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PL-REPRESENTATIONS OF ANOSOV FOLIATIONS

by Norikazu HASHIGUCHI

0. Introduction.

Let Σ_g be the closed oriented surface of genus $g(\geq 2)$ with a hyperbolic metric. The geodesic flow of the unit tangent vector bundle $T_1\Sigma_g$ of Σ_g is an Anosov flow. To study this flow, Fried gave a good Birkhoff section for it, and Ghys showed that it is obtained from the suspension flow of some hyperbolic toral automorphism by a certain Dehn surgery (see [F] and [Gh]). In other words, the geodesic flow on $T_1\Sigma_g$ restricted to the complement of the $4g+4$ closed orbits $\{\pm G_1, \pm G_2, \pm G_3, \ldots, \pm G_{2g+2}\}$ is topologically equivalent to the suspension of $\overline{A_g} : T^2 \to T^2$ restricted to the complement of the $4g+4$ closed orbits $\{O_l^m\}_{l=0,1,2,\ldots,2g+1, m=0,1}$ (see § 1). This toral automorphism $\overline{A_g}$ and the orbits $\{O_l^m\}$ are determined by the author as well as the type of the Dehn surgery (see [Ha1]).

Since the suspension of the (un)stable linear foliation of the torus by the hyperbolic toral automorphism is a transversally affine foliation, certain Dehn surgeries along leaf curves with nontrivial holonomy give rise to a transversally PL foliation (see §1). Since the (un)stable foliation of the geodesic flow a $T_1\Sigma_g$ is transverse to the fibre of the projection $T_1\Sigma_g \to \Sigma_g$, this transversally PL foliation can be seen as a PL foliated S^1-bundle. In other words, there exists a homomorphism

$$\Phi_g : \pi_1(\Sigma_g) \to PL_+(S^1)$$

such that the (un)stable foliation of the geodesic flow on $T_1\Sigma_g$ is topologically conjugate to the suspension of Φ_g (see [Gh]). Here $PL_+(S^1)$ is the

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group of orientation preserving homeomorphisms of S^1 which lift to piecewise linear homeomorphisms of $\mathbb{R}$. The elements in the image of Φ_g have almost everywhere defined derivatives which are multiple of a real quadratic λ_g. Ghys used this fact to show that the (extended) Godbillon-Vey invariant is not topologically invariant.

The purpose of this paper is to describe the homomorphism Φ_g concretely and study various properties of this homomorphism.

The organization of this paper is as follows. In §1, first we review how the Dehn surgery of transversally affine foliation gives rise to a transversally PL foliation. Secondly, we review the identification between $T_1\Sigma_g - \{\pm G_1, \pm G_2, \pm G_3, \dots, \pm G_{2g+2}\}$ and the suspension of $\overline{A_g}$ with $\{O_l^m\}$ deleted, and we see how the fibres of $T_1\Sigma_g \to \Sigma_g$ are in the suspension of $\overline{A_g}$. Then we determine the holonomy homomorphism Φ_g. In order to determine Φ_g, it is enough to determine only one PL homeomorphism f_g. Since this f_g satisfies

$$\left\{f_g \circ T\left(-\frac{1}{g+1}\right)\right\}^{g+1} = \left\{f_g \circ T\left(-\frac{1}{2(g+1)}\right)\right\}^{2(g+1)} = 1,$$

we can determine the holonomy on the generators of $\pi_1(\Sigma_g)$, and we verify that these holonomies on the generators respect the relations of $\pi_1(\Sigma_g)$. Here, $T(\theta)(\theta \in \mathbb{R}/\mathbb{Z} = S^1)$ denotes the rotation by θ.

In §2, we calculate the discrete Godbillon-Vey invariant of Φ_g. This invariant is defined by Ghys and Sergiescu (see [GS] and [Gh]). They gave the 2-cocycle representing the discrete Godbillon-Vey class $\overline{gv} \in H^2(PL_+(S^1); \mathbb{R})$. So we evaluate $\Phi_g^*(\overline{gv}) \in H^2(\pi_1(\Sigma_g); \mathbb{R})$ on the fundamental class of $\pi_1(\Sigma_g)$. The value is $-4(g+1)(\log \lambda_g)^2$. Let $\mathcal{D}_+(S^1)$ denote the homomorphisms of class P of S^1 (see [He]). The usual Godbillon-Vey class $gv \in H^2(\mathrm{Diff}_+^2(S^1); \mathbb{R})$ as well as $\overline{gv} \in H^2(PL_+(S^1); \mathbb{R})$ is extended to $H^2(\mathcal{D}_+(S^1); \mathbb{R})$ (see [Gh]). As a corollary of the above calculation, we obtain again the result of Ghys which says that each $\alpha gv + \beta \overline{gv} \in H^2(\mathcal{D}_+(S^1); \mathbb{R})\, (\alpha, \beta \in \mathbb{R}, \alpha^2 + \beta^2 \neq 0)$ is not a topological invariant.

