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CONSTRUCTION OF A CERTAIN SUPERHARMONIC MAJORANT

by Paul KOOSIS

1. Introduction.

Let $W(x) \geqslant 1$ be continuous for $-\infty < x < \infty$, and suppose that

$$(1) \qquad \int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\, dx \ < \ \infty.$$

In that event, we can form the functions

$$(2) \qquad F_a(z) \ = \ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|\Im z| \log W(t)}{|z-t|^2}\, dt - a|\Im z|, \quad z \in \mathbb{C},$$

and it is known that as long as $W(x)$ enjoys some mild regularity on $\mathbb{R}$, the existence, for $a > 0$, of a *finite superharmonic majorant* of $F_a(z)$ (on $\mathbb{C}$) is *equivalent* to that of a non-zero *entire function* $\varphi(z)$ *of exponential type* a making $\varphi(x)W(x)$ *bounded* on $\mathbb{R}$ (see [1] and [2], Ch. XI, §B.3).

It was shown by Beurling and Malliavin in 1961 (see [3]) that when either

i) $W(x) = |\Phi(x)|$ with $\Phi(z)$ an entire function of exponential type

or

ii) $\log W(x)$ is uniformly Lip 1 on $\mathbb{R}$,

$W(x)$ already has enough regularity for the equivalence just spoken of to obtain, and that then (1) is indeed enough to guarantee existence, for each

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$a > 0$, of an entire function $\varphi(z)$ having the enumerated properties. This statement (whether for case i) or for case ii)) is usually referred to as the *Beurling-Malliavin multiplier theorem*, and the entire functions $\varphi(z)$ whose existence it asserts are called *multipliers* (for $W(x)$).

The two forms of the multiplier theorem are actually equivalent. It was shown in [4] that the version for case ii) can be deduced from the one for case i) (see also [2], Ch. X §C.1 and [5]). Recently, the reverse implication has been noted. One has, in fact, the following result ([5]) :

If $\Phi(z)$ is an entire function of exponential type with $|\Phi(x)| \geqslant 1$ on $\mathbb{R}$ and

$$\int_{-\infty}^{\infty} \frac{\log|\Phi(x)|}{1+x^2}dx \ < \ \infty$$

there is, for any $\ell > 0$, an $f \geqslant 0$ defined on $\mathbb{R}$ with

$$\int_{-\infty}^{\infty} (f(x)/(1+x^2))dx \ < \ \infty,$$

$|f(x') - f(x)| \leqslant \ell|x' - x|$ *on $\mathbb{R}$ and $f(x) \geqslant \log|\Phi(x)|$ there.*

Since the volume containing [5] has not yet appeared (and may be hard to locate when it does), a proof of this statement is sketched in the appendix to the present paper.

Either version of the Beurling-Malliavin multiplier theorem can thus be obtained from the other, and a weakened form of it indeed implies both versions. The result just quoted shows that it is already enough to know that for any $W(x) \geqslant 1$ satisfying (1) and the relation $|\log W(x') - \log W(x)| \leqslant \ell|x' - x|$ on $\mathbb{R}$ we have a multiplier $\varphi(z)$ of exponential type $A\ell$, with A a numerical constant independent of W. If that is true, we can infer the version for case i), and from the latter the one for case ii) will follow.

As said above, one can ensure existence of a multiplier $\varphi(z)$ of exponential type $A\ell$ by verifying that $F_a(z)$, given by (2), has a finite superharmonic majorant when $a = A\ell$. One way of doing that is to use harmonic estimation (as in [1] and in [2], Ch. XI §§C.1-2), but it seems worthwhile to instead proceed by *exhibiting* such a superharmonic majorant. That is what we do in this paper.

I believe that the procedure followed below has some advantages :

1. It is more constructive than the previous ones. The superharmonic majorant in question is arrived at by solving a sequence of explicit functional relations, and from it a multiplier can be obtained in fairly straightforward fashion.

2. The proof of the multiplier theorem furnished by the following development makes no use of a certain quadratic functional (the *energy*) playing an essential rôle in all the earlier proofs (in [3], [4], [1], [2] and [5]) despite its apparent irrelevance for the matter under consideration. (The proof in [5] is based on a result about polynomials established with the help of that same functional in [6] and in [7], Ch. VIII §B.)

3. A good part of the construction given below can most likely be carried out in $\mathbb{R}^n \times \mathbb{R}$ as well as in $\mathbb{C}$, after replacing Lemma 2.4 by a suitable substitute based on a theorem of Sjögren ([15], [16]). This may well be useful for the study of Lipschitz domains.

Lemma 2.4 is actually the basis for our construction. It is a quantitative version of a theorem given by Beurling and Malliavin in 1967 and used by them in [9] to study the distribution of the real zeros of an entire function $\Phi(z)$ of exponential type with $\int_{-\infty}^{\infty}(\log^+|\Phi(x)|/(1+x^2))dx < \infty$. (See also [2], Ch.IX §E.2; the result was in fact almost surely known to Beurling in 1965, see [8].) Our proof of the lemma is like that of the original result published in 1967, and is a beautiful application of the Ahlfors-Carleman estimate for harmonic measure.

The theorem from [5] quoted above provided the real motivation for the following work, and it was also proved with the lemma's help. The latter may therefore be looked on as the workhorse for this whole subject.

I am grateful to L. Carleson for having pointed out to me a mistake in equation (58) and the computation leading to it in an earlier version of this paper, also to D. Drasin and the referee for having noticed some other errors.

2. Construction of a majorant on $\mathbb{R}$.

We start with a measurable function $f(t) \geqslant 0$ defined on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \frac{f(t)}{1+t^2}\, dt \;<\; \infty\,; \tag{3}$$

the generality of the following work is not affected if we also require that

$$f(t) \geqslant \eta > 0, \quad t \in \mathbb{R}, \tag{4}$$

for some constant η, and that property we henceforth assume.

NOTATION. — *For $y > 0$ we write*

$$J(x,y) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)dt}{(x-t)^2+y^2}\,. \tag{5}$$

$J(x,y)$ is just the ordinary Poisson integral of f for the upper half plane divided by y. It is clear from (3) and (4) that $J(x,y)$, for each real x, is a *strictly decreasing* function of $y > 0$, tending to ∞ when $y \to 0$ and to zero when $y \to \infty$.

DEFINITION. — *Given $\mu > 0$ and $x \in \mathbb{R}$, $Y_\mu(f,x)$ is the unique $y > 0$ for which $J(x,y) = \mu$.*

Our construction will make systematic use of the function $Y_\mu(f,x)$. The behaviour of this object is at first sight not very transparent; it certainly does not depend linearly or in convex fashion on f. However, the following properties are obvious :

$$Y_\mu(f,x) \leqslant Y_\mu(g,x) \quad \text{if} \quad f(t) \leqslant g(t) \text{ on } \mathbb{R}, \tag{6}$$

$$Y_\mu(f,x) < Y_\lambda(f,x) \quad \text{if} \quad \lambda < \mu. \tag{7}$$

We have in fact

2.1. LEMMA. — $Y_\lambda(f,x) \geqslant \sqrt{\mu/\lambda}\,Y_\mu(f,x)$ *for* $0 < \lambda < \mu$.

Proof. — Take $y = (\mu/\lambda)^{1/2}Y_\mu(f,x)$ in (5) with $0 < \lambda < \mu$; the value of $J(x,y)$ thus obtained will be $> \lambda$ according to the definition of $Y_\mu(f,x)$. So y must be made yet larger in order to bring $J(x,y)$ down to λ.

2.2. LEMMA. — *If* $|x'-x| \leqslant Y_\mu(f,x)$, $\quad Y_{\mu/3}(f,x') > Y_\mu(f,x)$.

Proof. — For $y > 0$ we have, by (5),

$$J(x',y) \;\geqslant\; \inf_{t\in\mathbb{R}} \frac{(x-t)^2+y^2}{(x'-t)^2+y^2}\cdot J(x,y).$$

In terms of $\xi = (x' - x)/y$ and the variable $\tau = (x' - t)/y$, the infimum on the right is

$$\inf_{\tau \in \mathbb{R}} \left| 1 - \frac{\xi}{\tau - i} \right|^2 = \left(\frac{|\xi + 2i| - |\xi|}{2} \right)^2 = \left(\frac{2}{|\xi + 2i| + |\xi|} \right)^2$$

(see [7], pp. 152-153). Assuming, then, that $|\xi| = |x' - x|/y \leqslant 1$, we have $J(x', y) > \frac{1}{3} J(x, y)$, so if $y = Y_\mu(f, x)$, $J(x', y) > \mu/3$, and $Y_{\mu/3}(f, x')$ must be $> Y_\mu(f, x)$.

2.3. Corollary. — *If* $\frac{1}{\pi} \int_{-\infty}^{\infty} (f(t)/(1 + t^2))dt \leqslant 1$, *we have* $Y_\mu(f, |x|) < \max(1, |x|)$ *for* $\mu \geqslant 3$.

Proof. — The condition on f makes $Y_1(f, 0) \leqslant 1$. If $Y_3(f, x) \geqslant |x|$ we have, putting $x' = 0$ in the last lemma,

$$Y_3(f, x) < Y_1(f, 0) \leqslant 1.$$

Otherwise $Y_3(f, x) \leqslant |x|$, so the corollary holds when $\mu = 3$ and hence for $\mu > 3$ by (7).

As stated in the introduction, the real basis for our work is the following result :

2.4. Lemma. — *When* $\frac{1}{\pi} \int_{-\infty}^{\infty} (f(t)/(1 + t^2))dt \leqslant 1$, *we have*

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{Y_\mu(f, x)}{1 + x^2} dx \leqslant \left(\frac{1}{2} + 2 \log \frac{12}{\pi} \right) \sqrt{\frac{3}{\mu}}$$

for $\mu \geqslant 3$.

Proof. — Consider first the case $\mu = 3$; the argument there is essentially that of [9].

In the domain $\mathcal{O} = \{z = x + iy;\ y > Y_3(f, x)\}$ the function

$$u(z) = 3y - yJ(x, y)$$

is harmonic and > 0; it is also zero on $\partial\mathcal{O}$ by definition of $Y_3(f, x)$. Our condition on f implies by (7) that $Y_3(f, 0) < Y_1(f, 0) \leqslant 1$, so $i \in \mathcal{O}$, and each circle $|z| = r > 1$ has on it an arc in $\mathcal{O}$ intersecting the positive imaginary axis. For such r, we denote by $\varphi(r)$ the *largest* angle $\vartheta < \pi/2$

such that $re^{i\vartheta}$ falls on the curve $y = Y_3(f,x)$, and by $\pi - \psi(r)$ the *smallest* angle $\vartheta > \pi/2$ for which $re^{i\vartheta}$ has that property. Then the arc

$$\Gamma_r = \{re^{i\vartheta}; \varphi(r) \leqslant \vartheta \leqslant \pi - \psi(r)\}$$

lies entirely in $\mathcal{O}$ except for its two endpoints. Given $R > 1$, we denote by $\mathcal{O}_R$ the set of $z \in \mathcal{O}$ lying *below* the arc Γ_R; $\partial\mathcal{O}_R$ consists of Γ_R and part of the curve $y = Y_3(f,x)$.

