.e., a solution of the system

$$[A_0,A] \qquad \frac{\mathrm{d}F}{\mathrm{d}x} = AF - FA_0$$

where $A = {}^{\widehat{F}}A_0$ denotes the transformed system.

III.2.1. Sums of $\widehat{F}$.

Let $\dot{\varphi} = (\dot{\varphi}_j)_{j\in J} \in \Gamma(\dot{\mathcal{V}}; \Lambda(A_0))$ be a 1-cocycle in the cohomology class $\exp_{\mu_0}(\widehat{F})$ (cf. Theorem I.2.1).

It is easy to prove (see [LR91], [Mal83]):

Proposition III.2.1. — *There exists a unique family of realizations $(F_j)_{j\in J}$ of $\widehat{F}$ over $\mathcal{V}$ such that $\dot{\varphi}_j = F_{j-1}F_j^{-1}$, where $j \in J$.*

By realizations we mean analytic matrices F_j on V_j which satisfy $[A_0, A]$ and are asymptotic to $\widehat{F}$ on V_j.

When $\mathcal{V}$ is the cyclic covering $\mathcal{U} = \mathcal{U}^{\leq k_r} = \{U_{\overline{\alpha}}^{\leq k_r}; \alpha \in \mathbb{A}\}$ (cf. Section II.3.1) and $\dot{\varphi}$ is in its Stokes form (cf. Section II.2), we call *sums of* $\widehat{F}$, the corresponding realizations F_α. Denoting by α^+ the nearest anti-Stokes direction on the right of α, one has $\dot{U}_\alpha = U_\alpha \cap U_{\alpha^+}$ the intersection of U_α «on the left» of α and U_{α^+} «on the right» of α. It is then natural to set:

Definition III.2.2.

$$S_{\overline{\alpha}}^{-}(\widehat{F}) := F_\alpha \quad \text{is the } \textit{sum of } \widehat{F} \textit{ on the left of } \alpha,$$

$$S_{\overline{\alpha}}^{+}(\widehat{F}) := F_{\alpha^+} \quad \text{is the } \textit{sum of } \widehat{F} \textit{ on the right of } \alpha.$$

Sums have been defined by different authors in many ways: by Martinet and Ramis [MR91], by Écalle [Ec92], [Br91-1], [Br91-2], by Malgrange and Ramis [MalR92], by Balser, Braaksma, Ramis and Sibuya [BBRS91], by Balser [Ba91],...

In all these definitions sums on the left and sums on the right of α are identic to ours sums $S_{\overline{\alpha}}^{-}(\widehat{F})$ and $S_{\overline{\alpha}}^{+}(\widehat{F})$. We sketch the comparison of our sums to those defined by Martinet and Ramis and to those defined by Écalle in Section III.2.4 (Theorem III.2.8) and we give in more details the comparison to those defined by Malgrange and Ramis in Section III.2.5 (Theorems III.2.13 and III.2.14).

III.2.2. k-summability.

In this section we introduce a few definitions which we use in the next section.

Let $\widehat{F}_0 \in \widehat{G}(A_0)$ and $A_1 = {}^{\widehat{F}_0}A_0$.

When they do not require $[A_0]$ to be a normal form, definitions in Chapter I make sense on $[A_1]$ without any change. This is namely the case for:

- $\widehat{G}(A_1)$, the set of transformations of $[A_1]$;
- $\Lambda(A_1)$, the sheaf of flat isotropies of $[A_1]$;
- $\exp_{\mu_1} : G \setminus \widehat{G}(A_1) \to H^1(S^1; \Lambda(A_1))$, the Malgrange-Sibuya isomorphism.

This is also the case for Stokes germs in $\Lambda(A_1)$ (Remark I.4.13).

The definition of the subsheaf $\Lambda^{\geq k}(A_1)$ of $\Lambda(A_1)$ of germs of level $\geq k$ needs a few additional justifications: let $\theta \in S^1$; the sets $\widehat{G}(A_0)$ and $\widehat{G}(A_1)$ are related by

$$\widehat{G}(A_1) = \widehat{G}(A_0)\widehat{F}_0^{-1} := \{\widehat{F} \in \mathrm{GL}(n; \mathbb{C}[[x]]\,[1/x]); \widehat{F}\widehat{F}_0 \in \widehat{G}(A_0)\}.$$

Then, to a realization F_0 of $\widehat{F}_0$ at θ, there is an isomorphism $\Lambda_\theta(A_0) \to \Lambda_\theta(A_1)$ given by $\varphi \mapsto F_0\varphi F_0^{-1}$. Since the possible realizations of $\widehat{F}_0$ are the $F_0 f_0$'s for $f_0 \in G_0(A_0)$ an isotropy of $[A_0]$, there is also the family of isomorphisms $\varphi \mapsto F_0 f_0 \varphi f_0^{-1} F_0^{-1}$. And since conjugacy by f_0 preserves $\Lambda_\theta^{\geq k}(A_0)$ (Proposition I.5.3 (i)), one can speak of *germs of level* $\geq k$ in $\Lambda(A_1)$. Notice however that we cannot this way define k-germs as conjugacy by f_0 does not preserve $\Lambda_\theta^k(A_0)$.

DEFINITION III.2.3.

(i) A 1-cocycle $\dot{\varphi} = (\dot{\varphi}_j)_{j\in J} \in \Gamma(\dot{\mathcal{V}}; \Lambda(A_1))$ is a *k-summable cocycle* when, for all $j \in J$, it satisfies the two conditions

$$\begin{cases} \dot{\varphi}_j \text{ is of level } \geq k & (\dot{\varphi}_j \in \Gamma(\dot{V}_j;\, \Lambda^{\geq k}(A_1))), \\ \text{the opening of } \dot{V}_j \text{ is } \pi/k. \end{cases}$$

(ii) A series $\widehat{F} \in \widehat{G}(A_1)$ is a *k-summable series* when $\exp_{\mu_1}(\widehat{F})$ contains a k-summable cocycle.

A k-summable cocycle in a cohomology class, if any, is unique up to extra trivial components. Realizations of such a k-summable cocycle defines then the *k-sums* of $\widehat{F}$ in an essentially unique way (Proposition III.2.1). When $[A_1]$ is a normal form (i.e. when $[A_1]$ is meromorphically equivalent to $[A_0]$) then $\widehat{F}$ is k-summable if and only if its Stokes cocycle $\dot{\varphi} = (\dot{\varphi}_\alpha)_{\alpha\in\mathbb{A}}$

belongs to $\Gamma(\dot{\mathcal{U}}^k;\, \Lambda^k)$, precisely, to the range of the natural injective map $\tau^k : \Gamma(\dot{\mathcal{U}}^k;\, \Lambda^k) \hookrightarrow \Gamma(\dot{\mathcal{U}};\, \Lambda)$ (cf. Section II.3.1). This occurs when the Stokes cocycle $\dot{\varphi}$ of $\widehat{F}$ bears the unique level k.

This definition of k-summability obviously agrees with the cohomological definition given by Malgrange and Ramis in the very general context of asymptotic series (see [MalR92], Def. 2.2 applied to $k = k_1 = \cdots = k_r$, and Definition III.2.10 below). Let us mention that it also agrees with the asymptotic definition given by Ramis in [Ra80] (see [MalR92], thm 1.6) and with the generalization to the level k of the Borel-summability given by Leroy (see [Le00], [MR89], Chap. 2).

The *Turrittin's problem*, that is, the question to know if a series solution of a linear differential equation is k-summable for a suitable k has received a negative answer in [RS89] (cf. also [LR90-2]). The different generalizations of k-sums quoted at the end of Section III.2.1 enables to give a precise answer to this question.

Definition III.2.4. — Let $\widehat{F} \in \widehat{G}(A_1)$ be a k-summable series and

$$(\dot{\varphi}_j)_{j\in J} \in \prod_{j\in J} \Gamma\big(\dot{V}_j;\, \Lambda(A_1)\big)$$

a k-summable cocycle in $\exp_{\mu_1}(\widehat{F})$. The bisecting directions of the $\dot{V}_j$'s corresponding to nontrivial $\dot{\varphi}_j$ are called *singular directions* for $\widehat{F}$.

III.2.3. Factorization.

The factorization theorem below (Theorem III.2.5) was first proved by Ramis (see [Ra85-1], [Ra85-3]) in a quite technical way using Gevrey estimates. The definition of k-summability used there was also the one based on asymptotics (see [Ra80], [LR90-2], [MalR92], Def. 1.5).

Our aim now is to show how to deduce this factorization theorem from our Stokes cocycle without any use of Gevrey estimates. Of course, we refer here to k-summability as defined in Definition III.2.3.

Both proofs are not constructive.

Theorem III.2.5 (Factorization theorem). — *Let $\widehat{F} \in \widehat{G}(A_0)$ be a transformation of the normal form $[A_0]$ and $\mathcal{K} = \{k_1 < k_2 < \cdots < k_r\}$ be the set of levels of $[A_0]$. Then $\widehat{F}$ can be factored in*

$$\widehat{F} = \widehat{F}_r\, \widehat{F}_{r-1} \cdots \widehat{F}_2 \widehat{F}_1$$

where, for $j = 1, \dots, r$, $\widehat{F}_j$ is k_j-summable with singular directions belonging to the set $\mathbb{A}^{k_j}$ of anti-Stokes directions of $[A_0]$ bearing the level k_j.

This factorization is essentially unique (cf. Proposition III.2.6 (iv)).

