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ON THE CHARACTERISTIC POWER SERIES OF THE U OPERATOR

by F.Q. GOUVÊA and B. MAZUR

Let p be a prime number, and let k be an integer. Atkin's U operator acts in a completely continuous manner on the p -adic space of overconvergent modular forms of weight k . The goal of this note is to show that the "Fredholm" characteristic power series of U varies " p -adically continuously" in the weight k , in the following sense. If $a_m(k)$ is the m -th coefficient of the characteristic power series of U acting on overconvergent forms of weight k , we show that if $k_1 \equiv k_2 \pmod{p^n(p-1)}$ then $a_m(k_1) \equiv a_m(k_2) \pmod{p^{n+1}}$ for every $m \geq 0$. We then extend this to "higher order differences" of the function $k \mapsto a_m(k)$, in the spirit of [Ser2], Thm. 14.

Our p -adic continuity result leads us to hope that there is a notion of "overconvergent p -adic modular form of weight k " not only for rational integers k , but for k in the p -adic space

$$\mathcal{X} = \varprojlim_n \mathbb{Z}/(p-1)p^n\mathbb{Z},$$

and that the U operator preserves overconvergence and is completely continuous (and therefore has a spectral theory) for all $k \in \mathcal{X}$. If so, our result would suggest that this spectral theory is uniformly continuous in k . At present, however, it is not evident to us how to define overconvergence for p -adic modular forms of general p -adic weights.

The methods we use are a direct extension of those in [Gou2], and our main result answers one of the "Further Questions" posed there. This

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paper fits into the general research project outlined in [Gou], and we refer our readers to that paper for further discussion of motivation.

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1. Introduction.

To describe our main result precisely, let p be a prime number, and assume $p \geq 5$. Fix a "tame level" N not divisible by p ; we will be working with p -adic modular forms of integral weight on $\Gamma_1(N)$. Let B be a p -adically complete and separated ring, and let $r \in B$. We will let $M_k(N, B; r)$ denote the space of r -overconvergent p -adic modular forms of weight k on $\Gamma_1(N)$ defined over B (for definitions and properties of these spaces, whose importance was first realized by Dwork, we refer to the accounts in [Kat] and [Gou2]). If B is a discrete valuation ring and K is its field of fractions, we write $M_k(N, K; r) = M_k(N, B; r) \otimes K$; this is a p -adic Banach space over K with respect to the norm determined by making $M_k(N, B; r)$ the unit ball. This space contains the classical spaces considered in [GM].

We fix a discrete valuation ring B , let K be its field of fractions, and write, for simplicity, $M_k(r) = M_k(N, K; r)$. When $0 < \text{ord}(r) < p/(p+1)$, the Atkin U operator is a completely continuous linear operator on the p -adic Banach space $M_k(r)$, and hence has a spectral theory. In particular we can consider the characteristic power series $P_k(t) = \det(1 - tU | M_k(r))$ and, for each rational number α , the "slope α subspace" $M_{k,\alpha}$ which is spanned by all the forms $f \in M_k(r)$ such that $(U - \lambda)^m(f) = 0$ for some integer $m > 0$ and some $\lambda \in \overline{K}$ such that $\text{ord}(\lambda) = \alpha$. It is a basic result in the spectral theory of the U operator that the space $M_{k,\alpha}$ is finite-dimensional and independent of the choice of r (provided $0 < \text{ord}(r) < p/(p+1)$).

We can now state our main result.

THEOREM 1. — *Let $p \geq 5$ be a prime number, N an integer not divisible by p , B a p -adically complete and separated discrete valuation ring, and K its field of fractions. Choose any $r \in B$ satisfying $0 < \text{ord}(r) < p/(p + 1)$. Let $P_k(t)$ be the characteristic power series of the U operator acting on the space $M_k(N, K; r)$ of r -overconvergent p -adic modular forms of weight k and level N . Write $P_k(t) = \sum a_m(k)t^m$. If k_1 and k_2 are integers such that*