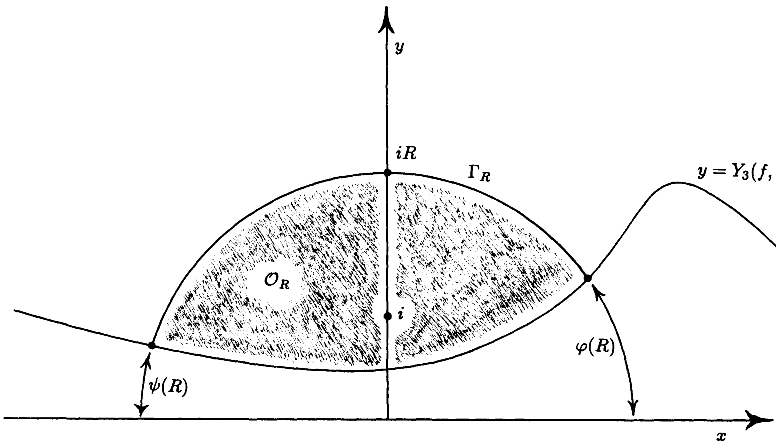

Fig. 1.

We have $u(z) \leqslant 3R$ on Γ_R, and $u(z) = 0$ on the rest of $\partial\mathcal{O}_R$. Therefore, since $u(i) = 3 - J(0,1) \geqslant 2$, we have

$$2 \leqslant 3R\,\omega_R(\Gamma_R, i), \tag{8}$$

where $\omega_R(\quad,\quad)$ denotes harmonic measure for $\mathcal{O}_R$.

For this harmonic measure, we can use the Ahlfors-Carleman estimate (see [10], p. 102). Writing $r\theta(r)$ for the *length* of the arc Γ_r, we can express the estimate thus :

$$\omega_R(\Gamma_R, i) \leqslant \left(\frac{8}{\pi} + o(1)\right) \exp\left(-\pi \int_1^R \frac{dr}{r\theta(r)}\right). \tag{9}$$

Here, $o(1)$ denotes a quantity tending to zero when the exponential does. The formula given in [10] has the coefficient 4 on the right; for its refinement to $(8/\pi) + o(1)$ see problem 32 on p. 101 of [2].

We have $\theta(r) = \pi - \varphi(r) - \psi(r)$, so

$$\frac{1}{\theta(r)} \geqslant \frac{1}{\pi} + \frac{\varphi(r)+\psi(r)}{\pi^2},$$

and thence, by (8) and (9),

$$2 \leqslant \left(\frac{8}{\pi} + o(1)\right) \cdot 3R \exp\left(-\int_1^R \left(1 + \frac{\varphi(r)+\psi(r)}{\pi}\right)\frac{dr}{r}\right).$$

Making $R \longrightarrow \infty$, this yields

$$\frac{1}{\pi}\int_1^\infty \frac{\varphi(r)+\psi(r)}{r}\,dr \leqslant \log\frac{12}{\pi}.$$

The integral on the left is just an expression in polar coordinates of

$$\frac{1}{\pi}\int\int_{\Omega\cap\{|z|>1\}} \frac{dx\,dy}{x^2+y^2},$$

where Ω is a subset of the *upper half plane including* (perhaps properly) the *complement of $\mathcal{O}$ therein*. We thus surely have

$$\frac{1}{\pi}\int_{|x|>1}\int_0^{Y_3(f,x)} \frac{dx\,dy}{x^2+y^2} < \log\frac{12}{\pi}.$$

By Corollary 2.3, $Y_3(f,x) < |x|$ for $|x| > 1$, so the denominator in the last integral is $< 2x^2$, and we get

$$\frac{1}{\pi}\int_{|x|>1} \frac{Y_3(f,x)}{x^2}\,dx < 2\log\frac{12}{\pi}.$$

Again by Corollary 2.3, $Y_3(f,x) < 1$ for $|x| < 1$, so

$$\frac{1}{\pi}\int_{-1}^1 \frac{Y_3(f,x)}{1+x^2}\,dx < \frac{1}{2}.$$

With the previous, this yields

$$\frac{1}{\pi}\int_{-\infty}^\infty \frac{Y_3(f,x)}{1+x^2}\,dx < \frac{1}{2} + 2\log\frac{12}{\pi},$$

and the lemma is proved for $\mu = 3$.

When $\mu > 3$, we have $Y_\mu(f,x) \leqslant \left(\frac{3}{\mu}\right)^{1/2} Y_3(f,x)$ by Lemma 2.1, and substitution of this relation into the preceding one yields our conclusion. We are done.

Remark 1. — With $f(t)$ known only to satisfy (3) and (4), we still have $\int_{-\infty}^{\infty}(Y_\mu(f,x)/(1+x^2))dx < \infty$, and this for *every* $\mu > 0$. Proof of this statement is by an argument similar to the one just made, with i replaced by $iY_{3\mu}(f,0)$ (which is at least *finite*). At the end, an appropriate substitute for Corollary 2.3 (based again on Lemma 2.2) is used.

Remark 2. — We record the numerical estimate

$$\frac{1}{2} + 2\log\frac{12}{\pi} \ < \ 3.2.$$

The functions $f(t)$ considered in this paper satisfy a uniform Lipschitz condition on $\mathbb{R}$ as well as (3) and (4). That implies some additional properties for the $Y_\mu(f,x)$.

2.5. Lemma. —*If $f(t)$ satisfies* (3), (4) *and*

$$|f(t') - f(t)| \ \leqslant \ |t' - t| \quad \text{for } t', t \in \mathbb{R},$$

we have $Y_\mu(f,x) \geqslant f(x)/4\mu$ for $\mu \geqslant 1/2$.

Proof. — The Lipschitz property makes $f(t) \geqslant f(x)/2$ for $|t-x| \leqslant f(x)/2$, and thence, by (5),

$$J(x,y) \ \geqslant \ \frac{f(x)}{2\pi}\int_{-f(x)/2}^{f(x)/2}\frac{d\tau}{\tau^2+y^2} \ = \ \frac{f(x)}{\pi y}\arctan\left(\frac{f(x)}{2y}\right).$$

If $\mu \geqslant 1/2$ we find, putting $y = f(x)/4\mu$ ($\leqslant f(x)/2$), that $(f(x)/\pi y)\arctan(f(x)/2y) \geqslant \mu$, so we must make $y \geqslant f(x)/4\mu$ in order to bring $J(x,y)$ down to μ. The lemma follows by definition of $Y_\mu(f,x)$.

Definition. — *$\tilde{Y}_\mu(f,x)$ is the* largest minorant *of $Y_\mu(f,x)$ having the property that*

$$|\tilde{Y}_\mu(f,x') - \tilde{Y}_\mu(f,x)| \ \leqslant \ \frac{1}{2}|x'-x| \quad \text{(sic!)}$$

for $x', x \in \mathbb{R}$.

2.6. Lemma. — *If, for f, satisfying* (3) *and* (4), *we have*

$$|f(t') - f(t)| \ \leqslant \ |t' - t| \quad \text{for } t', t \in \mathbb{R},$$

then $\tilde{Y}_\mu(f,x) \geqslant Y_{\mu+2}(f,x)$.

Proof. — Fix any x_0. In view of (7), we need only consider the case where $\tilde{Y}_\mu(f, x_0) \leqslant Y_\mu(f, x_0)$ is in fact $< Y_\mu(f, x_0)$. Then the point $(x_0, \tilde{Y}_\mu(f, x_0))$ lies on a *secant* of slope $1/2$ or $-1/2$ joining two points of the curve $y = Y_\mu(f, x)$ and lying *below* that curve. (The possibility, *conceivable*, of the secant's degeneration to a semi-infinite *ray* starting from a point on the curve cannot occur here on account of Remark 1 to Lemma 2.4.) Consider the case where the secant has slope $1/2$; treatment of the other is similar.

We have, then, $a < x_0 < b$ with $Y_\mu(f, b) = Y_\mu(f, a) + \frac{1}{2}(b - a)$, and

$$\tilde{Y}_\mu(f, x_0) \;=\; Y_\mu(f, a) + \frac{1}{2}(x_0 - a).$$

Taking $y = \tilde{Y}_\mu(f, x_0)$, we get, by (7) and (5), $J(a, y) \leqslant \mu$, i.e.,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\tau + a)}{\tau^2 + y^2}\, d\tau \;\leqslant\; \mu.$$

Thence,

$$J(x_0, y) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\tau + x_0)}{\tau^2 + y^2}\, d\tau \;\leqslant\; \mu + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\tau + x_0) - f(\tau + a)}{\tau^2 + y^2}\, d\tau.$$

By hypothesis, the last integral is $\leqslant ((x_0 - a)/\pi) \int_{-\infty}^{\infty} (\tau^2 + y^2)^{-1} d\tau = (x_0 - a)/y$. But $y \geqslant \frac{1}{2}(x_0 - a)$, so finally $J(x_0, y) \leqslant \mu + 2$ for $y = \tilde{Y}_\mu(f, x_0)$, making $\tilde{Y}_\mu(f, x_0) \geqslant Y_{\mu+2}(f, x_0)$ by definition. Done.

We proceed to our construction, starting with a function $F(t)$ satisfying (4),

$$(10) \qquad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{F(t)}{1 + t^2}\, dt \;\leqslant\; \frac{1}{2} \quad \text{(sic!)}$$

and

$$(11) \qquad |F(t') - F(t)| \;\leqslant\; \frac{1}{2}|t' - t| \quad \text{for } t', t \in \mathbb{R}.$$

Choose now and fix once and for all the value

$$(12) \qquad \mu = 123.$$

According to Lemma 2.4 and Remark 2 to it, *we will then have*

$$(13) \qquad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{Y_\mu(f, x)}{1 + x^2}\, dx \;<\; \frac{1}{2}$$

for any f satisfying (4) *with*

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{1+t^2}\,dt \leqslant 1.$$

Let us put successively

$$\begin{aligned} &f_0(t) = F(t), \quad Y_0(x) = Y_\mu(f_0, x);\\ &f_1(t) = F(t) + Y_0(t), \quad Y_1(x) = Y_\mu(f_1, x);\\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ &f_n(t) = F(t) + Y_{n-1}(t), \quad Y_n(x) = Y_\mu(f_n, x);\\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{aligned}$$

It is of course necessary to verify that the functions $Y_n(x)$ exist; that will follow if we show by induction that the f_n satisfy (3). We in fact have

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{Y_n(x)}{1+x^2}\,dx < \frac{1}{2} \tag{14}$$

for each n. For $n = 0$ this follows from (10) on putting $f = f_0 = F$ in (13). But then (3) holds for $f = f_1 = F + Y_0$, and indeed we have $(1/\pi)\int_{-\infty}^{\infty}(f_1(t)/(1+t^2))dt < 1$. Therefore $Y_1(x) = Y_\mu(f_1, x)$ exists and (13) implies (14) with $n = 1$. This reasoning may be repeated indefinitely.

The sequence of functions $Y_n(x)$ is increasing. Indeed, $f_1(t) \geqslant f_0(t)$, so $Y_\mu(f_1, x) \geqslant Y_\mu(f_0, x)$ by (6). In other words, $Y_1(x) \geqslant Y_0(x)$, but then $f_2(x) \geqslant f_1(x)$, so again $Y_\mu(f_2, x) \geqslant Y_\mu(f_1, x)$. This argument also may be repeated indefinitely.

