ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 43, nº 3 (1993), p. 655-712

<http://www.numdam.org/item?id=AIF_1993__43_3_655_0>

Ann. Inst. Fourier, Grenoble
43, 3 (1993), 655-712

ON THE COMPLEX ANALYTIC GEL'FAND-FUKS COHOMOLOGY OF OPEN RIEMANN SURFACES

by Nariya KAWAZUMI

Introduction.

The (continuous) cohomology theory of the Lie algebra of C^∞ vector fields on C^∞ manifold X was originated by Gel'fand and Fuks [GF][GF1]. Accordingly it has been called the Gel'fand-Fuks cohomology theory. Haefliger [Ha], Bott and Segal [BS] described the cohomology with trivial coefficient as the singular cohomology of the section space of a certain fiber space over X. The cohomologies with coefficients in the tensor fields on X were described in a similar way by Tsujishita [Ts].

Our purpose is to establish a complex analytic analogue of these theories. In the present paper we confine ourselves to the Lie algebra of complex analytic vector fields on an open Riemann surface.

Let M be an open Riemann surface and S a finite subset of M. We denote by $L(M,S)$ the Lie algebra of complex analytic vector fields on M which have zeroes at all points in S. $L(M,S)$ is a Fréchet space with respect to the topology of uniform convergence on compact sets. The bracket $[\,,\,]$ is continuous, and so $L(M,S)$ is a topological Lie algebra. For $S=\varnothing$ we abbreviate $L(M,S)$ to $L(M)$. In view of a theorem of Behnke and Stein [BeSt] M is a Stein manifold, and so we can utilize some topological methods. The uniformization theorem of Riemann surfaces supports our investigation.

Key words : Riemann surfaces – Vector fields – Gel-fand-Fuks cohomology – Spaces of holomorphic functions.
A.M.S. Classification : 30H05 – 57R32 – 14H15.

Recently, on the other hand, the relations of the Lie algebra $L(M)$ to the moduli space of Riemann surfaces via the Virasoro action have been clarified [ADKP]. To investigate these relations more closely we need the cohomology with coefficients in the complex analytic tensor fields of certain types on the product spaces M^n, on which $L(M)$ acts by the diagonal action. In this paper we study the cohomologies with such coefficients.

Before the result of Tsujishita [Ts], Rešetnikov [R1] determined the cohomology of the Lie algebra of C^∞ vector fields on the circle S^1 with coefficient in the C^∞ functions on S^1 in a classical way. Since we study more general coefficients than the tensor fields on M itself, we need to pursue Rešetnikov's classical method.

Fix a type $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{Z}^n$ arbitrarily. Consider complex analytic tensor fields on the product space M^n represented locally as $f(z_1, \ldots, z_n) d{z_1}^{\nu_1} \cdots d{z_n}^{\nu_n}$, where z_i is a local coordinate of the i-th component. Let $T(M^n)$ be the Fréchet space of complex analytic tensor fields on M^n of the fixed type ν. The Lie algebra $L(M)$ acts on $T(M^n)$ continuously by the diagonal action. Denote by $\mathcal{T}_{(x_1,\ldots,x_n)}$ the $L(M)$ module of germs of complex analytic tensor fields of the type ν at the point $(x_1, \ldots, x_n) \in M^n$. By a classical method coming from Rešetnikov, there exists a spectral sequence

$$E_2^{p,q} = H^p(M^n; \mathcal{H}^q)$$

converging to $H^*(L(M); T(M^n))$ (see §9), where $\mathcal{H}^q$ is a sheaf on M^n whose stalk at $(x_1, \ldots, x_n) \in M^n$ is given by

$$\mathcal{H}^q_{(x_1,\ldots,x_n)} = H^q(L(M); \mathcal{T}_{(x_1,\ldots,x_n)}).$$

Our main theorem in the present paper gives a decomposition of the cohomology group $H^q(L(M); \mathcal{T}_{(x_1,\ldots,x_n)})$ into the global part derived from the homology of M and the local part coming from the coefficient $\mathcal{T}_{(x_1,\ldots,x_n)}$. Denote by $W_1 := \mathbb{C}\{z\}\dfrac{d}{dz}$ the topological Lie algebra of germs of complex analytic vector fields at $0 \in \mathbb{C}$. Set $T := \{x_1, \ldots, x_n\} \subset M$. There exists a W_1 module of germs of tensor fields N_0^t for each $t \in T$ such that an $L(M)$ isomorphism

$$\mathcal{T}_{(x_1,\ldots,x_n)} = \bigotimes_{t\in T} N_0^t$$

holds, where N_0^t is acted on by $L(M)$ through a local parametrization centred at t. Here and throughout this paper $\otimes$ means the *completed* tensor product [G,PTT] [T].

THEOREM 5.3. — *Let M be a connected open Riemann surface whose first Betti number $b_1(M)$ is finite and T a finite subset of M. Suppose a W_1 module N^t satisfying the condition (5.2) in §5 is given for each $t \in T$. Then we have an isomorphism*

$$H^*(L(M); \bigotimes_{t\in T} N^t) \cong \bigwedge^*(\Sigma^3 H_1(M,T)) \otimes \bigotimes_{t\in T} H^*(W_1; N^t),$$

where $\Sigma^3 H_1(M,T)$ is the graded linear space concentrated to degree 2 given by the 3 times suspension of the first complex valued singular homology group $H_1(M,T)$, and $\bigwedge^*(\Sigma^3 H_1(M,T))$ is the free graded commutative algebra generated by the graded space.

Now our N_0^t's stated above satisfy the condition (5.2). Consequently the computation of $H^*(L(M); T(M^n))$ is reduced to

(1) computing the local part $H^*(W_1; N_0^t)$ and

(2) studying the topology of the configuration space M^n.

It seems very difficult to establish a general theory for computing such cohomology groups as $H^*(W_1; N_0^t)$ (cf, [FF]). It would be our next problem to investigate those groups with concretely fixed coefficients.

The outline of the paper is as follows. We explain the meaning of the condition (5.2) in the above theorem in §2 and §4. In §3 we construct the cohomology classes of degree 2 coming from $H_1(M,T)$ stated above. The classes are represented by cocycles taking the form of covariant derivatives. In §5 we formulate our main theorem and Addition Theorem of Bott-Segal type for the Lie algebra $L(M,S)$. The latter plays a fundamental role throughout this paper. It is proved in §8 by replacing partition-of-unity arguments in [BS] by Oka-Cartan's Theorem B. Addition Theorem for the formal vector fields is formulated by Feigin-Fuks [FF] and Retakh-Feigin [RF]. Using Addition Theorem, we compute the cohomology of $L(M,S)$ with trivial coefficient in §6 and §7. In §9 a spectral sequence coming from Rešetnikov is introduced, and two easy examples (the tensor fields on M itself and the functions on $\mathbb{C}^3$) are given.

The author would like to express his gratitude to Prof. Y. Matsumoto and Dr. K. Ahara for their constant encouragement and to Prof. H. Komatsu, Prof. T. Tsujishita and Prof. K. Iwasaki for their helpful suggestions.

Contents

1. Basic notations.

For a complex topological Lie algebra $\mathfrak{g}$, we mean by a $\mathfrak{g}$ *module* a complex topological vector space which $\mathfrak{g}$ acts on continuously. The continuous cochain complex of the topological Lie algebra $\mathfrak{g}$ with coefficients in a $\mathfrak{g}$ module N is denoted by

$$C^*(\mathfrak{g};N) = \bigoplus_{p\geq 0} C^p(\mathfrak{g};N),$$

whose cohomology group, i.e., the continuous cohomology group of $\mathfrak{g}$ with coefficients in N, is denoted by $H^*(\mathfrak{g};N)$. The complex $C^*(\mathfrak{g};N)$ is acted on by $\mathfrak{g}$ through the Lie derivative $\mathcal{L}(\cdot)$. When N is the trivial $\mathfrak{g}$ module $\mathbb{C}$, we abbreviate them to $C^*(\mathfrak{g})$ and $H^*(\mathfrak{g})$ respectively. (For details, see for example [HS].)

Let M be an open Riemann surface (1 dimensional non-compact complex manifold) and S a finite subset of M. We denote by $L(M,S)$ the Lie algebra of complex analytic vector fields on M which have zeroes at all points in S. $L(M,S)$ is a Fréchet space with respect to the topology of uniform convergence on compact sets. The bracket $[\,,\,]$ is continuous, and so $L(M,S)$ is a complex topological Lie algebra. When $S=\varnothing$, we abbreviate $L(M,S)$ to $L(M)$.

A typical example of $L(M)$ modules is the Fréchet space of complex analytic k-th covariant tensor fields on M :

$$T_k(M) = H^0(M; \mathcal{O}_M((T^*M)^{\otimes k})).$$

($\mathcal{O}_M$ denotes the structure sheaf of the complex manifold M.) $L(M)$ acts on it continuously by the (usual) Lie derivative $\mathcal{L}(\cdot)$. For $k = 0, 1, 2$, we denote $T_0(M) = F(M)$, $T_1(M) = K(M)$ and $T_2(M) = Q(M)$.

Let M' be an open Riemann surface, S' a finite subset of M' and $\phi : M \to M'$ a complex analytic immersion satisfying $\phi(S) \subset S'$. In an obvious way the pullback homomorphism

$$\phi^* : L(M', S') \to L(M, S)$$

is induced. If N is a $L(M,S)$ module, $L(M', S')$ acts on N through the homomorphism ϕ^*. The $L(M', S')$ module obtained in this way is denoted by $\phi_* N$. A natural cochain map

$$\phi_* : C^*(L(M,S); N) \to C^*(L(M', S'); \phi_* N)$$

is induced.

Denote by L_0 the Lie algebra of germs at the origin $0 \in \mathbb{C}$ of complex analytic vector fields on the complex line $\mathbb{C}$ which have a zero at the origin 0,

$$L_0 := z\mathbb{C}\{z\}\frac{d}{dz} = \varinjlim_{\epsilon \to 0} L(\{|z| < \epsilon\}, \{0\}),$$

which is endowed with the inductive limit locally convex topology [G, TVS] [K] [K1]. A 1-cocycle δ_0 of L_0 with trivial coefficient is defined by

$$\delta_0\left(f(z)\frac{d}{dz}\right) = f'(0),$$

which plays an important role in studying the cohomology of L_0.

A parametrization ϕ_s centered at $s \in M$ is a complex analytic homeomorphism of a neighbourhood of $0 \in \mathbb{C}$ onto a neighbourhood of $s \in M$ satisfying $\phi_s(0) = s$. If $s \in S$, the parametrization ϕ_s induces the pullback homomorphism

$$\phi_s{}^* : L(M,S) \to L_0.$$

Using the coordinate $z_s = {\phi_s}^{-1}$, we have

$$ {\phi_s}^*(X) = f(z)\frac{d}{dz} \in L_0, $$

where $X \overset{\text{loc.}}{=} f(z_s)\dfrac{d}{dz_s} \in L(M,S)$. For a L_0 module N, a $L(M,S)$ module ${\phi_s}_*N$ and a natural cochain map

$$ {\phi_s}_* : C^*(L_0;N) \to C^*(L(M,S);{\phi_s}_*N) $$

are induced. A 1-cocycle δ_0^s of $L(M,S)$ with trivial coefficient is defined by

$$ \delta_0^s := {\phi_s}_*\delta_0 \in C^1(L(M,S)). $$

When $X \in L(M,S)$ is represented locally by $X = f_s(z_s)\dfrac{d}{dz_s}$, we have

$$ \delta_0^s(X) = \frac{df_s}{dz_s}(0). \tag{1.1} $$

The complex number $\delta_0^s(X)$ does not depend upon the choice of a parametrization ϕ_s.

Denote by W_1 the Lie algebra of germs at the origin $0 \in \mathbb{C}$ of complex analytic vector fields on the complex line $\mathbb{C}$ with the inductive limit locally convex topology :

$$ W_1 := \mathbb{C}\{z\}\frac{d}{dz} = \varinjlim_{\epsilon \to 0} L(\{|z| < \epsilon\}). $$

A parametrization ϕ_t centered at $t \in M$ induces the pullback homomorphism

$$ {\phi_t}^* : L(M) \to W_1. $$

If N is a W_1 module, a $L(M)$ module ${\phi_t}_*N$ and a natural cochain map

$$ {\phi_t}_* : C^*(W_1;N) \to C^*(L(M);{\phi_t}_*N) $$

are induced in a similar way.

In the sequel we consider the following situation. Let M be an open Riemann surface and S and T disjoint finite subsets of M. A L_0 module N^s is given for each $s \in S$ and a W_1 module N^t for each $t \in T$. A parametrization ϕ_u centered at each $u \in S \cup T$ is fixed. Then a $L(M,S)$ module

$$N := \bigotimes_{u \in S \cup T} \phi_{u*} N^u$$

is defined. Here and in the sequel, the tensor product $\otimes$ means the completed projective tensor product [G,PTT][T]. Our purpose in the present paper is to give a description of the cohomology $H^*(L(M,S);N)$ using the natural map

$$\phi_* : \bigotimes_{s \in S} H^*(L_0; N^s) \otimes \bigotimes_{t \in T} H^*(W_1; N^t) \to H^*(L(M,S);N)$$

under some natural restriction of N^u $(u \in S \cup T)$.

2. Regular L_0 modules and the case of disk.

In this section we introduce a natural restriction on L_0 modules to study the case M is a disk and S is a one-point set.

Set

$$e_0 = z\frac{d}{dz} \in L_0.$$

The 1-dimensional subspace $\mathbb{C}e_0$ is a subalgebra of L_0. For a L_0 module N, a subcomplex $C^*(L_0, e_0; N)$ of the complex $C^*(L_0;N)$ is defined by

$$C^*(L_0, e_0; N) = \{c \in C^*(L_0;N); \operatorname{int}(e_0)c = \mathcal{L}(e_0)c = 0\}.$$

Its cohomology group, i.e., the relative cohomology group of the pair $(L_0, \mathbb{C}e_0)$, is denoted by $H^*(L_0, e_0; N)$ (see [HS]).

For $\epsilon > 0$, we denote

$$D_\epsilon := \{|z| < \epsilon\} \subset \mathbb{C}.$$

Using the coordinates $z : D_\epsilon \hookrightarrow \mathbb{C}$, we regard the algebra $L(D_\epsilon, \{0\})$ as a subalgebra of L_0. The algebra $L(D_\epsilon, \{0\})$ acts on a L_0 module in this way.

The multiplicative group $\mathbb{C}^\times = \mathbb{C} - \{0\}$ acts complex analytically on the Lie algebra L_0 as follows :

$$T_t\Big(f(z)\frac{d}{dz}\Big) = \frac{1}{t}f(tz)\frac{d}{dz},$$

where $t \in \mathbb{C}^\times$ and $f(z)\dfrac{d}{dz} \in L_0$. Then we have

$$t\frac{d}{dt}T_t = T_t e_0 = e_0 T_t.$$

DEFINITION 2.1. — *A L_0 module N is* regular *if the multiplicative group $\mathbb{C}^\times$ acts complex analytically on N :*

$$T_t : N \to N \quad (t \in \mathbb{C}^\times),$$

and satisfies the following :

$$T_t(Xn) = T_t(X)T_t(n) \qquad (X \in L_0, n \in N) \tag{1}$$

$$t\frac{d}{dt}T_t = T_t e_0 = e_0 T_t. \tag{2}$$

Example. — The following L_0 modules are regular :

(1) the trivial module $\mathbb{C}$.

(2) the L_0 module of germs at the origin $0 \in \mathbb{C}$ of complex analytic k-th tensor fields on the complex line $\mathbb{C}$ of order $\geq l$

$$z^l \mathbb{C}\{\{z\}\}(dz)^{\otimes k},$$

where $k, l \in \mathbb{Z}$.

(3) the (completed) tensor product of two regular L_0 modules.

(4) the L_0 module 1_n ($n \in \mathbb{Z}$) defined as follows. 1_n is a 1 dimensional complex vector space. By abuse of notation, we also denote its base by 1_n :

$$1_n = \mathbb{C}1_n.$$

L_0 acts on 1_n by

$$X \cdot 1_n = n\delta_0(X)1_n, \qquad X \in L_0.$$

Thus, when N is a L_0 module, for $t \in \mathbb{C}^\times$, a cochain map

$$T_t : C^*(L(D_\epsilon, \{0\}); N) \to C^*(L(D_{|t|\epsilon}, \{0\}); N)$$
$$(T_t c)(X_1, \ldots, X_p) = T_t(c({T_t}^{-1}X_1, \ldots, {T_t}^{-1}X_p))$$

is induced. This map satisfies

$$t\frac{d}{dt}T_t = T_t e_0 = e_0 T_t. \tag{2.2}$$

PROPOSITION 2.3. — *If N is a regular L_0 module, there is a natural isomorphism*

$$H^*(L(D_\epsilon, \{0\}); N) = H^*(L_0, e_0; N) \otimes H^*(L(D_\epsilon, \{0\})).$$

Here $H^*(L_0, e_0; N)$ denotes the relative cohomology group of the pair $(L_0, \mathbb{C}e_0)$ with coefficients in the module N ([HS]).

PROPOSITION 2.4.

$$H^*(L(D_\epsilon, \{0\})) = \mathbb{C} \oplus \mathbb{C}\delta_0,$$

where $\delta_0 = \delta_0^0$ is the 1-cocycle defined in (1.1).