$$k_1 \equiv k_2 \pmod{p^n(p - 1)},$$

then we have, for each m ,

$$a_m(k_1) \equiv a_m(k_2) \pmod{p^{n+1}}.$$

Much of the technical complication in the proof of such a result is due to the fact that there are two natural topologies on the Banach spaces $M_k(r)$. For the first topology, recall that elements of $M_k(r)$ can be interpreted as functions of “not too supersingular” elliptic curves E defined over some p -adically complete and separated B -algebra A . The restriction on the curve E is that $E_{p-1}(E, \omega)$ should be a divisor of $r \in B$. (See [Kat] and [Gou2] for details.) A “test-object of level N and growth condition r ” is simply such a curve together with a level structure. The first topology is just the natural topology on such “functions” : its norm $\|\cdot\|_{\text{mod}}$ is characterized by

$$\|f\|_{\text{mod}} \leq 1 \qquad \text{if and only if} \qquad f(E/A, \omega, \iota, Y) \in A$$

for any test-object $(E/A, \omega, \iota, Y)$ of level N and growth condition r . We call this topology the *modular topology*; its unit ball is precisely the space $M_k(N, B; r)$. The second topology, which we call the *q -expansion topology*, is induced by the q -expansion map; its norm $\|\cdot\|_{q\text{-exp}}$ can be described by saying that $\|f\|_{q\text{-exp}} \leq 1$ if and only if all the coefficients of the q -expansion of f are in B (i.e., are integral). It is a basic fact that the modular topology on $M_k(1)$ (i.e., for $r = 1$) coincides with the q -expansion topology, so that the unit ball in $M_k(r)$ with respect to the q -expansion topology can also be described as the intersection $M_k(r) \cap M_k(N, B; 1)$. (A proof can be found in [Kat].) This shows, in particular, that $M_k(1)$ is isomorphic to Serre’s space of p -adic modular forms of weight k , which is defined in [Ser2] in terms of limits of q -expansions.⁽¹⁾ We have an inclusion of the “closed” unit balls

$$M_k(N, B; r) \subset M_k(r) \cap M_k(N, B; 1),$$

⁽¹⁾ In other words, given a sequence of classical forms f_i whose q -expansions $f_i(q)$ converge, coefficient-by-coefficient, to $f(q) \in B[[q]]$, there always exists a form f in

but the set on the right is *unbounded* with respect to the modular topology.

It is sometimes convenient to use the q -expansion map to identify $M_k(r)$ with its image in $K \otimes B[[q]]$. (Except in the case when r is a unit in B , the image will not be closed with respect to the “natural” topology on $K \otimes B[[q]]$.) From this point of view, the “unit ball with respect to the q -expansion topology” is just the intersection $M_k(r) \cap B[[q]]$.

2. Proof of Theorem 1.

As usual, there are Hecke operators T_ℓ for each prime number $\ell \neq p$ which act on $M_k(\mathbb{N}, B; r)$; these have the expected action on q -expansions. (See [Kat] or [Gou2] for the definitions.) For $\ell = p$, however, the relevant operator is not T_p (even though $p \nmid \mathbb{N}$), but Atkin’s U operator, which acts on q -expansions by the formula

$$U\left(\sum a_n q^n\right) = \sum a_{np} q^n.$$

This is defined on $M_k(\mathbb{N}, K; r)$ as $1/p$ times the trace of the Frobenius operator Frob , which acts on q -expansions as

$$\text{Frob}\left(\sum a_n q^n\right) = \sum a_n q^{np}.$$

The theory of these two operators is described in detail in Chapter II of [Gou2]. We will recall here only the most important points for our purposes. To begin with,

PROPOSITION 1. — *If $\text{ord}(r) < 1/(p+1)$, then we have*

$$U(M_k(\mathbb{N}, K; r)) \subset M_k(\mathbb{N}, K; r^p).$$

See [Gou2] for a proof; we refer to this result by the code phrase “ U improves overconvergence.” As Dwork was the first to point out, the fact that U improves overconvergence implies that U is a *completely continuous* endomorphism of $M_k(\mathbb{N}, K; r)$ for any r satisfying $0 < \text{ord}(r) < p/(p+1)$. What this means is that for any integer n one can find a *finite-dimensional*

some $M_k(1)$ (here k may be a p -adic weight) whose q -expansion is $f(q)$. Conversely, any such form is obtained in this way. A form defined by such a limit *may or may not be overconvergent*, since it is an element of $M_k(1)$, which properly contains $M_k(r)$, and there seems to be no direct way of deciding if it is from the existence of such a construction.