These properties ensure the existence of $\Upsilon(x) = \lim_{n\to\infty} Y_n(x)$ and the *finiteness* of $\int_{-\infty}^{\infty}(\Upsilon(x)/(1+x^2))dx$; it is then easy to verify that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{F(t)+\Upsilon(t)}{(x-t)^2+(\Upsilon(x))^2}\,dt \leqslant \mu$$

for each $x \in \mathbb{R}$. We are not, however, assured of Υ's being uniformly Lip 1 on $\mathbb{R}$. For that reason, we modify the preceding construction as follows.

For each n, put

$$\tilde{Y}_n(x) = \tilde{Y}_\mu(f_n, x); \tag{15}$$

$\tilde{Y}_n(x)$ is by definition the *largest minorant* of $Y_n(x)$ for which $|\tilde{Y}_n(x') - \tilde{Y}_n(x)| \leqslant \frac{1}{2}|x'-x|$ on $\mathbb{R}$. Then, since $Y_n(x) \leqslant Y_{n+1}(x)$, we have

$$\tilde{Y}_n(x) \leqslant \tilde{Y}_{n+1}(x), \quad n = 0, 1, 2, \ldots .$$

Write for the moment

$$\tilde{f}_{n+1}(t) = F(t) + \tilde{Y}_n(t), \quad n = 0, 1, 2, \dots ;$$

then $\tilde{f}_{n+1}(t) \leqslant F(t) + Y_n(t) = f_{n+1}(t)$, so $Y_\mu(\tilde{f}_{n+1}, x) \leqslant Y_\mu(f_{n+1}, x) = Y_{n+1}(x)$ by (6), and we have

$$\tilde{Y}_\mu(\tilde{f}_{n+1}, x) \leqslant \tilde{Y}_{n+1}(x)$$

by (15). From (11) and the Lip 1 property of the $\tilde{Y}_n$, we have, for each n,

$$|\tilde{f}_{n+1}(t') - \tilde{f}_{n+1}(t)| \leqslant |t' - t|, \quad t', t \in \mathbb{R}.$$

Thence, by Lemma 2.6,

$$\tilde{Y}_\mu(\tilde{f}_{n+1}, x) \geqslant Y_{\mu+2}(\tilde{f}_{n+1}, x),$$

so by the above specification of $\tilde{f}_{n+1}$,

$$\tilde{Y}_{n+1}(x) \geqslant Y_{\mu+2}(F + \tilde{Y}_n, x),$$

i.e.,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{F(t) + \tilde{Y}_n(t)}{(x-t)^2 + (\tilde{Y}_{n+1}(x))^2}\,dt \leqslant \mu + 2 \tag{16}$$

for $x \in \mathbb{R}$.

The sequence $\{\tilde{Y}_n(x)\}$ has a limit, being, as we have seen, *increasing*.

NOTATION. — *We write*

$$H(x) = \lim_{n\to\infty} \tilde{Y}_n(x).$$

Then we can state the following

2.7. THEOREM. — *Let $F(t)$ satisfy* (4), (10) *and* (11). *Then the function $H(x)$ just constructed satisfies*

$$|H(x') - H(x)| \leqslant \frac{1}{2}|x' - x|, \quad x', x \in \mathbb{R},$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{H(t)}{1+t^2}\,dt \leqslant \frac{1}{2},$$

$$H(x) \geqslant F(x)/(4\mu + 8),$$

and finally

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{H(t)}{(x-t)^2+(H(x))^2}\,dt \;\leqslant\; \mu+2 \quad \textit{for} \quad x\in\mathbb{R},$$

where $\mu = 123$.

Proof. — The first relation holds because each of the $\tilde{Y}_n(x)$ has the same Lip 1 property. The second follows by (14) and monotone convergence, since $\tilde{Y}_n(x) \leqslant Y_n(x)$. For the third relation, we observe that

$$H(x) \geqslant \tilde{Y}_0(x) = \tilde{Y}_\mu(F,x) \geqslant Y_{\mu+2}(F,x)$$

by (11) and Lemma 2.6, with then $Y_{\mu+2}(F,x) \geqslant F(x)/(4\mu+8)$ by Lemma 2.5.

To obtain the last relation, make $n \to \infty$ in (16) and apply Fatou's lemma. One gets

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{F(t)+H(t)}{(x-t)^2+(H(x))^2}\,dt \;\leqslant\; \mu+2,$$

more than what is needed.

NOTATION. — *We write henceforth*

$$\sigma = \mu + 2. \tag{17}$$

Note that by (12), this makes

$$\sigma = 125.$$

3. The function $H(x)$ and its companion $Y(x)$.

NOTATION. — *The function* $Y_\sigma(H,x)$ *is henceforth denoted by* $Y(x)$.

We are, in other words, taking $Y(x)$ so as to have

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{H(t)}{(x-t)^2+(Y(x))^2}\,dt \;=\; \sigma, \quad x\in\mathbb{R}.$$

According to the last relation furnished by Theorem 2.7, to (7) and to (17), this makes

$$Y(x) \;\leqslant\; H(x). \tag{18}$$

At the same time, by the first relation from that theorem and Lemma 2.5, $Y(x) \geqslant H(x)/4\sigma$. This and the theorem's third relation yield

$$Y(x) \;\geqslant\; F(x)/16\sigma^2. \tag{19}$$

The function $Y(x)$ also enjoys some remarkable *smoothness properties.*

3.1. LEMMA. — *$Y(x)$ is C_1 and uniformly Lip 1, with*

$$|Y'(x)| \;\leqslant\; 2\pi\sigma, \quad x \in \mathbb{R}.$$

Proof. — Take the function $J(x,y)$ given by (5) with $f(t) = H(t)$; $Y(x)$ is then determined by the relation

$$J(x, Y(x)) = \sigma,$$

so $Y'(x)$ exists and

$$Y'(x) = -\frac{J_x(x,y)}{J_y(x,y)} \tag{20}$$

with y put equal to $Y(x)$ *as long as the denominator on the right is* $\neq 0$.

But for that denominator we have

$$J_y(x,y) \;=\; -\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{yH(x-\tau)}{(\tau^2+y^2)^2}\,d\tau,$$

and for $y = Y(x)$, the integral on the right, after change of sign, is

$$\geqslant\; \frac{1}{2\pi(Y(x))^3}\int_{-Y(x)}^{Y(x)} H(x-\tau)d\tau.$$

By (18) and the first relation from Theorem 2.7,

$$H(x-\tau) \;\geqslant\; H(x) - \frac{1}{2}Y(x) \;\geqslant\; \frac{1}{2}Y(x) \quad \text{for } |\tau| \leqslant Y(x),$$

so we find that

$$J_y(x,y) \;\leqslant\; -\frac{1}{2\pi Y(x)} \quad \text{for } y = Y(x). \tag{21}$$

The use of (20) is therefore legitimate, and we have, for the numerator in its right side,

$$J_x(x,y) \;=\; -\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{(x-t)H(t)}{((x-t)^2+y^2)^2}\,dt.$$

The integral here is in absolute value $\leqslant J(x,y)/y$, so for $y = Y(x)$,

$$|J_x(x,y)| \leqslant \frac{\sigma}{Y(x)}. \tag{22}$$

Our lemma now follows from (20), (21) and (22).

3.2. Lemma. — *$Y''(x)$ exists and is continuous, with*

$$|Y''(x)| \leqslant \frac{2 + 4\pi^2\sigma + 64\pi^3\sigma^3}{Y(x)}.$$

Proof. — Taking the function $J(x,y)$ used in the preceding proof, we see that the asserted qualitative properties of $Y''(x)$ follow from (20) and (21).

As to the estimate, (20) yields

$$Y''(x) = -\frac{J_{xx}}{J_y} + 2\frac{J_x}{J_y^2}J_{xy} + \frac{J_x}{J_y^2}J_{yy}Y'(x), \tag{23}$$

with the partial derivatives of $J(x,y)$ evaluated for $y = Y(x)$.

We have

$$J(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{H(x-\tau)}{\tau^2 + y^2}\,d\tau,$$

and, according to the first relation from Theorem 2.7, $H'(t)$ exists a.e., with

$$|H'(t)| \leqslant \frac{1}{2} \quad \text{a.e.} \tag{24}$$

The preceding formula can now be differentiated under the integral sign (that is justified by dominated convergence), and we get

$$J_x(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{H'(x-\tau)d\tau}{\tau^2 + y^2} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{H'(t)dt}{(x-t)^2 + y^2}. \tag{25}$$

Differentiating the second integral with respect to x and then using (24), we find that

$$|J_{xx}(x,y)| \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\tau|d\tau}{(\tau^2 + y^2)^2} = \frac{1}{\pi y^2},$$

making

$$|J_{xx}(x,Y(x))| \leqslant \frac{1}{\pi(Y(x))^2}. \tag{26}$$

By differentiating the second integral in (25) with respect to y and using (24) we see in like manner that

$$|J_{xy}(x, Y(x))| \leqslant \frac{1}{2(Y(x))^2}. \tag{27}$$

Finally, we have

$$J_{yy}(x,y) = -\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{H(t)dt}{((x-t)^2+y^2)^2} + \frac{8}{\pi}\int_{-\infty}^{\infty}\frac{y^2H(t)dt}{((x-t)^2+y^2)^3}.$$

When $y = Y(x)$, $J(x,y) = \sigma$, so the first integral on the right lies between 0 and $-2\sigma/(Y(x))^2$, and the second between 0 and $8\sigma/(Y(x))^2$. Thus,

$$|J_{yy}(x, Y(x))| \leqslant \frac{8\sigma}{(Y(x))^2}. \tag{28}$$

We now plug (21), (22), (26), (27) and (28) into (23), together with the inequality furnished by Lemma 3.1. The asserted inequality for $|Y''(x)|$ follows, and we are done.

NOTATION. — *We henceforth write*

$$\gamma = 2 + 4\pi^2\sigma + 64\pi^3\sigma^3, \tag{29}$$

with $\sigma = 125$.

In terms of the new constant γ, the result of Lemma 3.2 reads

$$|Y''(x)| \leqslant \frac{\gamma}{Y(x)}. \tag{30}$$

We shall see in §4 why it is useful to have a bound inversely proportional to $Y(x)$ on $|Y''(x)|$.

We now take the function

$$U(z) = \sigma y - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{yH(t)dt}{(x-t)^2+y^2}, \tag{31}$$

harmonic for $y > 0$, and look at its behaviour in the domain

$$\mathcal{D}_+ = \{z = x + iy;\ y > Y(x)\}$$

and on $\partial\mathcal{D}_+$.

With $J(x, y)$ as in the proofs of Lemmas 3.1 and 3.2, we have

$$U(z) = y(\sigma - J(x, y)),$$

so $U(z) > 0$ for $z \in \mathcal{D}_+$, and $U(z) = 0$ precisely on $\partial\mathcal{D}_+$, where $J(x, y) = J(x, Y(x)) = \sigma$.

