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## THE TOPOLOGY OF STEIN CR MANIFOLDS AND THE LEFSCHETZ THEOREM

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### 1. Preliminaries.

An abstract CR manifold is a triple  $(M, H, J)$  where  $M$  is a connected paracompact smooth real manifold,  $H$  is an even dimensional subbundle of the tangent bundle  $TM$ , and  $J$  a partial pseudocomplex structure on  $H$ ; i.e.  $J : H \rightarrow H$  is a smooth fiber preserving bundle isomorphism with  $J^2 = -I$ . We also require that  $J$  be formally integrable; i.e. that we have

$$(1.1) \quad [\tau^{0,1}M, \tau^{0,1}M] \subset \tau^{0,1}M,$$

where

$$\tau^{0,1}M = \{X + iJX \mid X \in \Gamma(M, H)\} \subset \Gamma(M, \mathbb{C}TM),$$

with  $\Gamma$  denoting smooth sections.

Let  $m$  be the real dimension of  $M$  and  $2n$  be the real dimension of the fiber of  $H$ . Then  $n$  is called the CR dimension of  $M$  and  $k = m - 2n$  is its CR codimension. In this case we say that  $M$  is of type  $(n, k)$ .

A CR map of a CR manifold  $(M_1, H_1, J_1)$  into a CR manifold  $(M_2, H_2, J_2)$  is a differentiable map  $\varphi : M_1 \rightarrow M_2$  such that

$$(1.3) \quad d\varphi(H_1) \subset H_2,$$

$$(1.4) \quad d\varphi(J_1X) = J_2d\varphi(X) \text{ for } X \in H_1.$$

A CR map  $f : M \rightarrow \mathbb{C}$  is called a CR function.

If, in addition,  $\varphi$  is a diffeomorphism and  $\varphi^{-1}$  is CR, then we say that  $\varphi$  is a CR isomorphism.

A CR manifold of the form  $(M, TM, J)$  is a complex manifold (type  $(n, 0)$ ) by the Newlander-Nirenberg theorem, and one of the form  $(M, 0, 0)$  is simply a real differentiable manifold (type  $(0, k)$ ).

Let  $M$  be a real submanifold of a complex manifold  $\widetilde{M}$ , with complex structure  $\widetilde{J}$ . For  $p \in M$  set

$$(1.5) \quad H_p M = T_p M \cap \widetilde{J}T_p M.$$

Then  $(M, HM, \widetilde{J}|_{HM})$  is a CR manifold provided that the spaces  $H_p M$  have constant dimension. In this case the embedding map  $\iota : M \rightarrow \widetilde{M}$  is a CR map of  $(M, HM, \widetilde{J}|_{HM})$  into  $(\widetilde{M}, T\widetilde{M}, \widetilde{J})$ .

An *embedding* (resp. *immersion*)  $\varphi$  of a CR manifold  $(M, H, J)$  into a complex manifold  $(\widetilde{M}, T\widetilde{M}, \widetilde{J})$  is a CR map  $\varphi : M \rightarrow \widetilde{M}$  which is an embedding (resp. immersion).

An immersion (resp. embedding) of a CR manifold  $(M, H, J)$  of type  $(n, k)$  into a complex manifold  $\widetilde{M}$  of complex dimension  $n + k$  is said to be *generic*.

**DEFINITION 1.** — *A Stein CR manifold is a CR manifold  $(M, H, J)$  such that*

$$(1.6) \quad (M, H, J) \text{ has a CR embedding as a closed CR submanifold of some Stein manifold } X.$$

*Because of the known results about the embedding of Stein manifolds [3] we could just as well replace (1.6) by the equivalent condition*

$$(1.7) \quad (M, H, J) \text{ has a CR embedding as a closed CR submanifold of } \mathbb{C}^N, \text{ for some } N.$$

