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SMOOTH NORMALIZATION OF A VECTOR FIELD NEAR A SEMISTABLE LIMIT CYCLE

by Sergey Yu. YAKOVENKO

0. Formulation of the main result.

We consider a neighborhood of a semistable (of multiplicity two) limit cycle for a C^∞-smooth vector field on the real plane. If γ is a smooth transversal to the cycle, endowed with a local chart $x \in (\mathbb{R}^1, 0)$, then the monodromy (otherwise called the first return map) $\Delta\ : (\mathbb{R}^1, 0) \to (\mathbb{R}^1, 0)$ is well defined and smooth germ,

$$\Delta(x) = x + cx^2 + \cdots, \qquad c \neq 0.$$

For such a germ the smooth normal form is well known.

THEOREM (F. Takens, 1973 [T]). — *There exists a local C^∞-smooth transformation conjugating the germ Δ with the time 1 map for the vector field* $\mathbf{p}$ *on the real line,*

$$\text{(1)} \qquad \mathbf{p} = p(x) \cdot \partial_x, \qquad p(x) = x^2 + ax^3, \quad x \in (\mathbb{R}^1, 0), \quad a \in \mathbb{R}^1.$$

The scalar parameter $a \in \mathbb{R}$ is an invariant of both smooth and formal classifications. □

Note. — If a coordinate system $(x, \ldots)$ on a manifold (the real line in the above case) is fixed, we use the symbol ∂_x for the corresponding coordinate unit vector field.

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COROLLARY. — *There exists a smooth diffeomorphism conjugating the vector field near a semistable limit cycle with the vector field on the cylinder* $C = (\mathbb{R}^1, 0) \times (\mathbb{R}/\mathbb{Z})$ *having the form*

$$R(x,\theta)((x^2 + ax^3) \cdot \partial_x + 1 \cdot \partial_\theta), \tag{2}$$

where R *is a certain smooth function, not vanishing (say, positive) on the circle* $S = \{x = 0\} \subseteq C$.

Proof. — A conjugacy $q : (\mathbb{R}^1, 0) \to (\mathbb{R}^1, 0)$ between the two monodromy maps (which exists by virtue of Takens theorem) may be extended to a diffeomorphism Q of a neighborhood of the cycle, sending orbits of the initial field into orbits of the field (2). Therefore the former field is sent by Q into the the field which differs from the standard one by a nonvanishing factor R. □

This corollary solves the problem of *orbital classification* of vector fields (in a neighborhood of a semistable cycle). For *flows* there is at least one additional invariant, namely, the period T of the cycle. It turns out that there is another one, and together with the formal invariant of the monodromy and the period, the three constitute a complete system of invariants of smooth classification of vector fields near limit cycles of multiplicity 2. The main result of the paper follows.

THEOREM. — *There exists a* C^∞*-smooth transformation taking the field* (2) *into the polynomial form*

$$(1 + bx)((x^2 + ax^3) \cdot \partial_x + T^{-1} \cdot \partial_\theta), \qquad a, b, T \in \mathbb{R} \tag{3}$$

where $T > 0$ *is the period of the semistable periodic solution* $x \equiv 0$, *and* $b \in \mathbb{R}$ *is another real invariant of smooth classification: two fields with different invariants* (a, b, T) *are not conjugate to each other.*

The proof of this theorem is given in Sections 1 through 5 below, and in Section 6 there is given a generalization of this result for limit cycles of any finite multiplicity.

1. Preliminary normalization.

From now on we are working with vector fields on the cylinder $C = \mathbb{R} \times \mathbb{S}^1$, $\mathbb{S}^1 = \mathbb{R} \bmod \mathbb{Z}$, endowed with the standard coordinates $(x, \theta \bmod 1)$, and consider only transformations which preserve the orbital structure of the field (2). Thus performing all these transformations, one has to control only the function $R(\cdot,\cdot)$.

LEMMA 1. — *The field* (2) *can be transformed to another one of the following form,*

$$(1 + x\rho(x,\theta))((x^2 + ax^3) \cdot \partial_x + T^{-1} \cdot \partial_\theta), \tag{4}$$

where ρ *is a smooth function on the cylinder.*

Proof. — We choose T being equal to the period of the cycle $\mathbb{S}^1 = \{x = 0\}$ for the field (2). Then there exists a transformation Θ of the circle $\mathbb{S}^1$ into itself conjugating the flow $R(0,\theta) \cdot \partial_\theta$ with $T^{-1} \cdot \partial_\theta$. If one extends such a transformation to the transformation of the form $(x,\theta) \mapsto (X(x,\theta), \Theta(\theta))$ preserving the orbital structure, then such a transformation will send the field (2) to the field coinciding with (4) on $\mathbb{S}^1$ and proportional to it everywhere else. □

Remark. — As it will be seen from the proof of Lemma 3, the coefficient b in the normal form (3) is the average of the function $\rho_0(\theta) = \rho(0,\theta)$ over the cycle $\mathbb{S}^1$: $b = \int_0^1 \rho_0(\theta)\, d\theta$.