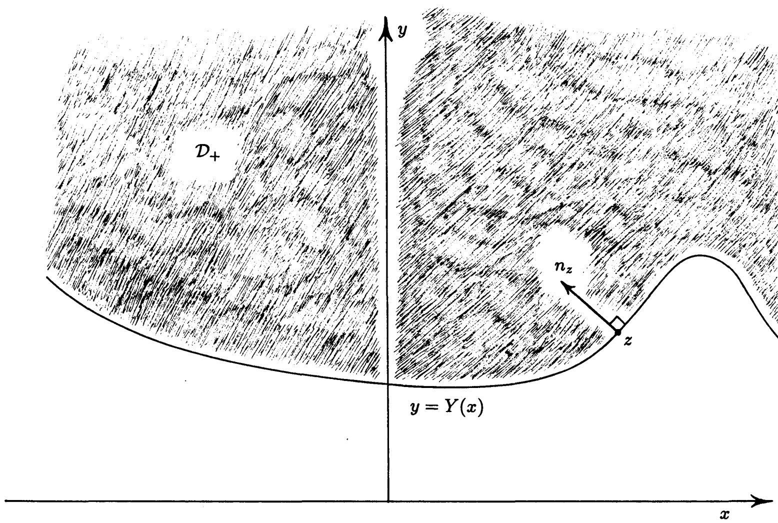

Fig. 2.

The boundary $\partial\mathcal{D}_+$ (where $y = Y(x)$) is *smooth* by Lemma 3.1, so we have at each of its points z a *well defined inner normal* n_z *of unit length* (pointing upwards into $\mathcal{D}_+$, see figure 2). At such points z the function $U(z)$ given by (31) is $\mathcal{C}_\infty$, and we can speak of the directional derivative $\partial U(z)/\partial n_z$ along n_z ; clearly

$$\frac{\partial U(z)}{\partial n_z} \geqslant 0.$$

In the present circumstances, we can say more. The following result will be crucial in our work.

3.3. THEOREM. — *At each z on $\partial\mathcal{D}_+$, we have*

$$\frac{\partial U(z)}{\partial n_z} \geqslant \frac{1}{2\pi}.$$

Proof. — Since $U(z) = 0$ along $\partial\mathcal{D}_+$, $\partial U(z)/\partial n_z$ is there equal to the magnitude of **grad** U. However, by (31),

$$\frac{\partial U(z)}{\partial y} = \sigma - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{H(t)dt}{(x-t)^2+y^2} + \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{y^2H(t)dt}{((x-t)^2+y^2)^2}.$$

For $y = Y(x)$, the first two terms on the right cancel each other out (!), and the third is $-\,Y(x)J_y(x, Y(x)) \geqslant 1/2\pi$ by (21). So the magnitude of **grad** U is surely $\geqslant 1/2\pi$ on $\partial\mathcal{D}_+$, where $y = Y(x)$. Done.

4. Construction in the strip $-\,Y(x) < y < Y(x)$.

NOTATION. — $\mathcal{S} = \{z = x + iy;\ |y| < Y(x)\}$.

The boundary $\partial\mathcal{S}$ of $\mathcal{S}$ consists of the two curves $y = Y(x)$, $y = -Y(x)$.

DEFINITION. — *For $z \in \overline{\mathcal{S}}$, we put*

$$S(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} G_{\mathcal{S}}(z,t)dt,$$

where $G_{\mathcal{S}}(\ ,\ )$ is the Green's function for $\mathcal{S}$.

4.1. LEMMA. — *For $-Y(x_0) \leqslant y \leqslant Y(x_0)$,*

$$S(x_0+iy) \leqslant \frac{2Y(x_0)}{\sin\alpha},$$

where

$$\alpha = \arctan\frac{\sqrt{3}}{4\pi\sigma}$$

with $\sigma = 125$ (see (17)).

Proof. — Fixing x_0, we see by Lemma 3.1 that the whole strip $\mathcal{S}$ is contained between 4 rays, of slopes $\pm 2\pi\sigma$, going out from the points $P = x_0 + iY(x_0)$ and $\overline{P} = x_0 - iY(x_0)$ (look at figure 3). An easy calculation shows that the 4 rays in question are tangent to the branches of the hyperbola

$$y^2 - \frac{16\pi^2\sigma^2}{3}(x-x_0)^2 = 4(Y(x_0))^2 \tag{32}$$

passing through the points $Q = x_0 + 2iY(x_0)$ and $\overline{Q} = x_0 - 2iY(x_0)$. The strip $\mathcal{S}$ is therefore *included* in the region $\mathcal{R}$ lying between the two branches of that hyperbola, so if $G_{\mathcal{R}}(\ ,\ )$ denotes the Green's function for $\mathcal{R}$,

$$G_{\mathcal{S}}(z,t) \leqslant G_{\mathcal{R}}(z,t), \quad z \in \mathcal{S},\ t \in \mathbb{R}. \tag{33}$$

The asymptotes to the hyperbola (32) make angles

$$\alpha \cdot = \arctan \frac{\sqrt{3}}{4\pi\sigma} \tag{34}$$

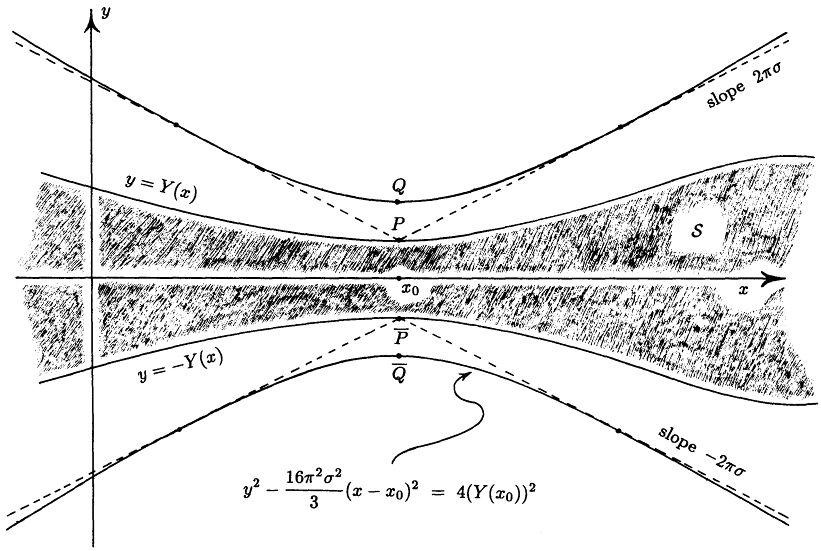

Fig. 3.

with the vertical. The linear transformation

$$z \longrightarrow w = \frac{i\cos\alpha}{2Y(x_0)}(z - x_0) \tag{35}$$

takes $\mathcal{R}$ conformally onto the region $\mathcal{E}$ of the w-plane bounded by the two branches of the hyperbola

$$\frac{u^2}{\cos^2\alpha} - \frac{v^2}{\sin^2\alpha} = 1$$

(we write $w = u + iv$); this mapping takes the x-axis of the z plane to the v-axis, and the vertical segment joining Q and $\overline{Q}$ to the interval

$[-\cos\alpha,\ \cos\alpha]$ of the u-axis. For $-Y(x_0) \leqslant y \leqslant Y(x_0)$ we therefore have, by (33) and (35),

$$\int_{-\infty}^{\infty} G_{\mathcal{S}}(x_0+iy,t)dt \quad \leqslant \quad \frac{2Y(x_0)}{\cos\alpha}\int_{-\infty}^{\infty} G_{\mathcal{E}}(u,iv)dv \tag{36}$$

with $-\cos\alpha < u < \cos\alpha$, $G_{\mathcal{E}}(\ ,\ )$ being the Green's function for $\mathcal{E}$.

A conformal mapping of $\{\Re\zeta > 0\}$ onto $\mathcal{E}$ is available, making it possible for us to express $G_{\mathcal{E}}(\ ,\ )$ in terms of the (known) Green's function for the right half plane. In terms of the parameter α given by (34), we put

$$p \;=\; \frac{\pi - 2\alpha}{\pi}; \tag{37}$$

then our mapping is given by

$$\zeta \longrightarrow w \;=\; \frac{1}{2}\left(i\zeta^p + \frac{1}{i\zeta^p}\right) \tag{38}$$

(composition of $\zeta \to i\zeta^p$ with the Joukowski transformation). In this mapping, points $\zeta = \xi > 0$ go to the points iv of the imaginary w-axis, where

$$v \;=\; \frac{1}{2}(\xi^p - \xi^{-p}),$$

and the points $e^{i\beta}$ with $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ go to the real points u, $-\cos\alpha < u < \cos\alpha$. In terms of ξ and β, we therefore have

$$G_{\mathcal{E}}(u,iv) \quad = \quad \log\left|\frac{e^{i\beta}+\xi}{e^{i\beta}-\xi}\right|$$

for such u and v, making

$$\int_{-\infty}^{\infty} G_{\mathcal{E}}(u,iv)dv \quad = \quad \frac{p}{2}\int_0^{\infty}\log\left|\frac{e^{i\beta}+\xi}{e^{i\beta}-\xi}\right|\frac{\xi^p+\xi^{-p}}{\xi}d\xi. \tag{39}$$

Differentiation shows that for each $\xi > 0$, $\log|(e^{i\beta}+\xi)/(e^{i\beta}-\xi)|$ attains its maximum on the interval $-\frac{\pi}{2} \leqslant \beta \leqslant \frac{\pi}{2}$ for $\beta = 0$. The right side of (39) is thus

$$\leqslant \quad \frac{p}{2}\int_0^{\infty}\log\left|\frac{1+\xi}{1-\xi}\right|\frac{\xi^p+\xi^{-p}}{\xi}d\xi \quad = \quad p\int_0^{\infty}\log\left|\frac{1+\xi}{1-\xi}\right|\xi^{p-1}d\xi,$$

for the substitution $\xi \longrightarrow 1/\xi$ leaves the second integral unchanged. The *value* of that second integral is *known*, and equal to

$$\pi \tan \frac{\pi p}{2} \quad = \quad \pi \cot \alpha$$

(see [11], p. 316). By (39) and (36) we thus have, for $-Y(x_0) \leqslant y \leqslant Y(x_0)$,

$$\int_{-\infty}^{\infty} G_{\mathcal{S}}(x_0 + iy, t)dt \quad \leqslant \quad \frac{2\pi Y(x_0)}{\sin \alpha}$$

which, with the above definition of $S(z)$, gives the lemma.

The argument just made can be applied to yield a useful result, best stated as a

COROLLARY. — *The function $S(z)$ is superharmonic in $\mathcal{S}$ and tends to zero when z tends to any point of $\partial\mathcal{S}$.*

Proof. — Both parts of the conclusion will follow from the *uniform convergence* of the integral $\int_{-\infty}^{\infty} G_{\mathcal{S}}(z,t)dt$ for z ranging over any compact subset of $\overline{\mathcal{S}}$ (sic!).

To verify that uniform convergence, look at (35), (36), (38), (39) and the uniform estimate for the right side of the last relation. Note that $p < 1$ by (37), and that $Y(x_0) > 0$ is a continuous function of x_0. Superharmonicity of $S(z)$ is now manifest from that function's definition, and we see that it is in fact *harmonic* at the points of $\mathcal{S}$ *off of* the real axis.