Proof of (2.3) *and* (2.4). — First of all we prove there is a natural isomorphism

$$H^*(L(D_\epsilon, \{0\}); N) = H^*(L(D_\epsilon, \{0\}), e_0; N) \otimes (\mathbb{C} \oplus \mathbb{C}\delta_0). \tag{2.5}$$

We denote by $C^*(L(D_\epsilon, \{0\}); N)^{e_0}$ the subcomplex of $C^*(L(D_\epsilon, \{0\}); N)$ consisting of all cochains invariant under e_0 (i.e., annihilated by the action $\mathcal{L}(e_0)$). Using the averaging operator

$$C^*(L(D_\epsilon, \{0\}); N) \to C^*(L(D_\epsilon, \{0\}); N)^{e_0}$$
$$c \mapsto \int_0^1 (T_{\exp 2\pi\sqrt{-1}\theta} c) d\theta,$$

one obtains a natural isomorphism

$$H^*(L(D_\epsilon, \{0\}); N) \cong H^*(C^*(L(D_\epsilon, \{0\}); N)^{e_0}).$$

Observe the interior product

$$\mathrm{int}(e_0) : C^*(L(D_\epsilon, \{0\}); N)^{e_0} \to C^{*-1}(L(D_\epsilon, \{0\}); N)^{e_0}$$

is a cochain map satisfying

$$\mathrm{int}(e_0)(\delta_0 \cup) + (\delta_0 \cup)\mathrm{int}(e_0) = 1.$$

Hence we obtain

$$C^*(L(D_\epsilon,\{0\});N)^{e_0} = C^*(L(D_\epsilon,\{0\}),e_0;N)\otimes(\mathbb{C}\oplus\mathbb{C}\delta_0),$$

which implies (2.5).

Next, for $0<\delta<1$, we consider the cochain map

$$\iota_\delta^* : C^*(L(D_{\epsilon\delta},\{0\}),e_0;N) \to C^*(L(D_\epsilon,\{0\}),e_0;N)$$

induced by the inclusion $\iota_\delta : L(D_\epsilon,\{0\}) \subset L(D_{\delta\epsilon},\{0\}) \subset L_0$. By (2.2), T_δ gives the inverse of the cochain map of ι_δ^*. Consequently we obtain

$$C^*(L(D_\epsilon,\{0\}),e_0;N) = C^*(L_0,e_0;N). \tag{2.6}$$

When N is the trivial module $\mathbb{C}$, (2.5) and (2.6) imply

$$H^*(L(D_\epsilon,\{0\}),e_0) = \mathbb{C}\oplus\mathbb{C}\delta_0,$$

which proves Proposition 2.4. Proposition 2.3 follows from (2.4), (2.5) and (2.6).

As is known,

$$H^*(W_1) = \bigwedge^{*}(\theta),$$

where the 3 cocycle θ is defined by

$$\theta\left(f(z)\frac{d}{dz}, g(z)\frac{d}{dz}, h(z)\frac{d}{dz}\right) = \det\begin{pmatrix} f(0) & f'(0) & f''(0)\\ g(0) & g'(0) & g''(0)\\ h(0) & h'(0) & h''(0)\end{pmatrix}$$

for $f(z)\frac{d}{dz}, g(z)\frac{d}{dz}, h(z)\frac{d}{dz} \in W_1$.

A parametrization ϕ_t centered at a point t in an open Riemann surface M induces a 3 cocycle

$$\theta^t := \phi_{t*}\theta \in C^3(L(M)). \tag{2.7}$$

The 3 *cocycle* θ^t does not depend upon the choice of parametrizations.

LEMMA 2.8. — *If M is contractible,*

$$\phi_{t*} : H^*(W_1) \to H^*(L(M))$$

is an isomorphism, i.e.,

$$H^*(L(M)) = \bigwedge^*(\theta^t).$$

Proof. — In view of the uniformization theorem of Riemann surfaces, we may assume $M = \{z \in \mathbb{C}; |z| < 1\}$ or $\mathbb{C}$ and the parametrization ϕ_s is induced by the inclusion $M \hookrightarrow \mathbb{C}$. By an argument similar to (2.4), one deduces

$$H^*(L(M)) = H^*(C^*(L(M))^{e_0}).$$

The complex $C^*(L(M))^{e_0}$ is equal to $C^*(W_1)^{e_0}$. Hence we have

$$H^*(L(M)) = H^*(W_1) = \bigwedge^*(\theta^t).$$

3. Covariant derivative cocycles.

On an open Riemann surface M, there exists a complex analytic nowhere zero vector field ∂. In fact, in view of a theorem of Behnke and Stein [BeSt], M is a Stein manifold, and hence all complex line bundles on M are *complex analytically* trivial.

For a non-negative integer k, we define a 1 cochain $\nabla_k^\partial \in C^1(L(M); T_k(M))$ by

$$\nabla_k^\partial(f\partial) = \frac{1}{(k+1)!}(\partial^{k+1}f)\partial^{-k} \qquad f \in F(M),$$

where $\partial^{-k} \in T_k(M)$ is the k-th tensor product of the dual of ∂. We call ∇_k^∂ the *covariant derivative cochain* associated to ∂. Using the Leibniz' rule, one deduces the following

Lemma 3.1.

$$d(\nabla_k^\partial) = \sum_{0<a<b<k, a+b=k} (b-a)\nabla_a^\partial \cup \nabla_b^\partial,$$

where the cocycle $\nabla_a^\partial \cup \nabla_b^\partial \in C^2(L(M); T_{a+b}(M))$ is given by contracting the coefficient in the cup product $\nabla_a^\partial \cup \nabla_b^\partial \in C^2(L(M); T_a(M) \otimes T_b(M))$. Especially $d\nabla_k^\partial = 0$ if and only if $k = 0, 1, 2$.

For $k = 0, 1, 2$, we call ∇_k^∂ the k-th *covariant derivative cocycle* associated to ∂.

The behavior of ∇_k^∂'s under the gauge transformation of TM needs to be clear. Let $g \in F(M)$ be a nowhere zero function on M. By straightforward calculations we have

LEMMA 3.2.

$$(1) \quad \nabla_1^{g^{-1}\partial} - \nabla_1^\partial = d\left(\frac{1}{2} d\log g\right) = d\left(\frac{1}{2}\frac{\partial g}{g}\partial^{-1}\right) \in C^1(L(M); K(M)).$$

$$(2) \ \nabla_2^{g^{-1}\partial} - \nabla_2^\partial = d\left(\frac{1}{3!}\left\{\frac{\partial^2 g}{g} - \frac{3}{2}\left(\frac{\partial g}{g}\right)^2\right\}\partial^{-2}\right) \in C^2(L(M); Q(M)).$$

Consequently the cohomology classes of ∇_1^∂ and ∇_2^∂ do not depend upon the choice of a nowhere zero vector field ∂. We denote their cohomology classes by ∇_1 and ∇_2, respectively.

Remark. — For local coordinates z and w, Lemma 3.2 (2) implies

$$\nabla_2^{\frac{d}{dw}} - \nabla_2^{\frac{d}{dz}} = d\left(\frac{1}{3!}\{w, z\} dz^{\otimes 2}\right),$$

where $\{w, z\}$ denoted the Schwarzian derivative of w with regard to z. Using this fact, one can obtain a function theoretic proof of the correspondence [ADKP] of the Virasoro 2-cocycle of $\mathrm{Vect}S^1$ to the Weil-Petersson Kähler form on the moduli space of Riemann surfaces via the Virasoro action.

Next we study the behavior of the cocycle ∇_0^∂. Observe that the holomorphic de Rham cohomology of a Stein manifold is naturally isomorphic to the (usual) de Rham cohomology. Thus we may identify $H^1(M) = K(M)/dF(M)$. A natural map

$$\kappa : K(M) \to C^1(L(M); F(M)), \quad \omega \mapsto (X \mapsto \omega(X))$$

induces an *injection* $\kappa : H^1(M) \to H^1(L(M); F(M))$, since the coboundary map on $C^0(L(M); F(M)) = F(M)$ is equal to the exterior derivative d.

LEMMA 3.3.

$$\nabla_0^{g^{-1}\partial} - \nabla_0^\partial = \kappa(d\log g) \in C^1(L(M); F(M)).$$

Consequently the set of all the cohomology classes induced by the 0-th covariant derivative cocycles ∇_0^∂ is just parametrized by the lattice $H^1(M; 2\pi\sqrt{-1}\mathbb{Z})$.

The covariant derivative cocycles ∇_0 and ∇_1 are utilized for constructing elements of $H^2(L(M,S))$ in the following way.

Let $H_1(M,S)$ denote the first complex valued singular homology group of the pair (M,S). By virtue of Stokes' theorem, for an element $\gamma \in H_1(M,S)$, the line integral along γ induces a well-defined $L(M,S)$-homomorphism

$$\int_\gamma : K(M) \to \mathbb{C} \text{ (the trivial module)}, \quad \omega \mapsto \int_\gamma \omega,$$

which induces a linear map

$$\int : H_1(M,S) \otimes H^*(L(M,S); K(M)) \to H^*(L(M,S)).$$

LEMMA 3.4. — *For $\gamma \in H_1(M,S)$ and $u \in H^1(M)$,*

$$\int_\gamma \kappa(u)\nabla_1 = 0 \in H^2(L(M,S)).$$

Proof. — Put $u = [h\partial^{-1}], h \in F(M)$. Define a 1-cocycle $c \in C^1(L(M,S))$ by $c = -\int_\gamma u\nabla_0^\partial$, that is,

$$c(f\partial) = -\frac{1}{2}\int_\gamma h(\partial f)\partial^{-1}, \quad f \in F(M).$$

Then easily we have $(\int_\gamma \kappa(u)\nabla_1)(f\partial, g\partial) = dc(f\partial, g\partial)$ for all $f, g \in F(M)$, which shows that the cocycle $\int_\gamma \kappa(u)\nabla_1$ is a coboundary.

From Lemma 3.4 follows

COROLLARY 3.5. — *The linear map*

$$\int \nabla_0\nabla_1 : H_1(M,S) \to H^2(L(M,S))$$

does not depend upon the choice of a nowhere zero vector field ∂.

We denote the image of this map by $\int_{H_1(M,S)} \nabla_0\nabla_1$.

The above construction may be generalized slightly.

Let s and t be two points on M. Denote by F^t (resp. F^s) the $L(M)$ module of germs at $t \in M$ (resp. $s \in M$) of complex analytic functions on M. Fix a path connecting s to t on M. Then the *indefinite* line integral along the path

$$\int_s^{t+\epsilon} : K(M) \to F^t, \quad \omega \mapsto \int_s^{t+\epsilon} \omega$$

and

$$\begin{aligned}\int_{s+\epsilon}^{t+\epsilon} &: K(M) \to F^t \otimes 1 + 1 \otimes F^s \subset F^t \otimes F^s \\ &\qquad \omega \mapsto \int_*^{t+\epsilon} \omega \otimes 1 - 1 \otimes \int_*^{s+\epsilon} \omega,\end{aligned}$$

(where $t+\epsilon$ varies near t on M,) are $L(M)$ and $L(M, \{s\})$ homomorphisms, respectively. Especially 2 cocycles

$$\int_s^{t+\epsilon} \nabla_0^\partial \nabla_1^\partial \in C^2(L(M, \{s\}); F^t)$$

and

$$\int_{s+\epsilon}^{t+\epsilon} \nabla_0^\partial \nabla_1^\partial \in C^2(L(M); F^t \otimes F^s)$$

are defined.

LEMMA 3.6. — *For $u \in K(M)$, we have*

$$\begin{aligned}\int_s^{t+\epsilon} \kappa(u)\nabla_1 = 0 \in H^2(L(M, \{s\}); F^t) \\ \int_{s+\epsilon}^{t+\epsilon} \kappa(u)\nabla_1 = 0 \in H^2(L(M); F^t \otimes F^s).\end{aligned}$$

Proof. — There exist $h_t \in F^t$ and $h_s \in F^s$ such that

$$\begin{aligned}u &= dh_t \quad \text{near } t, \\ &= dh_s \quad \text{near } s.\end{aligned}$$

Set

$$\begin{aligned}c' &= -\frac{1}{2}\int_s^{t+\epsilon} u\nabla_0^\partial + h_t\nabla_0^\partial \in C^1(L(M, \{s\}); F^t) \\ c'' &= -\frac{1}{2}\int_{s+\epsilon}^{t+\epsilon} u\nabla_0^\partial + (h_s \otimes 1 - 1 \otimes h_s)\nabla_0^\partial \in C^1(L(M); F^t \otimes F^s).\end{aligned}$$

Then one deduces $dc' = \int_s^{t+\epsilon} \kappa(u)\nabla_1^\partial$ and $dc'' = \int_{s+\epsilon}^{t+\epsilon} \kappa(u)\nabla_1^\partial$, as was to be shown.

Consequently the cohomology classes $\int_s^{t+\epsilon} \nabla_0\nabla_1$ and $\int_{s+\epsilon}^{t+\epsilon} \nabla_0\nabla_1$ do not depend on the choice of a nowhere zero vector field ∂.

4. Analytic preliminaries : (DFS) spaces and (DFG) spaces.

An inductive system of locally convex spaces $\{N_i, u_{i+1}^i : N_i \to N_{i+1}\}_{i\in\mathbb{N}}$ is *compact injective* (resp. *nuclear injective*) if each linear map

$$u_{i+1}^i : N_i \to N_{i+1} \quad (i \in \mathbb{N})$$

is compact (resp. nuclear) and injective. Since nuclear maps are always compact, every nuclear injective inductive system is compact injective. A locally convex space N is a (DFS) space (resp. a (DFG) space) if it can be represented as the locally convex inductive limit $\varinjlim N_i$ of a compact (resp. nuclear) injective inductive system $\{N_i, u_{i+1}^i\}_{i\in\mathbb{N}}$ (see [K]).

Our purpose in this paper is to compute the complex analytic Gel'fand Fuks cohomology with coefficients in the germs of tensor fields

$$\mathbb{C}\{z_1, \dots, z_p\} {dz_1}^{\nu_1} \dots {dz_p}^{\nu_p}.$$

This coefficient group is clearly a (DFG) space (see [T] §§50-51). Clearly a finite dimensional space is a (DFG) space.

We begin this section by summarizing some basic properties of (DFS) and (DFG) spaces following [K] and [K1]. Lemma 4.3 is essential to the definition of the support of an analytic functional valued in a (DFS) space (§8) and the computation of the cohomology of tensor fields (Appendix). Next we review the Künneth formula for topological complexes. Finally we treat the Mittag-Leffler lemma on the derived functor $\varprojlim^1$, which is utilized in the proof of Addition Theorem.

We use [G,TVS], [G,PTT], [G,DF], [K1], [T] and especially [K] as general references.

LEMMA 4.1. — *Every compact injective inductive system of locally convex spaces is equivalent to a compact injective inductive system of Banach spaces.*

Lemma 4.1 is proved in [K]III Prop. 9.2, p.302 or [K1] Lemma 2, p.369 and p.374. l.30.

LEMMA 4.2.

(1) *A (DFS) space is a complete reflexive Hausdorff (DF) space.*

(2) *A (DFG) space is a (DFS) space.*

(3) *The strong dual of a (DFG) space is a Fréchet nuclear space, and the strong dual of a Fréchet nuclear space is a (DFG) space.*

(4) *A closed subspace and a quotient space (by a closed subspace) of a (DFG) space are (DFG) spaces.*

(5) *A continuous linear bijection (= injective and surjective map) between (DFG) spaces is a topological isomorphism.*

(6) *The tensor product of two (DFG) spaces is a (DFG) space.*

Proof. — (1) See [K1] Lemma 3 and Theorem 6, p.372.

(2) A nuclear linear map is compact.

(3) See [K] III Theorem 9.8, p.309.

(4) See [K] III Theorem 11.3, p.330.

(5) Let $f : E \to F$ be a comtinuous linear mapping berween (DFG) spaces. Suppose f is injective and surjective. Then the strong dual $f^* : F^* \to E^*$ is continuous, injective and surjective ([T] Theorem 37.2, p.382). In view of Banach's Open Mapping Theorem, f^* is an isomorphism. Since E and F are reflexive, $f = f^{**}$ is an isomorphism.

(6) (DFG)$\otimes$(DFG) is (DF) ([G,PTT]I §1, Proposition 5.2, p.43), nuclear and complete. Hence, by [K] III Theorem 9.10, p.313, it is a (DFG) space.

Hom denotes the linear space consisting of all *continuous* linear mappings throughout this paper.

LEMMA 4.3. — *Let $\{N_i, u^i_{i+1}\}_{i\in\mathbb{N}}$ be a compact injective inductive system of locally convex spaces and F a Fréchet space. Then we have a natural isomorphism (of linear spaces)*

$$\mathrm{Hom}(F, \varinjlim N_i) \cong \varinjlim \mathrm{Hom}(F, N_i).$$

Proof. — Our proof imitates that of [K] III Theorem 9.3, p.303 or [K1] Lemma 3, p.372. We prove the natural map

$$\varinjlim \operatorname{Hom}(F, N_i) \to \operatorname{Hom}(F, \varinjlim N_i)$$

is injective and surjective. The injectivity follows from that of each u^i_{i+1}. For the rest the surjectivity is proved. We may assume each N_i is a Banach space by Lemma 4.1.

Fix an arbitrary $f \in \operatorname{Hom}(F, \varinjlim N_i)$. Using a distance d of the Fréchet space F, set

$$B_i := f(\{y \in F; d(0,y) < 2^{-i}\}).$$

We derive a contradiction under the hypothesis that B_{i-1} is not included in the image of any bounded subset of N_i for each i. Then we can construct $x_i \in B_i$ and an absolutely convex neighbourhood V_i of $0 \in N_i$ inductively satisfying the following conditions (cf. ibid. loc. cit.)