Proof. — Theorem III.2.5 follows immediately from Proposition III.2.6 below by descending induction on levels. □

Let $\dot{f} \in \Gamma(\dot{\mathcal{U}};\ \Lambda(A_0))$ be the Stokes cocycle associated to $\widehat{F}$ in $\exp_{\mu_0}(\widehat{F})$ and k be the maximal level beared by $\dot{f}$. From Corollary I.5.2 one can write in a unique way

$$\dot{f} = \dot{f}^{<k}\dot{g}^k \quad \text{and} \quad \dot{f} = \dot{f}^k\dot{f}^{<k},$$

where $\dot{f}^k, \dot{g}^k \in \Gamma(\dot{\mathcal{U}}^{\leq k};\ \Lambda^k(A_0))$ and $\dot{f}^{<k} \in \Gamma(\dot{\mathcal{U}}^{\leq k};\ \Lambda^{<k}(A_0))$.

The cohomology class of the Stokes cocycle $\dot{f}^{<k}$ belongs to $H^1(S^1; \Lambda(A_0))$. Then, from the Malgrange-Sibuya isomorphism theorem (Theorem I.2.1) there exists a transformation $\widehat{F}^{<k} \in \widehat{G}(A_0)$ such that $\dot{f}^{<k}$ belongs to $\exp_{\mu_0}(\widehat{F}^{<k})$; moreover $\widehat{F}^{<k}$ is uniquely determined up to a left meromorphic factor.

Let $[A_1 = {}^{\widehat{F}^{<k}}A_0]$ denote the system transformed from $[A_0]$ by $\widehat{F}^{<k}$ and let $\widehat{F}^k$ be defined by $\widehat{F}^{<k} = \widehat{F}(\widehat{F}^{<k})^{-1}$.

PROPOSITION III.2.6. — *One has:*

(i) $\widehat{F} = \widehat{F}^k\widehat{F}^{<k}$.

(ii) *$\widehat{F}^k$ is k-summable with singular directions belonging to $\mathbb{A}^k$.*

(iii) *The levels in the Stokes cocycle of $\widehat{F}^{<k}$ are $< k$.*

(iv) *The decomposition* (i) *with properties* (ii) *and* (iii) *is essentially unique, that is, unique up to an intermediate meromorphic factor. Let $\widehat{F} = \widehat{H}^k\widehat{H}^{<k}$ be another decomposition of $\widehat{F}$; then there is a matrix $h \in \mathrm{GL}(n; \mathbb{C}\{x\}[1/x])$ with meromorphic entries such that $\widehat{H}^{<k} = h\widehat{F}^{<k}$ and $\widehat{H}^k = \widehat{F}^k h^{-1}$.*

Proof. — Properties (i) and (iii) are obvious. In order to prove (ii) we have to prove that $\exp_{\mu_1}(\widehat{F}^k)$ contains a k-summable cocycle and this is the case of the twisted cocycle

$$\begin{aligned}\dot{\varphi} &= \big(S^+_{\alpha}(\widehat{F}^{<k})\ \dot{g}^k_{\alpha}\ S^+_{\alpha}(\widehat{F}^{<k})^{-1}\big)_{\alpha\in\mathbb{A}^{\leq k}}\\ &= \big(S^-_{\alpha}(\widehat{F}^{<k})\ \dot{f}^k_{\alpha}\ S^-_{\alpha}(\widehat{F}^{<k})^{-1}\big)_{\alpha\in\mathbb{A}^{\leq k}}.\end{aligned}$$

Indeed, for the first equality, consider the Stokes cocycle

$$(\dot f_\alpha)_{\alpha\in\mathbb{A}^{\le k}} = (\dot f_\alpha^{<k}\dot g_\alpha^k)_{\alpha\in\mathbb{A}^{\le k}}$$

associated to the product $\widehat F = \widehat F^k \widehat F^{<k}$:

$$\begin{aligned}\dot f_\alpha^{<k}\dot g_\alpha^k &= (F_\alpha^k F_\alpha^{<k})^{-1}(F_{\alpha+}^k F_{\alpha+}^{<k}) &&\text{(Proposition III.2.1)}\\ &= \dot f_\alpha^{<k}(F_{\alpha+}^{<k})^{-1}(F_\alpha^k)^{-1}F_{\alpha+}^k F_{\alpha+}^{<k}\\ &= \dot f_\alpha^{<k}(F_{\alpha+}^{<k})^{-1}\dot\varphi_\alpha F_{\alpha+}^{<k}.\end{aligned}$$

Using the previous notation $F_{\alpha+}^{<k} = S_\alpha^+(\widehat F^{<k})$ this gives the first equality. In order to get the second equality, use the formula $\dot f_\alpha = \dot f_\alpha^k \dot f_\alpha^{<k}$.

From these formulæ, we see that $\dot\varphi$ being conjugated to a Stokes cocycle of pure level k is of level $\ge k$. Moreover, it is a cocycle on $\dot{\mathcal U}^{\le k}$ whose arcs $\dot U_\alpha^{\le k}$ have an opening greater than π/k. Hence $\dot\varphi$ is k-summable with nontrivial components only when $\dot U_\alpha^{\le k}$ has an opening equal to π/k, i.e., when $\alpha\in\mathbb{A}^k$.

Property (iv) results from the fact that we must have

$$\exp_{\mu_0}(\widehat H^{<k}) = \exp_{\mu_0}(\widehat F^{<k}). \qquad \square$$

Writing also the twisted cocycle $\dot\varphi \in \exp_{\mu_1}(\widehat F^k)$ in terms of sums of $\widehat F^k$ over $\mathcal U^{\le k}$:

$$\dot\varphi_\alpha = S_\alpha^-(\widehat F^k)^{-1}\, S_\alpha^+(\widehat F^k)\ , \quad \alpha\in\mathbb{A}^{\le k}$$

we get the following corollary:

Corollary III.2.7. — *The factors*

$$\dot g^k = (\dot g_\alpha^k)_{\alpha\in\mathbb{A}^{\le k}} \quad\text{and}\quad \dot f^k = (\dot f_\alpha^k)_{\alpha\in\mathbb{A}^{\le k}}$$

in the decompositions $\dot f = \dot f^{<k}\dot g^k$ *and* $\dot f = \dot f^k \dot f^{<k}$ *introduced above satisfy*

$$\dot g_\alpha^k = S_\alpha^+(\widehat F^{<k})^{-1}\, S_\alpha^-(\widehat F^k)^{-1}\, S_\alpha^+(\widehat F^k)\, S_\alpha^+(\widehat F^{<k}),$$

$$\dot f_\alpha^k = S_\alpha^-(\widehat F^{<k})^{-1}\, S_\alpha^-(\widehat F^k)^{-1}\, S_\alpha^+(\widehat F^k)\, S_\alpha^-(\widehat F^{<k}).$$

III.2.4. Sums in the sense of Martinet and Ramis and sums in the sense of Écalle.

A series is said to be *multisummable by means of the levels* $k_1,\dots,k_r$ *in the sense of Martinet and Ramis* (see [MR91]) when it belongs to the differential algebra generated by the differential algebras of k_j-summable series for $j = 1, \ldots, r$ (see [Ra80], [LR90-2]).

Accelerosums in the sense of Écalle are defined in a very general context by iterated integral formulæ including kernels of acceleration. These kernels generalize the Laplace and inverse Laplace kernels in the classical Borel summation integrals (see [Ec92], [Br91-2], cf. [LR92] for formulæ useful in the linear case).

THEOREM III.2.8. — *The sums $S_\alpha^-(\widehat{F})$ and $S_\alpha^+(\widehat{F})$ coincide with the multisums in the sense of Martinet and Ramis and with the accelerosums of Écalle.*

Proof. — The factorization theorem of Ramis (see [Ra85-1], [Ra85-3], Theorem III.2.5 above) proves that $\widehat{F}$ is multisummable in the sense of Martinet and Ramis by means of the levels $k_1, \ldots, k_r$ of $[A_0]$. The multisums of $\widehat{F}$ are the products of the k_j-sums of the factors. Theorem 14 (i) of [MR91] states that, when $\widehat{F} = \widehat{F}_r \ldots \widehat{F}_1$, the $\widehat{F}_j$ being k_j-summable, the 1-cocycle built with the k_j-sums of the $\widehat{F}_j$'s is a Stokes cocycle. From the unicity of the Stokes cocycle (main Theorem II.2.1) and the unicity of the sums associated to a given cocycle (Proposition III.2.1) this proves that multisums in the sense of Martinet and Ramis coincide with our sums $S_\alpha^-(\widehat{F})$ and $S_\alpha^+(\widehat{F})$ (Definition III.2.2).

Theorem 9 (i) in [MR91] proves that the multisummmability of $\widehat{F}$ in the sense of Martinet and Ramis is a particular case of accelerosummability in the sense of Écalle. □

III.2.5. Sums in the sense of Malgrange and Ramis.

Our aim, in this section, is to compare the sums $S_\alpha^-(\widehat{F})$ and $S_\alpha^+(\widehat{F})$ of a given transformation $\widehat{F} \in \widehat{G}(A_0)$ to its $(k_1, \ldots, k_r)$-sums in the sense of Malgrange and Ramis using the Stokes cocycle of $\widehat{F}$.

Let us first recall and summarize the definition of a $(k_1, \ldots, k_r)$-sum of a series $\hat{f}$ (one assumes $0 < k_1 < k_2 < \cdots < k_r$) and its wild analytic interpretation following Malgrange and Ramis [MalR92].)