In order to deduce the second asserted property of $S(z)$, suppose that $z \rightarrow z_0 \in \partial\mathcal{S}$. Uniform convergence of the integral gives us an A such that $\int_{|t|\geqslant A} G_{\mathcal{S}}(z,t)dt \ < \ \epsilon$ for $z \in \mathcal{S}$ and $|z - z_0| < 1$ say. But then by Harnack there is an M with $G_{\mathcal{S}}(z,t) \leqslant MG_{\mathcal{S}}(z,0)$ for all $z \in \mathcal{S}$ sufficiently close to z_0 (and hence bounded away from $\mathbb{R}$!) when $-A \leqslant t \leqslant A$. Thence, by definition, $S(z) \leqslant (\varepsilon/\pi) + (2AM/\pi)G_{\mathcal{S}}(z,0)$ for such z, where $G_{\mathcal{S}}(z,0) \rightarrow 0$ as $z \rightarrow z_0$. This does it.

Remark. — The reasoning shows that the convergence of $S(z)$ to 0 is *uniform* when $z \in \mathcal{S}$ tends to any *bounded arc* of $\partial\mathcal{S}$.

For the parameter α given by (34) we have

$$\sin\alpha \;=\; \frac{\sqrt{3}}{\sqrt{16\pi^2\sigma^2+3}}$$

with, as we know, $\sigma = 125$. We bring in another numerical constant.

NOTATION.

$$\kappa \;=\; \frac{2}{\sin\alpha} \;=\; 2\left(\frac{16\pi^2\sigma^2}{3}\;+\;1\right)^{1/2}.$$

Then the conclusion of Lemma 4.1 reads thus : for $z = x+iy \in \mathcal{S}$,

$$(40) \qquad S(z) \;\leqslant\; \kappa Y(x).$$

In $\mathcal{S}$, the Riesz mass corresponding to the function $S(z) = (1/\pi)\int_{-\infty}^{\infty} G_{\mathcal{S}}(z,t)dt$, *superharmonic* there, is supported on the *real axis*, where it has the *linear* density $1/\pi$. At the same time, the function

$$|\Im z| \;=\; \frac{1}{\pi}\int_0^\infty \log\left|1-\frac{z^2}{t^2}\right| dt$$

is *subharmonic* in the *whole plane*, and *its* Riesz mass, *also* supported on $\mathbb{R}$, has by inspection *the same linear density*, but with *opposite sign*. The sum

$$(41) \qquad V(z) \;=\; |\Im z| \;+\; S(z)$$

is therefore *harmonic* in $\mathcal{S}$. According to the last corollary and the remark following it, $V(z)$ *extends continuously* up to $\partial\mathcal{S}$, and

$$(42) \qquad V(z) \;=\; Y(x) \quad \text{for} \quad z = x \pm iY(x) \text{ on } \partial\mathcal{S}.$$

Note that by (40) and (41) we have the *upper bound* $V(z) \;\leqslant\; (1+\kappa)Y(x)$ for $z = x+iy \in \mathcal{S}$. To get a *lower bound* is easier ; we will need

4.2. LEMMA. — *For real x, we have*

$$V(x) \;\geqslant\; \frac{Y(x)}{8\pi\sigma}.$$

Proof. — Fixing any real x_0, we get, from Lemma 3.1,

$$Y(x) \;\geqslant\; Y(x_0)/2 \quad \text{for} \quad |x-x_0| \;\leqslant\; Y(x_0)/4\pi\sigma.$$

Since $\sigma = 125$, $4\pi\sigma > 2$ (!), and the *square* Q of side $Y(x_0)/2\pi\sigma$ with centre at x_0 and sides parallel to the coordinate axes lies entirely in the strip $\mathcal{S}$.

By (41) the function $V(z)$, harmonic in $\mathcal{S}$, is $\geqslant |\Im z|$ there and hence $\geqslant Y(x_0)/4\pi\sigma$ on the *horizontal sides* of Q. Seen from x_0, those horizontal sides have, in Q, harmonic measure $1/2$, so, since at any rate $V(z) \geqslant 0$ on ∂Q, we have $V(x_0) \geqslant Y(x_0)/8\pi\sigma$. Done.

COROLLARY. — $V(x) \geqslant F(x)/128\pi\sigma^3$ *for* $x \in \mathbb{R}$.

Proof. — Refer to (19) near the beginning of §3.

We will need to know more about the behaviour of $S(z)$ near the points of $\partial\mathcal{S}$.

NOTATION. — *For* $z \in \partial\mathcal{S}$, ν_z *denotes the* unit inner normal *to* $\partial\mathcal{S}$ *at* z, *i.e., the unit normal pointing* into $\mathcal{S}$.

The reader should refer to figure 4 below; when z is on the curve $y = Y(x)$, ν_z is *opposite in direction* to the normal n_z shown in figure 2.

NOTATION. — *We write*

$$\overline{D}_{\nu_z} S(z) \quad = \quad \limsup_{\eta \to 0+} \frac{S(z + \eta\nu_z)}{\eta}$$

for $z \in \partial\mathcal{S}$.

(Actually, the directional derivative $\partial S(z)/\partial\nu_z$ *exists* at each $z \in \partial\mathcal{S}$; that follows from the $\mathcal{C}_2$ character of $\partial\mathcal{S}$ (Lemma 3.2) by Kellog's theorem ([12], p. 361; [13], p. 374), by the harmonicity of $S(z)$ for $0 < y < Y(x)$ and for $-Y(x) < y < 0$, and by $S(z)$'s vanishing on $\partial\mathcal{S}$. For our purposes this knowledge is not required.)

It is clear that $\overline{D}_{\nu_z} S(z) \geqslant 0$ for $z \in \partial\mathcal{S}$. What will be crucial for our construction is an *upper bound.*

4.3. THEOREM. — *At each* $z \in \partial\mathcal{S}$ *we have*

$$\overline{D}_{\nu_z} S(z) \quad \leqslant \quad \frac{2\kappa\gamma}{\log\gamma}(1 + 2\pi\sigma),$$

where κ *is the numerical constant specified just before* (40), γ *is given by* (29), and $\sigma = 125$.

Proof. — Let us look at a point $z_0 = x_0 + iY(x_0)$ on the *upper part* of $\partial\mathcal{S}$. That curve $y = Y(x)$ has at each of its points a *radius of curvature* equal to

$$\frac{(1+(Y'(x)^2)^{3/2}}{|Y''(x)|} \geqslant \frac{1}{|Y''(x)|},$$

with the right side $\geqslant Y(x)/\gamma$ by (30). According to (29), γ is *much larger* than $4\pi\sigma$. Therefore, if $|x - x_0| \leqslant Y(x_0)/\gamma$, we will have $Y(x) \geqslant Y(x_0)/2$ by Lemma 3.1, making the above radius of curvature $\geqslant Y(x_0)/2\gamma$.

Let Γ be a circle of radius $Y(x_0)/2\gamma$ with centre P *outside* $\mathcal{S}$, *tangent* to $\partial\mathcal{S}$ at z_0. At any abscissa x of the *horizontal diameter* of Γ, the *radius of curvature* of $\partial\mathcal{S}$ is $\geqslant Y(x_0)/2\gamma$ as we have just seen; that means however, that $\partial\mathcal{S}$ *can never pass inside* Γ, making *that circle's interior disjoint from* $\mathcal{S}$ (see second remark following this proof).

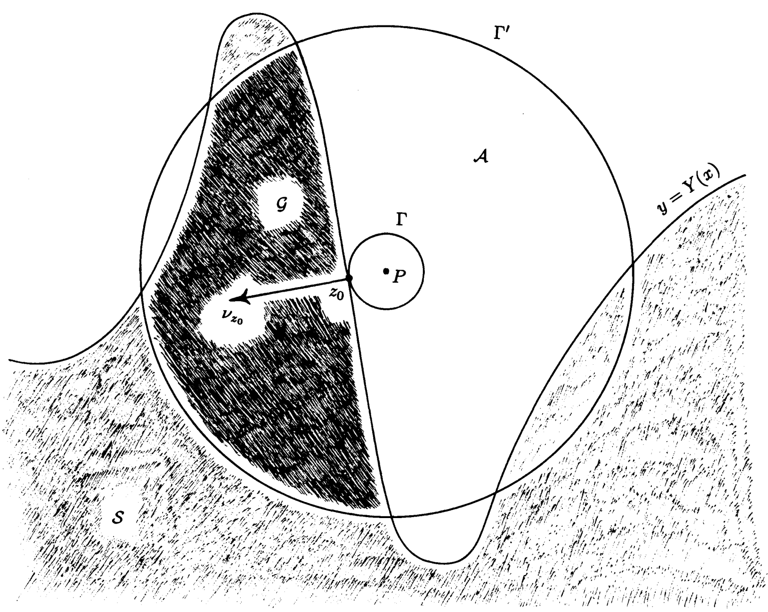

Fig. 4.

About P (which is *higher* than z_0, see figure 4) we describe a *larger* circle Γ', of radius $Y(x_0)/2$; the *lowest* point on Γ' then lies *above* the x-axis since the ordinate of z_0 is $Y(x_0)$. We denote by $\mathcal{A}$ the annulus bounded by Γ and Γ'. The function $S(z)$ is then *harmonic* in $\mathcal{S} \cap \mathcal{A}$; that intersection is certainly *not empty* because $z_0 \in \partial\mathcal{S}$ lies on Γ.

It is possible that $\mathcal{S} \cap \mathcal{A}$ consists of more than one component; in that event *one* of those has z_0 on its boundary and we denote that component by $\mathcal{G}$; at least *part* of the normal vector ν_{z_0} pointing into $\mathcal{S}$ from z_0 *lies* in $\mathcal{G}$. $\partial\mathcal{G}$ consists of an arc (or arcs) on $\partial\mathcal{S}$ and of one (or more) on Γ', lying in $\mathcal{S}$.

For z on Γ' (of diameter $Y(x_0)$), we surely have $|x - x_0| \leqslant Y(x_0)$, so $Y(x) \leqslant (1 + 2\pi\sigma)Y(x_0)$ by Lemma 3.1. Since $S(z) \leqslant \kappa Y(x)$ in $\mathcal{S}$ by (40), we see that

$$S(z) \;\leqslant\; (1 + 2\pi\sigma)\kappa Y(x_0) \quad \text{for} \quad z \in \Gamma' \cap \mathcal{S};$$

this holds in particular on $\partial\mathcal{G} \cap \Gamma'$. The other points of $\partial\mathcal{G}$ are on $\partial\mathcal{S}$ where $S(z) = 0$. Therefore, taking the function $h(z)$ harmonic in $\mathcal{A} \supseteq \mathcal{G}$ with constant boundary values equal to $(1 + 2\pi\sigma)\kappa Y(x_0)$ on Γ' and to *zero* on Γ, we have

$$(42) \qquad S(z) \;\leqslant\; h(z) \quad \text{for} \quad z \in \mathcal{G}$$

by the principle of maximum.