For example suppose  $\Omega$  is a domain of holomorphy in  $\mathbb{C}^\ell$ , with complex structure  $\widetilde{J}$ , and let  $(M, HM, \widetilde{J}|_{HM})$  be a closed CR submanifold of  $\Omega$ . Then it is a Stein CR manifold. Let  $(M, H, J)$  be a CR manifold of type  $(n, k)$ . Let  $H^0 \subset T^*M$  be the annihilator bundle of the bundle  $H$ . We consider the bundle  $T^{(0,1)}M = \{X + iJX | X \in H\}$ . Then the Levi form of  $(M, H, J)$  at  $\omega \in H_p^0$  is the Hermitian form on  $T_p^{(0,1)}M$  :

$$(1.8) \quad L(\omega, Z) = \text{id } \tilde{\omega}(Z, \bar{Z}) = -i\omega([\mathbf{Z}, \bar{\mathbf{Z}}]),$$

where  $\tilde{\omega} \in \Gamma(M, H^0)$  satisfies  $\tilde{\omega}(P) = \omega$  and  $\mathbf{Z} \in \tau^{0,1}M$  satisfies  $\mathbf{Z}(P) = Z$ .

The equality of the last two expressions shows that they do not depend on the choice of  $\tilde{\omega}$  and  $\mathbf{Z}$ , and therefore  $L$  is a function defined on the direct sum of the bundles  $H^0$  and  $T^{0,1}M$ .

**DEFINITION 2.** — *A weakly  $q$ -concave CR manifold ( $0 \leq q \leq n$ ) is a CR manifold  $(M, H, J)$  such that, for every  $\omega \in H^0 - \{0\}$ , the Levi form  $L(\omega, \cdot)$  has at least  $q$  eigenvalues that are  $\leq 0$ .*

Replacing the requirement  $\leq 0$  above by  $< 0$  we arrive at the standard definition of a  $q$ -concave CR manifold. Hence every  $q$ -concave CR manifold is a fortiori weakly  $q$  concave. Note that weak 0-concavity involves no condition at all on the manifold. We adopt the convention that a real differentiable manifold (type  $(0, k)$ ) is Stein (Whitney embedding theorem) and is also weakly 0-concave. A complex manifold (type  $(n, 0)$ ) is weakly  $n$ -concave. Any Levi flat ( $L \equiv 0$ ) CR manifold of type  $(n, k)$  is also weakly  $n$ -concave. It follows from the result of [2] that a weakly 1-concave Stein CR manifold cannot be compact.

### 1. The topology of weakly $q$ -concave CR manifolds.

**THEOREM 1.** — *Let  $(M, H, J)$  be a weakly  $q$ -concave Stein CR manifold of type  $(n, k)$ . Then  $M$  has the homotopy type of a CW-complex of dimension  $\leq 2n + k - q$ . In particular*

$$(2.1) \quad H_j(M; \mathbb{Z}) = 0 \quad \text{for } j > 2n + k - q$$

and

$$(2.2) \quad H_{2n+k-q}(M; \mathbb{Z}) \quad \text{has no torsion.}$$

Note that the above theorem interpolates between two classical results :

1. (type  $(0, k)$ ) Any real  $k$ -dimensional differentiable manifold has the homotopy type of a CW-complex of dimension  $\leq k$  [4], and

2. (type  $(n, 0)$ ) Any complex  $n$ -dimensional Stein manifold has the homotopy type of a CW-complex of dimension  $\leq n$ . Thus we obtain the classical result of Andreotti-Frankel [1].

*Proof.* — Assume for a moment the first assertion of the theorem. Then we have (2.1) as well as

$$(2.3) \quad H_j(M; K) = 0 \quad \text{for } j > 2n + k - q,$$

where  $K$  is an arbitrary field. The universal coefficient theorem

$$(2.4) \quad H_j(M; K) = H_j(M; \mathbb{Z}) \otimes K + \text{Tor}[H_{j-1}(M; \mathbb{Z}), K]$$

then yields (2.2).