In §3, we give remarks related to our result. The presentation of $\pi_1(\Sigma_g)$ we used to describe Φ_g is interesting in itself. We exhibit a fundamental domain in the Poincaré disk for this presentation. Then we geometrically show that Φ_g factors through a homomorphism

$$\phi_g : \Gamma_g \to PL_+(S^1),$$

where Γ_g is a triangle group $\Gamma(g+1, 2g+2, 2g+2)$.

We study the deformation of ϕ_g and Φ_g in [Ha2]. We saw the foliation obtained from transversally affine foliation by Dehn surgery along leaf curves with nontrivial holonomy is transversally PL. The Godbillon-Vey invariant and the discrete Godbillon-Vey invariant of a transversally affine foliation are 0. By the result of the above calculation of the discrete Godbillon-Vey invariant, one would conjecture that each $(1,1)$-Dehn surgery along the closed orbit $\{O_l^m\}$ decreases $\overline{gv}$ by $(\log \lambda_g)^2$. We will also see that this conjecture is true by using the result of Greenberg. We will show it in a future paper.

Finally, the author would like to thank Professor T. Tsuboi for helpful advice and continuous encouragement.

1. Geodesic flows on Σ_g.

Let Σ_g be a closed orientale surface of genus $g(\geq 2)$. We consider a Riemannian metric with constant negative curvature -1 on it. Let F_t denote the geodesic flow on the unit tangent vector bundle $T_1\Sigma_g$ and π, the projection $T_1\Sigma_g \to \Sigma_g$.

Fried constructed the Birkhoff section S for F_t in [F] as follows. Let $\pm G_i \subset T_1\Sigma_g (i = 1, 2, 3, \dots, 2g+2)$ be the oriented closed geodesics shown in Figure 1. Then $G_i = \pi(\pm G_i) \subset \Sigma_g$ is a closed geodesic. $\{G_1, G_2, G_3, \dots, G_{2g+2}\}$ divide Σ_g into four $2g+2$ gons P_1, P_2, P_3, P_4 where P_1 and P_2 are named so that they intersect at only $2g+2$ vertices. Let $p_i \in \Sigma_g$ be $G_i \cap G_{i+1}$ where $i = 1, 2, 3, \dots, 2g+2, G_{2g+3} = G_1$ (see Figure 1). For $i = 1, 2$, we choose a family C_i of convex smooth simple closed curves which fill the interior of P_i with one singularity o_i deleted. Let S be the closure of the set of unit vectors which are tangent to the curves belonging to C_i. $\partial S = \left(\bigcup_{i=1}^{2g+2} (+G_i)\right) \cup \left(\bigcup_{i=1}^{2g+2} (-G_i)\right)$, $\pi^{-1}(o_i) \subset S (i = 1, 2)$ and S is diffeomorphic to a torus with $4g+4$ open disks deleted. Let $b_1 \subset S$ denote $\pi^{-1}(o_1)$ and $b_2 \subset S$, a component of $S \cap \pi^{-1}(\underline{b_2})$ where $\underline{b_2}$ is the closed geodesic through o_1, p_1, o_2, p_{g+2} and o_1. If we take $\langle b_1, b_2 \rangle$ as the basis of S, then the first return map of F_t about S is semi conjugate to the hyperbolic toral automorphism induced by

$$A_g = \begin{pmatrix} 2g^2-1 & 2g(g-1) \\ 2g(g+1) & 2g^2-1 \end{pmatrix} \in SL(2, \mathbb{Z}).$$