However,

$$(43) \qquad h(z) \;=\; (1 + 2\pi\sigma)\kappa Y(x_0)\, \frac{\log|z - P| - \log(Y(x_0)/2\gamma)}{\log\gamma}$$

for $Y(x_0)/2\gamma \leqslant |z - P| \leqslant Y(x_0)/2$, and the unit normal ν_{z_0} to $\partial\mathcal{S}$ at z_0 is *colinear* with the *radius* of Γ' passing through z_0. By (42) and (43) we thus have, for small values > 0 of $2\eta\gamma/Y(x_0)$,

$$(44) \quad S(z_0 + \eta\nu_{z_0}) \;\leqslant\; \frac{(1 + 2\pi\sigma)\kappa Y(x_0)}{\log\gamma}\left\{\frac{2\eta\gamma}{Y(x_0)} \;+\; O\!\left(\frac{2\eta\gamma}{Y(x_0)}\right)^2\right\}$$
$$\;=\; \frac{2\gamma\kappa(1 + 2\pi\sigma)}{\log\gamma}\eta \;+\; \frac{O(\eta^2)}{Y(x_0)}.$$

This makes $\overline{D}_{\nu_z} S(z) \leqslant 2\kappa\gamma(1 + 2\pi\sigma)/\log\gamma$ at $z = z_0$ on the *upper curve* of $\partial\mathcal{S}$. The argument for a point on the *lower curve* of $\partial\mathcal{S}$ is the same, and we are done.

Remark 1. — The proof just given shows more than what the theorem asserts. Write

$$(45) \qquad L \;=\; \frac{2\kappa\gamma}{\log\gamma}(1 + 2\pi\sigma);$$

this is a *pure number*, like κ, γ and $\sigma = 125$. Then, corresponding to any $\varepsilon > 0$ we have a $\delta > 0$ *independent of* x_0 such that, for any z on the inner normal to $\partial\mathcal{S}$ at $z_0 = x_0 \pm iY(x_0)$ with

$$|z - z_0| \;<\; \delta\, Y(x_0),$$

we have

$$S(z) \;\leqslant\; (1+\varepsilon)L\; |z - z_0|.$$

This refinement, which will find use in the next §, follows from (42), (43) and the fact that an open disk of radius $Y(x_0)/2\gamma$ tangent to $\partial\mathcal{S}$ *from inside* $\mathcal{S}$ at z_0 lies *entirely within* $\mathcal{S}$. The last property is verified by reasoning like that at the beginning of the above proof.

Remark 2. — The statement about $\partial\mathcal{S}$ involving radius of curvature and made near the beginning of the last theorem's proof is geometrically evident and everybody "knows" it, but I have been unable to find a reference. Here is an easy way to check it.

We are given a circle Γ of radius $1/k$ tangent to the $\mathcal{C}_2$ curve $y = Y(x)$ at z_0, with centre at (a, b), say, and it is assumed that $|Y''(x)|/(1 + (Y'(x))^2)^{3/2} \leqslant k$ for $|x - a| \leqslant 1/k$. One wishes to show that the curve *never passes inside* Γ, and it is clearly enough to verify that for the points $(x, Y(x))$ with $|x - a| < 1/k$. There is no loss of generality in taking $z_0 = 0$ and assuming Γ to be tangent to the curve *from above* there, making $b > 0$.

Put $\theta(x) = \arctan Y'(x)$ (with $-\pi/2 < \theta(x) < \pi/2$), and consider the abscissae x with $0 \leqslant x < a + \dfrac{1}{k}$. In terms of $\theta(x)$, we have $Y''(x)/(1+(Y'(x))^2)^{3/2} = \theta'(x)\cos\theta(x)$, so our assumption on $Y(x)$ implies that $\sin\theta(x) - \sin\theta(0) \leqslant kx$ for $0 \leqslant x < a + \dfrac{1}{k}$. The given conditions on Γ make $\sin\theta(0) = -ak$, whence $\sin\theta(x) \leqslant k(x - a)$ for the x under consideration and, in terms of $Y'(x)$, $Y'(x) \leqslant k(x-a)/(1-k^2(x-a)^2)^{1/2}$. Thence, since $Y(0) = 0$, $(Y(x)-b)^2+(x-a)^2 \geqslant 1/k^2$ for $0 \leqslant x < a + \dfrac{1}{k}$. The argument for $a - \dfrac{1}{k} < x \leqslant 0$ is similar.

5. Formation of a superharmonic majorant.

The curves $y = \pm Y(x)$ divide the complex plane into 3 regions : $\mathcal{D}_+$, described shortly before Theorem 3.3 and shown in figure 2, $\mathcal{S}$, defined at the beginning of §4 (see figure 3), and

$$\mathcal{D}_- \;=\; \{z = x + iy;\ y < -Y(x)\}.$$

The functions $U(z)$, given by (31) for $\Im z > 0$, and $S(z)$, defined at the beginning of §4, *both vanish* on the curve $y = Y(x)$; $S(z)$ also vanishes for $y = -Y(x)$. We now put

$$G(z) \;=\; \begin{cases} -2\pi(L+1)U(z), & z \in \mathcal{D}_+, \\ S(z), & z \in \overline{\mathcal{S}}, \\ -2\pi(L+1)U(\overline{z}), & z \in \mathcal{D}_-, \end{cases} \tag{46}$$

where L is the numerical constant defined by (45). Then $G(z)$ is *continuous everywhere* (see the corollary to Lemma 4.1), and we have the

5.1. LEMMA. — *$G(z)$ is everywhere superharmonic.*

Proof. — $G(z)$ is superharmonic in $\mathcal{S}$ by the corollary to Lemma 4.1 and (46), and it is even harmonic in $\mathcal{D}_+$ and in $\mathcal{D}_-$ by (46) and (31). Since it is also continuous at the points of $\partial\mathcal{S}$, we need only verify that it has the mean value property at those points.

Let, wlog, $z_0 = x_0 + iY(x_0)$; since then $G(z_0) = 0$ by definition, it will suffice to show that

$$\int_{-\pi}^{\pi} G(z_0 + re^{i\vartheta})d\vartheta \;<\; 0 \tag{47}$$

for all sufficiently small $r > 0$. Verification of this is based on Theorem 3.3 and Remark 1 to Theorem 4.3; it is essentially an exercise in advanced calculus.

Take new cartesian coordinates with origin at z_0, one axis pointing along the *tangent* to $y = Y(x)$ at z_0, and the other directed along the *outward normal* to that curve there. The *polar coordinates* corresponding to these new axes are used to estimate the left side of (47). Computations are actually the same for *any* orientation of the new coordinate system, so we give them for the case of a *horizontal tangent* to $\partial\mathcal{S}$ at z_0 in order

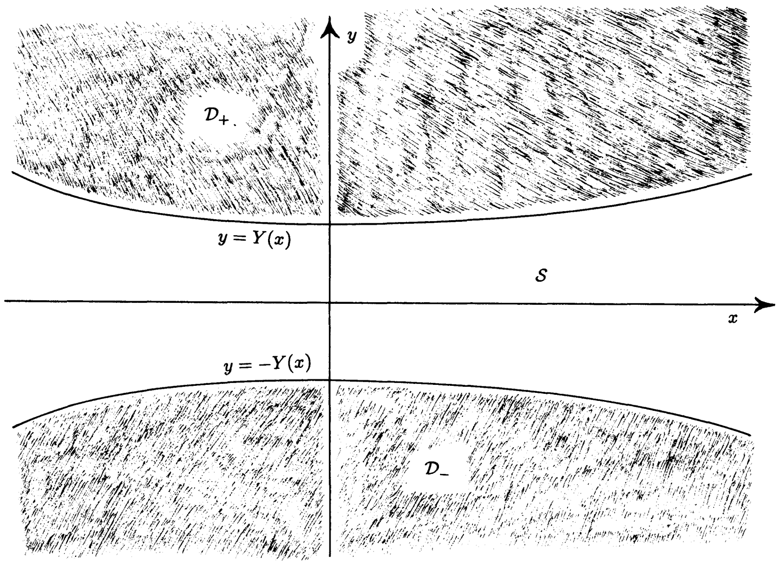

Fig. 5.

not to confuse the reader with additional symbols. That involves no loss in generality.

Assuming, then, that $Y'(x_0) = 0$, we fix a small $\beta > 0$ (according to a specification to be furnished presently), and break up the integral in (47) as

$$\left\{\int_{-\beta}^{\beta} + \int_{\pi-\beta}^{\pi+\beta} + \int_{\beta}^{\pi-\beta} + \int_{-\pi+\beta}^{-\beta}\right\} G(z_0 + re^{i\vartheta})d\vartheta. \tag{48}$$

If $r > 0$ is small and $z = z_0 + re^{i\vartheta}$ *lies in* $\mathcal{S}$ we have, by (46), Remark 1 to Theorem 4.3, and the continuity of $Y(x)$ near x_0,

$$G(z) \;=\; S(z) \;\leqslant\; (1+\varepsilon)L\cdot \operatorname{dist}(z, \partial\mathcal{S}), \tag{49}$$

where $\varepsilon > 0$ is *small* (with r) and $\operatorname{dist}(z, \partial\mathcal{S})$ denotes the shortest (i.e., *perpendicular*!) distance from z to the curve $y = Y(x)$. Because $Y'(x_0) = 0$, the intersection of that curve with a disk of small radius $r > 0$ about z_0 lies within the two sectors consisting of the points $z_0 + \rho e^{i\vartheta}$ with either $|\vartheta| \leqslant \eta$ or $|\vartheta - \pi| \leqslant \eta$ (and $0 \leqslant \rho \leqslant r$), where $\eta > 0$ is *also small* with r. This makes

$$\operatorname{dist}(z_0 + re^{i\vartheta}, \partial\mathcal{S}) \;=\; r|\sin\vartheta| + o(r)$$

for small r, which, substituted in (49), yields

$$(50) \qquad G(z_0 + re^{i\vartheta}) \leqslant Lr|\sin\vartheta| + o(r)$$

for $z_0 + re^{i\vartheta} \in \mathcal{S}$ with $r > 0$ small.

Consider now the points $z_0 + re^{i\vartheta}$ with $-\beta \leqslant \vartheta \leqslant \beta$ figuring in the *first integral* of (48). If such a point lies in $\mathcal{D}_+$, we of course have $G(z_0 + re^{i\vartheta}) < 0$ by (46) and (31). If, however, $z_0 + re^{i\vartheta} \in \mathcal{S}$, we have $G(z_0 + re^{i\vartheta}) \leqslant Lr\sin\beta + o(r)$ by (50); the *first* integral in (48) is therefore

$$\leqslant Lr\beta\sin\beta + o(r).$$

The same estimate holds for the *second integral* in (48).