subspace $V_n \subset M_k(N, k; r)$ such that the image of the unit ball $M_k(N, B; r)$ is contained in $V_n + p^n M_k(N, K; r)$. In our case, one can find V_n quite explicitly : it is generated by the p -adic modular forms obtained as quotients f/E_{p-1}^i , where f is a classical modular form of level N and weight $k+i(p-1)$, for $0 \leq i < (n+1)/((p-1)\text{ord}(r))$. It is straightforward to estimate that we have $\dim V_n = O(n^2)$ as n tends to infinity.⁽²⁾

The fact that U is overconvergent implies that it has a spectral theory, as explained in [Ser] and [Mon] (see also the discussion in [Gou2]). In particular, we emphasize the following three facts :

- (1) The U operator has a characteristic power series

$$P_k(t) = \det(1 - tU|M_k(r)) \in B[[t]]$$

which is independent of r and defines a p -adic entire function whose reciprocal roots are the eigenvalues of U on $M_k(r)$ and form a sequence tending to zero in B . In particular, we can write

$$P_k(t) = \prod_i (1 - \lambda_i t)$$

with λ_i ranging through the nonzero eigenvalues of U (taken in the algebraic closure of K). We know that $\text{ord}(\lambda_i) \geq 0$ and $\lambda_i \rightarrow 0$.

- (2) It is possible to define the exterior powers $\bigwedge^n U$ of any completely continuous operator; they are again completely continuous, hence have traces. Then, if we write

$$P_k(t) = \sum a_n(k)t^n,$$

we have

$$a_n(k) = \text{trace}\left(\bigwedge^n U\right).$$

See [Ser2], [Lan], Chapt. 15, §5 and [Gou2] for more information on this.

- (3) Fix $\alpha \geq 0$, and define $M_{k,\alpha}$ to be the subspace of $M_k(r)$ spanned by the forms f such that we have

$$(U - \lambda)^m(f) = 0$$

for some integer $m > 0$ and some $\lambda \in \overline{K}$ with $\text{ord}(\lambda) = \alpha$. $M_{k,\alpha}$ is then a finite-dimensional vector space, and there exists a closed Banach subspace

(2) After a conversation with G. Stolzenberg, we have come to think of an estimate for $\dim V_n$ as giving a “modulus of complete continuity” for our operator.

$F_{k,\alpha}$ such that we have a U -equivariant decomposition of $M_k(r)$ as a direct sum :

$$M_k(N, K; r) = M_{k,\alpha} \oplus F_{k,\alpha}.$$

We call $M_{k,\alpha}$ the *slope α eigenspace* for U acting on forms of weight k .

Recall that a \mathbb{Z}_p -lattice $D \subset V$ in a p -adic vector space V is a free \mathbb{Z}_p -submodule of V such that $D \otimes \mathbb{Q}_p = V$.

LEMMA 2. — *Let Φ_1 and Φ_2 be completely continuous operators on a p -adic Banach space V , and let $D \subset V$ be any \mathbb{Z}_p -lattice in V . If $\Phi_1(D) \subset D$, $\Phi_2(D) \subset D$ and*

$$(\Phi_1 - \Phi_2)(D) \subset p^n D,$$

then

$$P(t, \Phi_1) \equiv P(t, \Phi_2) \pmod{p^n},$$

where we understand the congruence coefficient-by-coefficient.

Proof. — Put

$$P(t, \Phi_1) = \sum a_i t^i \quad \text{and} \quad P(t, \Phi_2) = \sum b_i t^i.$$

We have $a_0 = b_0 = 1$, and we want to show that $a_i \equiv b_i \pmod{p^n}$ for each $i \geq 1$.