(i) $u^{i-1}_i(V_{i-1}) \subset V_i$

(ii) $x_1, \dots, x_i \notin u^{i+1}(\overline{u^i_{i+1}(V_i)}^{N_{i+1}})$

(iii) $\overline{u^i_{i+1}(V_i)}^{N_{i+1}}$ is compact.

Here $u^i : N_i \to \varinjlim N_i$ denotes the canonical injection. Then $V = \bigcup_{i=1}^{\infty} V_i$ is absolutely convex and absorbing because each V is a neighbourhood of $0 \in N_i$. Hence V is a neighbourhood of 0 in $\varinjlim N_i$. Since f is continuous, we have $x_i \in B_i \subset V$ for some i, which contradicts the condition (ii).

Consequently B_{i-2} is contained in a bounded set of X_{i-1} for some i. Especially $\overline{u^{i-1}_i(B_{i-2})}^{N_i}$ is compact. Since $\varinjlim N_i$ is Hausdorff by Lemma 4.2(1), $u^i : \overline{u^{i-1}_i(B_{i-2})}^{N_i} \to \varinjlim N_i$ is an imbedding and

$$(u^i)^{-1} \circ f : \{y \in F; d(y,0) < 2^{-i+2}\} \to N_i$$

is continuous. Therefore f is consisted in the image of $\operatorname{Hom}(F, N_i)$. This completes the proof of Lemma 4.3.

Next we review the Künneth formula for locally convex complexes, which is essential to the computations below. A short sequence of locally convex spaces and continuous linear mappings

$$0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0 \tag{4.4}$$

is *topologically exact* if i induces a topological isomorphism of A onto the subspace $\ker \pi$ and π induces a topological isomorphism of the quotient space $B/\ker \pi$ onto C.

LEMMA 4.5. — *In the topologically exact sequence (4.4), suppose B is a (DFG) space or a Fréchet nuclear space. Then the strong dual*

$$0 \to C^* \to B^* \to A^* \to 0$$

is topologically exact.

Proof. — When B is a (DFG) space, A, B and C are (DF) spaces (Lemma 4.2 (4)(2)(1)). Hence, by [G,DF] Proposition 5, p.76, the strong dual is topologically exact.

Suppose B is a Fréchet nuclear space. Then, by [G,PTT] II §2 no.1 Theorem 6 Corollary 1, p.38, B is a Fréchet Montel space. Similarly by [G, PTT] II §2 no.2 Theorem 9, p.47, A and C are Fréchet Montel spaces. Thus the strong dual is topologically exact by [K] p.268.

A cochain complex of a locally convex space (E, d) is *topological* if the differential d induces an isomorphism of the quotient space $E/\ker d$ onto the subspace $d(E) \subset E$.

LEMMA 4.6. — *A (DFG) cochain complex $\{C^*, d\}$ is topological if it is locally finite, i.e., for all $p \in \mathbb{N}$*

$$\dim H^p(C^*) < +\infty.$$

Proof (cf. [G,TVS] I§14, Ex.4, p.42). — Set $B^p = dC^{p-1}$ and $Z^p = \ker(d : C^p \to C^{p+1})$. Since Z^p/B^p is finite dimensional, a lift $\tilde{H}^p \subset Z^p$ of H^p is closed (from Hahn-Banach Theorem. See [G,TVS] I,§12,Theorem 7, Corollary 2, p.38). Hence $\tilde{H}^p \oplus (C^{p-1}/Z^{p-1}) \to Z^p$ is a continuous bijection of (DFG) spaces. By Lemma 4.2(5) it is a topological isomorphism. Consequently $C^{p-1}/Z^{p-1} \to B^p$ is a topological isomorphism.

LEMMA 4.7. — *Let E be a (DFG) (resp. Fréchet nuclear) topological cochain complex. Then E^* is a Fréchet nuclear (resp. (DFG)) topological* *cochain complex and its cohomology is topologically isomorphic to the strong dual of* $H(E)$:

$$H(E^*) = H(E)^*.$$

$H(E)$ *is a (DFG) (resp. Fréchet nuclear) space.*

One deduces Lemma 4.7 by usual arguments involved with Banach's Open Mapping Theorem (resp. Lemma 4.2(3)) and Lemma 4.5.

The following theorem is proved in [S].

THEOREM 4.8 (Künneth formula). — *Let* E *and* F *be Fréchet topological cochain complexes. Suppose* E *or* F *are nuclear. Then the tensor product* $E \otimes F$ *is a Fréchet topological cochain complex and its cohomology is topologically isomorphic to the tensor product* $H(E) \otimes H(F)$:

$$H(E \otimes F) = H(E) \otimes H(F).$$

Dualizing Theorem 4.8, we obtain

PROPOSITION 4.9 (Künneth formula). — *Let* E *and* F *be (DFG) topological cochain complexes. Then the tensor product* $E \otimes F$ *is a (DFG) topological cochain complex and its cohomology is topologically isomorphic to the tensor product* $H(E) \otimes H(F)$:

$$H(E \otimes F) = H(E) \otimes H(F).$$

Proof. — Applying Theorem 4.8 to the strong duals E^* and F^*, we have

$$\begin{aligned} H(E \otimes F) &= H((E^* \otimes F^*)^*) = H(E^* \otimes F^*)^* \\ &= (H(E^*) \otimes H(F^*))^* = H(E^*)^* \otimes H(F^*)^* \\ &= H(E) \otimes H(F) \end{aligned}$$

by Lemmata 4.2 and 4.7. Especially $H(E \otimes F)$ is a (DFG) space and Hausdorff. Hence the coboundaries $B(E \otimes F)$ is closed in the cocycles $Z(E \otimes F)$. It follows from Lemma 4.2 $E \otimes F$ is a topological complex.

In §9 we need another version of the Künneth formula

PROPOSITION 4.10 (Künneth formula). — *Let* E *be a Fréchet topological cochain complex and* F *a Fréchet nuclear space. Suppose there exists a projective system* $\{F_i, u_i^{i+1} : F_{i+1} \to F_i\}_{i=1}^{\infty}$ *of (DFG) spaces such that*

(1) $F = \varprojlim F_i$,

(2) *the natural projection $F \to F_i$ has a dense image for each i,*

(3) *the map $u_i^{i+1} : F_{i+1} \to F_i$ is compact.*

Then we have

$$H^*(\mathrm{Hom}(F, E)) = \mathrm{Hom}(F, H^*(E)).$$

Proof. — From [K]III Theorem 5.13, the strong dual of F is topologically isomorphic to $\varinjlim (F_i)^*$. Since F is nuclear, $\mathrm{Hom}(F, E) = F^* \otimes E = \varinjlim ({F_i}^* \otimes E)$, which implies

$$\begin{aligned} H^*(\mathrm{Hom}(F, E)) &= \varinjlim H^*({F_i}^* \otimes E) = \varinjlim {F_i}^* \otimes H^*(E) \\ &= F^* \otimes H^*(E) = \mathrm{Hom}(F, H^*(E)). \end{aligned}$$

Here we utilize Theorem 4.8.

It should be remarked $F = \bigwedge^p L(M, S)$ satisfies all the conditions of the above proposition.

Finally we review the Mittag-Leffler lemma. Let $\{A_n, \phi_n^{n+1} : A_{n+1} \to A_n\}_{n \in \mathbb{N}}$ be a projective system of linear spaces. Consider the linear map

$$\delta : \prod_n A_n \to \prod_n A_n, \quad (a_n) \mapsto (a_n - \phi_n^{n+1}(a_{n+1})).$$

By definition ([M]) we have

$$\varprojlim A_n := \ker \delta \quad \text{and} \quad {\varprojlim}^1 A_n := \mathrm{coker}\delta.$$

PROPOSITION 4.11 (Mittag-Leffler). — *For a compact injective inductive system of Banach spaces $\{B_n, \psi_n^{n+1}\}_{n \in \mathbb{N}}$, we have*

$${\varprojlim}^1 {B_n}^* = 0,$$

where $\{{B_n}^*, {\psi_n^{n+1}}^*\}_{n \in \mathbb{N}}$ is the dual projective system of $\{B_n, \psi_n^{n+1}\}$.

For the proof, see Palamodov [P] or Komatsu [K] III Lemma 10.2, p.318.

We need modify this proposition for the use in the proof of Addition Theorem.

PROPOSITION 4.12 (Mittag-Leffler). — *Let* $\{B_n, \psi_n^{n+1}\}_{n\in\mathbb{N}}$ *be an inductive system of locally convex spaces satisfying the conditions (1)-(3) :*

(1) $\{B_n, \psi_n^{n+1}\}_{n\in\mathbb{N}}$ *is nuclear injective.*

(2) B_n *is a (DFG) space if* n *is even.*

(3) B_n *is a Fréchet nuclear space if* n *is odd.*

Then we have

$$\varprojlim{}^1 \mathrm{Hom}(B_n, N) = 0$$

for an arbitrary (DFS) space N.

Proof. — Proposition 4.11 and Lemma 4.1 imply the sequence

$$0 \to \varprojlim B_{2n}{}^* \to \prod B_{2n}{}^* \xrightarrow{\delta} \prod B_{2n}{}^* \to 0 \tag{4.13}$$

is exact. From Banach's Open Mapping Theorem the sequence (4.13) is topologically exact.

Let C be a Fréchet space. Since $(\oplus B_{2n})^* = \prod(B_{2n}{}^*)$ is a Fréchet nuclear space, (this isomorphism is a topological one. See [K] or [K1]) we have $(\prod B_{2n}{}^*) \otimes C = \mathrm{Hom}(\oplus B_{2n}, C) \cong \prod \mathrm{Hom}(B_{2n}, C)$. Tensoring C to the sequence (4.13), one deduces $\delta : \prod \mathrm{Hom}(B_{2n}, C) \to \prod \mathrm{Hom}(B_{2n}, C)$ is surjective from Theorem 4.8. It follows from the cofinality of $\varprojlim{}^1$

$$\varprojlim{}^1 \mathrm{Hom}(B_{2n+1}, C) = 0. \tag{4.14}$$

N is represented as the locally convex inductive limit of a compact injective inductive system $\{N_i, u_{i+1}^i\}_{i\in\mathbb{N}}$ of Banach spaces. From (4.14) $\delta : \prod \mathrm{Hom}(B_{2n+1}, N_i) \to \prod \mathrm{Hom}(B_{2n+1}, N_i)$ is surjective. Since the inductive limit $\varinjlim$ commutes with the direct product $\prod$, $\delta : \prod \mathrm{Hom}(B_{2n+1}, N) \to \prod \mathrm{Hom}(B_{2n+1}, N)$ is surjective by Lemma 4.3. Thus we obtain $\varprojlim{}^1 \mathrm{Hom}(B_n, N) = 0$, as was to be shown.

5. Main theorems.

In the sequel we impose the following three conditions (5.1) on a L_0 module N :

(5.1.1) N is regular as a L_0 module (2.1).

(5.1.2) N is a (DFG) space (§4).

(5.1.3) The cohomology group $H^*(L_0, e_0; N)$ is locally finite (4.6).

The L_0 module of all (twisted) germs of tensor fields at the origin $(0, \ldots, 0) \in \mathbb{C}^p$

$$1_{\nu_0} \otimes \mathbb{C}\{z_1, \ldots, z_p\} dz_1{}^{\nu_1} \cdots dz_p{}^{\nu_p} \quad (p \in \mathbb{N}, \nu_0, \nu_1, \ldots, \nu_p \in \mathbb{Z})$$

satisfies the condition (5.1). The locally finiteness of its cohomology is proved in Appendix (Corollary A.2).

Let $F = \mathbb{C}\{z\}$ be the W_1 algebra of germs of complex ananlytic functions at the origin $0 \in \mathbb{C}$. We impose the following three conditions (5.2) on a W_1 module N. 1_1 is a 1 dimensional L_0 module defined by

$$X \cdot 1_1 = \delta_0(X) 1_1, \qquad X \in L_0.$$

(5.2.1) N satisfies the condition (5.1) as a L_0 module.

(5.2.2) $1_1 \otimes N$ satisfies the condition (5.1) as a L_0 module.

(5.2.3) The algebra F acts on N, and the action $F \otimes N \to N$ is a W_1 homomorphism, i.e., the Leibniz' rule

$$X(f \cdot n) = (Xf) \cdot n + f \cdot (Xn) \quad X \in W_1, f \in F, n \in N$$

holds.

The W_1 module of all germs of tensor fields at the origin $(0, \ldots, 0) \in \mathbb{C}^p$

$$\mathbb{C}\{z_1, \ldots, z_p\} dz_1{}^{\nu_1} \cdots dz_p{}^{\nu_p} \quad (p \in \mathbb{N}_{\geq 1}, \nu_1, \ldots, \nu_p \in \mathbb{Z})$$

satisfies the condition (5.2). Here F acts on it by

$$f(z) \cdot g(z_1, \ldots, z_p) dz_1{}^{\nu_1} \cdots dz_p{}^{\nu_p} = f(z_1) g(z_1, \ldots, z_p) dz_1{}^{\nu_1} \cdots dz_p{}^{\nu_p}.$$

Now we can formulate our main theorem in the present paper.

THEOREM 5.3. — *Let M be a connected open Riemann surface whose first Betti number $b_1(M)$ is finite, S and T disjoint finite subsets of M and ϕ_u a local parametrization centered at $u \in S \cup T$. Suppose a L_0 module N^s (resp. a W_1 module N^t) satisfying the condition (5.1) (resp. (5.2)) is given for each $s \in S$ (resp. $t \in T$). We define a $L(M, S)$ module N by*

$$N := \bigotimes_{u \in S \cup T} \phi_{u*} N^u.$$

Then the homomorphism

$$\phi_* := \bigotimes_{u\in S\cup T} \phi_{u*} : \bigotimes_{s\in S} H^*(L_0; N^s) \otimes \bigotimes_{t\in T} H^*(W_1; N^t) \to H^*(L(M,S); N)$$

induces an isomorphism

$$H^*(L(M,S); N) \cong \bigwedge^*(\Sigma^3 H_1(M, S\cup T)) \otimes \bigotimes_{s\in S} H^*(L_0; N^s) \otimes \bigotimes_{t\in T} H^*(W_1; N^t),$$

where $\Sigma^3 H_1(M, S\cup T)$ is the graded linear space concentrated to degree 2 given by the 3 times suspension of the first complex valued singular homology group $H_1(M, S\cup T)$, and $\bigwedge^*(\Sigma^3 H_1(M, S\cup T))$ is the free graded commutative algebra generated by the graded space.

Let $\gamma : ([0,1], \{0,1\}) \to (M, S\cup T)$ be a continuous path. Then we can give the cohomology class corresponding to $\Sigma^3[\gamma]$ explicitly by

$$\begin{cases} \int_\gamma \nabla_0\nabla_1 \in H^2(L(M,S)) & \text{if } \gamma(0)=\gamma(1) \text{ or } \gamma(\{0,1\})\subset S \\ \int_{\gamma(0)}^{\gamma(1)+\epsilon} \nabla_0\nabla_1 \in H^2(L(M,S); {\phi_{\gamma(1)}}_* F) & \text{if } \gamma(0)\in S \text{ and } \gamma(1)\in T \\ \int_{\gamma(0)+\epsilon}^{\gamma(1)+\epsilon} \nabla_0\nabla_1 \in H^2(L(M,S); {\phi_{\gamma(1)}}_* F \otimes {\phi_{\gamma(0)}}_* F) & \text{if } \gamma(\{0,1\})\subset T \end{cases}$$

using the covariant derivative cocycles (§3). Since each $N^t (t\in T)$ satisfies the condition (5.2.3), the cup product

$$\cup \int_{\gamma(0)}^{\gamma(1)+\epsilon} \nabla_0\nabla_1 : H^*(L(M,S); {\phi_{\gamma(1)}}_* N^{\gamma(1)}) \to H^{*+2}(L(M,S); {\phi_{\gamma(1)}}_* N^{\gamma(1)})$$

makes sense.

We consider the case $T = \varnothing$ first. We need formulate an addition theorem of Bott-Segal type for the Lie algebra $L(M,S)$, (for the formal vector fields, see Feigin-Fuks [FF] and Retakh-Feigin [RF]) which plays a fundamental role in this paper. We prove it in §8 by adding the cohomology vanishing theorem of Stein manifolds of Oka and Cartan to partition-of-unity arguments in the proof of Bott and Segal [BS] for the C^∞ case.

We begin by recalling a *simplicial cochain complex*

$$C : [q] \mapsto C_q,$$

i.e., a contravariant functor from the category of finite ordered sets to the category of cochain complexes. The normalized complex $\bar{C}_q$ is the quotient

$\bar{C}_q = C_q/C_q^{\mathrm{degen}}$, where C_q^{degen} is the sum of the images of the degeneracy operations. The total cochain complex $\{C_0 \leftarrow \bar{C}_1 \leftarrow \bar{C}_2 \leftarrow \cdots\}$ is denoted by $|C|$ and called the *realization* of the simplicial cochain complex C. The following lemma introduced by Bott and Segal is a basic tool for calculating realizations.