2. Homotopy method and homological equation.

In this section we use the homotopy method in the form exposed in [IY1]; other applications of this method one can find in [M]. From now on we fix the value of period T and denote for simplicity by $\mathbf{v}$ the field on the cylinder,

$$\mathbf{v} = (x^2 + ax^3) \cdot \partial_x + T^{-1} \cdot \partial_\theta, \tag{5}$$

and by $g^t : C \to C$ the one-parameter group of the flow maps for the field $\mathbf{v}$.

Let $R(x,\theta;\tau) > 0$ be a smooth function depending on an additional parameter $\tau \in [0,1]$, and $\widetilde{C} = C \times [0,1]$ the extended phase space endowed with the coordinates (x,θ,τ). Consider the vector field $V = R \cdot \mathbf{v} + 0 \cdot \partial_\tau$ on $\widetilde{C}$ tangent to the foliation $\tau = \text{const}$.

PROPOSITION (Homotopy lemma). — *Suppose that there exists a smooth vector field* H *on* $\widetilde{C}$ *having the form* $A(x,\theta;\tau) \cdot \partial_x + B(x,\theta;\tau) \cdot \partial_\theta + 1 \cdot \partial_\tau$*, which commutes with the field* V*. Assume that* $A(0,\cdot\,;\cdot) \equiv B(0,\cdot\,;\cdot) \equiv 0$.

Then the fields $w_0 = R(\cdot,\cdot;0)\cdot \mathbf{v}$ *and* $w_1 = R(\cdot,\cdot;1)\cdot\mathbf{v}$ *are smoothly conjugate.*

Proof. — The time 1 map for the field H sends $C_0 = C\times\{0\}$ to $C_1 = C\times\{1\}$ and conjugates restrictions of the field $w_i = V|_{C_i}$, $i=0,1$, with each other. The conditions on A and B guarantee that the time 1 map for the field H is well defined in a sufficiently small neighborhood of the cycle $\{x=0\}\subseteq C_0$. □

We are interested in the case of

$$R(x,\theta;\tau) = 1 + b(\tau)x + (1-\tau)x\rho(x,\theta), \qquad b(0)=0, \tag{6}$$

which would yield a smooth conjugacy between the field $w_0 = (1+x\rho(x,\theta))\cdot \mathbf{v}$ and $w_1 = (1+b(1)x)\cdot\mathbf{v}$. Let us look for a field H commuting with $V = R\cdot\mathbf{v}$ in the form

$$H = f\cdot V + 1\cdot\partial_\tau,$$

where $f : C\to\mathbb{R}$ is a smooth function. Then the commutativity relation $[H,V]=0$ takes the form

$$\begin{aligned} 0 = [1\cdot\partial_\tau + fV, V] &= \frac{\partial R}{\partial\tau}\mathbf{v} + [fV,V] \\ &= \frac{\partial R}{\partial\tau}\mathbf{v} + (L_V f)V + f[V,V] \\ &= \frac{\partial R}{\partial\tau}\mathbf{v} + R^2(L_{\mathbf{v}} f)\mathbf{v}, \end{aligned}$$

where $L_V f$ and $L_{\mathbf{v}} f$ stand for the Lie derivative of the function f along the fields V and $\mathbf{v}$ respectively. This equation is in fact scalar rather than vector one, being equivalent to the *homological equation*

$$L_{\mathbf{v}} f = r, \qquad r = -R^{-2}\frac{\partial R}{\partial\tau}, \tag{7}$$

which is to be solved with respect to f. Later on we establish necessary and sufficient conditions to be imposed on the right hand side part r of the homological equation (7), guaranteeing its solvability.

3. Solvability of homological equation.

Integrating (7) between two points of subsequent intersection of a phase curve of the field $\mathbf{v}$ with some transversal, say, $\theta=0$, one obtains the *discrete time homological equation*

$$f\circ\Delta - f = h, \qquad h = h(\cdot,\tau), \tag{8}$$

where $\Delta\ : (\mathbb{R}^1, 0) \to (\mathbb{R}^1, 0)$ is the time 1 map for the standard vector field $\mathbf{p}$ on the real line (1), and h is a smooth function defined by the formula

$$h = \int_0^T (r \circ g^t)\, dt, \tag{9}$$

where g^t is the flow of the field $\mathbf{v}$ on the cylinder, and T is the corresponding period, see (5) (for sake of simplicity we omit in this section the parameter τ in the notation, while keeping in mind its presence).