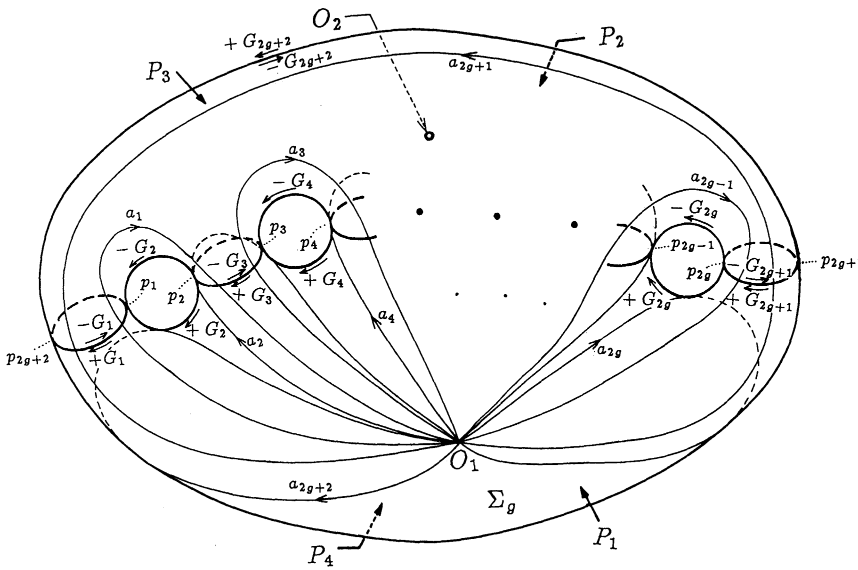


Figure 1

Similarly, we can construct the Birkhoff section S' for F_t over $P_3 \cup P_4$ from families C_3 and C_4 which are mapped on C_2 and C_1 by the reflection of a plane V, respectively. Here, V divides Σ_g into $P_1 \cup P_3$ and $P_2 \cup P_4$. F_t can be constructed from the matrix A_g as follows (see [Ha1]). The matrix A_g acts on T^2 as a diffeomorphism $\overline{A_g}$ and let M be the torus bundle over S' with monodromy $\overline{A_g}$, and ϕ_t, the suspension flow of $\overline{A_g}$. More explicitly, let

$$\widetilde{\phi}_t : \mathbb{R}^3 \to \mathbb{R}^3$$

be the flow defined by $\widetilde{\phi}_t(x, y, z) = (x, y, z + t)$. Consider the equivalence relation $\sim$ on $\mathbb{R}^3$ generated by $(x+1, y, z) \sim (x, y, z)$, $(x, y+1, z) \sim (x, y, z)$ and $(x, y, z + 1) \sim \left({}^t\left(A_g \begin{pmatrix} x \\ y \end{pmatrix} \right), z \right)$. Then M is the quotient space $\mathbb{R}^3/\sim$, and we obtain the induced Anosov flow

$$\phi_t : M \to M.$$

Let

$$q : \mathbb{R}^3 \to M$$

be the quotient map. We construct an Anosov flow

$$\varphi_t : T_1\Sigma_g \to T_1\Sigma_g$$

from the suspension flow ϕ_t by $(1,1)$-Dehn surgeries along $4g+4$ closed orbits $\{O_l^m\}$ of period 1. Here,

$$O_l^m = q\left(\left\{\left(\frac{l}{2(g+1)}, \frac{m}{2}, t\right) \in \mathbb{R}^3; t \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right\}\right)$$
$$(l = 0, 1, \dots, 2g+1, m = 0, 1).$$

These closed orbits correspond to $4g+4$ oriented closed geodesics

$$\{\pm G_1, \pm G_2, \dots, \pm G_{2g+2}\}$$

shown in Figure 1. Then in [Ha1], we showed F_t is topologically equivalent to φ_t.

We will show that the unstable foliation $\mathcal{F}^{u'}$ of φ_t is transversally piecewise linear. It is easy to see that the unstable foliation $\mathcal{F}^u$ of ϕ_t is induced from a linear foliation on a torus and transversally affine. More precisely, the holonomy pseudogroup of $\mathcal{F}^u$ is generated by the affine maps

$$y = (\lambda_g)^\sigma x + \tau(\sigma \in \mathbb{Z}, \tau \in \mathbb{R})$$

where $\lambda_g = 2g^2 - 1 + 2g\sqrt{g^2-1}$ is the larger eigenvalue of A_g. The leaves of $\mathcal{F}^u$ are diffeomorphic to $\mathbb{R}^2$ or $S^1 \times \mathbb{R}$. Each O_l^m is a leaf curve in a cylindrical leaf L_l^m and the holonomy of O_l^m is an affine map $y = \lambda_g^{-1}x$. There is a neighborhood N of O_l^m such that it is homeomorphic to $[-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon] \times S^1, \{0\} \times \{0\} \times S^1 = O_l^m$ and $\{0\} \times [-\varepsilon, \varepsilon] \times S^1$ is a component of $N \cap L_l^m$ where ε is a very small positive number and $S^1 = \mathbb{R}/\mathbb{Z}$. The holonomy of L_l^m along O_l^m is $y = \lambda_g^{-1}x$ and the holonomy of L_l^m along the leaf curve $\{(0,t,0) \in N; -\varepsilon \le t \le \varepsilon\}$ is the identity.