Since  $M$  is Stein we can assume that  $M$  is embedded as a closed submanifold of  $\mathbb{C}^N$ . Following Andreotti-Frankel we take, for  $P_0 \in \mathbb{C}^N - M$ , the square of the Euclidean distance

$$(2.5) \quad \varphi(P) = |P - P_0|^2, \quad P \in M.$$

By a standard argument using Sard's theorem, we choose the point  $P_0$  so that  $\varphi(P)$  is a Morse function on  $M$ ; i.e.  $\varphi$  has only isolated nondegenerate critical points. By Morse theory (see [4], p. 20)  $M$  has the homotopy type of a CW-complex obtained by attaching an  $r$ -cell for each critical point having Morse index  $r$ . Hence it will suffice to show that  $\varphi$  has no critical point with Morse index  $r > 2n + k - q$ .

Let  $P \in M$  be a critical point of  $\varphi$ . By an affine orthogonal change of coordinates we may assume  $P = 0$  and that  $M$  is described in a neighborhood of 0 by equations of the form

$$(2.6) \quad \begin{cases} y_j = h_j(x_1, \dots, x_k, w_1, \dots, w_n) & 1 \leq j \leq k \\ \zeta_s = g_s(x_1, \dots, x_k, w_1, \dots, w_n) & 1 \leq s \leq \ell \end{cases}$$

where  $x = (x_1, \dots, x_k)$  and  $y = (y_1, \dots, y_k)$  are real coordinates, and  $z = x + iy$ ,  $w = (w_1, \dots, w_n)$ ,  $\zeta = (\zeta_1, \dots, \zeta_\ell)$  are complex holomorphic coordinates, with  $N = k + n + \ell$ . Here the smooth functions  $h_j$  are real valued, the  $g_s$  are complex valued, and all vanish to second order at 0. The tangent space to  $M$  at 0 is  $T_0M = \{(x, w, 0)\}$ . As the line segment  $\overline{PP_0}$  is orthogonal to  $M$ , the point  $P_0$  has coordinates  $P_0 = (ia_1, \dots, ia_k, 0, \dots, 0, \lambda_1, \dots, \lambda_\ell)$  with the  $a_j \in \mathbb{R}$  and the  $\lambda_s \in \mathbb{C}$ . A point  $Q \in M$  near  $P = 0$  has coordinates  $Q = (x + ih(x, w), w, g(x, w))$ . We have

$$\left\{ \begin{aligned} \varphi(Q) &= \sum_1^k x_j^2 + \sum_1^k (h_j - a_j)^2 \\ &+ \sum_1^n |w_r|^2 + \sum_1^\ell |g_s - \lambda_s|^2 \\ &= \varphi(0) + \sum_1^k x_j^2 + \sum_1^n |w_r|^2 - 2 \sum_1^k a_j h_j \\ &- 2\text{Re} \sum_1^\ell \bar{\lambda}_s g_s + O(3). \end{aligned} \right.$$

The  $g_s$  are CR functions on  $M$  near 0 since they are the restrictions to  $M$  of the holomorphic coordinates  $\zeta_s$ . Therefore the formal Taylor expansion of each  $g_s$ , about 0 is an expansion in terms of the  $x_j + ih_j$  and the  $w_r$ . Thus

$$(2.8) \quad g_s = \sum_1^k a_s^{ij} x_i x_j + \sum_1^k \sum_1^n b_s^{jr} x_j w_r + \sum_1^n c_s^{rt} w_r w_t + O(3).$$

We write

$$(2.9) \quad \varphi(Q) = \varphi(0) + B(x, w, \bar{w}) + O(3)$$

where  $B$  is a quadratic form in  $x, w, \bar{w}$ . To prove our contention, it is enough to show that the quadratic form  $B(0, w, \bar{w})$  has at most  $2n - q$  negative eigenvalues. In view of (2.8) we can write

$$(2.10) \quad B(0, w, \bar{w}) = \operatorname{Re} A(w) + L(\omega, w) + \sum_1^n |w_r|^2$$

where  $A$  is a holomorphic quadratic form in  $w$ , and  $L$  is the Levi form of  $M$  at  $\omega = \pm 2 \sum_1^k a_j dx_j|_0$ .