LEMMA 5.4 (= Proposition (5.9) [BS]). — *If $C \to C'$ is a morphism of simplicial cochain complexes such that $C_q \to C'_q$ is a cohomology equivalence for each q, then $|C| \to |C'|$ is a cohomology equivalence.*

Let M, S, ϕ_s and N^s $(s \in S)$ be as in Theorem 5.3. Suppose T is empty. Let $\mathfrak{U} = \{U_j\}_{j=1}^J$ be a finite open covering of M. Set $I := \{j; 1 \leq j \leq J\}$. Consider the simplicial open Riemann surface M_Σ associated to the covering $\mathfrak{U}$:

$$M_\Sigma : [q] \mapsto M_{\Sigma,q} = \coprod_{\sigma_0 \supseteq \sigma_1 \supseteq \cdots \supseteq \sigma_q} [\sigma_0, \sigma_1, \cdots, \sigma_q; U_{\sigma_0}]$$

where σ_0 is a finite subset of I, $U_{\sigma_0} = \bigcap_{j \in \sigma_0} U_j$, and $[\sigma_0, \sigma_1, \cdots, \sigma_q; U_{\sigma_0}]$ is a copy of the Riemann surface U_{σ_0}. We denote by ϖ_q the natural projection

$$\varpi_q : M_{\Sigma,q} \to M \quad [\sigma_0, \sigma_1, \cdots, \sigma_q; U_{\sigma_0}] \to U_{\sigma_0}.$$

For $s \in S$ and $q \in \mathbb{N}_{\geq 0}$, denote by s_q the point in $[\sigma_s, \cdots, \sigma_s; U_{\sigma_s}]$ corresponding to s with labelling $[\sigma_s, \cdots, \sigma_s]$:

$$s_q = [\sigma_s, \cdots, \sigma_s; s] \in [\sigma_s, \cdots, \sigma_s; U_{\sigma_s}] \subset M_{\Sigma,q},$$

where $\sigma_s = \{j \in I; s \in U_j\}$. We define a L_0 module N^u $(u \in {\varpi_q}^{-1}(S))$ by

$$N^u = \begin{cases} N^s, & \text{if } u = s_q,\ s \in S \\ \mathbb{C} \text{ (trivial)}, & \text{if } u \notin \{s_q; s \in S\} \end{cases}$$

and a $L(M_{\Sigma,q}, {\varpi_q}^{-1}(S))$ module N_q by

$$N_q := \bigotimes_{u \in {\varpi_q}^{-1}(S)} \phi_{u*} N^u,$$

where ϕ_u is the local parametrization centered at $u \in M_{\Sigma,q}$ induced by $\phi_{\varpi_q(u)}$. We obtain a simplicial cochain complex

$$[q] \mapsto C^*(L(M_{\Sigma,q}, {\varpi_q}^{-1}(S)); N_q),$$

because of the naturality of the pullback cochain map (§1).

THEOREM 5.5 (Addition Theorem). — *Under the above situation, the augumentation*

$$|C^*(L(M_{\Sigma,q}, {\varpi_q}^{-1}(S)); N_q)| \to C^*(L(M,S); N)$$

induces a cohomology equivalence.

The proof is given in §8.

Part of the main theorem follows from Addition Theorem immediately.

PROPOSITION 5.6. — *Let M, S, ϕ_s, N^s ($s \in S$) and the homomorphism ϕ_* be as in Theorem 5.3. Suppose T is empty. Then the homomorphism ϕ_* induces an $H^*(L(M,S))$ isomorphism*

$$H^*(L(M,S); N) \cong H^*(L(M,S)) \otimes \bigotimes_{s\in S} H^*(L_0, e_0; N^s),$$

where $N = \bigotimes_{s\in S} \phi_{s*} N^s$.

Proof. — The local parametrization ϕ_s is a complex analytic homomorphism of $D_{\epsilon_s} := \{|z| < \epsilon_s\}$ ($\epsilon_s > 0$) onto an open subset U_s of M satisfying $\phi_s(0) = s$.

Since the first Betti number of M is finite, there exists a finite open covering $\{U_a\}_{a\in I}$ of $M - S$ such that the union

$$\mathfrak{U} := \{U_a\}_{a\in I} \cup \{U_s\}_{s\in S}$$

is a finite *contractible* open covering of M.

Consider the simplicial cochain complex

$$[q] \mapsto C^*(L(M_{\Sigma,q}, {\varpi_q}^{-1}(S)); N_q)$$

associated to the covering $\mathfrak{U}$. We have an isomorphism

$$\begin{aligned}
&C^*(L(M_{\Sigma,q}, {\varpi_q}^{-1}(S)); N_q) \\
&\cong \bigotimes_{s\in S} C^*(L(U_s, \{s\}); \phi_{s*}N^s) \\
&\otimes \bigotimes_{u\in {\varpi_q}^{-1}(S), u\neq s_q} C^*(L(U_{\varpi_q(u)}, \{\varpi_q(u)\})) \otimes \bigotimes C^*(L(U_{\sigma_0})) \\
&\cong \bigotimes_{s\in S} C^*(L(D_{\epsilon_s}, \{0\}); N^s) \\
&\otimes \bigotimes_{u\in {\varpi_q}^{-1}(S), u\neq s_q} C^*(L(D_{\epsilon_{\varpi_q(u)}}, \{0\})) \otimes \bigotimes C^*(L(U_{\sigma_0})),
\end{aligned}$$

where the third $\bigotimes$ runs over the set of $[\sigma_0, \sigma_1, \cdots, \sigma_q; U_{\sigma_0}]$ satisfying $I \cup S \supseteq \sigma_0 \supseteq \sigma_1 \supseteq \cdots \supseteq \sigma_q$ and $\sigma_0 \cap I \neq \varnothing$. $C^*(L(D_{\epsilon_s}, \{0\}); N^s)$, $C^*(L(D_{\epsilon_{\varpi_q(u)}}, \{0\}))$ and $C^*(L(U_{\sigma_0}))$ are (DFG) complexes because N^s are (DFG) spaces (5.1.2). Since each N^s is regular as a L_0 module (5.1.1), Proposition 2.3 implies

$$H^*(L(D_{\epsilon_s}, \{0\}); N^s) \cong H^*(L(D_{\epsilon_s}, \{0\})) \otimes H^*(L_0, e_0; N^s).$$

Especially this space is locally finite by (5.1.3) and Proposition 2.4. Hence $C^*(L(D_{\epsilon_s}, \{0\}); N^s)$ is a topological complex from Lemma 4.6. Similarly $C^*(L(D_{\epsilon_{\varpi_q(u)}}, \{0\}))$ and $C^*(L(U_{\sigma_0}))$ are topological complexes from Proposition 2.4 and Lemma 2.8 respectively. Consequently we have an isomorphism

$$\begin{aligned} &H^*(L(M_{\Sigma,q}, {\varpi_q}^{-1}(S)); N_q) \\ &\cong \bigotimes_{s\in S} H^*(L(U_s, \{s\}); \phi_{s*}N^s) \\ (5.7) \qquad &\otimes \bigotimes_{u\in {\varpi_q}^{-1}(S), u\neq s_q} H^*(L(U_{\varpi_q(u)}, \{\varpi_q(u)\})) \otimes \bigotimes H^*(L(U_{\sigma_0})) \end{aligned}$$

by Künneth formula (4.9).

Using the cochain maps ϕ_{s*} and $\alpha_s : H^*(L_0, e_0; N^s) \to C^*(L_0, e_0; N^s)$ satisfying $d\alpha_s(u) = 0$ and $[\alpha_s(u)] = u$ for all $u \in H^*(L_0, e_0; N^s)$, we obtain a homomorphism of $C^*(L(M_{\Sigma,q}, {\varpi_q}^{-1}(S)))$ simplicial cochain complexes

$$\begin{aligned} \alpha : C^*(L(M_{\Sigma,q}, {\varpi_q}^{-1}(S)) \otimes \bigotimes_{s\in S} H^*(L_0, e_0; N^s) \\ \to C^*(L(M_{\Sigma,q}, {\varpi_q}^{-1}(S)); N_q). \end{aligned}$$

(5.7) means α is a cohomology equivalence. It follows from Lemma 5.4 and Theorem 5.5 the realization of α is an isomorphism

$$H^*(L(M, S)) \otimes \bigotimes_{s\in S} H^*(L_0, e_0; N^s) \cong H^*(L(M, S); N),$$

which completes the proof of Proposition 5.6.

Following Bott and Segal [BS], we prove in §6

PROPOSITION 6.2.

$$H^*(L(M)) \cong H^*(\mathrm{Map}(M, S^3); \mathbb{C}) = \bigwedge^*(\omega_1, \ldots, \omega_{b_1}, \theta^t),$$

where $\deg \omega_i = 2$, $1 \leq i \leq b_1 = b_1(M)$, $\mathrm{Map}(M, S^3)$ *denotes the space of continuous maps of* M *to* S^3 *with the compact open topology,* θ^t *is the 3 cocycle defined in (2.7), and* $\bigwedge^*(\cdot)$ *means the free graded commutative algebra over* $\mathbb{C}$ *generated by* $(\cdot)$.

For the C^∞ vector fields of holomorphic type on an open Riemann surface and the algebraic vector fields on an affine algebraic curves, the analogues of this proposition are known (cf. Feigin [F]).

Furthermore we prove

THEOREM 6.1.

$$H^*(L(M)) = \bigwedge^* \left(\int_{H_1(M)} \nabla_0 \nabla_1, \theta^t \right).$$

In §7 we utilize Proposition 5.6 and Theorem 6.1 to prove

THEOREM 7.1. — *If* S *is non empty,*

$$H^*(L(M,S)) = \bigwedge^* \Big(\int_{H_1(M,S)} \nabla_0 \nabla_1,\, \delta_0^s \,(s \in S) \Big).$$

Assembling these results, we can prove our main theorem :

Proof of Theorem 5.3. — The theorem for the case $T = \varnothing$ is proved immediately. In fact, by Proposition 5.6. and Theorem 7.1, we have

$$\begin{aligned} H^*(L(M,S);N) &= H^*(L(M,S)) \otimes \bigotimes_{s \in S} H^*(L_0, e_0; N^s) \\ &= \bigwedge^* \left(\int_{H_1(M,S)} \nabla_0 \nabla_1,\, \delta_0^s \,(s \in S) \right) \otimes \bigotimes_{s \in S} H^*(L_0, e_0; N^s) \\ &= \bigwedge^* (\Sigma^3 H_1(M,S)) \otimes \bigotimes_{s \in S} H^*(L_0; N^s). \end{aligned}$$

Next we consider the case $S \neq \varnothing$ and $T \neq \varnothing$.

Fix an element $s_0 \in S$ and a path connecting s_0 to each $t \in T$. From the case $T = \varnothing$ and the fact

$$\int_{s_0}^{t+\epsilon} \nabla_0 \nabla_1 = \int_{s_0}^{t} \nabla_0 \nabla_1 + \int_{t}^{t+\epsilon} \nabla_0 \nabla_1$$

$$\int_{s_0}^{t} \nabla_0 \nabla_1 \in H^2(L(M, S \cup T)), \quad \int_{t}^{t+\epsilon} \nabla_0 \nabla_1 \in \phi_{t*} H^2(L_0; F),$$

we obtain an isomorphism

(5.8)

$$H^*(L(M,S\cup T);N)$$
$$\cong \bigwedge^{*}\Big(\int_{H_1(M,S)} \nabla_0\nabla_1\Big)\otimes \bigwedge^{*}\Big(\int_{s_0}^{t+\epsilon} \nabla_0\nabla_1, (t\in T)\Big)\otimes \bigotimes_{u\in S\cup T} H^*(L_0;N^u)$$

where $\int_{s_0}^{t+\epsilon}\nabla_0\nabla_1 \in H^2(L(M,S);\phi_{t*}F^t)$ acts on $H^*(L(M,S);N)$ and $H^*(L(M,S\cup T);N)$ using the F structure on N^t.

Consider the Hochschild-Serre spectral sequence of the pair $(L(M,S), L(M,S\cup T))$. Since the cohomology class $\int_{s_0}^{t+\epsilon}\nabla_0\nabla_1$ lifts to $H^2(L(M,S);\phi_{t*}F)$, the transgression d_r vanishes on the class. Therefore it follows from (5.8)

$$\begin{aligned} &H^*(L(M,S);N)\\ &\cong \bigwedge^{*}\Big(\int_{H_1(M,S)} \nabla_0\nabla_1\Big)\otimes \bigwedge^{*}\Big(\int_{s_0}^{t+\epsilon} \nabla_0\nabla_1, (t\in T)\Big)\\ &\quad\otimes \bigotimes_{s\in S} H^*(L_0;N^s)\otimes \bigotimes_{t\in T} H^*(W_1;N^t)\\ &\cong \bigwedge^{*}(\Sigma^3 H_1(M,S\cup T))\otimes \bigotimes_{s\in S} H^*(L_0;N^s)\otimes \bigotimes_{t\in T} H^*(W_1;N^t), \end{aligned}$$

as was to be shown.

Finally we prove the case $S=\varnothing$ and $T\neq\varnothing$.

Fix a point $s_0\in M-T$. Set $S=\{s_0\}$. The Hochschild Serre spectral sequence of the pair $(L(M), L(M,S))$ induces an exact sequence

(5.9)

$$\cdots \to H^q(L(M);N)\to H^q(L(M,S);N)\xrightarrow{d_1} H^q(L(M,S);1_1\otimes N)\to\cdots,$$

where 1_1 is a 1 dimensional $L(M,S)$ module defined by

$$X\cdot 1_1 = {\delta_0}^{s_0}(X)1_1, \qquad X\in L(M,S).$$

Fix a point $t_0\in T$ and a complex analytic nowhere vector field ∂ on M. The theorem for the case $S\neq\varnothing$ and $T\neq\varnothing$ implies

(5.10)

$$H^*(L(M,S);N)$$
$$= \bigwedge^{*}(\Sigma^3 H_1(M,T))\otimes \bigwedge^{*}\Big(\int_{s_0}^{t_0+\epsilon} \nabla_0\nabla_1\Big)\otimes H^*(L_0)\otimes \bigotimes_{t\in T} H^*(W_1;N^t)$$

and

(5.11)

$$H^*(L(M,S);1_1\otimes N) = \overset{*}{\bigwedge}(\Sigma^3 H_1(M,T))\otimes \overset{*}{\bigwedge}\left(\int_{s_0}^{t_0+\epsilon}\nabla_0\nabla_1\right)\otimes H^*(L_0;1_1)\otimes \bigotimes_{t\in T} H^*(W_1;N^t).$$

We claim

$$d_1\left(\int_{s_0}^{t_0+\epsilon}\nabla_0\nabla_1\right)=\delta_0^{s_0}\epsilon_1\in H^2(L(M,S);1_1\otimes \phi_{t_0*}F), \tag{5.12}$$

where the cocycle $\epsilon_1\in C^1(L(M,S);1_1)$ is defined by

$$\epsilon_1(f\partial)=(\partial^2 f)(s_0)1_1,\qquad f\partial\in L(M,S).$$

Using ∂, we define a projector

$$P: L(M)\to L(M,S),\quad f\partial\mapsto (f-f(s_0))\partial.$$

For $X\in L(M,S)$, we have $P([\partial,X])=[\partial,X]-\delta_0^{s_0}(X)\partial$. For $X,Y\in L(M,S)$, integrating the cocycle condition $d\nabla_0^\partial\nabla_1^\partial=0$ along the path from s_0 to t_0, we obtain

$$\begin{aligned}0=&-\delta_0^{s_0}\epsilon_1(X,Y)+(\nabla_0^\partial\nabla_1^\partial)(X,Y)(t_0+\epsilon)\\&-\left(\int_{s_0}^{t_0+\epsilon}\nabla_0\nabla_1\right)([\partial,X],Y)-\left(\int_{s_0}^{t_0+\epsilon}\nabla_0\nabla_1)(X,[\partial,Y]\right).\end{aligned}$$

Therefore

$$\begin{aligned}d_1\left(\int_{s_0}^{t_0+\epsilon}\nabla_0\nabla_1\right)(X,Y)=&\left\{\left(d\left(\int_{s_0}^{t_0+\epsilon}\nabla_0^\partial\nabla_1^\partial\right)\circ P\right)(\partial,X,Y)\right\}\cdot 1_1\\=&\delta_0^{s_0}\epsilon_1(X,Y).\end{aligned}$$

(5.12) follows.

As is proved in Lemma 7.4 (without the use of this theorem),

$$d_1\delta_0=-\epsilon_1. \tag{5.13}$$

Consequently the transgression d_1 induces an isomorphism

$$\overset{*}{\bigwedge}\left(\int_{s_0}^{t_0+\epsilon}\nabla_0\nabla_1\right)\otimes H^*(L_0)\cong \overset{*}{\bigwedge}\left(\int_{s_0}^{t_0+\epsilon}\nabla_0\nabla_1\right)\otimes H^*(L_0;1_1)$$

for degree ≥ 1 under the isomorphisms (5.10) and (5.11). It follows from (5.9)

$$H^*(L(M); N) \cong \bigwedge^*(\Sigma^3 H_1(M,T)) \otimes \bigotimes_{t\in T} H^*(W_1; N^t).$$

This completes all the proof of Theorem 5.3.

6. Explicit description of $H^*(L(M))$.

Let M be a connected open Riemann surface. Assume the first Betti number of M is finite. In this section, we give an explicit description of $H^*(L(M))$:

THEOREM 6.1.

$$H^*(L(M)) = \bigwedge^* \left(\int_{H_1(M)} \nabla_0\nabla_1, \theta^t \right),$$

where $t \in M$, θ^t is the 3 cocycle defined in (2.7), and $\bigwedge^*(\cdot)$ means the free graded commutative algebra over $\mathbb{C}$ generated by $(\cdot)$.