LEMMA 2. — *The discrete homological equation* (8) *admits a smooth solution* f *smoothly depending on the parameter* τ, *if and only if*

$$\forall \tau \in [0,1] \qquad h(0) = h'(0) = 0.$$

Proof. — We prove this statement in two steps : first we prove solvability of (8) in the class of formal power series (with coefficients depending smoothly on τ), and then we show that there exists a flat solution for the right hand side part h which is flat in x. Both steps are standard.

A. Formal solvability.

Expand h in the Taylor series of the form

$$h \sim \sum_{k=0}^{\infty} h_k\, x^k, \qquad h_k = h_k(\tau) \in \mathbb{R}, \quad \tau \in [0,1], \quad h_0 \equiv h_1 \equiv 0,$$

and prove that there exists a formal series of the same form (but starting with a first order term), satisfying (8). Since the map Δ has no zero order term, the result of substitution of it into the formal power series is well defined series of the same form. Here and below dots after the addition sign stay for terms of orders higher than those explicitly written.

An elementary computation shows that $\Delta^k(x) = x^k(1 + kx + \cdots)$ and therefore

$$f \circ \Delta - f = \sum_k f_k\, (kx^{k+1} + \cdots),$$

whence comes the system of equations determining the coefficients f_k :

$$kf_k + \ \text{(linear combination of } f_j,\ j < k) = h_{k+1}, \qquad k = 1, 2, \ldots$$

Being triangular, such a system always admits a formal solution, that is an infinite sequence of smooth functions $f_k = f_k(\tau)$.

B. Flat solutions.

Suppose that the function h is flat in x, that is

$$\forall \alpha, \beta \in \mathbb{N} \quad \exists K_{\alpha,\beta} \in \mathbb{R} : \qquad \left| \frac{x^{-\beta} \partial^\alpha h}{\partial x^\alpha} \right| < K_{\alpha,\beta}.$$

Then the discrete homological equation admits a smooth solution with the same property.

Indeed, one can write the explicit formulas,

$$\begin{aligned} f &= -\sum_{s=0}^{\infty} h \circ \Delta^{[s]} \qquad \text{for} \quad x \leqslant 0, \\ f &= \sum_{s=-1}^{-\infty} h \circ \Delta^{[s]} \qquad \text{for} \quad x \geqslant 0, \end{aligned} \tag{10}$$

where $\Delta^{[s]}$ stands for s times iterated standard map, $\Delta^{[0]} = \mathrm{id}$. We prove that these series converge to smooth flat functions in a sufficiently small neighborhood of the origin (sl is sufficient to take a closed segment not containing fixed points of Δ other than that at the origin). Obviously, it is sufficient to prove this fact only for any of the two formulas.

If $x \leqslant 0$, then on $[-1/2|a|, 0]$ one has the uniform estimate $|\Delta^{[s]}(x)| \leqslant O(1/s)$ (since

$$\int_{-1/2|a|}^{\Delta^{[s]}(x)} \frac{du}{p(u)} \geqslant \int_{x}^{\Delta^{[s]}(x)} \frac{du}{p(u)} = s, \qquad p(u) = u^2 + au^3,$$

by definition of the standard map Δ). Therefore for any β

$$|h \circ \Delta^{[s]}(x)| < \widetilde{K}_{0,\beta}\, s^{-\beta}$$

uniformly on $[-1/2|a|, 0]$, and the series (10) converges to a continuous function, if $\beta \geqslant 2$, the sum decreasing more rapidly than any power of $|x|$ as $x \to 0^-$.

To prove differentiability, we note that the derivative of $\Delta^{[s]}$ satisfies the identity

$$\big(\Delta^{[s]}\big)'(x) \cdot p(x) = \big(p \circ \Delta^{[s]}\big)(x),$$

since $\Delta^{[s]}$ is the automorphism of the standard field $\mathbf{p} = p(x) \cdot \partial_x$ on the line, thus formal differentiation of (10) yields the series

$$f' = -\sum_{s=0}^{\infty} \frac{(h' \circ \Delta^{[s]}) \cdot (p \circ \Delta^{[s]})}{p},$$

which is of the same form as above, but with the function h replaced by $h' \cdot p$, up to the common factor p^{-1}. Since for a flat h the product $h' \cdot p$ is also flat, the latter series also converges to a flat function, and this flatness of the sum is not affected by the division by p. Iterating these arguments, one proves infinite differentiability and flatness of the series (10). Differentiability in τ is even more easy to see, since Δ does not depend on the parameter while all derivatives of h with respect to τ are flat in x.