Let $\Psi = \Phi_1 - \Phi_2$. Clearly, Ψ is completely continuous, and $\Psi(D) \subset p^n D$ implies that every eigenvalue of Ψ is divisible by p^n . Hence we have

$$\text{trace}(\Phi_1) - \text{trace}(\Phi_2) = \text{trace}(\Psi) = \sum \lambda \equiv 0 \pmod{p^n},$$

where the sum is over the eigenvalues of Ψ . Since $a_1 = \text{trace}(\Phi_1)$ and $b_1 = \text{trace}(\Phi_2)$, this proves the first congruence.

For the remaining congruences, recall that we have

$$a_m = \text{trace} \left(\bigwedge^m \Phi_1 \right) \quad \text{and} \quad b_m = \text{trace} \left(\bigwedge^m \Phi_2 \right),$$

so we need to look at $\Psi = \bigwedge^m \Phi_1 - \bigwedge^m \Phi_2$. These are operators on $\bigwedge^m V$, which contains the \mathbb{Z}_p -lattice $D' = \bigwedge^m D$. Then, noting that

$$\bigwedge^m \Phi_1 - \bigwedge^m \Phi_2 = \left(\bigwedge^{m-1} \Phi_1 \right) \wedge (\Phi_1 - \Phi_2) + \left(\bigwedge^{m-1} \Phi_1 - \bigwedge^{m-1} \Phi_2 \right) \wedge \Phi_2,$$

we prove by induction that $\Psi(D')$ is contained in $p^n D'$. Thus, $a_m \equiv b_m \pmod{p^n}$, as claimed. □

Now assume $k_1 \equiv k_2 \pmod{p^n(p-1)}$, and let $\mathcal{E} : M_{k_1}(\mathbb{N}, K; r) \rightarrow M_{k_2}(\mathbb{N}, K; r)$ denote multiplication by $E_{p-1}^{(k_2-k_1)/(p-1)}$. This is easily seen to be an isomorphism of Banach spaces. (One needs only check that the inverse map preserves overconvergence; for this, note that if $f \in M_{k_2}(B, \mathbb{N}; r)$ then one sees directly from the definition that $r^{(k_2-k_1)/(p-1)} \mathcal{E} f \in M_{k_1}(B, \mathbb{N}; r)$.)

Write U_k for the U operator acting on forms of weight k . We consider the operators

$$\Phi = U_{k_1} \quad \text{and} \quad \Psi = \mathcal{E}^{-1} U_{k_2} \mathcal{E},$$

both acting on $M_{k_1}(\mathbb{N}, K; r)$. Note, first, that both are completely continuous, because both U operators are. Furthermore, our two series may be computed using them :

$$P_1(t) = \det(1 - tU|M_{k_1}(\mathbb{N}, K; r)) = \det(1 - t\Phi)$$

and, since conjugate operators have the same characteristic series,

$$\begin{aligned} P_2(t) &= \det(1 - tU|M_{k_2}(\mathbb{N}, K; r)) \\ &= \det(1 - t(\mathcal{E}^{-1}U\mathcal{E})|M_{k_1}(\mathbb{N}, K; r)) = \det(1 - t\Psi). \end{aligned}$$

Now we are in position to invoke Lemma 2. We take

$$D = M_{k_1}(\mathbb{N}, K; r) \cap M_{k_1}(\mathbb{N}, B; 1) = \{f \in M_{k_1}(\mathbb{N}, K; r) | f(q) \in B[[q]]\}.$$

This is a lattice in $M_k(r)$, since the q -expansions of modular forms have bounded denominators. To apply the lemma, we need to see that $(\Phi - \Psi)D \subset p^n D$.

LEMMA 3. — *Let W be a vector space over K , and let L be a lattice in W . Suppose $E : W \rightarrow W$ satisfies $E = I + p^t T$, where I is the identity map and $T : W \rightarrow W$ is a linear map stabilizing L . Set $F = E^{-1}$.*

If $\Upsilon : W \rightarrow W$ is a linear operator mapping L into vL for some $v \in K$, then the linear operator $F\Upsilon E - \Upsilon$ maps L into $p^t vL$.