Following Bott and Segal [BS], we show first

PROPOSITION 6.2.

$$H^*(L(M)) \cong H^*(\mathrm{Map}(M, S^3); \mathbb{C}) = \bigwedge^* \left(\omega_1, \ldots, \omega_{b_1}, \theta^t\right),$$

where $\deg \omega_i = 2$, $1 \leq i \leq b_1 = b_1(M)$, and $\mathrm{Map}(M, S^3)$ denotes the space of continuous maps of M to S^3 with the compact open topology.

For the C^∞ vector fields of holomorpic type on an open Riemann surface and the algebraic vector fields on an affine algebraic curves, the analogues of this proposition are known (cf. Feigin [F]).

Next we prove

PROPOSITION 6.3.

$$H^2(L(M)) = \int_{H_1(M)} \nabla_0\nabla_1.$$

Theorem 6.1 follows from Propositions 6.2 and 6.3 immediately.

To prove Proposition 6.2, fix a complex analytic nowhere zero vector field ∂ on M.

LEMMA 6.4. — *If an open set O in M satisfies $H^1(O) = 0$, there exists a complex analytic immersion $z : O \to \mathbb{C}$ satisfying $\partial = \dfrac{d}{dz}$.*

Proof. — For a local coordinate w, consider the equation

$$\frac{dz}{dw} = \frac{1}{f(w)} (\neq 0), \tag{6.5}$$

where $f(w) = \partial w$. Each solution z of (6.5) is an immersion into $\mathbb{C}$ and satisfies $\dfrac{d}{dz} = \partial$. The integral constants of (6.5) form a Čech 1 cocycle on O. Hence the condition $H^1(O) = 0$ implies the existence of a desired immersion $z : O \to \mathbb{C}$.

Proof of Proposition 6.2. — The fundamental map in our case is easier than that in [BS].

Since the first Betti number of M is finite, there exists a finite contractible covering $\mathfrak{U} = \{U_a\}_{a \in I}$ of M of covering dimension 1. A cosimplicial space M^Σ is defined by

$$[q] \mapsto \coprod_{\sigma = \sigma_0 \supset \cdots \supset \sigma_q} U^\sigma,$$

where σ is a finite subset of I such that $U_\sigma \neq \varnothing$, and

$$U^\sigma = \bigcup_{a \in \sigma} U_a.$$

Observe U^σ is contractible because the covering $\mathfrak{U}$ is contractible. Hence, applying Lemma 6.4 to U^σ, we obtain a complex analytic immersion $z_\sigma : U^\sigma \to \mathbb{C}$ satisfying $\partial = \dfrac{d}{dz_\sigma}$. For $p \in U^\sigma$, we define a map $f_{\sigma,p} : U_\sigma \to \mathbb{C}$ by

$$f_{\sigma,p}(\cdot) = z_\sigma(\cdot) - z_\sigma(p).$$

By Lemma 2.8, the homomorphism

$$(f_{\sigma,p})_* : C^*(L(U_\sigma)) \to C^*(L(\mathbb{C}))$$

is a cohomology equivalence because U_σ is contractible.

Denote by $A^*(U^\sigma : C^*(L(\mathbb{C})))$ the double complex of C^∞ differential forms on U^σ with values in $C^*(L(\mathbb{C}))$. Since U^σ is contractible and $(f_{\sigma,p})_*$ is a cohomology equivalence, a cochain map

$$\hat{f}_\sigma : C^*(L(U_\sigma)) \to \Omega^*(U^\sigma : C^*(L(\mathbb{C})))$$

defined by

$$\alpha \mapsto (\partial^{-1}) \otimes (f_{\sigma,p})_* \mathrm{int}(\partial)\alpha + (f_{\sigma,p})_*\alpha$$

is a cohomology equivalence. By virtue of Lemma 5.4, the homomorphism of simplicial cochain complexes

$$\hat{f} : C^*(L(M_{\Sigma,q})) \to A^*(M^{\Sigma,q}; C^*(L(\mathbb{C})))$$

induces a cohomology equivalence on their realizations. Therefore, by Theorem 5.5, we obtain an isomorphism

$$H^*(L(M)) \cong H^*(|A^*(M^\Sigma; C^*(L(\mathbb{C})))|),$$

whose RHS is isomorphic to $H^*(\mathrm{Map}(M, S^3)) \cong H^*(\mathrm{Map}(\bigvee^{b_1} S^1, S^3))$ because of [BS] Corollary 4.8.

Proof of Proposition 6.3. — Our proof proceeds inductively on $b_1(M)$.

The case $b_1(M) = 1$. — In this case M is an annulus, and so we may assume $M = \{r_1 < |z| < r_2\}$, $0 \le r_1 < r_2 \le +\infty$. Fix a 1 cycle γ on M such that $\int_\gamma z^{-1}dz = 1$. It suffices to show $[\int_\gamma \nabla_0\nabla_1] \neq 0 \in H^2(L(M))$. Assume $[\int_\gamma \nabla_0\nabla_1] = 0 \in H^2(L(M))$, i.e., there exists a 1 cochain $c \in C^1(L(M))$ satisfying

$$\int_\gamma (f'g'' - f''g')dz = c\left((fg' - f'g)\frac{d}{dz}\right) \tag{6.6}$$

for all $f, g \in F(M)$. Substituting $f = 1$ and $g = z^{k+1}$, $(k \neq -1)$ to (6.6), we obtain $c\left(z^k \frac{d}{dz}\right) = 0$ for $k \neq -1$, which implies

$$c\left(f\frac{d}{dz}\right) = \lambda \int_\gamma f dz, \quad (f \in F(M)) \tag{6.7}$$

for some constant $\lambda \in \mathbb{C}$. Substituting $f = z^3, g = z^{-1}$ to (6.6) and (6.7), a contradiction is derived. Thus $[\int_\gamma \nabla_0\nabla_1] \neq 0 \in H^2(L(M))$, as was to be shown.

The case $b_1(M) \geq 2$. — There exists an open covering $\{U_1, U_2\}$ of M such that $U_1 \cap U_2$ is contractible, and U_1 is an annulus. By inductive assumption,

$$H^2(L(U_2)) = \int_{H_1(U_2)} \nabla_0 \nabla_1.$$

The proposition for the case $b_1(M) = 1$ implies

$$H^2(L(U_1)) = \int_{H_1(U_1)} \nabla_0 \nabla_1.$$

Since $H_1(M) = H_1(U_1) \oplus H_1(U_2)$, the proposition follows from :

LEMMA 6.8. — *Let M be a connected open Riemann surface of finite Betti number and $\{U_1, U_2\}$ an open covering of M such that $U_1 \cap U_2$ is contractible. Then the inclusion homomorphism*

$$H^2(L(U_1)) \oplus H^2(L(U_2)) \to H^2(L(M)) \tag{6.9}$$

is an isomorphism.

Proof. — Denote by M_Σ the simplicial open Riemann surface associated to the covering $\{U_1, U_2\}$. We regard $H^*(L(M_{\Sigma,q}))$ as a cochain complex with trivial differentials. Since $H^*(L(U_1)), H^*(L(U_2))$, and $H^*(L(U_1 \cap U_2))$ are free graded commutative algebras, there exists a morphism of simplicial cochain algebras $H^*(L(M_{\Sigma,q})) \to C^*(L(M_{\Sigma,q}))$, which is a cohomololgy equivalence for each q. Hence, by Theorem 5.5, we have an isomorphism $H^*|H^*(L(M_\Sigma))| \cong H^*(L(M))$. Easily one deduces

$$\begin{aligned} H^2(L(M_{\Sigma,0})) \hookrightarrow & |H^*(L(M_\Sigma))|^2 \\ & = H^2(|H^*(L(M_\Sigma))|) \cong H^2(L(M)), \end{aligned}$$

which implies the injectivity of (6.9). Since the dimensions of both sides of (6.9) are equal, (6.9) is an isomorphism.

7. Explicit description of $H^*(L(M, S))$.

As in §6, let M be a connected open Riemann surface of finite Betti number and S a finite subset of M. Our purpose in this section is to show an explicit description of $H^*(L(M, S))$ analogous to Theorem 6.1 :

THEOREM 7.1. — *If S is non empty,*

$$H^*(L(M,S)) = \bigwedge^{*}\left(\int_{H_1(M,S)} \nabla_0\nabla_1,\ \delta_0^s\,(s\in S)\right).$$

To prove it inductively, fix a point $t \in M - S$ and set $S_1 = S \cup \{t\}$. For the rest of this section, we abbreviate

$$L = L(M,S), \quad L_1 = L(M,S_1).$$

We have to compute $H^*(L_1)$ explicitly under the assumption (7.1) holds for L. The Hochschild-Serre spectral sequence of the pair (L, L_1) [HS] gives us an exact sequence of $H^*(L)$ modules

$$\cdots \to H^q(L) \xrightarrow{r} H^q(L_1) \xrightarrow{d_1} H^q(L_1;1_\nu) \to H^{q+1}(L) \to \cdots, \tag{7.2}$$

where 1_ν is a L_1 module defined by

$$X\cdot 1_\nu = \delta_0^t(X)1_\nu, \quad X \in L_1.$$

Fix a complex analytic nowhere zero vector field ∂ on M. A 1 cochain $\epsilon_1 \in C^1(L_1, 1_\nu)$ defined by

$$\epsilon_1(f\partial) = (\partial^2 f)(t)1_\nu, \quad f\partial \in L_1$$

is a 1 cocycle. By virtue of Proposition 5.6 and the fact

$$H^*(L_0, e_0; 1_1) = \mathbb{C}\epsilon_1,$$

its cup product

$$\bigcup_{\epsilon_1} : H^q(L_1) \to H^{q+1}(L_1;1_\nu)$$

is an isomorphism. Substituting it to the exact sequence (7.2), we obtain an exact sequence of $H^*(L)$ modules

$$\cdots \to H^q(L) \xrightarrow{r} H^q(L_1) \xrightarrow{\Delta} H^{q-1}(L_1) \xrightarrow{E} H^{q+1}(L) \to \cdots. \tag{7.3}$$

LEMMA 7.4.

$$\Delta(\delta_0^t) = -1 \quad \in H^0(L_1).$$

Proof. — Set $V = h\partial$, where $h \in F(M)$ is a function satisfying $h(t) = 1$ and $h(s) = 0$ for $s \in S$. Then a linear map $P : L \to L_1$ defined by

$$P(f\partial) = f\partial - f(t)V, \quad f\partial \in L$$

is a projector onto L_1. We have

$$P[V,X] = [V,X] - \delta_0^t(X)h\partial = (h\partial f - f\partial h - \delta_0^t(X)h)\partial. \tag{7.5}$$

Using (7.5), we obtain

$$\begin{aligned}(d_1\delta_0^t)(X) &= d(\delta_0^t \circ P)(V,X) = -(\partial^2 f)(t) - \delta_0^t(X)(\partial h)(t),\\ &= (-\epsilon_1 - d(\partial h(t)\cdot 1_\nu))(X),\end{aligned}$$

which proves Lemma 7.4.

Consider the case $S = \varnothing$ first. By the definition of E,

$$E(\delta_0^t) = \theta^t \in H^3(L). \tag{7.6}$$

LEMMA 7.7.

$$H^*(L(M,\{t\})) = \bigwedge^{*}\left(\int_{H_1(M)} \nabla_0\nabla_1,\ \delta_0^t\right).$$

Proof. — As is proved in Proposition 2.4, $H^*(L(D_\epsilon,\{t\})) = \bigwedge^*(\delta_0^t)$. Tensoring the algebra $A^* := \bigwedge^*(\int_{H_1(M)} \nabla_0\nabla_1)$ to the exact sequence (7.3) for the disk D_ϵ, we obtain a long exact sequence

$$\begin{aligned}\cdots \to \left(A^*\otimes \bigwedge^{*}(\theta^t)\right)^q &\to \left(A^*\otimes \bigwedge^{*}(\delta_0^t)\right)^q\\ &\to \left(A^*\otimes \bigwedge^{*}(\delta_0^t)\right)^{q-1} \to \left(A^*\otimes \bigwedge^{*}(\theta^t)\right)^{q+1} \to \cdots.\end{aligned} \tag{7.8}$$

A homomorphism of the sequence (7.8) to the sequence (7.3) is constructed in an obvious way. Applying the 5-lemma to the homomorphism, we can prove $H^q(L(M,\{t\})) \cong H^q(L) \cong (A^*\otimes \bigwedge^*(\delta_0^t))^q$ inductively on the degree q.

Next consider the case $S \neq \varnothing$. Fix $s \in S$ and a path connecting s to t on M. The line integral along the path defines a cocycle $\alpha_0 \in C^2(L_1)$:

$$\alpha_0 = \int_s^t \nabla_0\nabla_1.$$

LEMMA 7.9.

$$\Delta(\alpha_0) = -\delta_0^t.$$

Proof. — Let $V \in L$ and $P : L \to L_1$ be as in the proof of Lemma 7.4. We have the following formulae :

$$\int_s^t \mathcal{L}(V)\omega = \omega(V)(t), \quad \text{and} \quad \int_s^t \mathcal{L}(X)\omega = 0$$

for any $\omega \in K(M)$ and any $X \in L_1$. Let X and Y be arbitrary elements of L_1. Integrating the cocycle condition $d(\nabla_0\nabla_1)(V, X, Y) = 0$ and using the above formulae, we have

$$\left\{\int_s^t (\nabla_0\nabla_1)([V,X],Y) + \int_s^t (\nabla_0\nabla_1)(X,[V,Y])\right\} \cdot 1_\nu$$
$$= \delta_0^t \epsilon_1(X,Y) + c([X,Y]),$$

where we define a 1 cocycle $c \in C^1(L_1, 1_\nu)$ by

$$c(X) = \int_s^t (\nabla_0\nabla_1)(V,X) \cdot 1_\nu,$$

for $X \in L_1$. It follows

$$\begin{aligned} d_1\left(\int_s^t \nabla_0\nabla_1\right)(X,Y) &= \left\{d\Big(\Big(\int_s^t \nabla_0\nabla_1\Big) \circ P\Big)(V,X,Y)\right\} \cdot 1_\nu \\ &= \Big\{ - \Big(\int_s^t \nabla_0\nabla_1\Big)([V,X] - \delta_0^t(X)V, Y) \\ &\quad - \Big(\int_s^t \nabla_0\nabla_1\Big)(X, [V,Y] - \delta_0^t(Y)\Big\} \cdot 1_\nu \\ &= -\delta_0^t\epsilon_1(X,Y) - dc(X,Y), \end{aligned}$$

namely, $d_1(\int_s^t \nabla_0\nabla_1) = -\delta_0^t\epsilon_1$, as was to be shown.

From Lemmata 7.4 and 7.10 follow

$$\Delta({\alpha_0}^n) = -n{\alpha_0}^{n-1}\delta_0^t \quad \text{and} \quad \Delta({\alpha_0}^n\delta_0^t) = -{\alpha_0}^n. \tag{7.10}$$

Conclusion of the proof of Theorem 7.1.

We consider a free graded commutative algebra $R^* = \bigwedge^*(\alpha, \delta)$ with $\deg \alpha = 2$ and $\deg \delta = 1$, and a derivative D on R^* defined by

$$D(\alpha^n) = -n\alpha^n\delta \quad \text{and} \quad D(\alpha^n\delta) = -\alpha^n.$$

By abuse of notation, we denote by $\mathbb{C}$ the trivial graded algebra defined by $(\mathbb{C})^0 = \mathbb{C}$ and $(\mathbb{C})^q = 0$ for $q \geq 1$. Then we have an exact sequence

$$\cdots \to (\mathbb{C})^q \hookrightarrow R^q \xrightarrow{D} R^{q-1} \xrightarrow{0} (\mathbb{C})^{q+1} \hookrightarrow \cdots.$$

Tensoring it by $H^* := H^*(L)$, an exact sequence

$$\cdots \to H^q \hookrightarrow (H^* \otimes R^*)^q \xrightarrow{D} (H^* \otimes R^*)^{q-1} \xrightarrow{0} H^{q+1} \hookrightarrow \cdots$$

is obtained. From (7.10), the correspondence defined by $\alpha \mapsto \alpha_0$ and $\delta \mapsto \delta_0^t$ induces a morphism of long exact sequences

$$\begin{array}{ccccccc} \to & H^q & \to & (H^* \otimes R^*)^q & \to & (H^* \otimes R^*)^{q-1} & \to \\ & \| & & \downarrow & & \downarrow & \\ \to & H^q & \to & H^q(L_1) & \to & H^{q-1}(L_1) & \to \end{array}$$

By the 5-lemma the map $H^* \otimes R^* \to H^*(L_1)$ is an isomorphism. This means

$$H^*(L_1) = H^*(L) \otimes \bigwedge \left(\int_s^t \nabla_0 \nabla_1, \delta_0^t \right).$$

Consequently now we can prove Theorem 7.1 inductively on $\sharp S$. The theorem for the case $\sharp S = 1$ is already shown in Lemma 7.7.

8. Proof of Addition Theorem.

This section is devoted to our proof of Addition Theorem (5.5), which is a translation of that for C^∞ manifolds in [BS] into the case for open Riemann surfaces. The notation in §5 is retained. For an arbitrary complex manifold Ω, we denote by $F(\Omega)$ the Fréchet space of complex analytic functions on Ω with the topology of uniform convergence on compact sets.

First of all we need to define the Gel'fand-Fuks filtration.