When $r > 0$ is small, the points $z_0 + re^{i\vartheta}$ with $-\pi + \beta \leqslant \vartheta \leqslant -\beta$ all lie in $\mathcal{S}$, so (50) can be used to estimate $G(z_0 + re^{i\vartheta})$ for such ϑ. We see in that way that the *fourth integral* in (48) is

$$\leqslant 2Lr\cos\beta + o(r).$$

At the same time, the points $z_0 + re^{i\vartheta}$ with $\beta \leqslant \vartheta \leqslant \pi - \beta$ lie in $\mathcal{D}_+$, making $G(z_0 + re^{i\vartheta}) = -2\pi(L+1)U(z_0 + re^{i\vartheta})$ by (46). Since $U(z_0) = 0$ and $U(z)$ is $\mathcal{C}_\infty$ near z_0, U's partial derivatives at z_0 can be used to approximate $U(z_0 + re^{i\vartheta})$ for small r. Here, since $U(z)$ vanishes along $\partial\mathcal{S}$ and the tangent to that boundary at z_0 is *horizontal*, we have $U_x(z_0) = 0$. For the same reason, the unit outward normal n_{z_0} to $\partial\mathcal{S}$ at z_0 has the direction of the positive y-axis, making $U_y(z_0) \geqslant 1/2\pi$ by Theorem 3.3. Using these relations, we get $U(z_0 + re^{i\vartheta}) \geqslant \dfrac{1}{2\pi} r\sin\vartheta + o(r)$ for $0 < \vartheta < \pi$, or, in terms of $G(z)$,

$$G(z_0 + re^{i\vartheta}) \leqslant -(L+1)r\sin\vartheta + o(r), \quad \beta \leqslant \vartheta \leqslant \pi - \beta.$$

From this, we find that the *third integral* in (48) is

$$\leqslant -2(L+1)r\cos\beta + o(r).$$

The estimates just obtained for the four integrals in (48) are now combined, and that yields the upper bound

$$-2r\cos\beta + 4rL\beta\sin\beta + o(r)$$

on the integral in (47) when r is small. Taking $\beta > 0$ small enough at the beginning to make $\cos\beta - 2L\beta\sin\beta > 0$ will thus ensure (47)'s validity for all sufficiently small values of $r > 0$. The lemma is proved.

The superharmonic function $G(z)$ has been defined by (46) in terms of $Y(x)$, $U(z)$ and $S(z)$. $Y(x)$ and $U(z)$ were in turn constructed from $H(x)$ ($Y(x)$ at the beginning of §3 and $U(z)$ near the end, by (31)), while $S(z)$ was formed from $Y(x)$ at the beginning of §4. $H(x)$, finally, was constructed from our *initial* function $F(t)$ near the end of §2. Between $G(z)$ and the function $F(t)$ from which it originates there holds a relation, provided by the following

5.2. LEMMA. — *We have*

$$G(z) \quad \geqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|}{|z-t|^2}\cdot\frac{F(t)}{128\pi\sigma^3}\,dt \quad - \quad 2\pi\sigma(L+1)|\Im z|,$$

with the right side interpreted as $F(x)/128\pi\sigma^3$ *for* $z = x \in \mathbb{R}$.

Proof. — Denote the *right* side of the inequality to be proved by $W(z)$. Then, for $z \notin \overline{\mathcal{S}}$, say wlog for $z \in \mathcal{D}_+$, we have by (46) and (31),

$$G(z) \;=\; -2\pi(L+1)U(z) \;=\; 2\pi(L+1)\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z H(t)}{|z-t|^2}\,dt \;-\; \sigma\Im z\right\},$$

with the right side certainly $\geqslant W(z)$ because $H(t) \geqslant F(t)/4\sigma$ by Theorem 2.7 and (17). The desired relation therefore holds for $z \in \mathcal{D}_+$ and similarly for $z \in \mathcal{D}_-$. By continuity, it continues to be valid on $\partial\mathcal{S}$.

What remains is the verification of our relation in $\mathcal{S}$. There, $G(z) = S(z)$ by (46), so it is enough, by symmetry, to show that

$$(51) \qquad S(z) \;\geqslant\; W(z) \quad \text{for} \quad z = x+iy \quad \text{with} \quad 0 \leqslant y \leqslant Y(x).$$

When $y = 0$, $S(z) = S(x)$ reduces to $V(x)$ which is in turn $\geqslant F(x)/128\pi\sigma^3$ by (41) and the corollary to Lemma 4.2; (51) thus *holds* for $y = 0$. For $y = Y(x)$, $S(z)$ vanishes (continuously) by the corollary to Lemma 4.1 whereas $W(z) \leqslant G(z) = 0$ as we have seen; (51) therefore holds in this case also. The difference $S(z) - W(z)$ is *harmonic* in the strip

$$\mathcal{S}_+ \;=\; \{z = x+iy;\quad 0 < y < Y(x)\}$$

by the discussion before and after (41); it is also continuous up to $\partial\mathcal{S}_+$ and $\geqslant 0$ there, as we have just verified. In order to deduce (51) from this, a Phragmén-Lindelöf argument is needed.

Since $S(z) \geqslant 0$ in $\mathcal{S}$ we have, at any rate,

$$(52) \qquad S(z) - W(z) \;\geqslant\; -F(z) \quad \text{for} \quad z \in \mathcal{S}_+,$$

where

$$F(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\, F(t)dt. \tag{53}$$

Taking a large x_0, we look at the bounded region

$$\mathcal{S}_+(x_0) = \{z = x+iy; \quad -x_0 < x < x_0,\ 0 < y < Y(x)\}$$

shown in figure 6.

On $\partial\mathcal{S}_+(x_0) \cap \partial\mathcal{S}_+$ we know that $S(z)-W(z) \geqslant 0$. The rest of $\partial\mathcal{S}_+(x_0)$ consists of the two vertical segments $J(x_0)$, $J(-x_0)$ shown in figure 6, and on *them*, by (52),

$$\begin{aligned} S(z) - W(z) &\geqslant -M(x_0), \quad z \in J(x_0), \\ S(z) - W(z) &\geqslant -M(-x_0), \quad z \in J(-x_0), \end{aligned}$$

where $M(x_0)$ is the supremum of $F(z)$ on $J(x_0)$ and $M(-x_0)$ that function's supremum on $J(-x_0)$.

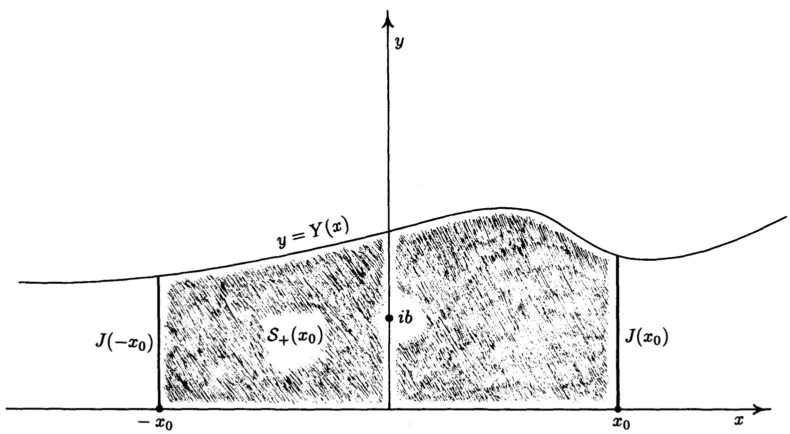

Fig. 6.

Denote *harmonic measure* for $\mathcal{S}_+(x_0)$ by $\omega_{x_0}(\ ,\ )$. Then, for $z' \in \mathcal{S}_+(x_0)$ we have, by the above relations,

$$S(z') - W(z') \geqslant -M(x_0)\omega_{x_0}(J(x_0), z') - M(-x_0)\omega_{x_0}(J(-x_0), z'); \tag{54}$$

we will be done if we can show that the right side tends to zero for each fixed z' when $x_0 \to \infty$.

To estimate $M(x_0)$ we take $z = x_0 + iy$ with $0 \leqslant y \leqslant Y(x_0)$ in (53), and break up the integral occurring there as

$$\frac{1}{\pi}\left\{\int_{|t-x_0|\geqslant x_0/2} + \int_{x_0/2}^{3x_0/2}\right\} \frac{yF(t)}{(t-x_0)^2+y^2}\,dt. \tag{55}$$

For large x_0, the *first* integral in (55) is

$$\leqslant \quad \text{const. } y\int_{-\infty}^{\infty} \frac{F(t)}{t^2+x_0^2}\,dt \quad \leqslant \quad \text{const. } Y(x_0)$$

by (10). By Lemma 3.1, $Y(x_0) \leqslant Y(0) + 2\pi\sigma x_0$, so the last expression is $\leqslant$ const. x_0 when x_0 is large. In the *second* integral of (55), where t ranges over $[x_0/2,\ 3x_0/2]$, we have $F(t) \leqslant F(0) + 3x_0/4$ by (11). This makes that second integral $\leqslant$ const. x_0 for large x_0, and finally,

$$F(x_0 + iy) \leqslant \text{const. } x_0, \quad 0 \leqslant y \leqslant Y(x),$$

making

$$M(x_0) \leqslant \text{const. } x_0 \tag{56}$$

when x_0 is large; for $M(-x_0)$ we clearly have the same estimate.

We turn to the examination of $\omega_{x_0}(J(x_0), z')$ and $\omega_{x_0}(J(-x_0), z')$ for x_0 tending to ∞ with z' fixed. According to Harnack, the asymptotic behaviour of these quantities for any given z' is governed by that for the special case $z' = ib$ where $b = Y(0)/2$, and it will suffice to work out the behaviour in that particular situation. For that purpose, one may use a version of the Ahlfors-Carleman formula for *horizontal curvilinear strips* which, for $J(x_0)$, reads as follows :

$$\omega_{x_0}(J(x_0), ib) \quad \leqslant \quad \text{abs. const. } \exp\left(-\pi\int_0^{x_0} \frac{dx}{Y(x)}\right). \tag{57}$$

Regarding this formula, see pp. 7-8 of [14]; it may be derived from the more accessible polar version used in proving Lemma 2.4 by taking polar coördinates with origin at $-K$ on $\mathbb{R}$ and making $K \to \infty$.

By Schwarz, we have

$$\left(\int_0^{x_0} \frac{dx}{\sqrt{x^2+1}}\right)^2 \leqslant \int_0^{x_0} \frac{dx}{Y(x)} \cdot \int_0^{x_0} \frac{Y(x)}{1+x^2}\,dx.$$

Here the left side is $\sim (\log x_0)^2$ for large x_0, while $\int_0^\infty (Y(x)/(1+x^2))dx < \infty$ by (18) and Theorem 2.7. Therefore, $\pi \int_0^{x_0} (1/Y(x))dx \geqslant c(\log x_0)^2$ for large x_0 (with a constant $c > 0$) and this, substituted in (57), yields

$$\omega_{x_0}(J(x_0), ib) \leqslant \text{abs. const. } e^{-c(\log x_0)^2} \tag{58}$$

for x_0 tending to ∞. The same estimate of course holds for $\omega_{x_0}(J(-x_0), ib)$.

As already noted, $\omega_{x_0}(J(x_0), z')$ and $\omega_{x_0}(J(-x_0), z')$ have, for any fixed $z' \in \mathcal{S}_+$, the same general behaviour when $x_0 \to \infty$. We thus see from (56), (58) and their analogues for $-x_0$ that the *right side* of (54) *tends, for any given* $z' \in \mathcal{S}_+$, *to zero* (like $x_0 e^{-c(\log x_0)^2}$ at least) *when* $x_0 \to \infty$. But this means that $S(z') - W(z') \geqslant 0$, i.e., that (51) holds in $\mathcal{S}_+$. That is what we needed to finish proving the lemma, and we are done.