Let $\mathcal{A}$ denote the sheaf over S^1 of holomorphic germs with an asymptotic expansion at 0 in directions $\theta \in S^1$ and $\mathcal{A}^{\leq -k}$ denote the subsheaf of germs in $\mathcal{A}$ with an exponential decay of order k.

DEFINITION III.2.9. — A *k-quasi-function* is a global section over S^1 of the quotient sheaf $\mathcal{A}/\mathcal{A}^{\leq -k}$. Hence, it can be represented by a 0-cochain

$(f_j) \in \prod \Gamma(V_j;\, \mathcal{A})$ on a cyclic covering $\mathcal{V} = \{V_j; j \in J\}$ such that the successive differences $f_{j+1} - f_j$ are exponentially small of order k (that is, $\exists\, C, a > 0$ such that $|f_{j+1}(x) - f_j(x)| \leq C \exp(-a/|x|^k)$ on $V_{j+1} \cap V_j$).

Due to a theorem of Ramis (see [MalR92], cor. 1.8 to thm 1.6) a series $\hat{f}$ of k-Gevrey type ($\hat{f} \in \mathbb{C}[[x]]_{1/k}$) is the asymptotic expansion of a unique k-quasi-function f.

Definition III.2.10.

(i) Let I be a closed arc with opening $|I| = \pi/k$. A series $\hat{f}$ of k-Gevrey type is said to be *k-summable on I* if its k-quasi-function has no jump on I: precisely, one can represent f as a 0-cochain $(f_j) \in \prod \Gamma(V_j;\, \mathcal{A})$ with exponentially small $f_{j+1} - f_j$ of order k such that at least one V_j, say V_{j_0}, contain I.

(ii) $\hat{f}$ is said to be *k-summable* when it is k-summable on all arcs I but a finite number.

When $\hat{f}$ is k-summable on I, its k-sum $f_{j_0|I}$ is uniquely defined whatever is the choice of the 0-cochain (f_j). As already mentioned in Section III.2.2, the k-sums in this sense coincide with the k-sums in the sense of Leroy [Le00] or in the sense of Ramis [Ra80]. The generalization to the case of several levels given by Malgrange and Ramis is as follows.

Definition III.2.11 (See [MalR91], Déf. 2.2.). — A $(k_1, \ldots, k_r)$-*sum* of a series $\hat{f} \in \mathbb{C}[[x]]_{1/k_r}$ is a sequence $(f_0, f_1, \ldots, f_r) \in \Gamma(S^1; \mathcal{A}/\mathcal{A}^{\leq -k_1}) \times \Gamma(I_1; \mathcal{A}/\mathcal{A}^{\leq -k_2}) \times \cdots \times \Gamma(I_{r-1}; \mathcal{A}/\mathcal{A}^{\leq -k_r}) \times \Gamma(I_r; \mathcal{A})$ of quasi-functions on closed arcs $S^1 \supset I_1 \supset \cdots \supset I_r$ with opening $|I_1| = \pi/k_1, \ldots, |I_r| = \pi/k_r$, which are compatible in the obvious possible sense: $f_j \bmod \mathcal{A}^{\leq -k_j} = f_{j-1|I_j}$.

Given a normal form $[A_0]$ bearing the levels $k_1 < k_2 < \cdots < k_r$, Malgrange and Ramis proved (see [MalR92], thm 4.1) that a transformation $\widehat{F} \in \widehat{G}(A_0)$ is $(k_1, \ldots, k_r)$-summable in the previous sense provided that the bisecting directions of the I_j's are not anti-Stokes directions bearing the level k_j. Before comparing these sums to our sums $S_\alpha^-(\widehat{F})$ and $S_\alpha^+(\widehat{F})$ let us describe their wild analytic interpretation.

The geometrical background of the wild π_1 is the following: in the analytic manifold $\mathbb{C}$, one introduces at 0 a wild neighbourhood of 0 made of a closed disc $\Delta_R = \{z = \rho e^{i\theta}; |z| \leq R\}$, $R > k_r$, and the wild analytic sheaf $\widetilde{\mathcal{A}}$ whose fiber at a point $z=\rho\,\mathrm{e}^{i\theta}\in\Delta_R$ is

$$\begin{cases} \mathbb{C}[[x]]=\mathcal{A}_\theta/\mathcal{A}_\theta^{<0}, & \text{if } \rho<k_1; \\ \mathcal{A}_\theta/\mathcal{A}_\theta^{\leq -k_j}, & \text{if } k_j\leq\rho<k_{j+1} \text{ for } j=1,\dots,r \text{ (set } k_{r+1}:=R); \\ \mathcal{A}_\theta & \text{if } \rho=R. \end{cases}$$

Denote

- $\tilde{I}_j:=\{z=\rho\,\mathrm{e}^{i\theta};\rho\leq k_{j+1},\ \theta\in I_j\}$, for $j=1,\dots,r$;
- $\tilde{I}:=\{|z|\leq k_1\}\cup\tilde{I}_1\cup\cdots\cup\tilde{I}_r$;
- i_j the intersection of the bisecting line of I_j with $\tilde{I}_j\setminus\tilde{I}_{j-1}$ ($\tilde{I}_0=\emptyset$);
- $\gamma_{i_1,\dots,i_r}$ the path in Δ_R which connects the rays $i_1,\dots,i_r$ as shown in figure III.1.

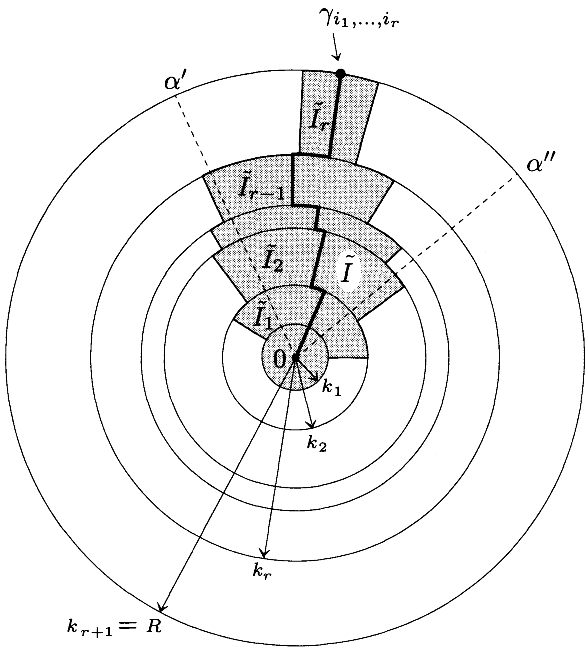

Figure III.1

Now, Definition III.2.11 can be stated:

Definition III.2.12. — A $(k_1,\dots,k_r)$-*sum* of a series $\hat{f}\in\mathbb{C}[[x]]_{1/k_r}$ is a section of the wild analytic sheaf $\widetilde{\mathcal{A}}$ on $\tilde{I}$ or equivalently is the analytic continuation of $\hat{f}$ along the path $\gamma_{i_1,\dots,i_r}$ in the sense of $\widetilde{\mathcal{A}}$.

Let us now achieve the comparison of the sums $S^-_\alpha(\widehat{F})$ and $S^+_\alpha(\widehat{F})$ of a given transformation $\widehat{F} \in \widehat{G}(A_0)$ to its $(k_1,\dots,k_r)$-sums $(\phi_0,\phi_1,\dots,\phi_r)$ on arcs $(I_1,I_2,\dots,I_r)$, precisely to ϕ_r on I_r. Of course, $k_1 < \cdots < k_r$ are no longer arbitrary positive numbers but the levels of $[A_0]$. Viewing a $(k_1,\dots,k_r)$-sum of $\widehat{F}$ as a wild analytic continuation of $\widehat{F}$ (Definition III.2.12) the comparison is given by a wild Cauchy theorem around singular anti-Stokes directions and it is mainly the geometric translation of formulae in Corollary III.2.7.

THEOREM III.2.13. — *When $\gamma_{i_1,\dots,i_r}$ lies in between two successive anti-Stokes directions α' and α'', say α' on the left and α'' on the right, then $\phi_r = S^+_{\alpha'}(\widehat{F})_{|I_r} = S^-_{\alpha''}(\widehat{F})_{|I_r}$.*

Proof. — Consider $\widehat{F}$ in the factored form $\widehat{F} = \widehat{F}_r \cdots \widehat{F}_1$ as given in Theorem III.2.5. For all $j = 1,\dots,r$, the k_j-quasi-function of $\widehat{F}_j$ can be represented by a 0-cochain equal to $S^+_{\alpha'}(\widehat{F}_j) = S^-_{\alpha''}(\widehat{F}_j)$ on I_j. The products of these k_j-quasi-functions in an obvious sense give the $(k_1,\dots,k_r)$-sum of $\widehat{F}$ along $\gamma_{i_1,\dots,i_r}$. Hence $\phi_r = S^+_{\alpha'}(\widehat{F}_r)|_{I_r} \cdots S^+_{\alpha'}(\widehat{F}_1)_{|I_r} = S^+_{\alpha'}(\widehat{F})_{|I_r}$. □

Thus, $(k_1,\dots,k_r)$-sums are preserved up to analytic continuation in the usual sense when moving the path $\gamma_{i_1,\dots,i_r}$ without crossing any anti-Stokes direction. Under the conditions of Theorem III.2.13, one can change $\gamma_{i_1,\dots,i_r}$ into any ray in between α' and α''.