The $(1,1)$-Dehn surgery along O_l^m used in [F] is as follows. Let a torus $T_l^m = \{(\theta, u) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}\}$ be the blowing-up of O_l^m with a flow $\phi_t^{(l,m)}$ on T_l^m induced from ϕ_t. We divide T_l^m into the circles

$$S_r^1 = \left\{(\theta, r) \in T_l^m; 0 \le \theta \le \frac{1}{2}\right\} \cup \left\{(\theta, 2\theta + r - 1) \in T_l^m; \frac{1}{2} \le \theta \le 1\right\}$$
$$(0 \le r \le 1).$$

We can perturb the latter part of S_r^1 in order that S_r^1 is a smooth circle and transverse to $\phi_t^{(l,m)}$. Contracting each S_r^1 to a point, we obtain a closed orbit $O_l^{m'}$ in a cylindrical leaf $L_l^{m'}$. There is also a neighborhood N' of $O_l^{m'}$ such that it is induced from N and it has the same properties as N. The holonomy of $L_l^{m'}$ along $O_l^{m'}$ is $y = \lambda_g^{-1}x$. The holonomy of $L_l^{m'}$ along $\{(0,t,0) \in N'; -\varepsilon \leq t \leq \varepsilon\}$ is piecewise linear as follows. By the choice of S_0^1, the rectangle $\{(s,t,0) \in N'; 0 \leq s \leq \varepsilon, -\varepsilon \leq t \leq \varepsilon\}$ is induced from the rectangle $\{(s,t,0) \in N; 0 \leq s \leq \varepsilon, -\varepsilon \leq t \leq \varepsilon\}$. And the rectangle $\{(s,t,0) \in N'; -\varepsilon \leq s \leq 0, -\varepsilon \leq t \leq \varepsilon\}$ is induced from

$$\begin{aligned} &\{(s,t,0) \in N; -\varepsilon \leq s \leq 0, -\varepsilon \leq t \leq 0\} \\ &\cup \{(s,0,\theta) \in N; -\varepsilon \leq s \leq 0, -1 \leq \theta \leq 0\} \\ &\cup \{(s,t-1) \in N; -\varepsilon \leq s \leq 0, 0 \leq t \leq \varepsilon\}. \end{aligned}$$

So the holonomy along $\{(0,t,0) \in N'; -\varepsilon \leq t \leq \varepsilon\}$ is

$$y = \begin{cases} x & \text{for } x \geq 0 \\ \lambda_g x & \text{for } x \leq 0. \end{cases}$$

Hence, $\mathcal{F}^{u\prime}$ is transversally piecewise linear in the neighborhood of $O_l^{m'}$. It implies that $\mathcal{F}^{u\prime}$ is transversally piecewise linear.

Similarly in general, operating $(1,n)$-Dehn surgeries along finite leaf curves of a transversally affine foliation, we obtain a transversally piecewise linear foliation.

Since the unstable foliation of F_t is transverse to the fibre of the projection $\pi : T_1\Sigma_g \to \Sigma_g$, the transversally piecewise linear foliation $\mathcal{F}^{u\prime}$ can be seen as a PL-foliated S^1 bundle. So the total holonomy of $\mathcal{F}^{u\prime}$

$$\Phi_g : \pi_1(\Sigma_g) \to PL_+(S^1)$$

is determined. Now we consider the spaces

$$\mathbb{L}^3 = [0,1] \times [0,1] \times \left[-\frac{1}{2}, \frac{1}{2}\right] \subset \mathbb{R}^3$$

and

$$\mathbb{L}_*^3 = \mathbb{L}^3 - \left(\bigcup_{l=0}^{2g+1} q^{-1}(O_l^0)\right) \cup \left(\bigcup_{l=0}^{2g+1} q^{-1}(O_l^1)\right).$$

By the above construction, $T_1\Sigma_{g*} = T_1\Sigma_g - \{\pm G_1, \pm G_2, \dots, \pm G_{2g+2}\}$ is obtained from $\mathbb{L}^3_*$ by identifying opposite faces of $\mathbb{L}^3_*$ by the equivalence relation $\sim$. We note that, in particular, for each $i \in \{-g, -(g-1), \dots, 3g+1, 3g+2\}$, two segments in $\partial\mathbb{L}^3_*$

$$\left\{\left(x, \frac{g+1}{g}\left(x - \frac{i}{2(g+1)}\right) + \frac{1}{2}, -\frac{1}{2}\right); x \in \mathbb{R}\right\} \cap \mathbb{L}^3_*$$

and

$$\left\{\left(x, -\frac{g+1}{g}\left(x - \frac{i}{2(g+1)}\right) + \frac{1}{2}, \frac{1}{2}\right); x \in \mathbb{R}\right\} \cap \mathbb{L}^3_*$$

are identified.