Let  $W$  be a maximal real subspace of  $\mathbb{C}^n$  on which  $B(0, w, \bar{w})$  is negative definite. On  $W \cap \sqrt{-1}W$  the quadratic form  $L(\omega, w) + \sum_1^n |w_r|^2$  is negative definite. Indeed if  $w$  and  $\sqrt{-1}w$  belong to  $W$  we have

$$(2.11) \quad \operatorname{Re} A(w) + L(\omega, w) + \sum_1^n |w_r|^2 < 0$$

and

$$(2.11) \quad -\operatorname{Re} A(w) + L(\omega, w) + \sum_1^n |w_r|^2 < 0,$$

as  $\operatorname{Re} A(\sqrt{-1}w) = -\operatorname{Re} A(w)$ . It follows, using our hypothesis of weak  $q$  concavity, that  $W \cap \sqrt{-1}W$  has complex dimension  $\leq n - q$  and therefore  $W$  has dimension  $\leq 2n - q$ . The proof of the theorem is complete.

### 3. The Lefschetz theorem on hyperplane sections.

**DEFINITION 3.** — A projective CR manifold is a CR manifold  $(M, H, J)$  which has a closed CR embedding in  $\mathbb{C}P^N$ , for some  $N$ .

It follows from the results of [2] that a weakly 1-concave projective CR manifold, when embedded in  $\mathbb{C}\mathbb{P}^N$ , for some  $N$ , intersects every hyperplane. Let  $\Sigma$  be such a hyperplane, and set  $M_0 = M \cup \Sigma$ . We call the closed subset  $M_0$  a hyperplane section of  $M$ .

**THEOREM 2.** — *Let  $(M, H, J)$  be an orientable weakly  $q$ -concave projective CR manifold of type  $(n, k)$ , and  $M_0$  be a hyperplane section of  $M$ . Then the natural homomorphism*

$$(3.1) \quad H^j(M; \mathbb{Z}) \rightarrow H^j(M_0; \mathbb{Z})$$

is an isomorphism for  $j < q - 1$ . It is injective for  $j = q - 1$ .

**THEOREM 2'.** — *Dropping the assumption of orientability, the same results are valid with  $\mathbb{Z}_2$  coefficients.*

*Proof.* — Since  $M_0$  is closed in  $M$ , we have the exact cohomology sequence

$$(3.2) \quad \begin{aligned} \cdots \rightarrow H_K^j(M - M_0; \mathbb{Z}) \rightarrow H^j(M; \mathbb{Z}) \rightarrow H^j(M_0; \mathbb{Z}) \rightarrow \\ \rightarrow H_K^{j+1}(M - M_0; \mathbb{Z}) \rightarrow \cdots, \end{aligned}$$

where the subscript  $K$  denotes compact supports.

By Poincaré duality we obtain  $H_K^j(M - M_0; \mathbb{Z}) \cong H_{2n+k-j}(M - M_0; \mathbb{Z}) = 0$  for  $j < q$ . Hence the result follows by (3.2). For the case where  $M - M_0$  is not orientable, we apply Poincaré duality for  $\mathbb{Z}_2$  coefficients and argue as above.

**THEOREM 3.** — *Under the same hypotheses as Theorem 2, the natural homomorphism*

$$(3.3) \quad H_j(M_0; \mathbb{Z}) \rightarrow H_j(M; \mathbb{Z})$$

is an isomorphism for  $j < q - 1$  and is surjective for  $j = q - 1$ . For the homomorphism

$$(3.4) \quad H_j(M; \mathbb{Z}) \rightarrow H_j(M, M - M_0; \mathbb{Z}),$$

we obtain an isomorphism if  $j > 2n + k - q + 1$ , and an injection for  $j = 2n + k - q + 1$ .

*Proof.* — We consider the exact homology sequence for the pair  $(M, M_0)$

$$(3.5) \quad \begin{aligned} \cdots \rightarrow H_{j+1}(M, M_0; \mathbb{Z}) \rightarrow H_j(M_0; \mathbb{Z}) \rightarrow H_j(M; \mathbb{Z}) \rightarrow \\ \rightarrow H_j(M, M_0; \mathbb{Z}) \rightarrow \cdots \end{aligned}$$

But the Lefschetz duality theorem asserts that

$$H_j(M, M_0; \mathbb{Z}) \cong H^{2n+k-j}(M - M_0; \mathbb{Z}),$$

and the latter group is zero for  $j < q$ , again by Theorem 1.