The comparison in the general case results by path composition from the two elementary following cases:

- Let $\alpha' < \alpha < \alpha''$ be three successive anti-Stokes directions and $\gamma^-_{\alpha,j}$, $\gamma^+_{\alpha,j}$ be two paths $\gamma_{i_1,\dots,i_r}$ in between α' and α''. The path $\gamma^-_{\alpha,j}$ is assumed to lie on the left of α but the part i_j lying on the right of α and the path $\gamma^+_{\alpha,j}$ is assumed to lie on the right of α but the part i_j lying on the left of α (cf. Fig. III.2).
- Let $\dot{f} = (\dot{f}_\alpha)_{\alpha\in\mathbb{A}}$ be the Stokes cocycle of $\widehat{F}$ and denote

$$\dot{f}_\alpha = \dot{f}^{k_j}_\alpha \dot{f}^{\neq k_j}_\alpha = \dot{g}^{\neq k_j}_\alpha \dot{g}^{k_j}_\alpha,$$

where $\dot{f}^{\neq k_j}_\alpha$ and $\dot{g}^{\neq k_j}_\alpha$ do not contain the level k_j.

THEOREM III.2.14.

(i) *The $(k_1,\dots,k_r)$-sum of $\widehat{F}$ along the path $\gamma^-_{\alpha,j}$ satisfies:*

$$\phi_r = S^-_\alpha(\widehat{F})\dot{f}^{k_j}_\alpha{}_{|I_r} = S^+_\alpha(\widehat{F})(\dot{f}^{\neq k_j}_\alpha)^{-1}{}_{|I_r}.$$

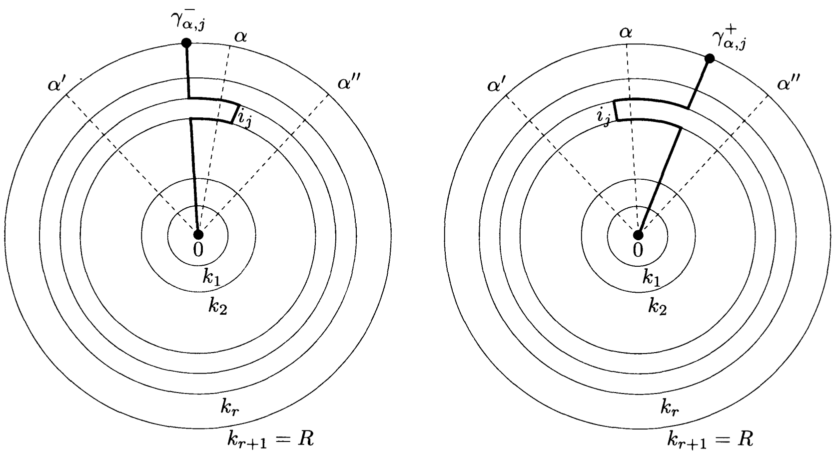

Figure III.2

(ii) *The* $(k_1, \ldots, k_r)$*-sum of* $\widehat{F}$ *along the path* $\gamma^+_{\alpha,j}$ *satisfies*:

$$\phi_r = S^+_\alpha(\widehat{F})(\dot{g}^{k_j}_\alpha)^{-1}{}_{|I_r} = S^-_\alpha(\widehat{F})\dot{g}^{\neq k_j}_\alpha{}_{|I_r}.$$

Proof. — Property (i): let $\widehat{F} = \widehat{F}_r \ldots \widehat{F}_1$ be the decomposition of $\widehat{F}$ in the factorization theorem (Theorem III.2.5). One has

$$\phi_r = S^-_\alpha(\widehat{F}_r \cdots \widehat{F}_{j+1})\, S^+_\alpha(\widehat{F}_j)\, S^-_\alpha(\widehat{F}_{j-1} \cdots \widehat{F}_1)_{|I_r}$$

and from Corollary III.2.7 applied to the transformation $\widehat{F}_j(\widehat{F}_{j-1} \cdots \widehat{F}_1)$ of maximal level k_j,

$$S^+_\alpha(\widehat{F}_j)\, S^-_\alpha(\widehat{F}_{j-1} \cdots \widehat{F}_1) = S^-_\alpha(\widehat{F}_j)\, S^-_\alpha(\widehat{F}_{j-1} \cdots \widehat{F}_1)\dot{f}^{k_j}_\alpha$$

($\dot{f}^{k_j}_\alpha$ is not affected by the commutation with factors of greater levels).

Property (ii) can be proved similarly. □

III.3. Differential Galois theory.

III.3.1. Definitions.

Let $\big(K = \mathbb{C}\{x\}[1/x],\ \partial = \mathrm{d}/\mathrm{d}x\big)$ be the differential field of meromorphic series at 0; it has $\mathbb{C}$ as subfield of constants, that is, of elements y such that $\partial y = 0$.

Let Sol_A denote a n-dimensional $\mathbb{C}$-vector space of solutions of a system $[A]$ which can be either the $\mathbb{C}$-vector space $\mathrm{Sol}_A(U)$ of analytic solutions of $[A]$ on a sector U with vertex 0 in $\mathbb{C}$ or the $\mathbb{C}$-vector space $\widehat{\mathrm{Sol}}_A$ of formal solutions of $[A]$ at 0. In a given space Sol_A of solutions, $Y = [Y_1 \cdots Y_n]$ denotes a fundamental solution, i.e., a matrix, the columns $Y_1, \ldots, Y_n$ of which are n linearly independent solutions of $[A]$ in Sol_A. According to whether $\mathrm{Sol}_A = \widehat{\mathrm{Sol}}_A$ or $\mathrm{Sol}_A = \mathrm{Sol}_A(U)$, the solution Y will be denoted by $\widehat{Y}$ or $Y(U)$.

One calls *differential Galois group* $\mathrm{Gal}_K(A)$ of a system $[A]$ the group of differential K-automorphisms (i.e. field automorphisms leaving the elements of K invariant and commuting with ∂) of any Picard-Vessiot extension of K relative to $[A]$. It is well-known (see [Ko73]) that, in the case when the constant field of (K, ∂) is algebraically closed with characteristic zero, the Picard-Vessiot extension is well-defined up to isomorphism. In particular, one can choose the Picard-Vessiot extension to be $(K\langle Y\rangle, \partial)$ where $K\langle Y\rangle = K(y_{(j,\ell)};\ 1 \leq j, \ell \leq n)$ is the differential field generated over K by the entries of a fixed fundamental solution $Y = [y_{(j,\ell)}]$ either in $\mathrm{Sol}_A = \widehat{\mathrm{Sol}}_A$ or in $\mathrm{Sol}_A = \mathrm{Sol}_A(U)$ for any U, and where $\partial = \mathrm{d}/\mathrm{d}x$ is the usual derivative with respect to x.

By defining the action of an automorphism $\sigma \in \mathrm{Gal}_K(A)$ on vector-matrices $Y = {}^t[y_1 \ldots y_n]$ by $\sigma(Y) = {}^t[\sigma(y_1) \ldots \sigma(y_n)]$ one obviously defines an action on Sol_A and then a representation

$$\rho : \begin{cases} \mathrm{Gal}_K(A) & \longrightarrow \quad \mathrm{GL}_{\mathbb{C}}(\mathrm{Sol}_A) \\ \quad \sigma & \longmapsto \quad \rho(\sigma) : [Y_1 \cdots Y_n] \to [\sigma(Y_1) \cdots \sigma(Y_n)] \end{cases}$$

of the differential Galois group $\mathrm{Gal}_K(A)$. If useful, ρ will be denoted by $\hat{\rho}$ when $\mathrm{Sol}_A = \widehat{\mathrm{Sol}}_A$ and by ρ_U when $\mathrm{Sol}_A = \mathrm{Sol}_A(U)$. These representations are faithful but they are not surjective in general (Proposition III.3.5).

DEFINITION III.3.1. — A $\mathbb{C}$-linear automorphism of Sol_A is said to be *Galoisian* when it belongs to the range of ρ.

Let $\widehat{Y} = \widehat{F} X_0$ be a formal fundamental solution of $[A]$ in which $\widehat{F} \in \mathrm{GL}(n, \mathbb{C}[[x]][1/x])$ and $X_0 = x^L\, \mathrm{e}^{Q(1/x)}$ is a fundamental solution of the normal form $[A_0]$.

The substitution $x \mapsto x\, \mathrm{e}^{2i\pi}$ defines a differential K-automorphism of the Picard-Vessiot extension $K\langle X_0\rangle$ as well as of the Picard-Vessiot extension $K\langle \widehat{Y}\rangle$ (see [Mi91]).

Definition III.3.2. — The *formal monodromy* of $[A]$ is the $\mathbb{C}$-linear isomorphism $\widehat{\mathcal{M}} \in \mathrm{GL}_{\mathbb{C}}(\widehat{\mathrm{Sol}}_A)$ induced by the substitution $x \mapsto x\,\mathrm{e}^{2i\pi}$ in the space $\widehat{\mathrm{Sol}}_A$ of formal solutions. Its matrix, in the basis $\widehat{Y} = \widehat{F}X_0$, is

$$\widehat{M} = X_{0,\tilde{\theta}}^{-1} X_{0,\tilde{\theta}+2\pi} = \mathrm{e}^{-Q(1/x)}\, \mathrm{e}^{2i\pi L}\, \mathrm{e}^{Q(1/x\,\mathrm{e}^{2i\pi})} \in \mathrm{GL}(n, \mathbb{C}).$$

(Here $\tilde{\theta}$ denotes an arbitrary determination of the argument on U.)