Proof. — Simply note that

$$F\Upsilon E - \Upsilon = F\Upsilon(E - I) + (F - I)\Upsilon,$$

that both $E - I$ and $F - I$ map L to $p^t L$, and that F preserves L . □

In our situation, we take $W = K \otimes B[[q]]$, $L = B[[q]]$, $E = \mathcal{E}$ to be multiplication by $E_{p-1}^{(k_2 - k_1)/(p-1)}$, and $\Upsilon = U$, so that $v = 1$. Applying the lemma, we get

$$(\Phi - \Psi)B[[q]] \subset p^n B[[q]].$$

Since we already know that the operator $\Phi - \Psi$ preserves $M_{k_1}(r)$, it follows that $(\Phi - \Psi)(D) \subset p^n D$, as claimed.

Thus, the hypotheses of Lemma 2 are satisfied, and this completes the proof of the theorem.

3. Higher order differences.

Given what has just been proved, it is natural to ask whether the coefficients $a_m(k)$ are Iwasawa functions, i.e., if there exist power series $A_m \in \mathbb{Z}_p[[T]]$ such that we have $a_m(k) = A_m((1 + p)^k - 1)$. We cannot yet answer this question. We can, however, move a few more steps in the direction of an answer by obtaining further congruence relations among the coefficients $a_m(k)$. In fact, Iwasawa functions can be completely characterized (as in [Ser2], Theorem 14) in terms of congruence properties; what we will show is that at least some of the congruences in Serre's characterization are indeed satisfied.

To state these congruences, let $k \mapsto a(k)$ be any function from \mathbb{Z} to \mathbb{Z}_p . Fix an n , set $s = p^n(p - 1)$, and construct difference functions as follows :

$$\begin{aligned} \delta_1(a, k) &= a(k + s) - a(k) \\ \delta_2(a, k) &= \delta_1(a, k + s) - \delta_1(a, k) \\ &= a(k + 2s) - 2a(k + s) + a(k) \end{aligned}$$

and, in general, for $i > 1$,

$$\delta_i(a, k) = \delta_{i-1}(a, k + s) - \delta_{i-1}(a, k).$$

What Serre shows is that if there exists a power series $A \in \mathbb{Z}_p[[T]]$ such that $a(k) = A((1 + p)^k - 1)$ for all $k \equiv k_0 \pmod{p - 1}$, then we must have

$$\delta_i(a, k_0) \equiv 0 \pmod{p^{i(n+1)}}.$$

Theorem 1 is the special case of $a(k) = a_m(k)$ and $i = 1$. The basic idea of the proof, however, easily extends to handle the general case, as follows.

THEOREM 2. — *Let $p \geq 5$ be a prime number, N an integer not divisible by p , B a p -adically complete and separated discrete valuation ring, and K its field of fractions. Let $P_k(t)$ be the characteristic power series of the U operator acting on the space $M_k(N, K; r)$ of r -overconvergent p -adic modular forms of weight k and level N . Write $P_k(t) = \sum a_m(k)t^m$. Let δ_i be as above; then we have, for each m and k ,*

$$\delta_i(a_m, k) \equiv 0 \pmod{p^{i(n+1)}}.$$

Proof. — Fix an integer m , and recall that $a_m(k)$ is the trace of the exterior power $\bigwedge^m U$ acting on (the m -th exterior power of) forms of weight k . We use this fact to express $\delta_i(a_m, k)$ as the trace of an operator.

Consider first the case when $i = 2$. Let E be the map $\bigwedge^m M_k(r) \rightarrow \bigwedge^m M_{k+s}(r)$ which is the m -th exterior power of the map given by multiplication by $E_{p-1}^{p^n}$. Then, as we saw above,

$$\delta_1(a_m, k) = \text{trace} \left(E^{-1} \circ \bigwedge^m U \circ E - \bigwedge^m U \right).$$

Similarly, we have

$$\delta_2(a_m, k) = \text{trace} \left(E^{-2} \circ \bigwedge^m U \circ E^2 - 2E^{-1} \circ \bigwedge^m U \circ E + \bigwedge^m U \right).$$