In order to determine Φ_g, we need to study the "bundle" structure $\mathbb{L}^3_*/\sim \to \Sigma_g$ corresponding to $\pi|_{T_1\Sigma_*} : T_1\Sigma_{g*} \to \Sigma_g$. So we see how the fibre of $\pi|_{T_1\Sigma_*}$ are in $\mathbb{L}^3_*$. This correspondence is only determined up to parallel transformations in x-direction. From the above construction, $\mathrm{Int}(S)$, which is a $4g+4$ punctured torus, corresponds to $[0,1] \times [0,1] \times \{0\} \cap \mathbb{L}^3_*$. More precisely, $\mathrm{Int}(S) \cap \pi^{-1}(P_i) (i = 1, 2)$ corresponds to

$$R_i = [0,1] \times \left[\frac{i-1}{2}, \frac{i}{2}\right] \times \{0\} \cap \mathbb{L}^3_*.$$

$\mathrm{Int}(S') \cap \pi^{-1}(P_3)$ corresponds to

$$R_3 = \left\{\left({}^t\left(B_g\begin{pmatrix} x \\ y \end{pmatrix}\right), \frac{1}{2}\right) \in \mathbb{R}; (x, y) \in [0,1] \times \left[\frac{1}{2}, 1\right]\right\},$$

or

$$R'_3 = \left\{\left({}^t\left(B_g^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right), -\frac{1}{2}\right) \in \mathbb{R}; (x, y) \in [0,1] \times \left[\frac{1}{2}, 1\right]\right\},$$

and $\mathrm{Int}(S') \cap \pi^{-1}(P_4)$ corresponds to

$$R_4 = \left\{\left({}^t\left(B_g\begin{pmatrix} x \\ y \end{pmatrix}\right), \frac{1}{2}\right) \in \mathbb{R}; (x, y) \in [0,1] \times \left[0, \frac{1}{2}\right]\right\},$$

or

$$R'_4 = \left\{\left({}^t\left(B_g^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right), -\frac{1}{2}\right) \in \mathbb{R}; (x, y) \in [0,1] \times \left[0, \frac{1}{2}\right]\right\},$$

where

$$B_g = \begin{pmatrix} -g & -(g-1) \\ -(g+1) & -g \end{pmatrix}$$

satisfying that $(B_g)^2 = A_g$. Since $b_1 = \pi^{-1}(o_1)$ is the base of S, it corresponds to the center line of R_1, *i.e.*,

$$L_1 = \left\{ \left(x, \frac{1}{4}, 0\right) \in \mathbb{L}^3_*; x \in [0,1] \right\} \quad \text{(see Figures 1 and 2).}$$

Similarly, $\pi^{-1}(o_2)$ corresponds to the center line of R_2

$$L_2 = \left\{ \left(x, \frac{3}{4}, 0\right) \in \mathbb{L}^3_*; x \in [0,1] \right\}.$$

Since b_2 is another base of S, it corresponds to $\left\{ \frac{1}{4(g+1)} \right\} \times [0,1] \times \{0\} \cap \mathbb{L}^3_*$.

The correspondence between closed orbits $\{\pm G_i\}$ and $\{O^m_\ell\}$ is as follows. From the definition of $b_2, \partial(\mathrm{Int}(S) \cap \pi^{-1}(P_1)) \cap b_2 \cap \pi^{-1}(p_1)$ is between $q_2^+ = \partial(\mathrm{Int}(S) \cap \pi^{-1}(P_1)) \cap (+G_2)$ and $q_1^+ = \partial\,(\mathrm{Int}(S) \cap \pi^{-1}(P_1)) \cap (+G_1)$. So, operating $(1,1)$-Dehn surgery along O^0_0(resp. O^0_1), we obtain $+G_2$(resp. $+G_1$). Similarly, since $\partial(\mathrm{Int}(S) \cap \pi^{-1}(P_1)) \cap b_2 \cap \pi^{-1}(p_{g+2})$ is between $q^+_{g+2} = \partial(\mathrm{Int}(S) \cap \pi^{-1}(P_1)) \cap (-G_{g+2})$ and $q^+_{g+3} = \partial(\mathrm{Int}(S) \cap \pi^{-1}(P_1)) \cap (-G_{g+3}), O^1_0$(resp. O^1_1) corresponds to $-G_{g+3}$(resp. $-G_{g+2}$). $\partial(\mathrm{Int}(S) \cap \pi^{-1}(P_1))$ intersects

$$+G_2 + G_1, +G_{2g+2}, \dots, +G_3$$

$$(\text{resp.} -G_{g+3}, -G_{g+2}, \dots, -G_2, -G_1, -G_{2g+2}, \dots, -G_{g+4})$$