Obviously $\widehat{\mathcal{M}}$ is Galoisian ($\widehat{\mathcal{M}} \in \hat{\rho}(\mathrm{Gal}_K(A))$).

Let $p_1, \dots, p_\eta$ be a $\mathbb{Z}$-basis of the lattice generated in $\mathbb{C}[1/t]$ by the polynomials $q_1, \dots, q_n$ of $\mathrm{diag}(Q)$ and set

$$q_j = \sum_{\ell=1}^{\eta} \beta_{j,\ell}\, p_\ell.$$

(Recall $t = x^{1/p}$ where p is the degree of a ramification of Q.)

The substitution $\mathrm{e}^{p_j} \mapsto \lambda_j\, \mathrm{e}^{p_j}$ (where $\lambda_j \in \mathbb{C}^*$) defines a differential K-automorphism τ_{λ_j} of the Picard-Vessiot extension $K\langle X_0\rangle$ as well as of the Picard-Vessiot extension $K\langle \widehat{Y}\rangle$ (see [Mi91]).

Definition III.3.3. — The *exponential torus* $\mathcal{T}$ is the subgroup of $\hat{\rho}(\mathrm{Gal}_K(A)) \subset \mathrm{GL}_{\mathbb{C}}(\widehat{\mathrm{Sol}}_A)$ generated by the $\mathbb{C}$-linear isomorphisms $\hat{\tau}_{\lambda_j} := \hat{\rho}(\tau_{\lambda_j})$, where $\lambda_j \in \mathbb{C}^*$ and $j = 1, \dots, \eta$.

The matrix $T_{\lambda_1,\dots,\lambda_\eta}$ of the $\mathbb{C}$-linear isomorphism $\hat{\tau}_{\lambda_1,\dots,\lambda_\eta} := \hat{\tau}_{\lambda_1} \circ \cdots \circ \hat{\tau}_{\lambda_\eta}$ in the basis $\widehat{Y}$ is the diagonal matrix

$$T_{\lambda_1,\dots,\lambda_\eta} = \mathrm{diag}\big(\lambda_1^{\beta_{1,1}} \cdots \lambda_\eta^{\beta_{1,\eta}}, \dots, \lambda_1^{\beta_{n,1}} \cdots \lambda_\eta^{\beta_{n,\eta}}\big).$$

Let $Y^1 = F^1 X_{0,\tilde{\theta}}$ and $Y^2 = F^2 X_{0,\tilde{\theta}}$ be two realizations of a formal fundamental solution $\widehat{Y} = \widehat{F}X_0$ on a sector U.

Definition III.3.4.

(i) The $\mathbb{C}$-linear isomorphism $Y^1 \mapsto Y^2 = Y^1 C$ in $\mathrm{GL}_{\mathbb{C}}(\mathrm{Sol}_A(U))$ is called a *Stokes automorphism*. Its matrix C in the basis Y^1 is called a *Stokes matrix*.

(ii) When associated to the Stokes cocycle $(\dot{\varphi}_\alpha)_{\alpha \in \mathbb{A}}$ of $\widehat{F}$ a Stokes automorphism and a Stokes matrix are respectively called a *Stokes-Ramis automorphism* and a *Stokes-Ramis matrix*. Precisely, setting

$$Y^1 = S_\alpha^-(\widehat{F})X_{0,\tilde{\alpha}} \quad \text{and} \quad Y^2 = S_\alpha^+(\widehat{F})X_{0,\tilde{\alpha}}$$

(we recall that $\dot{\varphi}_\alpha = S_\alpha^-(\widehat{F})^{-1}S_\alpha^+(\widehat{F})$, cf. Definition III.2.2), the $\mathbb{C}$-linear isomorphism

$$u_\alpha : Y^1 \longmapsto Y^2 = Y^1 C_{\widetilde{\alpha}}$$

in $\mathrm{GL}_{\mathbb{C}}(\mathrm{Sol}_A(\dot{U}_\alpha))$ is a *Stokes-Ramis automorphism at* α and its matrix $C_{\widetilde{\alpha}}$ in the basis Y^1, the *Stokes-Ramis matrix* at α. It is also the matrix of the representation of $\dot{\varphi}_\alpha$ in the sheat $\widetilde{\alpha}$ (cf. Definition I.4.7).

In the next section, we prove that — unlike the formal monodromy and automorphisms of the exponential torus — Stokes automorphisms are not always Galoisian. We prove further that Stokes-Ramis automorphisms are Galoisian.

III.3.2. A non-Galoisian Stokes automorphism.

PROPOSITION III.3.5. — *The representation ρ is not surjective in general. Precisely, there exist non-Galoisian Stokes automorphisms.*

Proof. — Let us consider the companion system

$$[E] \qquad \frac{\mathrm{d}Y}{\mathrm{d}x} = EY \quad \text{where} \quad E = \begin{bmatrix} 0 & 1 \\ x^{-3} & -(x^{-2}+x^{-1}) \end{bmatrix}$$

is associated to Euler's equation $x^2y' + y = x$. We choose:

- a formal fundamental solution $\widehat{Y} = \widehat{F}X_0$ where

$$X_0 = \begin{bmatrix} e^{1/x} & 0 \\ 0 & 1 \end{bmatrix}, \quad \widehat{F} = \begin{bmatrix} -1 & \hat{g} \\ x^{-2} & \hat{g}' \end{bmatrix},$$

$\hat{g}$ denoting the Euler series

$$\hat{g} = \sum_{n\geq 0}(-1)^n n!\, x^{n+1}$$

and $\hat{g}'$ its derivative;

- the sector $U = \{x \in \mathbb{C}; \mathrm{Re}\, x > 0\}$;
- and the realizations $Y^1 = F_0X_0$ and $Y^2 = F_cX_0$ of $\widehat{Y}$ where, for $c \in \mathbb{C}$,

$$F_c = \begin{bmatrix} -1 + cge^{-1/x} & g \\ x^{-2} + cg'e^{-1/x} & g' \end{bmatrix},$$

g denoting a realization of $\hat{g}$ on U.

The matrix $C = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$ is the Stokes matrix of the Stokes automorphism $Y^1 \mapsto Y^2$ in the basis Y^1 and it is non-Galoisian as soon as $c \neq 0$: indeed, assume that C is Galoisian. Denote Y^1 by

$$Y^1 = \begin{bmatrix} y_1 & z_1 \\ y_2 & z_2 \end{bmatrix}$$

and let $\sigma \in \mathrm{Gal}_K(A)$ be the differential K-automorphism of the Picard-Vessiot extension $K(y_1, y_2, z_1, z_2)$ represented by the matrix C in the basis Y^1. The functions $y_1 = -\,\mathrm{e}^{1/x}$ and $y_2 = x^{-2}\,\mathrm{e}^{1/x}$ satisfy the relation $x^2 y_1 + y_2 = 0$ while $\sigma(y_1) = y_1 + cz_1$ and $\sigma(y_2) = y_2 + cz_2$ satisfy $x^2\sigma(y_1) + \sigma(y_2) \neq 0$ as soon as $c \neq 0$. Thus σ is not a homomorphism of algebras, a fortiori of differential fields. This gives the contradiction. □

Remark III.3.6. — The anti-Stokes directions in the Euler system $[E]$ are $\alpha = 0$ and $\alpha = \pi$ and the corresponding Stokes-Ramis matrices are

$$C_0 = I \quad \text{and} \quad C_\pi = \begin{bmatrix} 1 & 2i\pi \\ 0 & 1 \end{bmatrix}.$$

These Stokes-Ramis matrices are not of the previous non-Galoisian type. Actually, they are Galoisian as it will be shown in the next section.

III.3.3. Stokes-Ramis automorphisms are Galoisian.

This section is devoted to the following theorem:

THEOREM III.3.7. — *The Stokes-Ramis automorphisms u_α of a system $[A]$ are Galoisian, i.e., $u_\alpha \in \rho(Gal_K(A))$ for all $\alpha \in \mathbb{A}$.*

The definition we use here of the Stokes-Ramis automorphisms u_α (Definition III.3.4) is an obvious subproduct of our Stokes cocycle (Theorem II.2.1) and the proof we give, uses the Tannakian method suggested by Deligne. Thus, unlike proofs previously done by Ramis (see [Ra85-3], thm 4.2.v, [Ra85-2], thm 1.v) using the factorization theorem (see [Ra85-3], thm 2.1, [Ra85-1], thm 1.1) or by Martinet and Ramis (see [MR90]) using their theory of multisums as injective homomorphisms of differential algebras, there is no need in our proof for asymptotics with Gevrey estimates or for a theory of multisummability.

Although Stokes matrices are transcendental invariants of the system, our arguments are all algebraic but one. This nonalgebraic argument is the main asymptotic existence theorem used in the proofs of the Malgrange-Sibuya isomorphism theorem. Notice however, that the main asymptotic existence theorem requires only ordinary asymptotics in the sense of Poincaré and not asymptotics with Gevrey estimates. It is responsible for the fact that the Stokes cocycles are nonconstructive from the system itself but only from asymptotic solutions or from 1-cocycles.

In the Tannakian method, to each connection are associated its infinitely many constructions and algebraic properties of a connection are expressed as linear properties on the package of its constructions.