Let $\partial_0 \in L(M, S)$ be a non-zero complex analytic vector fields on M whose divisor is equal to $\sum_{s \in S} s$. ∂_0 induces an isomorphism

$$F(M) \cong L(M, S), \quad f \mapsto f\partial_0. \tag{8.1}$$

For Stein manifolds Ω_1 and Ω_2, the algebraic tensor product $F(\Omega_1) \otimes^{\mathrm{alg}} F(\Omega_2)$ is dense in $F(\Omega_1 \times \Omega_2)$. In fact, by the approximation theorem of Weil and Oka ([C] Exposé 9, Théorème 4), it is reduced to the case for polynomials. Hence the completed tensor product $F(\Omega_1) \otimes F(\Omega_2)$ is topologically isomorphic to $F(\Omega_1 \times \Omega_2)$ ([T] Theorem 51.6). Thus (8.1) induces an isomorphism

$$\begin{aligned} C^p(L(M, S); N) &\cong (\mathrm{Hom}(F(M^p), N))^{\mathfrak{S}_p} \\ c &\mapsto (f_1, \ldots, f_p \mapsto c(f_1\partial_0, \ldots, f_p\partial_0)), \end{aligned} \tag{8.2}$$

where the symmetric group $\mathfrak{S}_p$ acts on $\mathrm{Hom}(F(M^p), N)$ by the permutation of the arguments twisted by the signature. In the sequel, we regard a cochain $c \in C^p(L(M,S); N)$ as an element of $\mathrm{Hom}(F(M^p), N)$, i.e., an analytic functional on M^p (with value in N).

Recall the notion of the *support* of analytic functionals on a complex manifolds. For a compact subset K of a complex manifold Ω, a topological linear space $F_\Omega(K)$ is defined by the inductive limit of locally convex spaces (see [K])

$$F_\Omega(K) = \varinjlim_{K \subset U} F(U),$$

where U runs over all the open neighbourhoods of K in Ω. An analytic functional $c \in \mathrm{Hom}(F(\Omega), N)$ has its *support* in K :

$$\mathrm{supp}_\Omega c \subset K,$$

if c is contained in the image of the inclusion homomorphism $\mathrm{Hom}(F_\Omega(K), N) \to \mathrm{Hom}(F(\Omega), N)$, namely, c is contained in the image of the inclusion homomorphism $\mathrm{Hom}(F(U), N) \to \mathrm{Hom}(F(\Omega), N)$ for any open neighbourhood U of K in Ω. For a subset A of Ω,

$$\mathrm{supp}_\Omega c \subset A$$

means that c has its support in some compact subset in A. It should be remarked any analytic functional has its support in some compact subset in Ω as is shown in the following lemma.

LEMMA 8.3. — *Let Ω be a Stein manifold and N a (DFS) space (§4). Then any analytic functional $c \in \mathrm{Hom}(F(\Omega), N)$ has its support in some compact subset of Ω.*

Proof. — By definition (§4) N is represented as the locally convex inductive limit of a compact injective inductive system of Banach spaces $\{N_i, u^i_{i+1} : N_i \to N_{i+1}\}_{i \in \mathbb{N}} : N = \varinjlim N_i$. Since $F(\Omega)$ is a Fréchet space, c is represented as a composite

$$c = u^i \circ c'$$

for some suffix i and some $c' \in \mathrm{Hom}(F(\Omega), N_i)$ by Lemma 4.3. Here $u^i : N_i \to N$ is the canonical embedding. N_i is a Banach space with a norm p. $p \circ c'$ is a continuous semi-norm on $F(\Omega)$. Hence $p \circ c'$ is bounded by the maximum norm on some compact subset K of Ω.

Since Ω is Stein, we may assume that K is $F(\Omega)$-convex. In view of the approximation theorem ([H] Corollary 5.2.9), $F(\Omega)$ is dense in $F_\Omega(K)$. Consequently c' extends to an analytic functional on $F_\Omega(K)$ with values in N, which proves Lemma 8.3.

We define

$$M_k^p = \{(x_1, \dots, x_p) \in M^p; \sharp(S \cup \{x_1, \dots, x_p\}) \leq k + \sharp S\}$$

$$(M_{\Sigma,q})_k^p = \{(x_1, \dots, x_p) \in (M_{\Sigma,q})^p; \\ \sharp(\{s_q; s \in S\} \cup \{x_1, \dots, x_p\}) \leq k + \sharp S\}$$

and

$$C_k^p = \left\{ c \in C^p(L(M, S); N); \operatorname*{supp}_{M^p} c \subset M_k^p \right\}$$

$$C_{k,q}^p = \left\{ c \in C^p(L(M_{\Sigma,q}, {\varpi_q}^{-1}(S)); N_q); \operatorname*{supp}_{(M_{\Sigma,q})^p} c \subset (M_{\Sigma,q})_k^p \right\}.$$

$C_k^* = \bigoplus_p C_k^p$ is a subcomplex of $C^*(L(M, S); N)$ and $C_{k,q}^* = \bigoplus_p C_{k,q}^p$ is a subcomplex of $C^*(L(M_{\Sigma,q}, {\varpi_q}^{-1}(S)); N_q)$. The filtrations $\{C_k^*\}_k$ and $\{C_{k,q}^*\}_k$ are called the Gel'fand Fuks filtrations. Thus we obtain the simplicial cochain complex

$$C_{k,*}^* : [q] \mapsto C_{k,q}^* = C_k^*(L(M_{\Sigma,q}, {\varpi_q}^{-1}(S)); N_q).$$

The theorem is reduced to the following two assertions.

ASSERTION 8.A. — *The simplicial linear space $C_{k,*}^p$ is degenerate above dimension dk, where d is the covering dimension of $\mathfrak{U}$.*

ASSERTION 8.B.

$$H_*(C_{k,*}^p) = C_k^p(L(M, S); N) \quad (\textit{in dim. } 0).$$

In fact, by virtue of (8.A), the realization $|C_{k,*}^p|$ is equal to the total complex

$$\{C_{k,0}^* \leftarrow \bar{C}_{k,1}^* \leftarrow \cdots \leftarrow \bar{C}_{k,kd}^*\},$$

the spectral sequence associated to which converges. It follows from (8.B) the augmentation

$$|C_{k,*}^*| \to C_k^*(L(M, S); N)$$

is a cohomology equivalence. Taking the union for all k, we obtain Addition Theorem (5.5).

The proof of (8.A). — Is similar to that in [BS] §8.

The rest of this section is devoted to

The proof of (8.B). — Fix p and k. We define a simplicial linear space B_* by

$$[q] \mapsto B_q = \{c \in \mathrm{Hom}(F((M_{\Sigma,q})^p), N); \mathrm{supp} c \subset (M_{\Sigma,q})^p_k\}.$$

From (8.2) $C^p_{k,*}$ is isomorphic to the simplicial linear space $[q] \mapsto {B_q}^{\mathfrak{S}_p}$ as a simplicial linear space. Set $B_{-1} = \{c \in \mathrm{Hom}(F(M^p), N); \mathrm{supp} c \subset M^p_k\}$. Then it suffices to show

ASSERTION 8.B.1.

$$H_*(B_*) = B_{-1} \quad \text{(in dim. 0)}.$$

Denote by $\widehat{K}_\Omega$ the analytic hull of a compact set K in a complex manifold Ω. Let Σ denote the nerve of the covering $\mathfrak{U}$:

$$\Sigma = \{\sigma \subset \{1, 2, \ldots, J\}; U_\sigma \neq \varnothing\}.$$

The partial order $\leq$ on the product Σ^p is defined by

$$\begin{aligned} &\pi = (\sigma_1, \ldots, \sigma_p) \leq \pi' = ({\sigma_1}', \ldots, {\sigma_p}') \\ &\Leftrightarrow \sigma_i \supset {\sigma_i}' \quad \text{for each } i. \end{aligned}$$

Then we have

$$(M_{\Sigma,q})^p = \coprod_{\pi_0 \leq \cdots \leq \pi_q} [\pi_0, \ldots, \pi_q; U_{\pi_0}].$$

We can construct sequences of compact subsets L_i and $L_{i,j}$, $i \in \mathbb{N}$, $1 \leq j \leq J$, in M satisfying

(8.4) $L_i = \bigcup_{j=1}^{J} L_{i,j}$,

(8.5) $U_j = \bigcup_{i=1}^{\infty} L_{i,j}$,

(8.6) $\widehat{L_{i,j}}_{U_j} = L_{i,j}$,

(8.7) Every compact subset in U_j is included by some $L_{i,j}$.

We define a simplicial compact subspace $K_{i,*} : [q] \mapsto K_{i,q}$ of M_Σ by

$$K_{i,q} = \left(\coprod_{\sigma_0 \supset \cdots \supset \sigma_q, \sigma_0 \in \Sigma} [\sigma_0, \ldots, \sigma_q; L_{i,\sigma_0}] \right)^p \cap (M_{\Sigma,q})^p_k,$$

where $L_{i,\sigma_0} = \bigcap_{j \in \sigma_0} L_{i,j}$. For each q, we have

(8.8) Every compact subset in $(M_{\Sigma,q})^p_k$ is included in some $K_{i,q}$.

(8.9) $\widehat{K_{i,q}}_{(M_{\Sigma,q})^p} = K_{i,q}$.

When we set $K_i = (L_i)^p \cap M^p_k$, we have

(8.10) Every compact subset in M^p_k is included in some K_i.

(8.11) $\widehat{K_i}_{M^p} = K_i$.

Here it should be observed that M^p_k and $(M_{\Sigma,q})^p_k$ are analytic subsets in M^p and $(M_{\Sigma,q})^p$ respectively.

The inductive limit is an exact functor. Hence, by (8.8) and (8.10), Assertion 8.B.1 is reduced to

ASSERTION 8.B.2. — *The augumented cochain complex*

$$\widetilde{B}_{i,*} := 0 \leftarrow \mathrm{Hom}(F_{M^p}(K_i), N) \leftarrow \mathrm{Hom}(F_{(M_{\Sigma,0})^p}(K_{i,0}), N) \leftarrow \mathrm{Hom}(F_{(M_{\Sigma,1})^p}(K_{i,1}), N) \leftarrow \cdots$$

is acyclic for each i.

For the rest we may fix the suffix i. Let U'_j be a relatively compact open neighborhood of $L_{i,j}$ in U_j for $1 \leq j \leq J$. For $\pi_0 \leq \pi_1 \leq \ldots \leq \pi_q$, $\pi_\alpha \in \Sigma^p$, set

$$D_{\pi_0,\ldots,\pi_q} = [\pi_0, \cdots, \pi_q : U_{\pi_0}] \cap (M_{\Sigma,q})^p_k$$
$$D'_{\pi_0,\ldots,\pi_q} = [\pi_0, \cdots, \pi_q : U'_{\pi_0}] \cap (M_{\Sigma,q})^p_k$$

where

$$U_{\pi_0} = U_{\sigma_0} \times \cdots \times U_{\sigma_p}$$
$$U'_{\pi_0} = U'_{\sigma_0} \times \cdots \times U'_{\sigma_p}, \quad \pi_0 = (\sigma_1, \ldots, \sigma_p) \in \Sigma^p.$$

Renumber S : $S = \{s_1, \ldots, s_m\}$. For $\pi \in \Sigma^p$, define an open set W_π in M by

$$W_\pi = U_\pi - \bigcup_{\rho \not\geq \pi} \overline{U'_\rho} - \iota^{-1} \left(\bigcup_{g\tilde{\pi} = \tilde{\pi}, g \in \mathfrak{S}_{p+m}} (M^{p+m}) \right)^g,$$

where $\tilde{\pi} = (\pi, \sigma_{s_1}, \dots, \sigma_{s_m}) \in \Sigma^{p+m}$ and $\iota : M^p \to M^{p+m}$ is defined by

$$\iota(x_1, \dots, x_p) = (x_1, \dots, x_p, s_1, \dots, s_m).$$

$\{W_\pi\}_{\pi \in \Sigma^p}$ is an open covering of M^p (see [BS] §8).

LEMMA 8.12. — *If there exists an element x of W_π satisfying $[\pi_0, \cdots, \pi_q : x] \in D'_{\pi_0,\dots,\pi_q}$, then $\pi \leq \pi_0$ and $[\pi, \pi_0, \dots, \pi_q : x] \in D_{\pi,\pi_0,\dots,\pi_q}$.*

Let $\{\varphi_\pi\}_{\pi \in \Sigma^p}$ be a partition of unity on M^p subject to the open covering $\{W_\pi\}_{\pi \in \Sigma^p}$. Fix a real valued C^∞ function ψ_π on M^p such that $\psi_\pi|\mathrm{supp}_{\varphi_\pi} \equiv 1$, $\mathrm{supp}\psi_\pi \subset W_\pi$, and $0 \leq \psi_\pi \leq 1$. Define a C^∞ map Ψ_π

$$\begin{aligned} \Psi_\pi : (M_{\Sigma,q})^p &\to \Sigma^{p(q+2)} \times M^p \times [0,1] \\ [\pi_0, \dots, \pi_p : x] &\mapsto (\pi, \pi_0, \dots, \pi_q, x, \psi_\pi(x)). \end{aligned}$$

If $(M_{\Sigma,q+1})^p$ is regarded as a subset of $\Sigma^{p(q+2)} \times M^p$ in an obvious way, from Lemma 8.12 follows

LEMMA 8.13.

(1) $\Psi_\pi(K_{i,q}) \subset K_{i,q+1} \times [0,1] \cup \Sigma^{p(q+2)} \times M^p \times \{0\}$.

(2) If $\pi \not\leq \pi_0$,

$$\Psi_\pi([\pi_0, \dots, \pi_q : U'_{\pi_0}]) \subset \Sigma^{p(q+2)} \times M^p \times \{0\}.$$

From a theorem of Grauert ([H] Theorem 5.1.6, Theorem 5.2.10) we have

LEMMA 8.14. — *Let K be a compact subset of a Stein manifold Ω such that $\widehat{K}_\Omega = K$. Then, for any open neighbourhood O of K, there exists a Stein open set O_1 satisfying $K \subset O_1 \subset O$.*

Consequently K_i (resp.$K_{i,q}$) has a fundamental neighbourhood system consisting of Stein open sets in M^p (resp. $(M_{\Sigma,q})^p$) by (8.11) (resp. (8.9)).

Set $K_{i,-1} = K_i$ and $M_{\Sigma,-1} = M$. Provide a distance d_q for each space $(M_{\Sigma,q})^p$. What we have seen so far implies

LEMMA 8.15. — *There exists a relatively compact open neighbourhood $V_{n,q}$ of $K_{i,q}$ in $(M_{\Sigma,q})^p$ for $n \geq 1$ and $q \geq -1$ such that*

(1) *$V_{n,q}$ is Stein.*

(2) *For each face map* $\partial_j : (M_{\Sigma,q})^p \to (M_{\Sigma,q-1})^p$ $(0 \leq j \leq q)$,

$$\partial_j(V_{n,q}) \subset V_{n,q-1}.$$

(3) $\pi_0(V_{n,q}, V_{n+1,q}) = 0.$

(4) $d_q(K_{i,q}, (M_{\Sigma,q})^p - V_{n,q}) \leq \dfrac{1}{n}.$

(5) *For any* $\pi \in \Sigma^p$,

$$\Psi_\pi(V_{n+1,q}) \subset V_{n,q+1} \times [0,1] \cup \Sigma^{p(q+2)} \times M^p \times [0,1[.$$

(6) $V_{n,q} \cap [\pi_0, \cdots, \pi_q : U_{\pi_0}] \subset U'_{\pi_0}.$

It follows from (4)

$$\widetilde{B}_{i,q} = \operatorname{Hom}(F_{(M_{\Sigma,q})^p}(K_{i,q}), N) = \varprojlim_n \operatorname{Hom}(F(V_{n,q}), N). \tag{8.16}$$

A result on the projective limit of chain complexes need to be recalled.

LEMMA 8.17 (Milnor [M]). — *Let* $\{C_*^n\}_{n\in\mathbb{N}}$ *be a projective system of chain complexes satisfying* $\varprojlim_n{}^1 C_q^n = 0$ *for each* q. *Then we have an exact sequence*

$$0 \to \varprojlim{}^1 H_{q+1}(C_*^n) \to H_q(\varprojlim C_*^n) \to \varprojlim H_q(C_*^n) \to 0.$$

Here $\varprojlim{}^1$ *is the derived functor of the projective limit.*

In view of Proposition 4.12, Lemma 8.15(3)(4) implies

$$\varprojlim_n{}^1 \operatorname{Hom}(F(V_{n,q}), N) = 0.$$

Hence we have an exact sequence

$$0 \to \varprojlim{}^1 H_{q+1}(\operatorname{Hom}(F(V_{n,*}), N)) \to H_q(\widetilde{B}_{i,*}) \to \varprojlim H_q(\operatorname{Hom}(F(V_{n,*}), N)) \to 0$$

by Lemma 8.17. Consequently Assertion 8.B.2 is reduced to

ASSERTION 8.B.3. — *The inclusion homomorphism*

$$H_*(\operatorname{Hom}(F(V_{n+1,*}), N)) \to H_*(\operatorname{Hom}(F(V_{n,*}), N))$$

is a zero map for all n.

Denote by $A^a(\Omega)$ the C^∞ differential forms of $(0,a)$ type on a complex manifold Ω. Consider the augumented double chain complex $C^n_{*,*} = \{C^n_{a,q}\}_{a\geq 0, q\geq -1}$, $C^n_{a,q} = \mathrm{Hom}(A^a(V_{n,q}), N)$. Using the column filtration $F'_pC := \bigoplus_{q\leq p} C^n_{*,q}$ and applying Oka-Cartan's Theorem B to the Stein manifolds $V_{n,q}$, we obtain

$$H_*(\mathrm{Total}C^n_{*,*}) = H_*(\mathrm{Hom}(F(V_{n,*}), N)). \tag{8.18}$$

The spectral sequence associated to the row filtration $F''_pC := \bigoplus_{a\leq p} C^n_{a,*}$ is given by

$$E^{1,n}_{a,q} = H_q(\mathrm{Hom}(A^a(V_{n,*}), N)).$$

Hence Assertion 8.B.3 follows from

LEMMA 8.19. — *The inclusion homomorphism*

$$H_*(\mathrm{Hom}(A^a(V_{n+1,*}), N)) \to H_*(\mathrm{Hom}(A^a(V_{n,*}), N)))$$

is a zero map.