in this order. Hence,

$$O^0_2, O^0_3, \dots, O^0_{2g+1}, O^1_2, O^1_3, \dots, O^1_{2g+1}$$

correspond to

$$+G_{2g+2}, +G_{2g+1}, \dots, +G_3, -G_{g+1}, -G_g, \dots, -G_1, -G_{2g+2}, \dots, -G_{g+4},$$

respectively. $\pi^{-1}(p_i) \cap \partial(\mathrm{Int}(S) \cap \pi^{-1}(P_1))$ is two open intervals (q_2^+, q_1^+) and $(q_2^-, q_1^-) \subset \partial(\mathrm{Int}(S) \cap \pi^{-1}(P_1))$ where $q_i^- = \partial(\mathrm{Int}(S) \cap \pi^{-1}(P_1)) \cap (-G_i)(i = 1, 2)$. The other part of $\pi^{-1}(p_1)$ is $\pi^{-1}(p_1) \cap \partial(\mathrm{Int}(S') \cap \pi^{-1}(P_3))$ which are two open intervals. Because of the correspondence between $\mathrm{Int}(S) \cap \pi^{-1}(P_1)$(resp. $\mathrm{Int}(S') \cap \pi^{-1}(P_3)$) and R_1(resp. $R_3 \subset$

$\left\{\left(x, y, \frac{1}{2}\right); x, y \in \mathbb{R}\right\}$), $\pi^{-1}(p_1)$ can be considered to be corresponding to the union of four segments

$$\begin{aligned}
S_1 = &\left\{(x,0,0) \in \mathbb{L}^3_*; x \in \left(0, \frac{1}{2(g+1)}\right)\right\} \\
&\cup \left\{\left(x, \frac{g+1}{g}\left(x - \frac{1}{2}\right) + \frac{1}{2}, -\frac{1}{2}\right) \in \mathbb{L}^3_*; x \in \left(\frac{1}{2(g+1)}, \frac{1}{2}\right)\right\} \\
&\cup \left\{\left(x, \frac{1}{2}, 0\right) \in \mathbb{L}^3_*; x \in \left(\frac{1}{2}, \frac{1}{2} + \frac{1}{2(g+1)}\right)\right\} \\
&\cup \left\{\left(x, -\frac{g+1}{g}\left(x - \frac{1}{2} - \frac{1}{2(g+1)}\right) + \frac{1}{2}, \frac{1}{2}\right) \in \mathbb{L}^3_*;\right. \\
&\qquad \left. x \in \left(\frac{1}{2} + \frac{1}{2(g+1)}, 1\right)\right\},
\end{aligned}$$