Let us recall that a construction is a connection of the form

$$\mathcal{C}(V) = \bigoplus_{\{m,p\}\text{ finite}} \Big[\Big(\bigotimes_K^m V\Big) \otimes_K \Big(\bigotimes_K^p V^*\Big)\Big],$$

$$\mathcal{C}(\nabla) = \bigoplus_{\{m,p\}\text{ finite}} \big(\nabla^{\otimes^m} \otimes (\nabla^*)^{\otimes^p}\big),$$

where (V^*, ∇^*) denotes the dual connection of (V, ∇). Constructions extend canonically to the spaces of solutions ($\mathcal{C}(V)^{\mathrm{sol}} \cong \mathcal{C}(V^{\mathrm{sol}})$) (see [Be85] IV.2.a, lemme fondamental) and to the automorphisms ($\mathcal{C}(u) \in \mathrm{GL}_{\mathbb{C}}(\mathcal{C}(V^{\mathrm{sol}}))$).

For the proof of Theorem III.3.7, we will use the direct part of the following theorem of Chevalley.

Chevalley's Theorem (direct and converse part; see [Be85], IV.2 and [MR89], thm 4.4).

Let (V, ∇) be a meromorphic connection. Let H be a subgroup of $\mathrm{GL}(V^{\mathrm{sol}})$ and $\mathcal{F}_H$ be the family, for all constructions $(\mathcal{C}(V), \mathcal{C}(\nabla))$ of (V, ∇), of the subspaces $\mathcal{W}$ of $\mathcal{C}(V^{\mathrm{sol}})$ which are invariant under H.

The family $\mathcal{F}_H$ characterizes the Zariski closure of H. In particular, $\overline{H} = \rho(\mathrm{Gal}_K(\nabla))$ if and only if $\mathcal{F}_H$ is the family of the subspaces W^{sol} of solutions of all subconnections $(W, \mathcal{C}(\nabla)|_W)$ of the constructions $(\mathcal{C}(V), \mathcal{C}(\nabla))$.

We will also use the two following lemmas:

Lemma III.3.8. — *Let $[A] : \mathrm{d}X/\mathrm{d}x = AX$ be a system with a blocked matrix*

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}.$$

Then $[A]$ has a formal fundamental solution $\widehat{F}x^L\,\mathrm{e}^{Q(1/x)}$ the first columns of which are made of a formal fundamental solution $\widehat{F}_1x^{L_1}\,\mathrm{e}^{Q_1(1/x)}$ of $[A_1] : \mathrm{d}X/\mathrm{d}x = A_1X$ extended by zeroes entries.

Proof. — We have to prove that $[A]$ has a formal fundamental solution $\widehat{F}x^L\,\mathrm{e}^{Q(1/x)}$ in which

$$\widehat{F} = \begin{bmatrix} \widehat{F}_1 & \widehat{F}_2 \\ 0 & \widehat{F}_4 \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} L_1 & L_2 \\ 0 & L_4 \end{bmatrix}$$

are blocked like A with a block of zeroes in the left lower corner.

We claim first that there exists a formal fundamental solution of the form $\Phi\,\mathrm{e}^Q$ where

$$\Phi = \begin{bmatrix} \Phi_1 & \Phi_2 \\ 0 & \Phi_4 \end{bmatrix}$$

is formal-logarithmic with a block of zeroes in the left lower corner: an arbitrary formal fundamental solution in the form

$$\widehat{X} = \begin{bmatrix} \varphi_1 & \varphi_2 \\ \varphi_3 & \varphi_4 \end{bmatrix} \mathrm{e}^Q \quad \text{and} \quad Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_4 \end{bmatrix}$$

satisfies the system

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}(\varphi_1\,\mathrm{e}^{Q_1}) = \cdots, & \dfrac{\mathrm{d}}{\mathrm{d}x}(\varphi_2\,\mathrm{e}^{Q_4}) = \cdots, \\ \dfrac{\mathrm{d}}{\mathrm{d}x}(\varphi_3\,\mathrm{e}^{Q_1}) = A_4\varphi_3\,\mathrm{e}^{Q_1}, & \dfrac{\mathrm{d}}{\mathrm{d}x}(\varphi_4\,\mathrm{e}^{Q_4}) = A_4\varphi_4\,\mathrm{e}^{Q_4}, \end{cases}$$

and one can assume that φ_4 is invertible (permute the columns of $\widehat{X}$ to get a suitable $\widehat{X}\sigma = (\Phi\sigma)\,\mathrm{e}^{\sigma^{-1}Q\sigma}$ if necessary). Then $\varphi_4\,\mathrm{e}^{Q_4}$ is a formal fundamental solution of the system $[A_4] : \mathrm{d}X/\mathrm{d}x = A_4X$ and, since $\varphi_3\,\mathrm{e}^{Q_1}$ also satisfies the system $[A_4]$, there exists a constant matrix C such that $\varphi_3\,\mathrm{e}^{Q_1} = \varphi_4\,\mathrm{e}^{Q_4}C$. The equality $\varphi_4^{-1}\varphi_3 = \mathrm{e}^{Q_4}C\,\mathrm{e}^{-Q_1}$ in which the left-hand side is purely formal-logarithmic and the right-hand side purely exponential implies that both sides are constant. Thus, there exists a constant matrix C' such that $\mathrm{e}^{Q_4}C = C'\,\mathrm{e}^{Q_1}$ and the fundamental solution

$$\begin{aligned} \widehat{X}\begin{bmatrix} I & 0 \\ -C & I \end{bmatrix} &= \begin{bmatrix} \varphi_1\,\mathrm{e}^{Q_1} - \varphi_2\,\mathrm{e}^{Q_4}C & \varphi_2\,\mathrm{e}^{Q_4} \\ 0 & \varphi_4\,\mathrm{e}^{Q_4} \end{bmatrix} \\ &= \begin{bmatrix} (\varphi_1 - \varphi_2C')\,\mathrm{e}^{Q_1} & \varphi_2\,\mathrm{e}^{Q_4} \\ 0 & \varphi_4\,\mathrm{e}^{Q_4} \end{bmatrix} \end{aligned}$$

has the required form if we set $\Phi_1 := \varphi_1 - \varphi_2C'$, $\Phi_2 := \varphi_2$, $\Phi_4 := \varphi_4$.

To end the proof, it suffices to put this matrix Φ in the form $\Phi = \widehat{F}x^L$ with blocks of zeroes both in the left lower corner of $\widehat{F}$ and L: like $\widehat{X} = \Phi\,\mathrm{e}^Q$, the formal monodromy matrix $\widehat{M} = \widehat{X}(x)^{-1}\widehat{X}(x\,\mathrm{e}^{2i\pi})$ has such a block of zeroes. Again, the equality

$$\Phi^{-1}(x)\Phi(x\,\mathrm{e}^{2i\pi}) = \mathrm{e}^{Q(1/x)}\widehat{M}e^{-Q(1/x\,\mathrm{e}^{2i\pi})}$$

implies $\Phi(x\,\mathrm{e}^{2i\pi}) = \Phi(x)\widehat{M}'$ with $\widehat{M}'$ an invertible constant matrix with a block of zeroes in the left lower corner. Choose a logarithm L of $\widehat{M}'$ with a block of zeroes of the same type ($\mathrm{e}^{2i\pi L} = \widehat{M}'$). The matrix $\Phi(x)x^{-L}$ being formal-logarithmic without formal monodromy has formal series entries and, obviously, has the required block of zeroes in the left lower corner. Henceforth the matrix $\widehat{F} = \Phi(x)x^{-L}$ answers the question. □

Lemme III.3.9. — *Let $\widehat{F}x^L\,\mathrm{e}^Q$ be a formal fundamental solution of a system $[A]$ with block of zeroes in the left lower corner as in Lemma III.3.8:*

$$\widehat{F} = \begin{bmatrix} \widehat{F}_1 & \widehat{F}_2 \\ 0 & \widehat{F}_4 \end{bmatrix}, \quad L = \begin{bmatrix} L_1 & L_2 \\ 0 & L_4 \end{bmatrix}.$$

Then, the sums $S_\alpha^-(\widehat{F})$ and $S_\alpha^+(\widehat{F})$ for every anti-Stokes direction α also have the same block of zeroes in the left lower corner.

Proof. — Choose realizations of $\widehat{F}$ with a block of zeroes like $\widehat{F}$. The associated cocycle in $\exp_\mu(\widehat{F})$ has then the same property which is preserved by the algorithm of reduction to the Stokes cocycle (Theorem II.2.1 and Section II.3.4). In this reduction, initial realizations are changed in the required sums with the same block of zeroes. □

Proof of theorem III.3.7. — The system $[A]$ being fixed, let (V,∇) be a connection represented in a suitable K-basis of V by $D = \mathrm{d}/\mathrm{d}x - A$. According to Chevalley's theorem, to prove that a Stokes-Ramis automorphism u_α is Galoisian, we only have to check that the subspaces W^{sol} of subconnections of constructions are all invariant under u_α.

Let (W,∇_W) be a subconnection of (V,∇). In a K-basis of V which completes a K-basis of W, the differential operator D writes

$$D = \frac{\mathrm{d}}{\mathrm{dx}} - B \quad \text{where} \quad B = \begin{bmatrix} B_1 & B_2 \\ 0 & B_4 \end{bmatrix} \quad \text{and} \quad D_{|W} = \frac{\mathrm{d}}{\mathrm{dx}} - B_1.$$

The invariance under u_α of the subspace W^{sol} of V^{sol} is an immediate consequence of Lemmas III.3.8 and III.3.9.