But since

$$\begin{aligned} E^{-2} \circ \bigwedge^m U \circ E^2 - 2E^{-1} \circ \bigwedge^m U \circ E + \bigwedge^m U \\ = E^{-1} \circ \left(E^{-1} \circ \bigwedge^m U \circ E - \bigwedge^m U \right) \circ E - \left(E^{-1} \circ \bigwedge^m U \circ E - \bigwedge^m U \right), \end{aligned}$$

we can apply Lemma 3 twice : once with

$$\Upsilon = \bigwedge^m U \quad \text{and} \quad v = 1,$$

and once with

$$\Upsilon = E^{-1} \circ \bigwedge^m U \circ E - \bigwedge^m U \quad \text{and} \quad v = p^{n+1}.$$

We conclude that $E^{-2} \circ \bigwedge^m U \circ E^2 - 2E^{-1} \circ \bigwedge^m U \circ E + \bigwedge^m U$ maps $\bigwedge^m D$ to $p^{2(n+1)} \bigwedge^m D$, and therefore that its trace is congruent to zero modulo $p^{2(n+1)}$, as desired.

The general case follows in an analogous way, by repeated application of Lemma 3. \square

4. Open questions.

What about the other congruences given by Serre in [Ser2]? Specifically, we would like to know the answer to the following :

QUESTION. — Let c_{ij} be defined by the equation

$$Y(Y-1)\cdots(Y-j+1) = \sum c_{ij}Y^i,$$

and, with notations as above, let

$$\gamma_j(a, k_0) = \sum_{i=1}^j c_{ij}p^{-i(n+1)}\delta_i(a, k_0).$$

Is it true that we have

$$\text{ord}_p(\gamma_j(a_m, k)) \geq \text{ord}_p(j!)$$

for every m and k ?

The point is that, according to [Ser2], this extra series of congruences, along with the congruences already proven, would be sufficient to guarantee that the $a_m(k)$ are Iwasawa functions of k .

There is a connection between Theorem 1 and the conjectures about “ p -adic families” of modular eigenforms which we proposed in [GM]. In that paper, we considered the classical spaces $\mathbf{M}_k(K, Np)$ of modular forms of weight k on $\Gamma_1(N) \cap \Gamma_0(p)$. On these spaces, there is an action of the U operator; thus, for each rational number $\alpha \geq 0$ we can look at the subspace $\mathbf{M}_{k,\alpha}$ spanned by the eigenforms for the U operator whose eigenvalues had valuation α . We write $d(k, \alpha)$ for the dimension of this space. In [GM], we made the following conjecture :

CONJECTURE 1. — Let k_1 and k_2 be integers. Suppose both k_1 and k_2 are bigger than $2\alpha + 2$, and that $k_1 \equiv k_2 \pmod{p^n(p-1)}$ for some integer $n \geq \alpha$. Then $d(k_1, \alpha) = d(k_2, \alpha)$.

In attempting to prove this conjecture, it seems natural to embed the classical spaces into the corresponding spaces of overconvergent p -adic

modular forms, which should be the “correct” context for studying p -adic properties of modular forms. Recall that we have an inclusion

$$\mathbf{M}_k(K, Np) \hookrightarrow M_k(B, N; r) \otimes K,$$

which therefore gives an inclusion $\mathbf{M}_{k,\alpha} \hookrightarrow M_{k,\alpha}$ of the slope α subspaces. Writing $d_p(k, \alpha) = \dim M_{k,\alpha}$ for the dimension of the p -adic slope α subspace, one might then consider a p -adic variant of our conjecture :

CONJECTURE 2. — *Let k_1 and k_2 be integers such that $k_1 \equiv k_2 \pmod{p^n(p-1)}$ for some integer $n \geq \alpha$. Then $d_p(k_1, \alpha) = d_p(k_2, \alpha)$.*

Both of these conjectures are known to be true when $\alpha = 0$, by the work of Hida on ordinary modular forms.⁽³⁾ In Hida’s work, proving Conjecture 2 is the first step in the proof of Conjecture 1, so that it is not unreasonable to expect the two conjectures to be similarly connected in general.

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(3) The case $\alpha = 0$ of Conjecture 2 also follows from the main theorem in this note, as one can see by considering the Newton polygons of characteristic power series in weights k_1 and k_2 . The dimension in question is the length of the first (horizontal) segment of the polygon, which is clearly the same when the coefficients of two power series are congruent modulo p .

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