Proof. — By the condition (6) of Lemma 8.15, the map

$$\begin{aligned} h : \mathrm{Hom}(A^a(V_{n+1,q}), N) &\to \mathrm{Hom}(A^a(V_{n,q+1}), N) \\ [\pi_0, \cdots, \pi_q : f] &\mapsto \sum_\pi [\pi, \pi_0, \cdots, \pi_q : f \cdot \varphi_\pi] \end{aligned}$$

is well defined. By usual calculations h is a contracting homotopy of the inclusion homomorphism.

This completes the proof of Addition Theorem.

9. Rešetnikov spectral sequence.

In this section we introduce a spectral sequence related to the Lie algebra $L(M,S)$ and originated by Rešetnikov [R] for C^∞ case, and we calculate two examples.

Let E be a complex analytic vector bundle over a Stein manifold Ω. For an open subset U in Ω, $E(U)$ denotes the Fréchet space of complex analytic sections of E over U with the topology of compact uniform convergence, and $A^q(U; E)$ denotes the Fréchet space of C^∞ sections of

$E \otimes \bigwedge^q \bar{T}^*\Omega$ over U with C^∞ topology. $\mathcal{O}_\Omega(E)$ is a sheaf of topological linear spaces. In view of Oka-Cartan's Theorem B ([H] Th.7.4.3),

$$H^*(A^*(\Omega; E)) = E(\Omega) \quad \text{(in dim 0)}.$$

Hence, for any Fréchet nuclear space F satisfying the conditions of Proposition 4.10, we have

$$\text{(9.1)} \qquad H^*(\mathrm{Hom}(F, A^*(\Omega; E))) = \mathrm{Hom}(F, E(\Omega)) \quad \text{(in dim 0)},$$

where Hom denotes the complex continuous linear maps.

Denote by $\mathrm{Hom}(F, \mathcal{O}_\Omega(E))$ the sheaf

$$U \overset{\text{open}}{\subset} \Omega \mapsto \mathrm{Hom}(F, E(U))$$

and by $\mathrm{Hom}(F, \mathcal{A}^q(E))$ the sheaf

$$U \overset{\text{open}}{\subset} \Omega \mapsto \mathrm{Hom}(F, A^q(U; E)).$$

Applying (9.1) to a fundamental neighbourhood system consisting of Stein open subsets and taking the inductive limit, we obtain a fine resolution of the sheaf $\mathrm{Hom}(F, \mathcal{O}_\Omega(E))$

$$0 \to \mathrm{Hom}(F, \mathcal{O}_\Omega(E)) \to \mathrm{Hom}(F, \mathcal{A}^0(E)) \to \cdots \to \mathrm{Hom}(F, \mathcal{A}^n(E)) \to 0.$$

Using (9.1) once more, we have

LEMMA 9.2. — *For any Fréchet space F satisfying the conditions of Proposition 4.10,*

$$H^*(\Omega, \mathrm{Hom}(F, \mathcal{O}_\Omega(E))) = \mathrm{Hom}(F, E(\Omega)) \quad \text{(in dim 0)}.$$

This Lemma allows us to apply a standard result of the sheaf cohomology theory (see [B] IV.2.5.Theorem) to our situation.

THEOREM 9.3 (Rešetnikov spectral sequence). — *Let M be an open Riemann surface and S a finite subset of M. Suppose the Lie algebra $L(M, S)$ acts on the sheaf of topological linear spaces $\mathcal{O}_\Omega(E)$ continuously. Then there exists a spectral sequence*

$$E_2^{p,q} = H^p(\Omega; \mathcal{H}^q)$$

converging to $H^{p+q}(L(M,S);E(\Omega))$, *where* $\mathcal{H}^q$ *is a sheaf over* Ω *whose stalk at* $x \in \Omega$ *is given by*

$$\mathcal{H}^q_x = H^q(L(M,S);\mathcal{O}_\Omega(E)_x).$$

For the rest we give two examples.

The first example is the cohomology group

$$H^*(L(M);T_n(M)),$$

where $T_n(M)$ is the $L(M)$ module of complex analytic n-th covariant tensor fields on a connected open Riemann surface M of finite Betti number. Denote by T_n the L_0 module of germs at $0 \in \mathbb{C}$ of complex analytic n-th covariant tensor fields. Let ϕ_s be a local parametrization of M centered at s. By Theorem 9.3 we have a spectral sequence

$$E_2^{p,q} = H^p(M;H^q(L(M);\phi_{s*}T_n)) \tag{9.4}$$

converging to $H^*(L(M);T_n(M))$.

From Theorem 5.3, the homomorphism ϕ_{s*} induces an isomorphism

$$H^*(L(M);\phi_{s*}T_n) \cong \bigwedge^{*}\Big(\int_{H_1(M)} \nabla_0\nabla_1\Big) \otimes H^*(W_1;T_n). \tag{9.5}$$

In view of a theorem of Goncharova [Go][V], the dimension of $H^q(W_1;T_n) \cong H^q(L_0;1_n)$ is equal to

$$\begin{cases} 1 & \text{if } n = e(\pm q) \text{ or } e(\pm(q-1)), \\ 0 & \text{otherwise,} \end{cases}$$

where $e(q) = \dfrac{1}{2}(3q^2+q)$.

Thus, if $n \notin e(\mathbb{Z})$, from (9.4) and (9.5) follows

$$H^*(L(M);T_n(M)) = 0.$$

In the sequel we assume

$$n = e(\pm q_0), \quad q_0 \in \mathbb{N}_0.$$

Fix a base φ_n of $H^{q_0}(L_0;1_n)$.

Denote by $C^*(L_0;1_n)^{e_0}$ the 0 eigenspace of $C^*(L_0;1_n)$ under the action of $e_0 = z\dfrac{d}{dz}$. Using the interior product $\mathrm{int}(e_0)$ as a homotopy, we have an isomorphism

$$H^*(C^*(L_0;1_n)^{e_0}) \cong H^*(L_0;1_n).$$

Consider the k-th covariant derivative cochain ∇_k^∂ associated to a nowhere zero complex analytic vector field ∂ (§3). Lemma 3.1 means the assignment $\delta_k \otimes 1_k \mapsto \nabla_k^\partial$ defines a cochain map

$$\nabla^\partial : C^*(L_0;1_n)^{e_0} \to C^*(L(M);T_n(M)),$$

where δ_k is given by $\delta_k\left(z^{l+1}\dfrac{d}{dz}\right) = \delta_{k,l}$ (Kronecker delta).

We denote by $\Phi_n \in H^{q_0}(L(M);T_n(M))$ the image of φ_n under the map ∇^∂. For each $s \in M$ the image of Φ_n under the restriction

$$\begin{aligned} H^*(L(M);T_n(M)) &\to H^*(L(M);\phi_{s*}T_n) \\ &\to H^*(L(M,\{s\});\phi_{s*}(T_n/zT_n)) = H^*(L(M,\{s\});1_n) \end{aligned}$$

coincides with $\phi_{s*}(\varphi_n)$. Therefore we have

$$H^*(L(M);\phi_{s*}T_n) = \Phi_n \cup \overset{*}{\bigwedge}\Big(\int_{H_1(M)} \nabla_0\nabla_1, \nabla_0\Big). \tag{9.6}$$

Since the RHS of (9.6) lifts to $H^*(L(M);T_n(M))$,

$$E_2^{p,q} = H^p(M) \otimes \Big(\Phi_n \cup \overset{*}{\bigwedge}\Big(\int_{H_1(M)} \nabla_0\nabla_1, \nabla_0\Big)\Big)^q$$

collapses. Consequently we obtain

$$\begin{aligned} H^*(L(M);T_n(M)) &\cong \Phi_n \cup H^*(L(M);F(M)) \\ &= \Phi_n \cup \Big(H^*(M) \otimes \overset{*}{\bigwedge}\Big(\int_{H_1(M)} \nabla_0\nabla_1, \nabla_0\Big)\Big). \end{aligned} \tag{9.7}$$

The second example treats the configuration space of the complex line :

$$P_n = \{(z_1,\dots,z_n) \in \mathbb{C}^n; z_i \neq z_j \quad (i \neq j)\} \quad (n \geq 1)$$

which is a Stein manifold. For an arbitrary complex manifold Ω, we denote by $F(\Omega)$ the Fréchet space of complex analytic functions on Ω with the topology of uniform convergence on compact sets. By the diagonal action the Lie algebra $L(\mathbb{C})$ acts on the Fréchet spaces $F(P_n)$ and $F(\mathbb{C}^n)$ continuously.

PROPOSITION 9.8.

$$H^*(L(\mathbb{C}); F(P_n)) = H^*(P_n) \otimes \bigwedge^* \left(\int_{z_i}^{z_{i+1}} \nabla_0\nabla_1, \nabla_0{}^j \quad (1 \le i < n, 1 \le j \le n) \right).$$

Here the (holomorphic) de Rham cohomology algebra of P_n, $H^*(P_n)$, is mapped into the algebra $H^*(L(\mathbb{C}); F(P_n))$ in an obvious way. The cocycle $\int_{z_i}^{z_{i+1}} \nabla_0\nabla_1$ is defined by

$$\left(\int_{z_i}^{z_{i+1}} \nabla_0\nabla_1 \right) \left(f(z)\frac{d}{dz}, g(z)\frac{d}{dz} \right) = F(z_{i+1}) - F(z_i)$$

$$\frac{d}{dz}F(z) = f'(z)g''(z) - f''(z)g'(z)$$

and $\nabla_0{}^j$ by

$$\nabla_0{}^j \left(f(z)\frac{d}{dz} \right) = f'(z_j).$$

Proof. — Consider the Rešetnikov spectral sequence (9.3)

$$E_2^{p,q} = H^p(P_n; H^q(L(\mathbb{C}); \mathcal{O}_{P_n(z_1,\ldots,z_n)})) \Rightarrow H^{p+q}(L(\mathbb{C}); F(P_n)).$$

Theorem 5.3 implies

$$H^*(L(\mathbb{C}); \mathcal{O}_{P_n(z_1,\ldots,z_n)}) = \bigwedge^* \left(\nabla_0{}^j, \int_{z_i+\epsilon}^{z_{i+1}+\epsilon} \nabla_0\nabla_1 \right),$$

which the fundamental group $\pi_1 P_n$ acts on trivially. Hence we have $E_2^{p,q} = E_2^{p,0} \otimes E_2^{0,q}$. Since every element of $E_2^{0,q}$ lifts to $H^*(L(\mathbb{C}); F(P_n))$, the E_2 term collapses. This completes the proof.

COROLLARY 9.9. — *The inclusion homomorphism*

$$\iota_n : H^*(L(\mathbb{C}); F(\mathbb{C}^n)) \to H^*(L(\mathbb{C}); F(P_n))$$

is surjective.

Proof. — By a result of Arnol'd [A], $H^*(P_n)$ is isomorphic to the subalgebra of the holomorphic de Rham algebra of P_n generated by the 1 forms

$$\frac{dz_i - dz_j}{z_i - z_j} \quad (1 \leq i < j \leq n).$$

Every such form lifts to an element of $H^1(L(\mathbb{C}); F(\mathbb{C}^n))$ as follows

$$f(z)\frac{d}{dz} \mapsto \frac{f(z_i) - f(z_j)}{z_i - z_j}.$$

It follows from Proposition 9.8 ι_n is surjective.

It would be very interesting if one can determine whether the surjective homomorphism ι_n is isomorphic or not. This question is answered affirmatively for $n = 2, 3$.

PROPOSITION 9.10. — *The homomorphisms ι_2 and ι_3 are isomorphic.*

Proof. — If $n = 2$, by Feigin and Fuks [FF], the Poincaré polynomial of $H^*(L(\mathbb{C}); F(\mathbb{C}^2))$ is equal to $(1-t^2)^{-1}(1+t)^3$. This is equal to that of $H^*(L(\mathbb{C}); F(P_2))$. Consequently ι_2 is isomorphic.

To show the proposition for $n = 3$, we consider the spectral sequence abutting to $H^*(L_0; F \otimes F) \cong H^*(L(\mathbb{C}); F(\mathbb{C}^3))$ associated to the decreasing filtration $(z^n F) \otimes F$, $n \in \mathbb{Z}_{\geq 0}$, whose E_1 term is equal to $H^*(L_0; 1_n \otimes F)$. In view of a theorem of Feigin and Fuks (Lemma 3.1. [FF]), the Poincaré polynomial of $H^*(L_0; 1_n \otimes F)$ is given by

$$\begin{array}{lll}
t^q(1+t)^2(1-t)^{-1} & \text{if} & n = e(q) \\
t^q(1+t) + t^q(1+t)^2(1-t)^{-1} & \text{if} & n = e(-q) \\
t^{q+1}(1+t) & \text{if} & e(q) < n < e(-q-1) \\
0 & \text{if} & e(-q) < n < e(q)
\end{array}$$

with $q \in \mathbb{Z}_{\geq 0}$. Hence the Poincaré polynomial of the E_1 term is equal to $(1+t)\left(1 + \sum_{q=1}^{\infty}(6q-1)t^q\right)$ = the Poincaré polynomial of $H^*(L(\mathbb{C}); F(P_3))$. This completes the proof of the proposition.

Appendix.

Fix $p \in \mathbb{N}$ and $\nu_0, \nu_1, \ldots, \nu_p \in \mathbb{Z}$. Let N be the L_0 module of twisted germs of tensor fields :

$$N := 1_{\nu_0} \otimes \mathbb{C}\{z_1, \ldots, z_p\} dz_1{}^{\nu_1} \cdots dz_p{}^{\nu_p}.$$

Define a L_0 submodule $F_k N$ $(k \in \mathbb{Z})$ of N by

$$F_k N := \{1_{\nu_0} \otimes f(z) dz_1{}^{\nu_1} \cdots dz_p{}^{\nu_p} \in N;$$
$$\text{the total degree of each component of } f(z) \geq k - \nu\},$$

where $\nu = \sum_{i=0}^{p} \nu_i$. $\{F_k N\}_{k \in \mathbb{Z}}$ is a decreasing L_0-filtration of N. The purpose of this appendix is to prove the following theorem.

THEOREM A.1. — *For any $q \in \mathbb{N}$, there exists an integer $k_q = k_q(N)$ such that*

$$H^q(L_0, e_o; F_k N) = 0$$

for all $k \geq k_q$. Here $H^*(L_0, e_0; \cdot)$ denotes the relative (continuous) cohomology group of the pair of Lie algebras $(L_0, \mathbb{C}e_0)$.

We prove this theorem by a method of successive approximation. A result of Vainshtein [V] plays a role of an a priori estimate. In the sequel we regard $C^*(L_0, e_0; F_k N)$ as a subcomplex of $C^*(L_0, e_0; N)$ in an obvious way.

COROLLARY A.2. — *$H^*(L_0, e_0; N)$ is locally finite, i.e., for all $q \in \mathbb{N}$*

$$\dim H^q(L_0, e_0; N) < +\infty.$$

Proof. — From Theorem A.1 follows

$$H^q(L_0, e_0; N) \cong H^q(L_0, e_0; N/F_{k_q + k_{q+1}} N).$$

Since the complex $C^*(L_0, e_0; N/F_{k_q + k_{q+1}} N)$ is finite dimensional, the RHS is finite dimensional. Therefore $H^*(L_0, e_0; N)$ is locally finite.

COROLLARY A.3. — *The cohomology group $H^*(L_0, e_0; N)$ is isomorphic to the projective limit of the system $\{H^*(L_0, e_0; N/F_k N)\}_{k \in \mathbb{Z}}$:*

$$H^*(L_0, e_0; N) \cong \varprojlim_k H^*(L_0, e_0; N/F_k N).$$

The RHS for $p=1$ is computed by Feigin and Fuks [FF].

To prove the theorem, we denote

$$B_n := \{(z_1,\ldots,z_n)\in\mathbb{C}^n; |z_i|<1 \quad (\forall i)\}$$
$$F_n := \mathbb{C}\{z_1,\ldots,z_n\} = \varinjlim_{\rho\to 0} F(\rho B_n)$$

for $n\in\mathbb{N}$. $F(\cdot)$ denotes the Fréchet space consisting of all complex analytic functions on $\cdot$. We endow F_n with the locally convex inductive limit topology. W_1 acts on F_n by the diagonal action :

$$\mathcal{L}\Big(\xi(z)\frac{d}{dz}\Big)f(z_1,\ldots,z_n) = \sum_{i=1}^{n}\xi(z_i)\frac{d}{dz_i}f(z_1,\ldots,z_n).$$

We need investigate the spaces of all e_0-invariant continuous linear mappings of F_q to $1_\nu\otimes F_p$ $(q,p\in\mathbb{N},\nu\in\mathbb{Z})$.