In the case of a subconnection of a construction $(\mathcal{C}(V),\mathcal{C}(\nabla))$, the invariance follows in the same way from the fact that $\mathcal{C}(u_\alpha)$ is also a Stokes-Ramis automorphism. □

Remark III.3.10. — One can apply the criterion of Chevalley to Euler's system $[E]$ to prove that the Stokes automorphisms considered in Section III.3.2 are non-Galoisian: let (V,∇) be a meromorphic connection represented in a K-basis $\underline{e}=(e_1,e_2)$ of V by $D^{\underline{e}}=\mathrm{d}/\mathrm{d}x-E$ and let $W=K\varepsilon_1$ be the K-subspace of V generated by $\varepsilon_1=-e_1+x^{-2}e_2$. As W is invariant under ∇ one can consider the subconnection $(W,\nabla_{|W})$ of (V,∇). In the K-basis $\underline{\varepsilon}=(\varepsilon_1,e_2)$ of V which completes a K-basis of W, the connection (V,∇) is represented by $D^{\underline{\varepsilon}}=\mathrm{d}/\mathrm{d}x-E^{\underline{\varepsilon}}$ where

$$E^{\underline{\varepsilon}}=\begin{bmatrix}-x^{-2} & x^{-1}\\ 0 & x^{-1}\end{bmatrix},$$

the subconnection $(W,\nabla_{|W})$ by $D^{\underline{\varepsilon}}{}_{|W}=\mathrm{d}/\mathrm{d}x+x^{-2}$ and the space W^{sol} of solutions of $(W,\nabla_{|W})$ is the subspace of V^{sol} generated by

$$\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}=\begin{bmatrix}\mathrm{e}^{1/x}\\ 0\end{bmatrix}.$$

Thus, in the K-basis $\underline{e}$, the space W^{sol} is the subspace of V^{sol} generated by

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}-\mathrm{e}^{1/x}\\ x^{-2}\,\mathrm{e}^{1/x}\end{bmatrix}$$

and V^{sol} is generated by the columns of

$$Y^1=\begin{bmatrix}y_1 & g\\ y_2 & g'\end{bmatrix}.$$

Obviously, the Stokes automorphism with matrix $C=\begin{bmatrix}1 & 0\\ c & 1\end{bmatrix}$ in the basis Y^1 of V^{sol} sends the solution $\begin{bmatrix}y_1\\ y_2\end{bmatrix}$ outside W^{sol} as soon as $c\neq 0$. And then, by Chevalley's theorem, such an automorphism is non-Galoisian. □

III.3.4. Differential Galois groups.

In Section III.3.1, we defined the formal monodromy and the automorphisms belonging to the exponential torus as elements in $\mathrm{GL}_{\mathbb{C}}(\widehat{\mathrm{Sol}}_A)$, while Stokes-Ramis automorphisms u_α were defined as elements in $\mathrm{GL}_{\mathbb{C}}(\mathrm{Sol}_A(\dot{U}_\alpha))$.

For a fixed choice of a determination $\widetilde{\alpha}$ of the anti-Stokes direction α, the $\mathbb{C}$-linear isomorphism $\widehat{\mathrm{Sol}}_A\to\mathrm{Sol}_A(\dot{U}_\alpha)$ such that $\widehat{F}X_0\mapsto S_\alpha^-(\widehat{F})X_{0,\widetilde{\alpha}}$, where $X_{0,\tilde{\alpha}}$ has been defined in Section I.4, conjugates the Stokes-Ramis automorphism u_α to a $\mathbb{C}$-linear isomorphism $\hat{u}_{\tilde{\alpha}}$ in $\widehat{\mathrm{Sol}}_A$.

When $\tilde{\alpha}$ runs through the set of all possible determinations of α, the automorphism $\hat{u}_{\tilde{\alpha}}$ runs through its orbit under the action of the formal monodromy (cf. Remarks I.4.9 (1) and Def. III.3.2). In this way, Stokes-Ramis automorphisms are well-defined in $\mathrm{GL}_{\mathbb{C}}(\widehat{\mathrm{Sol}}_A)$ up to conjugacy by the powers of the formal monodromy. The definition of $X_{0,\tilde{\alpha}}$ in Section I.4 has been given with a rigid choice $\mathrm{e}^{Q(1/|x|\,\mathrm{e}^{i\tilde{\alpha}})}$ of a realization of the exponential matrix e^Q. One could have chosen as well $\mathrm{e}^{Q(1/|x|\,\mathrm{e}^{i\tilde{\alpha}})}T$ for T the matrix in the basis $\widehat{Y}$ of a nontrivial automorphism in the exponential torus $\mathcal{T}$ (cf. Remarks I.4.9 and Def. III.3.3). In this way, Stokes-Ramis automorphisms are well-defined up to conjugacy by elements of the group generated in $\mathrm{GL}_{\mathbb{C}}(\widehat{\mathrm{Sol}}_A)$ by the formal monodromy and the exponential torus. Any of these definitions can be used in what follows.

Let us now show how the Tannakian method, as indicated by Deligne, can be used to prove the following theorem of Ramis (see [Ra85-3], thm 4.2.v, [Ra85-2], thm 1.v).

THEOREM III.3.11 (Ramis). — *The representation $\hat{\rho}(\mathrm{Gal}_K(A))$ of the differential Galois group $\mathrm{Gal}_K(A)$ of a system $[A]$ in $\mathrm{GL}_{\mathbb{C}}(\widehat{\mathrm{Sol}}_A)$ is the Zariski closure of the group $\mathcal{H}$ generated by the formal monodromy $\widehat{\mathcal{M}}$, the exponential torus $\mathcal{T}$ and the Stokes-Ramis automorphisms $\hat{u}_{\tilde{\alpha}}$, $\alpha \in \mathbb{A}$.*

We already proved the inclusion $\overline{\mathcal{H}} \subset \hat{\rho}(\mathrm{Gal}_K(A))$ in Theorem III.3.7 by checking the direct part of the criterion of Chevalley on the Stokes-Ramis automorphisms.

We will now prove the converse inclusion $\hat{\rho}(\mathrm{Gal}_K(A)) \subset \overline{\mathcal{H}}$ by checking the converse part of the criterion of Chevalley: briefly, we prove that an invariant subspace $\mathcal{W}$ of the space of solutions $\mathcal{C}(V)^{\mathrm{sol}}$ of a construction $(\mathcal{C}(V), \mathcal{C}(\nabla))$ is the space of solutions of a subconnection of $(\mathcal{C}(V), \mathcal{C}(\nabla))$ by *explicitely* building this subconnection.

A geometric variant of this proof, in the case of a singular regular connection on a compact Riemann surface, can be found in [MR89], theorem 6.8, second proof. This latter proof is based on the so-called Riemann-Hilbert correspondence which asserts the equivalence of Tannakian categories between the category of singular regular meromorphic connections on a compact Riemann surface X and the category of the conjugacy classes of finite dimensional linear representations of $\pi_1(X \setminus S)$ (where S is the subset of singular points in X). With this equivalence, to an invariant subspace in a representation of the π_1, corresponds *automatically* a subconnection. Such a proof can be extended to the general case of an irregular singular meromorphic connection by substituting the wild π_1 to the ordinary one.

To prove Theorem III.3.11 we will use the following lemma and, from now on, without further mention, the notation $\begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}$ for the block decomposition of a $n \times n$ matrix U in which U_1 is a s-dimensional square block.

LEMMA III.3.12. — *Let $X_0 = x^L \mathrm{e}^Q$ be a normal solution of a normal form $[A_0]$ and $X_0' = X_0 P$ where the matrix $P \in \mathrm{GL}(n, \mathbb{C})$ has a block decomposition of the form $\begin{bmatrix} I & 0 \\ P_3 & I \end{bmatrix}$. Let $\widehat{\mathcal{W}}_0$ be the subspace of $\widehat{\mathrm{Sol}}_{A_0}$ generated by the s first columns of X_0'.*

(i) If $\widehat{\mathcal{W}}_0$ is invariant under all automorphism in the exponential torus $\mathcal{T}$ then P and e^Q commute.

(ii) If, moreover, $\widehat{\mathcal{W}}_0$ is invariant under the formal monodromy $\widehat{\mathcal{M}}$ then $\mathrm{e}^{2i\pi P^{-1}LP}$ has a $(n-s) \times s$ block of zeroes in the left lower corner.

Proof. — Assertion (i). Let $\hat{\tau} \in \mathcal{T}$.

Let T be the matrix of $\hat{\tau}$ in the $\mathbb{C}$-basis X_0 of $\widehat{\mathrm{Sol}}_{A_0}$ and $T' = P^{-1}TP$ its matrix in the $\mathbb{C}$-basis X_0'. The matrix $T = \begin{bmatrix} T_1 & 0 \\ 0 & T_4 \end{bmatrix}$ is diagonal and $T' = \begin{bmatrix} T_1 & 0 \\ -P_3T_1 + T_4P_3 & T_4 \end{bmatrix}$. Thus $\hat{\tau}$ leaves $\widehat{\mathcal{W}}_0$ invariant if and only if $-P_3T_1 + T_4P_3 = 0$. The matrix $Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_4 \end{bmatrix}$ is diagonal and

$$P^{-1}\,\mathrm{e}^{Q(1/x)}P = \begin{bmatrix} \mathrm{e}^{Q_1(1/x)} & 0 \\ -P_3\,\mathrm{e}^{Q_1(1/x)} + \mathrm{e}^{Q_4(1/x)}P_3 & \mathrm{e}^{Q_4(1/x)} \end{bmatrix}.$$

If $-P_3T_1 + T_4P_3 = 0$ for all $\hat{\tau} \in \mathcal{T}$ then $-P_3\,\mathrm{e}^{Q_1(1/x)} + \mathrm{e}^{Q_4(1/x)}P_3 = 0$ for all $x \neq 0$. Hence the identity $P^{-1}\,\mathrm{e}^Q P = \mathrm{e}^Q$ holds.