C. Smooth solution.

To prove that (8) admits a smooth solution, we find a formal one $\widehat{f}$ and then extend it to a smooth function $\widetilde{f}$, using Borel theorem. The extension satisfies the homological equation up to an error $\widetilde{h} = \widetilde{f} \circ \Delta - \widetilde{f} - h$ which has the identically zero Taylor series, being therefore a flat function. Since the homological equation is linear, such an error can be eliminated by subtracting from $\widetilde{f}$ any solution to the homological equation with the flat right hand side part $\widetilde{h}$.

D. Necessity of the solvability condition.

If the equation (8) admits a C^1-smooth solution, then

$$h(0) = f(\Delta(0)) - f(0) = 0, \quad h'(0) = f'(0) \cdot \Delta'(0) - f'(0) = f'(0) - f'(0) = 0.$$

□

4. Construction of an appropriate homotopy.

It remains to prove that one can construct a homotopy $R(\cdot\,; \tau)$ in such a way that the solvability conditions for the homological equation (8) would be satisfied.

LEMMA 3. — *There exists a path $\tau \mapsto b(\tau) \in \mathbb{R}$ such that the homotopy* (6) *would yield the right hand side in* (7) *satisfying the solvability condition established in Lemma 2.*

Proof. — The condition $h(0) = 0$ is automatically satisfied because of the preliminary normalization : $g^t(\mathbb{S}^1) = \mathbb{S}^1$, and $r|_{\mathbb{S}^1} \equiv 0$, where g^t is the flow of the field $\mathbf{v}$.

We compute the derivative $h'(0)$. Evidently, for $t < T$ one has

$$g^t(x, \theta) = (x + O(x^2), \theta + tx + O(x^2)),$$

$$R \circ g^t(x,\theta) = 1 + O(x), \qquad \frac{\partial R}{\partial \tau} \circ g^t(x,\theta) = x(b'(\tau) - \rho(0, t/T)) + O(x^2),$$

and therefore the linear part of the integral (9) is

$$h'(0) = b'(\tau) - \int_0^1 \rho(0,\theta)\, d\theta.$$

Finally we conclude that $h'(0) = 0$, if $b(\tau) = \tau \cdot \langle \rho(0,\cdot) \rangle$ (the averaged value of $\rho(0,\cdot)$). Thus the explicit value of the parameter $b(1)$ in the normal form is determined. □

5. Necessary conditions : uniqueness of the normal form.

Let there be two systems of the form (3) but with different parameters a, b, T. Then they are C^2-smoothly equivalent if and only if the corresponding triplets coincide.

In the part concerning the values a and T this is evident : the parameter a is the formal invariant of the monodromy map therefore it is defined in a unique way, see [T]. The period of the solution $x = 0$ is also invariant even with respect to topological classification. So it remains only to prove that the parameter b is uniquely defined.

PROPOSITION. — *Let* $\mathbf{v}_i = (1 + b_i x) \cdot \mathbf{v}$, $i = 1, 2$, *be two vector fields on the cylinder. They are C^2-conjugate if and only if* $b_1 = b_2$.

Proof. — The "if" part is trivial. So assume that $H : C \to C$ is a C^2-smooth conjugacy between the two fields. Without loss of generality one may consider it being identical on the circle $\mathbb{S}^1$ (otherwise a parallel translation $\theta \mapsto \theta + \text{const}$ should be performed after H). Denote by g_i^t, $i = 1, 2$, the flows of the two fields, and by $\left(g_i^T\right)_* |_{\mathbb{S}^1} = J_i(\theta) \in \mathrm{GL}\,(2, \mathbb{R})$ the Jacobian matrices of their time T maps. Then

$$J_i(\theta) \equiv \begin{pmatrix} 1 & 0 \\ -b_i & 1 \end{pmatrix}.$$

Since H conjugates the flows, $H \circ g_1^T = g_2^T \circ H$, so the linearization $H_* = H_*(\theta)$ satisfies the equation

$$H_* J_1 = J_2 H_*,$$

whence we conclude that the ratio of two diagonal elements of H_* must be equal to b_1/b_2, while the field ∂_θ is invariant by H_*. Therefore in the coordinates x, θ one has

$$H_*|_{\mathbb{S}^1} = \begin{pmatrix} b_1/b_2 & 0 \\ * & 1 \end{pmatrix}.$$

Now note that such a diffeomorphism when applied to the vector field $(x^2+\cdots)\cdot\partial_x+(\cdots)\cdot\partial_\theta$, takes it into the form $(kx^2+\cdots)\cdot\partial_x+(\cdots)\cdot\partial_\theta$, $k=b_1/b_2$, hence the condition $b_1=b_2$ comes. □

This proposition finishes the proof of the main theorem.