From the preceding two lemmas, we now have, without further ado :

5.3. Theorem. — *Given* $F(t)$ *satisfying* (4), (10) *and* (11), *construct the function* $G(z)$ *using* (46). *Then* $128\pi\sigma^3 G(z)$ *is a (finite!) superharmonic majorant (in the whole complex plane) for*

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|}{|z-t|^2} F(t)dt \quad - \quad 256\pi\sigma^4(L+1)|\Im z|$$

(*this last interpreted as* $F(x)$ *for* $z = x \in \mathbb{R}$).

We now specify one more numerical constant.

Notation. — *With* $\sigma = 125$ *and* L *given by* (45), *we put*

$$A \quad = \quad 512\pi\sigma^4(L+1). \tag{59}$$

In terms of the constants γ, defined by (29), and κ, specified just before (40), we have

$$A \quad = \quad 512\pi\sigma^4\left(\frac{2\kappa\gamma}{\log\gamma}(1+2\pi\sigma) \; + \; 1\right).$$

This quantity is very large; computation with a sliderule yields $A \approx 3.03 \times 10^{27}$.

5.4. Theorem. — *If* $f(t) \geqslant 0$ *satisfies* (3) *and*

$$|f(t') - f(t)| \leqslant \ell|t'-t|, \quad t', t \in \mathbb{R}, \tag{60}$$

the function

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\Im z|}{|z-t|^2}f(t)dt \;-\; A\ell|\Im z| \tag{61}$$

(interpreted as $f(x)$ for $z = x \in \mathbb{R}$) has a finite superharmonic majorant in the complex plane.

Proof. — Consider first the case where $\ell = 1/2$. In that event, we take M large enough in the formula

$$F(t) \;=\; \frac{1}{4} + \max(f(t), M) - M$$

to make

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{F(t)}{1+t^2}dt \;\leqslant\; \frac{1}{2};$$

the term $1/4$ is added to ensure (4) for F.

The function $F(t)$ satisfies (11) because $f(t)$ does. The conclusion of Theorem 5.3 therefore holds for this F and the function $G(z)$ constructed from it. Then, however, $128\pi\sigma^3 G(z) + M - (1/4)$ is a finite superharmonic majorant of (61) (with $\ell = 1/2$).

In the general case, form F using $f(t)/2\ell$ instead of $f(t)$ and multiply afterwards by 2ℓ.

From this last result we obtain the

COROLLARY (Theorem of Beurling and Malliavin, 1961). — *Let $\Phi(z)$ be entire and of exponential type, with*

$$\int_{-\infty}^{\infty}\frac{\log^+|\Phi(x)|}{1+x^2}dx \;<\; \infty.$$

Then there are entire functions $\varphi(z) \not\equiv 0$ of arbitrarily small exponential type such that $\Phi(x)\varphi(x)$ is bounded on $\mathbb{R}$.

Proof. — One can, as in §2 of [5], use Akhiezer's version of the Riesz-Fejér theorem to reduce the general situation to one where $|\Phi(x)|$ is even and $\geqslant 1$ on $\mathbb{R}$ and all the zeros of $\Phi(z)$ lie in $\{\Im z < 0\}$ (see [7], p. 556 and pp. 55-58). Then, however, a result from §4 of [5] (whose proof is sketched in the appendix to this paper) gives us, for *any* $\ell > 0$, an $f(x) \geqslant \log|\Phi(x)|$ satisfying both (3) and (60).

Theorem 5.4 now guarantees that (61) and a fortiori

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\Im z|}{|z-t|^2}\log|\Phi(t)|dt \;-\; A\ell|\Im z|$$

have a finite superharmonic majorant in $\mathbb{C}$. By a known result ([1], pp. 68-69 and p. 84, [2], pp. 388-389), this implies the existence of entire functions $\varphi(z)\not\equiv 0$ of exponential type $A\ell$ with $\Phi(x)\varphi(x)$ bounded on $\mathbb{R}$. Hence, since $\ell>0$ is arbitrary, the corollary holds.

Remark. — Under the circumstances arranged for at the beginning of the corollary's proof, $f(x)$, like $|\Phi(x)|$, will be even; that will then be the case for the function $F(t)$ formed from f in proving Theorem 5.4, and thus finally for the superharmonic function $G(z)$ corresponding to that F figuring in Theorem 5.3. This even superharmonic function $G(z)$ *can already be used to construct a multiplier* $\varphi(z)$ for $|\Phi(x)|$.

To see how that is done, look first at the *Riesz mass* associated with $G(z)$ which, according to (46) and the discussion around (41), must all be located on the *real axis* and on the *two curves* $y=\pm Y(x)$. On the real axis this mass has constant linear density equal to $1/\pi$, and on the *curves* its linear density is at least *bounded.* The latter observation follows from Theorem 4.3 and the *upper bound* 3σ for $\partial U(z)/\partial n_z\geqslant 0$ on the curve $y=Y(x)$ (see end of §3) which is easily checked directly. These properties permit us to obtain a global Riesz representation for $G(z)$ analogous to the one for $(\mathfrak{M}F)(z)$ discussed on pp. 376-388 of [2], and, in a sense, *simpler* than the latter on account of the evenness of $G(z)$.

In order to get a multiplier φ, one first projects G's Riesz mass along circles about 0 from the curves $y=\pm Y(x)$ with $x>0$ onto the positive real axis. The total Riesz mass on those curves contained between the circles $|z|=r$ and $|z|=r+\triangle r$ (in the right half plane) is in other words transported to the segment $[r,\ r+\triangle r]$ of the x-axis. That projected mass is *combined* with the original Riesz mass (equal to $\triangle r/\pi$) *already present* on that segment, and the *total* mass distribution thus obtained on $[0,\infty)$ is denoted by $d\nu(t)$.

Write $\nu(t)=\int_0^t d\nu(\tau)$, and let $[\nu(t)]$ denote the greatest integer $\leqslant\nu(t)$. It suffices to take the entire function $\varphi(z)$ with

$$\log|\varphi(z)| \;=\; \int_a^{\infty}\log\left|1-\frac{z^2}{t^2}\right|d[\nu(t)],$$

where $a > 0$ is chosen to make $\nu(a) = 2$ (say). Cf. [2], pp. 162-4. The details of the construction just sketched are left to the reader.

APPENDIX

Let us show how to obtain the result from [5] used in proving the corollary at the end of the last paragraph.

THEOREM. — *Let $\Phi(z)$, entire, of exponential type, and in modulus $\geqslant 1$ on $\mathbb{R}$, have all its zeros in $\Im z < 0$. Then, given any $\ell > 0$, there is an $f(x) \geqslant \log|\Phi(x)|$ satisfying* (3) *and* (60).

Proof. — For a suitable constant $c > 0$, $cy + \log|\Phi(x+iy)|$ is an *increasing* function of y in the upper half plane; this may be verified by performing logarithmic differentiation on the Hadamard product for $\Phi(z)$ and then taking account of the convergence of $\sum_n \Im(1/z_n)$ for the zeros z_n of Φ. (That convergence follows in turn from a theorem of Lindelöf; see [7], p. 20.) We may just as well work with $e^{-icz}\Phi(z)$ instead of $\Phi(z)$ since both have the same modulus on the real axis. Changing, then, our notation so as to have $\Phi(z)$ designate the *former* product, we have the function $\log|\Phi(z)|$, *harmonic* for $\Im z > 0$ and an *increasing* function of y there. It is > 0 in the upper half plane because $|\Phi(x)| \geqslant 1$ on $\mathbb{R}$, and, being continuous up to the real axis, has the Poisson representation

$$(62)\qquad \log|\Phi(z)| \;=\; B\Im z \;+\; \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z \log|\Phi(t)|}{|z-t|^2}\,dt, \quad \Im z > 0,$$

where (wlog) $B > 0$. See [7], pp. 41-42 and pp. 47-48 – our present hypothesis implies in particular that

$$(63)\qquad \int_{-\infty}^{\infty}\frac{\log|\Phi(t)|}{1+t^2}\,dt \;<\; \infty.$$

We now take the *smallest* function $f_+(x) \geqslant \log|\Phi(x)|$ with

$$f_+(x') - f_+(x) \;\leqslant\; \ell(x'-x) \quad \text{whenever } x' > x, \quad x', x \in \mathbb{R},$$

and denote by $f_-(x)$ the smallest such function with

$$f_-(x') - f_-(x) \;\geqslant\; -\ell(x'-x) \quad \text{whenever } x' > x, \quad x', x \in \mathbb{R}.$$

Putting then

$$f(x) = \max(f_+(x),\ f_-(x)),$$

it is clear that $f(x)$ satisfies (60) and is $\geqslant \log|\Phi(x)|$; we therefore need only verify (3). That in turn follows if $f_+(x)$ and $f_-(x)$ *each* satisfy (3), and we show this for $f_+(x)$, the treatment for $f_-(x)$ being analogous.

The function $f_+(x)$ actually *coincides* with $\log|\Phi(x)|$ except over certain disjoint intervals $(a_k,\ b_k)$, where

$$f_+(x) = \log|\Phi(a_k)| + \ell(x-a_k), \qquad a_k < x < b_k. \tag{64}$$

Although it is conceivable that $b_k = \infty$ for some k, that cannot happen here, for if it *did*, we would have $\limsup\limits_{x\to\infty}(\log|\Phi(x)|/x) \geqslant \ell > 0$ which is impossible for entire functions $\Phi(z)$ of exponential type satisfying (63) (cf. [7], p. 174). Each b_k is therefore finite, and we have

$$\log|\Phi(b_k)| = \log|\Phi(a_k)| + \ell(b_k - a_k). \tag{65}$$

Fixing our attention on any interval $(a_k,\ b_k)$ we put

$$y_k = \frac{1}{2B}\log|\Phi(b_k)|.$$

Since $\log|\Phi(b_k+iy)|$ increases with y when $y>0$, we get, from (62),

$$u(b_k+iy_k) \geqslant \frac{1}{2}\log|\Phi(b_k)| = By_k,$$

where

$$u(z) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z\log|\Phi(t)|}{|z-t|^2}\,dt, \qquad \Im z > 0. \tag{66}$$

But by (65), since $\log|\Phi(a_k)| \geqslant 0$,

$$y_k \geqslant \frac{\ell}{2B}(b_k - a_k).$$

Therefore, the harmonic function $u(z)$ given by (66) being positive, we have

$$u(x+iy_k) \geqslant cy_k \quad \text{for } a_k \leqslant x \leqslant b_k$$

by Harnack, where c is a certain constant >0 depending only on B and ℓ. In terms of the notation introduced at the beginning of §2, this means that

$$Y_c(\log|\Phi|, x) \geqslant cy_k = \frac{c}{2B}\log|\Phi(b_k)| \quad \text{for} \quad a_k \leqslant x \leqslant b_k.$$

Thence, in view of (63) and Remark 1 to Lemma 2.4,

$$\sum_k \int_{a_k}^{b_k} \frac{\log|\Phi(b_k)|}{1+x^2}\,dx \quad < \quad \infty,$$

so by (64) and (65),

$$\sum_k \int_{a_k}^{b_k} \frac{f_+(x)}{1+x^2}\,dx \quad < \quad \infty.$$

Since $f_+(x) = \log|\Phi(x)|$ for $x \notin \bigcup_k (a_k,\ b_k)$, the last relation and (63) make

$$\int_{-\infty}^{\infty} \frac{f_+(x)}{1+x^2}\,dx \quad < \quad \infty.$$

We are done.

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