Assertion (ii). The formal monodromy $\widehat{\mathcal{M}}$ has

$$\widehat{M} = \mathrm{e}^{-Q(1/x)}\,\mathrm{e}^{2i\pi L}\,\mathrm{e}^{Q(1/x\,\mathrm{e}^{2i\pi})}$$

as matrix in the basis X_0. Since P and $\mathrm{e}^{Q(1/x)}$ commute for all x, its matrix $\widehat{M}' = P^{-1}\widehat{M}P$ in the basis X_0' takes the form

$$\widehat{M}' = \mathrm{e}^{-Q(1/x)}\,\mathrm{e}^{2i\pi P^{-1}LP}\,\mathrm{e}^{Q(1/x\,\mathrm{e}^{2i\pi})}.$$

The hypothesis that $\widehat{\mathcal{M}}$ leaves $\widehat{\mathcal{W}}_0$ invariant means that $\widehat{M}'$, and then also $\mathrm{e}^{2i\pi P^{-1}LP}$, has a $(n-s)\times s$ block of zeroes in the left lower corner. □

Proof of theorem III.3.11. — As already mentioned, the inclusion $\overline{\mathcal{H}} \subset \hat{\rho}(\mathrm{Gal}_K(A))$ results from Theorem III.3.7. To prove the converse inclusion $\hat{\rho}(\mathrm{Gal}_K(A)) \subset \overline{\mathcal{H}}$ we check that the converse part of the criterion of Chevalley is satisfied: let $(\mathcal{C}(V), \mathcal{C}(\nabla))$ be a meromorphic connection represented in a K-basis $\underline{e}$ of $\mathcal{C}(V)$ by $\nabla^{\underline{e}} = \mathrm{d}/\mathrm{d}x - A$ and let $\widehat{\mathcal{W}}$ be a s-dimensional $\mathbb{C}$-subspace of $\widehat{\mathrm{Sol}}_A$ which is invariant under $\mathcal{H}$. We only have to prove that $\widehat{\mathcal{W}}$ is the space of formal solutions of a subconnection of $(\mathcal{C}(V), \mathcal{C}(\nabla))$.

We denote by:

- $\widehat{X} = \widehat{F}x^L\,\mathrm{e}^Q$ a formal fundamental solution of the system $[A] : \mathrm{d}X/\mathrm{d}x = AX$;
- $[A_0] : \mathrm{d}X/\mathrm{d}x = A_0X$ the normal form of $[A]$ with fundamental solution $X_0 = x^L\,\mathrm{e}^Q$ (thus $\widehat{X}$ is a $\mathbb{C}$-basis of $\widehat{\mathrm{Sol}}_A$, X_0 a $\mathbb{C}$-basis of $\widehat{\mathrm{Sol}}_{A_0}$);
- $(\mathcal{C}(V), \nabla_0)$ the connection represented in the K-basis $\underline{e}$ of $\mathcal{C}(V)$ by $\nabla_0^{\underline{e}} = \mathrm{d}/\mathrm{d}x - A_0$;
- $\mathcal{H}_0 \subset \mathrm{GL}(\widehat{\mathrm{Sol}}_{A_0})$ the group generated by the formal monodromy and the exponential torus of $[A_0]$.

The map $\widehat{F}. : \widehat{\mathrm{Sol}}_{A_0} \to \widehat{\mathrm{Sol}}_A$ such that $Y \mapsto \widehat{F}Y$ is a $\mathbb{C}$-linear isomorphism and we denote by $\widehat{\mathcal{W}}_0$ the inverse image of $\widehat{\mathcal{W}}$. This map induces a $\mathbb{C}$-linear isomorphism $\mathrm{GL}(\widehat{\mathrm{Sol}}_{A_0}) \to \mathrm{GL}(\widehat{\mathrm{Sol}}_A)$, $\varphi \mapsto \widehat{F}\varphi\widehat{F}^{-1}$. With this isomorphism, we can identify the formal monodromy and the exponential torus of $[A_0]$ in $\mathrm{GL}(\widehat{\mathrm{Sol}}_{A_0})$ to the formal monodromy and the exponential torus of $[A]$ in $\mathrm{GL}(\widehat{\mathrm{Sol}}_A)$ and we can thus identify $\mathcal{H}_0$ to a subgroup of $\mathcal{H}$. Obviously $\widehat{\mathcal{W}}_0$ is invariant under $\mathcal{H}_0$.

We first claim that, with the invariance of $\widehat{\mathcal{W}}_0$ under $\mathcal{H}_0$, there exists a subconnection $(W, \nabla_{0|W})$ of the normal form $(\mathcal{C}(V), \nabla_0)$, the space of formal solutions of which is $\widehat{\mathcal{W}}_0$:

a permutation σ on the columns of X_0 changes X_0 in $X_0\sigma = \sigma x^{\sigma^{-1}L\sigma}\,\mathrm{e}^{\sigma^{-1}Q\sigma}$. The same permutation $\underline{e}\sigma$ on the K-basis $\underline{e}$ of V changes $X_0\sigma$ in $\sigma^{-1}X_0\sigma = x^{\sigma^{-1}L\sigma}\,\mathrm{e}^{\sigma^{-1}Q\sigma}$. Up to such permutations, we can thus assume that $\widehat{\mathcal{W}}_0$ and the $(n-s)$ last columns of X_0 generate $\widehat{\mathrm{Sol}}_{A_0}$. There exists then a matrix $P = \begin{bmatrix} I & 0 \\ P_3 & I \end{bmatrix}$ such that the s first columns of $X_0' = X_0P$ generate $\widehat{\mathcal{W}}_0$.

From Lemma III.3.12, we know that $X'_0 = Px^{P^{-1}LP}\,\mathrm{e}^Q$ and that $\mathrm{e}^{2i\pi P^{-1}LP}$ has a block of zeroes in the left lower corner. Let $2i\pi L'$ be a logarithm of $\mathrm{e}^{2i\pi P^{-1}LP}$ with a block of zeroes in the left lower corner. The matrix $Px^{P^{-1}LP}x^{-L'}$ is both an analytic function on the Riemann surface of $\log x$ and a formal-logarithmic series without monodromy. It is then a meromorphic function f and we get $X'_0 = fx^{L'}\,\mathrm{e}^Q$ where L' has a block of zeroes in the left lower corner. In the K-basis $\underline{e}' = \underline{e}f$, X'_0 writes $X''_0 = x^{L'}\,\mathrm{e}^Q$ and the new matrix B_0 of the normal form, which writes now $B_0 = (\mathrm{d}X''_0/\mathrm{d}x)X''^{-1}_0$, has the required block of zeroes:

$$B_0 = \begin{bmatrix} B_{0,1} & B_{0,2} \\ 0 & B_{0,4} \end{bmatrix}.$$

Hence, the space W generated by the s first elements of the K-basis $\underline{e}'$ satisfies the claim.

Next, we claim that, if $\widehat{\mathcal{W}}$ is moreover invariant under the Stokes-Ramis automorphisms $\hat{u}_{\tilde{\alpha}}$, then the restriction to W of the connection $(\mathcal{C}(V), \mathcal{C}(\nabla))$ itself makes sense and has $\widehat{\mathcal{W}}$ as space of formal solutions:

with the previous changes of basis, the $\mathbb{C}$-basis $\widehat{X}$ of $\widehat{\mathrm{Sol}}_A$ becomes $\widehat{X}' = \widehat{F}'X''_0$ and $\widehat{\mathcal{W}}$ is generated by the s first columns of $\widehat{X}'$.

Let $(\dot{\varphi}_\alpha)_{\alpha\in\mathbb{A}}$ be the Stokes cocycle of $\widehat{X}'$. One has $\dot{\varphi}_\alpha = X''_{0,\tilde{\alpha}} C_{\tilde{\alpha}} X''^{-1}_{0,\tilde{\alpha}}$ where $C_{\tilde{\alpha}}$ is the matrix of $\hat{u}_{\tilde{\alpha}}$ in the basis $\widehat{X}'$ (cf. definition III.3.4). Due to the invariance of $\widehat{\mathcal{W}}$ under the $\hat{u}_{\tilde{\alpha}}$'s, the matrix $C_{\tilde{\alpha}}$ has a $s\times(n-s)$ block of zeroes in the left lower corner. And since this property holds for X''_0 it holds also for $\dot{\varphi}_\alpha$. The block $\dot{\varphi}_{\alpha,1}$ in the left upper corner of $\dot{\varphi}_\alpha$ is then the Stokes cocycle relative to $\widehat{X}'_1 = \widehat{F}'_1X''_{0,1}$, the block in the left upper block of $\widehat{X}'$. In other words, one can restrict the connection $\mathcal{C}(\nabla)$ to the space W generated by the s first elements of the K-basis $\underline{e}'$. Moreover, this connection $\nabla_1 = \mathcal{C}(\nabla)_{|W}$ has $\widehat{X}'_1$ as formal fundamental solution i.e. $\widehat{\mathcal{W}} = \widehat{\mathrm{Sol}}_{\nabla_1}$. This ends the proof. □

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