Next we consider the exact homology sequence for the pair  $(M, M - M_0)$

$$(3.6) \quad \begin{aligned} \cdots \rightarrow H_j(M - M_0; \mathbb{Z}) \rightarrow H_j(M; \mathbb{Z}) \rightarrow H_j(M, M - M_0; \mathbb{Z}) \rightarrow \\ \rightarrow H_{j-1}(M - M_0; \mathbb{Z}) \rightarrow \cdots \end{aligned}$$

By Theorem 1,  $H_j(M - M_0; \mathbb{Z}) = 0$  for  $j > 2n + k - q$ . Hence the desired conclusion follows.

*Remark.* — When  $M$  is a smooth projective algebraic variety (type  $(n, 0)$ ) we recover the classical Lefschetz theorem on hyperplane sections along the lines of the Morse theoretic proof given by Andreotti-Frankel [1].

#### 4. Homotopy of projective CR manifolds.

Following Milnor [4] we prove

**THEOREM 4.** — *Let  $(M, H, J)$  be a weakly  $q$ -concave projective CR manifold of type  $(n, k)$ , and  $M_0$  be a hyperplane section of  $M$ . Then*

$$(4.1) \quad \pi_j(M, M_0) = 0 \quad \text{for } j < q.$$

*Proof.* — (We fix a base point  $x_0 \in M_0$ .) We have a closed CR embedding of  $M$  in  $\mathbb{C}\mathbb{P}^N$ , for some  $N$ , and  $M_0 = M \cap \Sigma$  with  $\Sigma$  given by  $z_0 = 0$  in homogeneous coordinates. Let  $\mathcal{N}(\Sigma, \delta)$  and  $\mathcal{N}(M, \varepsilon)$  be tubular neighborhoods of  $\Sigma$  and  $M$ , of radius  $\delta$  and  $\varepsilon$ , respectively, with respect to the Fubini-Study metric. For  $\delta_0 > 0$  sufficiently small, the geodesic flow determines a deformation retract

$$(4.2) \quad F_\delta : I \times \mathcal{N}(\Sigma, \delta) \rightarrow \mathcal{N}(\Sigma, \delta)$$

of  $\mathcal{N}(\Sigma, \delta) \rightarrow \Sigma$  if  $0 < \delta < \delta_0$ . For  $\varepsilon > 0$  sufficiently small, we have a retraction

$$(4.3) \quad g : \mathcal{N}(M, \varepsilon) \rightarrow M.$$

Since  $\mathcal{N}(M, \varepsilon)$  is open,  $F_\delta^{-1}(\mathcal{N}(M, \varepsilon))$  is open in  $I \times \mathcal{N}(\Sigma, \delta)$ , and contains  $I \times M_0$ . Then we can find  $\eta$ ,  $0 < \eta < \delta$ , such that  $F_\delta(I \times M \cap \mathcal{N}(\Sigma, \eta)) \subset \mathcal{N}(M, \varepsilon)$ .



We have  $\mathbf{CP}^N - \Sigma \cong \mathbf{C}^N$  and take  $\varphi$  as in (2.5). Set

$$\psi = \begin{cases} 0 & \text{on } \Sigma \\ \frac{1}{\varphi} & \text{on } M - \Sigma. \end{cases}$$

Then  $\psi$  is continuous on  $M$  and is a Morse function on  $\psi^{-1}([r, \infty))$  for  $r > 0$ . The critical points of  $\psi$  all have Morse index  $\geq q$ . Therefore  $M$  has the homotopy type of  $\psi^{-1}([0, r])$  with finitely many cells of dimension  $\geq q$  attached. Then, for  $j < q$ , every continuous pointed map  $f : (I^j, \partial I^j) \rightarrow (M, M_0)$  can be deformed to a continuous pointed map  $f_1 : (I^j, \partial I^j) \rightarrow (\psi^{-1}([0, r], M_0))$ . If  $r$  is sufficiently small, then  $\psi^{-1}([0, r]) \subset M \cap \mathcal{N}(\Sigma, \eta)$ , and  $g \circ F_\delta(t, f_1(p))$  gives a homotopy of  $f_1$  to a continuous pointed map  $f_2 : (I^j, \partial I^j) \rightarrow (M, M_0)$ . The proof is complete.