Remark. — The formal solvability of the homological equation (8) can be interpreted as the existence of a *semiformal* conjugacy

$$(x,\theta)\mapsto\Big(\sum_{k=1}^{\infty}\Theta_k(\theta)\,x^k,\theta\Big)$$

sending the field (4) into the normal form (3) : the functions $\Theta_k(\theta)$ are smooth, but no convergence assumption is made about the series. The remaining part of the proof of Lemma 2 is actually establishing the implication "formal conjugacy $\Longrightarrow$ smooth conjugacy" which is known to be true in some cases of local dynamical systems, in particular, for quasihyperbolic germs of diffeomorphisms at fixed points, see [B], Theorem 2.3.2. On the other hand, there exist examples where such an implication does not hold [IY2]. Our case is nonlocal.

6. Concluding remarks.

In a similar way one can prove that for a limit cycle of multiplicity $k\geqslant 3$, with the monodromy map of the form $x\mapsto x+cx^k+\cdots$, $c\neq 0$, $k>2$, the normal form is

$$P_{k-1}(x)((x^k+ax^{2k-1})\cdot\partial_x+T^{-1}\cdot\partial_\theta),$$

where P_{k-1} is a polynomial of degree $k-1$ with $P(0)=1$. All the details of the above proof go with only minor changes :

(1) the first return map is smoothly conjugate to the time one map for the field $\mathbf{p}=(x^k+ax^{2k-1})\cdot\partial_x$ on the real line [T], therefore the representation (2) holds, and the homological equation can be written down exactly as before;

(2) the discrete homological equation (8) admits a formal solution if and only if the function h has no terms of order less than k;

(3) for a flat function h the flat solution of (8) always exists and is given by the same formulas (10) : only the estimates proving the differentiability are changed, for example, on the interval $(-1/2a^{1/(k-1)},0)$ one has

$$|\Delta^{[s]}(x)|\leqslant O(s^{1/(k-1)}),$$

which still makes possible all the upper estimates of Section 3B;

(4) the homotopy of the form

$$V(\cdot,\tau) = R(\cdot,\tau)\cdot \mathbf{v}, \qquad \mathbf{v} = (x^k + ax^{2k-1})\cdot \partial_x + T^{-1}\cdot \partial_\theta,$$

$$R(x,\theta,\tau) = 1 + \sum_{j=1}^{k-1} b_j(\tau)\, x^j + (1-\tau)x\rho(x,\theta)$$

may be chosen in such a way that the solvability conditions for the equation (8) will be met;

(5) the polynomial P appears as $1 + \sum b_j(1)\, x^j$.

To clarify the nature of such a polynomial, proving at the same time its uniqueness, we need the following construction. Let γ be any smooth transversal to the cycle. Then with such a transversal, besides the monodromy Δ, also the return time function $\mathbf{T}\ : \gamma \to \mathbb{R}$, $x \mapsto \mathbf{T}(x)$, is associated. This construction is persistent under deformations splitting the fixed point at the origin into $\leqslant k$ points, generically nondegenerate. The values taken by the function $\mathbf{T}$ at these points, are invariants of smooth classification. Hence one concludes that the limit object $\mathbf{T} \bmod (\Delta - \mathrm{id})$ is invariantly associated with the field; this object is the equivalence class of the function $\mathbf{T}(\cdot)$ in the local algebra $C^\infty_x / \langle \Delta(x) - x\rangle$, see [AVG]. Such a class contains a unique polynomial P_{k-1} of degree $\leqslant k-1$, the truncated Taylor polynomial of the function $\mathbf{T}(\cdot)$, and is determined only by the unperturbed function.

A similar problem of finding a smooth normal form can be posed also for a family of vector fields, generically unfolding a field with a multiple limit cycle. In such a case the Takens theorem does not hold any more, and one can show that even in the C^1-smooth classification of unfoldings of diffeomorphisms tangent to identity, functional moduli appear, see the paper [IY2] in which a smooth classification for the case $k = 2$ was obtained. This implies that the orbital classification of such families already possesses functional moduli. But still there is a hypothesis that taking into consideration the dynamics, one adds to the functional moduli only a finite number of scalar invariants in the same manner as above.

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