LEMMA A.4. — *The correspondence*

$$\theta\in\mathrm{Hom}(F_q,1_\nu\otimes F_p)^{e_0}\mapsto\sum_{i_1,\ldots,i_q}\theta({\zeta_1}^{i_1}\cdots{\zeta_q}^{i_q}){z_1}^{i_1}\cdots{z_q}^{i_q}\in V_{\nu,q,p}$$

is a well defined linear isomorphism $\mathrm{Hom}(F_q,1_\nu\otimes F_p)^{e_0}\cong V_{\nu,q,p}$. *Here* $V_{\nu,q,p}$ *denotes the closed subspace*

$$\Big\{\sum a_{i_1\ldots i_q j_1\ldots j_p}{z_1}^{i_1}\cdots{z_q}^{i_q}{w_1}^{j_1}\cdots{w_p}^{j_p}\in F_q\otimes F_p = F_{p+q};$$
$$a_{i_1\ldots i_q j_1\ldots j_p}=0 \text{ if } i_1+\cdots+i_q\neq\nu+j_1+\cdots+j_p\Big\}$$

of $F_{p+q}=\varinjlim F(\rho B_{p+q})$.

Proof. — We reconstruct the inverse of the given correspondence. Fix

$$\alpha=\sum_{i_1+\cdots+i_q=\nu+j_1+\cdots+j_p} a_{i_1\ldots i_q j_1\ldots j_p}{z_1}^{i_1}\cdots{z_q}^{i_q}{w_1}^{j_1}\cdots{w_p}^{j_p}\in V_{\nu,q,p}.$$

Since α is an element of $F_{p+q}=\varinjlim F(\rho B_{p+q})$, we have

$$+\infty>\sum|a_{i_1\ldots i_q j_1\ldots j_p}|{\rho_\alpha}^{i_1+\cdots+i_q+j_1+\cdots+j_p}$$
$$={\rho_\alpha}^{\nu}\sum|a_{i_1\ldots i_q j_1\ldots j_p}|{\rho_\alpha}^{2(j_1+\cdots+j_p)}$$

for some $\rho_\alpha > 0$. Hence the series

$$\alpha(z^{-1}, w) = \alpha\Big(\frac{1}{z_1}, \dots, \frac{1}{z_q}, w_1, \dots, w_p\Big)$$
$$= \sum a_{i_1 \dots i_q j_1 \dots j_p} {z_1}^{-i_1} \cdots {z_q}^{-i_q} {w_1}^{j_1} \cdots {w_p}^{j_p}$$

converges absolutely and uniformly on any compact subsets of the domain $\{(z_1, \dots, z_q, w_1, \dots, w_p) \in \mathbb{C}^{p+q}; |w_k| < {\rho_\alpha}^2 |z_l| \quad (\forall k, \forall l)\}$. Thus a continuous linear map $\phi_\alpha : F(2\rho B_q) \to 1_\nu \otimes F({\rho_\alpha}^2 \rho B_p)$ is defined by

$$\phi_\alpha(f)(w_1, \dots, w_p) := 1_\nu \otimes \left(\frac{1}{2\pi\sqrt{-1}}\right)^q \int_{|z|=\rho} \alpha(z^{-1}, w) f(z) \frac{dz}{z}$$
$$= 1_\nu \otimes \left(\frac{1}{2\pi\sqrt{-1}}\right)^q \int \alpha\left(\frac{1}{z_1}, \dots, \frac{1}{z_q}, w_1, \dots, w_p\right) f(z_1, \dots, z_q) \frac{dz_1 \cdots dz_q}{z_1 \cdots z_q}.$$

Passing to the limit $\rho \to 0$, we obtain the continuous linear map $\phi_\alpha : F_q \to 1_\nu \otimes F_p$ such that

$$\phi_\alpha({z_1}^{i_1} \cdots {z_q}^{i_q}) = 1_\nu \otimes \sum a_{i_1 \dots i_q j_1 \dots j_p} {w_1}^{j_1} \cdots {w_p}^{j_p}.$$

Clearly $\phi_\alpha \in \mathrm{Hom}(F_q, 1_\nu \otimes F_p)^{e_0}$ and the linear map

$$\phi : V_{\nu,q,p} \to \mathrm{Hom}(F_q, 1_\nu \otimes F_p)^{e_0} \quad \alpha \mapsto \phi_\alpha$$

is injective.

By Lemma 4.3, an arbitrary $\theta \in \mathrm{Hom}(F_q, 1_\nu \otimes F_p)^{e_0}$ may be regarded as a continuous linear map $\theta : F(B_q) \to 1_\nu \otimes F(\rho B_p)$ for some $\rho > 0$. The series $\sum (\zeta_1 z_1)^{i_1} \cdots (\zeta_q z_q)^{i_q} = \dfrac{1}{1 - \zeta_1 z_1} \cdots \dfrac{1}{1 - \zeta_q z_q}$ with parameter $\zeta = (\zeta_1, \dots, \zeta_q) \in \dfrac{1}{2} B_q$ converges in $F(B_q)$ uniformly with respect to the parameter $\zeta \in \dfrac{1}{2} B_q$. Hence the series

$$1_{-\nu} \otimes \sum \theta({z_1}^{i_1} \cdots {z_q}^{i_q}) {\zeta_1}^{i_1} \cdots {\zeta_q}^{i_q} =: \alpha(\zeta_1, \dots, \zeta_q, w_1, \dots, w_p)$$

converges in $F(\rho B_p)$ uniformly with respect to the parameter $\zeta \in \dfrac{1}{2} B_q$, which implies $\alpha \in V_{\nu,q,p}$. By our construction we have $\phi_\alpha = \theta$.

Consequently ϕ is a linear isomorphism and its inverse is equal to the given correspondence. This completes the proof of Lemma A.4.

Consider the inner product on the space $\mathbb{C}[z_1, \dots, z_n]$ defined by

$$(f, g)_\rho = \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_{|z_1|=\rho} \cdots \int_{|z_n|=\rho} f(z_1, \dots, z_n)\overline{g(z_1, \dots, z_n)} \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n} \qquad f, g \in \mathbb{C}[z_1, \dots, z_n].$$

We denote by $H_{n,\rho}$ the Hilbert space obtained by completing $\mathbb{C}[z_1, \dots, z_n]$ with respect to the norm $\|f\|_\rho := ((f, f)_\rho)^{1/2}$. Since $F(\rho_1 B_n) \subset H_{n,\rho} \subset F(\rho B_n)$ for $\rho_1 > \rho$, we have

$$F_n = \varinjlim_{\rho \to 0} H_{n,\rho}. \tag{A.5}$$

By Lemma A.4 we may regard $C^q(L_0, e_0; F_k N)$ as a closed subset of $V_{\nu,q,p} \subset F_{p+q}$, where $\nu = \sum_{i=0}^{p} \nu_i$. The space

$$H_{q,k,\rho} := H_{p+q,\rho} \cap C^q(L_0, e_0; F_k N)$$

together with the inner product $(\cdot, \cdot)_\rho$ is a Hilbert space. Observe

$$C^q(L_0, e_0; F_k N) = \varinjlim_{\rho \to 0} H_{q,k,\rho}.$$

Theorem A.1 reduces to the following assertion.

ASSERTION A.6. — *For every q there exists an integer $k_0 = k_0(q)$ with the property that each cocycle $c \in H_{q,k,\rho}$ is represented as the coboundary of a cochain $b \in H_{q-1,k,\rho}$: $c = db$ for any $k \geq k_0$ and any $0 < \rho \leq 1/2$.*

In fact, take an arbitrary cocycle $c \in C^q(L_0, e_0; F_k N)$. By (A.5) c is contained in $H_{q,k,\rho}$ for some $\rho \leq 1/2$. But c is a coboundary because of Assertion A.6. Therefore any cocycle of $C^q(L_0, e_0; F_k N)$ is a coboundary. Theorem A.1 follows.

For the rest we prove Assertion A.6. Set

$$C_k^q := \left\{ c \in C^q(L_0, e_0; N); c(e_{i_1}, \dots, e_{i_q}) = 0 \text{ if } \sum_{a=1}^{q} i_a \neq k \right\},$$

where $e_i = z^{i+1}\dfrac{d}{dz} \in L_0 \quad (i \in \mathbb{N})$. Then we have an orthogonal decomposition

$$H_{q,k,\rho} = C_k^q \oplus H_{q,k+1,\rho}.$$

Furthermore the sum $\bigoplus_{l\geq k} C_l^q$ is dense in $H_{q,k,\rho}$.

The cochain map $d : C^q(L_0, e_0; N) \to C^{q+1}(L_0, e_0; N)$ is decomposed as follows :

$$\begin{aligned} d &= d_0 - \beta \\ (d_0c)(X_0, \ldots, X_q) &= \sum_{i<j} (-1)^{i+j} c(([X_i, X_j], X_0, ..\hat{i}..\hat{j}.., X_q) \\ (\beta c)(X_0, \ldots, X_q) &= \sum_i (-1)^{i+1} \mathcal{L}(X_i) c(X_0, ..\hat{i}.., X_q) \end{aligned}$$

where $c \in C^q(L_0, e_0; N)$, $X_0, \cdots, X_q \in L_0$. It should be remarked $d_0{}^2 = 0$, $d_0(C_k^*) \subset C_k^*$, and $\beta(C^q(L_0, e_0 : F_kN)) \subset C^{q+1}(L_0, e_0; F_{k+1}N)$.

We denote by

$$\partial = \partial_{q+1,k,\rho} : C_k^{q+1} \to C_k^q$$

the adjoint operator of $d_0 : C_k^q \to C_k^{q+1}$ with respect to the inner product $(\cdot, \cdot)_\rho$ and by

$$\triangle = \triangle_{q,k,\rho} = d_0\partial + \partial d_0 : C_k^q \to C_k^q$$

the Laplacian of d_0 with respect to the inner product $(\cdot, \cdot)_\rho$. The following result due to Vainshtein [V] plays a role of an a priori estimate in our proof.

THEOREM (Vainshtein [V]). — *The eigenvalues of $\triangle_{q,k,1}$ is given by*

$$E(i_1, \ldots, i_q) = \sum_{a=1}^{q} \binom{i_a}{3} - \sum_{1\leq a<b\leq q} i_a \cdot i_b = \frac{1}{6}\left(2k - 3k^2 + \sum_{a=1}^{q} i_a{}^3\right).$$

Here $i_1, \ldots, i_q$ are positive integers satisfying

$$i_{r+1} \geq i_r + 3 \quad \text{and} \quad \sum i_a = k.$$

As a corollary there exist an integer $k_1(q)$ and a constant $C_{1,q} > 0$ such that $\|(\triangle_{q-1,k,\rho})^{-1}\|_\rho \leq k^{-3} C_{1,q}$ for all $k \geq k_1(q)$, because $\triangle_{q-1,k,\rho} = \triangle_{q-1,k,1}$. Here $\|\cdot\|_\rho$ denotes the operator norm. Using the inequality $\|\triangle^{-1}\partial u\|_\rho^2 = (\triangle^{-1}u, d_0\triangle^{-1}\partial u)_\rho \leq (\triangle^{-1}u, u)_\rho$ for $u \in C_k^q$, we obtain for all $k \geq k_1(q)$

$$\|(\triangle_{q-1,k,\rho})^{-1}\partial_{q,k,\rho}\|_\rho \leq k^{-3/2} C_{1,q}. \tag{A.7}$$

We introduce another Hilbert space. We define an inner product $(\cdot,\cdot)^1_\rho$on the space $\mathbb{C}[z_1,\dots,z_q,w_1,\dots,w_p]$ by

$$(f,g)^1_\rho = \left(\sum_{i=1}^q z_i\frac{d}{dz_i}f(z_1,\dots,z_q,w), \sum_{i=1}^q z_i\frac{d}{dz_i}g(z_1,\dots,z_q,w)\right)_\rho$$
$$f,g\in\mathbb{C}[z_1,\dots,z_q,w_1,\dots .w_p].$$

Under the identification induced by (A.4), $(\cdot,\cdot)^1_\rho|_{C^q_k} = k^2(\cdot,\cdot)_\rho|_{C^q_k}$. Denote by $W_{q,\rho}$ the Hilbert space obtained by completing $\mathbb{C}[z_1,\dots,z_q,w_1,\dots,w_p]$ with respect to the norm $\|f\|^1_\rho := ((f,f)^1_\rho)^{1/2}$. We have

$$F(\rho_1 B_{p+q})\subset W_{q,\rho}\subset H_{p+q,\rho}\subset F(\rho B_{p+q})$$

for $\rho<\rho_1$.

$$W_{q,k,\rho} := W_{q,\rho}\cap C^q(L_0,e_0;F_kN)$$

is a closed subset of $W_{q,\rho}$.

(A.7) implies the following.

LEMMA A.8. — *There exist an integer $k_2(q)$ and a constant $C_{2,q}>0$ such that, for all $k\ge k_2(q)$, the operator*

$$(\triangle_{q,\rho})^{-1}\partial_{q+1,\rho} : H_{q+1,k,\rho}\to W_{q,k,\rho}$$

is a well defined continuous mapping and its operator norm is bounded by $C_{2,q}k^{-1/2}$.

In terms of $W_{q,k,\rho}$ we can estimate the operator β.

LEMMA A.9. — *The operator β defines a continuous mapping*

$$\beta : W_{q,k,\rho}\to H_{q+1,k,\rho},$$

whose operator norm is bounded by a constant independent of k and $0<\rho\le 1/2$.

Proof. — $\delta_k\in C^1(L_0)$ is defined by $\delta_k(e_l)=\delta_k\left(z^{l+1}\frac{d}{dz}\right)=\delta_{k,l}$. We have $-\beta(c\otimes n)=\sum_{k=1}^\infty(\delta_k\cup c)\otimes\mathcal{L}(e_k)n$ for $c\in C^*(L_0)$ and $n\in N$. Hence

$$\beta(f(z,w))dw_1{}^{\nu_1}\cdots dw_p{}^{\nu_p} = \sum_{i,j}\mathcal{L}\left(\frac{z_iw_j}{1-z_iw_j}\frac{d}{dw_j}\right)(f(z,w)dw_1{}^{\nu_1}\cdots dw_p{}^{\nu_p})$$

for $f(z,w) \in C^q(L_0, e_0; N) \subset V_{\nu,q,p}$. Clearly the RHS is estimated by the first derivatives of $f(z,w)$ and $\max(1-|z_i w_j|)^{-2} = (1-\rho^2)^{-2} \leq 16/9$.

From Lemmata A.8 and A.9 follows

LEMMA A.10. — *There exists an integer $k_3(q)$ such that*

$$\beta(\triangle_{q-1,\rho})^{-1}\partial_q : H_{q,k,\rho} \to H_{q,k,\rho}$$

is a bounded operator and its operator norm is $\leq \frac{1}{2}$: $\|\beta(\triangle_{q-1,\rho})^{-1}\partial_q\|_\rho \leq \frac{1}{2}$ for any $k \geq k_3(q)$ and any $\rho \leq 1/2$. Especially the operator

$$(1-\beta\triangle^{-1}\partial)^{-1} = \sum_{j=0}^{\infty}(\beta\triangle^{-1}\partial)^j$$

is a continuous mapping of $H_{q,k,\rho}$ to itself.

The operator

$$P := d_0\triangle^{-1}\partial : H_{q,k,\rho} \to H_{q,k,\rho}$$

is a well defined continuous mapping. In fact P is equal to the projection onto $\operatorname{im} d_0$ with respect to the Hodge decomposition

$$H_{q,k,\rho} = \operatorname{im}\partial \oplus \operatorname{im} d_0 \oplus \ker\triangle.$$

Let $u \in H_{q,k,\rho}$ $\quad (k \geq k_3(q))$ be a cocycle. We define a sequence $\{u_i\}_{i=0}^{\infty}$ in $H_{q,k,\rho}$ inductively by $u_0 = u$ and

$$u_{i+1} = u_i - (d_0 - \beta)\triangle^{-1}\partial u_i.$$

Observe

$$\text{(A.11.i)} \qquad u_i \in H_{q,k+i,\rho}.$$

In fact, clearly $u_0 \in H_{q,k,\rho}$. Assume (A.11.i). Let w_i be the C^q_{k+i} component of u_i. Since $(d_0 - \beta)u_i = (d_0 - \beta)u = 0$, we have $d_0 w_i = 0$ and $u_{i+1} \equiv w_i - d_0\triangle^{-1}\partial w_i = 0 \mod H_{q,k+i+1,\rho}$. (A.11.i+1) follows. Therefore the sequence $\{u_i\}_{i=0}^{\infty}$ converges to 0 in the weak topology.

On the other hand we have

$$u_i = (\beta\triangle^{-1}\partial)^i u + (1-P)\sum_{j=0}^{i-1}(\beta\triangle^{-1}\partial)^j u.$$

Hence the sequence $\{u_i\}_{i=0}^{\infty}$ converges to $(1-P)(1-\beta\triangle^{-1}\partial)^{-1}u$ by Lemma A.10. Consequently

$$(1-P)(1-\beta\triangle^{-1}\partial)^{-1}u=0. \tag{A.12}$$

Set

$$v:=\triangle^{-1}\partial(1-\beta\triangle^{-1}\partial)^{-1}u\in W_{q-1,k,\rho}\subset H_{q-1,k,\rho}.$$

Then by (A.12) we obtain

$$u-dv=(1-P)(1-\beta\triangle^{-1}\partial)^{-1}u=0,$$

which completes the proof of Assertion A.6 and that of Theorem A.1.

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Manuscrit reçu le 21 août 1992.

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Notes added in proof : After submission of this paper the author proved that the homomorphism ι_n in Cor.9.9 is isomorphic for all $n \geq 2$. For details, see the author's preprint "An application of the second Riemann continuation theorem to cohomology of the Lie algebra of vector fields on the complex line" UTMS93-18. Univ. of Tokyo.