We consider next the exact homotopy sequence of the pair  $(M, M_0)$  :

$$(4.5) \quad \begin{aligned} \cdots \rightarrow \pi_j(M_0) \rightarrow \pi_j(M) \rightarrow \pi_j(M, M_0) \rightarrow \pi_{j-1}(M_0) \rightarrow \cdots \\ \cdots \rightarrow \pi_1(M, M_0) \rightarrow \pi_0(M_0) \rightarrow \pi_0(M). \end{aligned}$$

Therefore if  $M_0$  is a hyperplane section of a weakly  $q$ -concave projective CR manifold  $(M, H, J)$ , then the natural map

$$(4.6) \quad \pi_j(M_0) \rightarrow \pi_j(M)$$

is an isomorphism for  $j < q - 1$ , and is surjective for  $j = q - 1$ . In particular, for  $q \geq 2$ , every hyperplane section of  $M$  is arcwise connected (as  $M$  is connected), and for  $q \geq 3$ , every hyperplane section of  $M$  has the same fundamental group as  $M$ . Finally we remark that a generically chosen hyperplane section  $M_0$  of  $M$  is a smooth submanifold.

### 5. A remark on the embedding dimension of projective CR manifolds.

Let  $(M, H, J)$  be a projective CR manifold of type  $(n, k)$ . Consider a closed CR embedding of it into  $\mathbf{CP}^N$ , for some  $N$ . Then it may be possible to reduce the embedding dimension  $N$  as follows :

**THEOREM 5.** — *With  $(M, H, J)$  as above we have*

*$(k = 1)$  : It has a global closed CR embedding in  $\mathbf{CP}^{2n+2}$ , and a global closed CR immersion in  $\mathbf{CP}^{2n+1}$ .*

*$(k \geq 2)$  : It has a global closed CR embedding in  $\mathbf{CP}^m$ , where  $m = [2n + (3/2)k]$  (greatest integer in).*

*Remark.* — For compact Stein CR manifolds of type  $(n, k)$ , precisely the same results hold, with complex projective space replaced by complex Euclidean space. For non compact Stein CR manifolds of type  $(n, k)$ , the same results hold, with the word “closed” removed, and the word “embedding” replaced by “one-to-one immersion”.

*Proof.* — Let  $M' = \{(p, q) \in M \times \mathbb{C}\mathbb{P}^N \mid q \in \mathbb{C}\mathbb{P}^N \text{ tangent to } M \text{ at } p\}$ . This is a smooth submanifold of  $M \times \mathbb{C}\mathbb{P}^N$  of real dimension  $4n + 3k$ . The map

$$(5.1) \quad M' \ni (p, q) \mapsto q \in \mathbb{C}\mathbb{P}^N$$

is smooth. By Sard’s theorem its image has measure zero in  $\mathbb{C}\mathbb{P}^N$  if  $N > 2n + (3/2)k$ . Choosing a point  $Q_0 \notin \{\text{its range}\} \cup M$ , and projecting from this point into a hyperplane  $\Sigma$  not containing  $Q_0$ , we obtain a CR closed immersion into a  $\mathbb{C}\mathbb{P}^{N-1}$ .

Next we consider  $M'' = \{(p, q, r) \mid (p, q) \in M \times M - \Delta, r \in \mathbb{C}\mathbb{P}^N \text{ and } p, q, r \text{ are collinear}\}$ . It is a smooth manifold of real dimension  $4n + 2k + 2$ . The map

$$(5.2) \quad M'' \ni (p, q, r) \mapsto r \in \mathbb{C}\mathbb{P}^N$$

is smooth and, again by Sard’s theorem, its image has measure zero if  $N > 2n + k + 1$ . If  $N$  satisfies both inequalities, the above CR immersion can be chosen to be globally one-to-one.

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