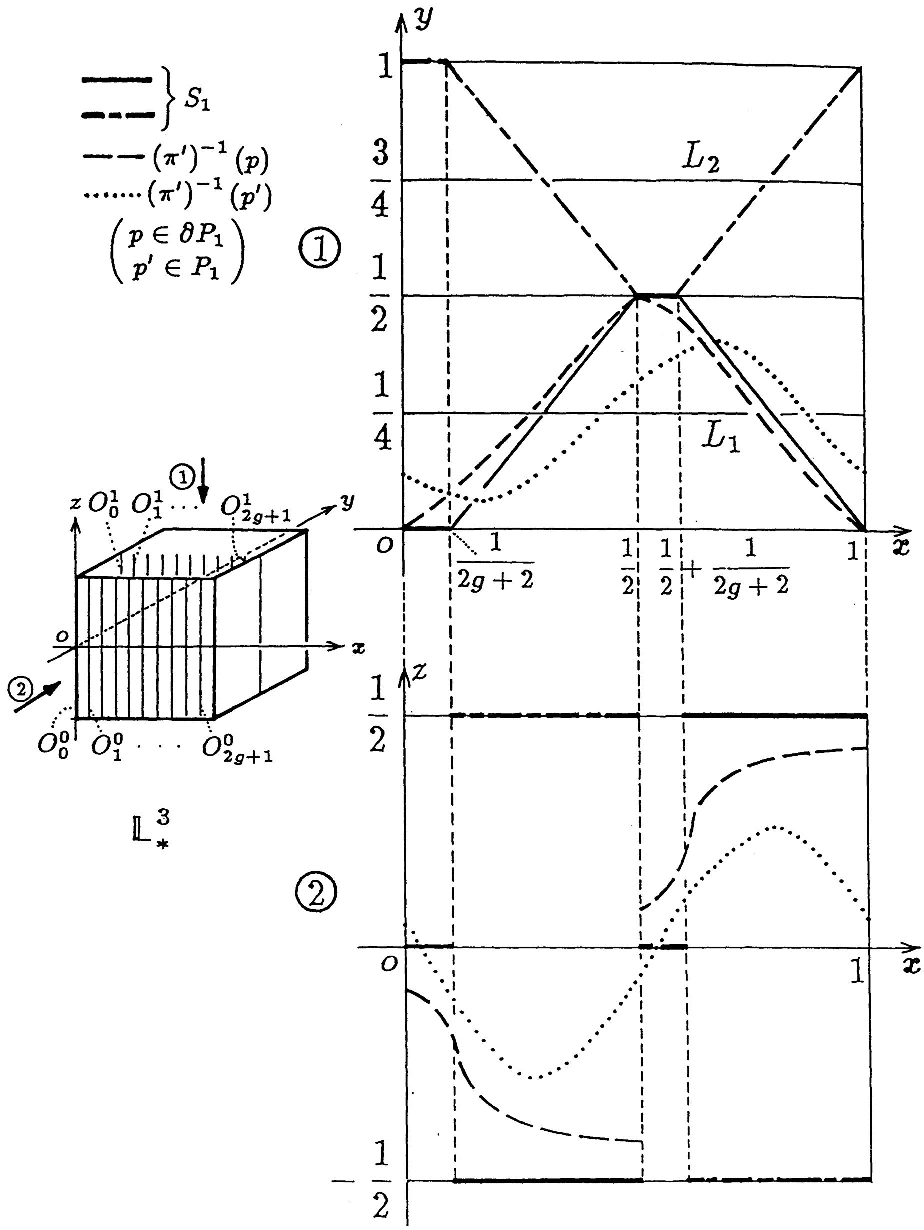
or

$$\begin{aligned}
S_1' = &\left\{(x,1,0) \in \mathbb{L}^3_*; x \in \left(0, \frac{1}{2(g+1)}\right)\right\} \\
&\cup \left\{\left(x, -\frac{g+1}{g}\left(x - \frac{1}{2}\right) + \frac{1}{2}, \frac{1}{2}\right) \in \mathbb{L}^3_*; x \in \left(\frac{1}{2(g+1)}, \frac{1}{2}\right)\right\} \\
&\cup \left\{\left(x, \frac{1}{2}, 0\right) \in \mathbb{L}^3_*; x \in \left(\frac{1}{2}, \frac{1}{2} + \frac{1}{2(g+1)}\right)\right\} \\
&\cup \left\{\left(x, \frac{g+1}{g}\left(x - \frac{1}{2} - \frac{1}{2(g+1)}\right) + \frac{1}{2}, -\frac{1}{2}\right) \in \mathbb{L}^3_*;\right. \\
&\qquad \left. x \in \left(\frac{1}{2} + \frac{1}{2(g+1)}, 1\right)\right\},
\end{aligned}$$

(see Figure 2).

In the same way as the case of $\pi^{-1}(p_1)$, we can see that each $\pi^{-1}(p_i)(i = 2, 3, 4, \ldots, 2g+2)$ corresponds to

$$\left\{\left(x - \frac{i-1}{2g+2}, y, z\right); (x,y,z) \in S_1\right\} \equiv S_i \subset \mathbb{L}^3_*,$$



S_1
$(\pi')^{-1}(p)$
$(\pi')^{-1}(p')$
$\begin{pmatrix} p \in \partial P_1 \\ p' \in P_1 \end{pmatrix}$
①
②
z O^1_0 O^1_1 $\cdots$ O^1_{2g+1}
y
o
x
O^0_0 O^0_1 $\cdots$ O^0_{2g+1}
$\mathbb{L}^3_*$
y
1
$\frac{3}{4}$
$\frac{1}{2}$
$\frac{1}{4}$
L_2
L_1
o
$\frac{1}{2g+2}$
$\frac{1}{2}$
$\frac{1}{2}+\frac{1}{2g+2}$
1
x
z
$\frac{1}{2}$
$-\frac{1}{2}$

Figure 2

or

$$\left\{\left(x-\frac{i-1}{2g+2},y,z\right);(x,y,z)\in S_1'\right\}\equiv S_i'\subset \mathbb{L}_*^3$$
$$(\text{mod } \mathbb{Z} \text{ in } x- \text{ coordinate}).$$

Let $l_1 \subset \Sigma_g$ be a geodesic arc between o_1 and p_1. $\pi^{-1}(l_1)$ is an annulus and it corresponds to a helicoid $H_1 \subset \mathbb{L}_*^3$ such that its center line is L_1 and its edge is S_1 which is a spiral around L_1. Similarly, let $l_i(\text{resp.}\, l_i') \subset \Sigma_g$ be the geodesic arc between $o_1(\text{resp.}\, o_2)$ and $p_i (i = 1, 2, 3, \ldots, 2g+2)$, then $\pi^{-1}(l_i)(\text{resp.}\pi^{-1}(l_i'))$ corresponds to a helicoid $H_i(\text{resp.}H_i') \subset \mathbb{L}_*^3$ such that its center line is $L_1(\text{resp.}L_2)$ and its edge is $S_i(\text{resp.}S_i')$. We only need H_i and H_i' to calculate Φ_g. Other fibres correspond to some curves in $\mathbb{L}_*^3$ as follows (see Figure 2). For $p \in \text{Int}(G_i \cap \partial P_j)(i = 1, 2, 3, \ldots, 2g+2, j = 1, 2), \pi^{-1}(p)$ corresponds to the union of two curves and for $p' \in \text{Int}P_j(j = 1, 2), \pi^{-1}(p')$ corresponds to a compatibly oriented spiral around L_j. Curves corresponding to fibres over $\text{Int}(P_3)(\text{resp.Int}(P_4))$ are the spirals around the center lines of R_3 or $R_3'(\text{resp.}R_4$ or $R_4')$.

To sum up, we obtain a continuous map

$$\pi' = (\pi|_{\mathbb{L}_*^3/\sim}) \circ (q|_{\mathbb{L}_*^3}) : \mathbb{L}_*^3 \to \Sigma_g$$

such that $(\pi')^{-1}(l_i) = H_i, (\pi')^{-1}(l_i') = H_i'(i = 1, 2, 3, \ldots, 2g+2)$ and $(\pi')^{-1}(P_j)(j = 1, 2)$ is $(2g+2 \text{ gon}) \times [0, 1]$ which twists around L_j. $(\pi')^{-1}(P_3)(\text{resp. } (\pi')^{-1}(P_4))$ is $(2g+2 \text{ gon}) \times [0, 1]$ which twists around the center line of R_3 or $R_3'(\text{resp.}R_4$ or $R_4')$.

To calculate the piecewise linear total holonomy of a foliated S^1 bundle over Σ_g, a presentation of the fundamental group $\pi_1(\Sigma_g)$ is adopted here. That is,

$$\pi_1(\Sigma_g) = \langle a_1, a_2, a_3, \ldots, a_{2g+2}; \\ a_1a_2a_3\ldots a_{2g+2} = a_1a_3\ldots a_{2g+1} = a_2a_4\ldots a_{2g+2} = 1\rangle$$

(see Figure 1 and Proposition 2 in §3).

Let α_i be the loop in Σ_g which starts o_1 and passes p_i, o_2, p_{i+1} and reaches o_1 such that α_i represents a_i in $\pi_1(\Sigma_g)$. It is easy to see that

$$(\pi')^{-1}(\alpha_i) = H_i \cup H_i' \cup H_{i+1}' \cup H_{i+1}.$$

Now we calculate the PL total holonomy ϕ_g of the unstable foliation $\mathcal{F}^{u'}$ of φ_t. The total holonomy of the stable foliation is conjugate to Φ_g

by $T(1/2)$. An eigenvector of A_g corresponding to λ_g is $\left(1, \sqrt{\dfrac{g+1}{g-1}}\right)$. Let $\Pi_s (s \in \mathbb{R})$ be the plane

$$\left\{ \left(x, \sqrt{\frac{g+1}{g-1}} x + s, z \right) \in \mathbb{R}^3; x, z \in \mathbb{R} \right\}.$$

The leaves of $\mathcal{F}^{u'}|_{\mathbb{L}^3_*/\sim}$ are made of

$$\left\{ \Pi_s \cap \mathbb{L}^3_*; s \in \left(-\sqrt{\frac{g+1}{g-1,1}} \right) \right\}.$$

Now, for $p \in (\pi')^{-1}(o_1) = L_1$, there exists $s \in \left(-\sqrt{\dfrac{g+1}{g-1}}, 1 \right)$ such that $p \in \Pi_s$ and we move p on

$$\Pi_s \cap (H_1 \cup H_1' \cup H_2' \cup H_2)$$

along α_1 in $\mathbb{L}^3_*$ following the next rule; if a point reaches a face of the cube $\mathbb{L}^3_*$ then the point is moved in the opposite face by the equivalence relation $\sim$ and starts from the face into $\mathrm{Int}(\mathbb{L}^3_*)$. During this move, p passes S_1, L_2, S_2 one by one. Finally, p returns to L_1. But any points which pass some $q^{-1}(O_l^m)$ cannot return to L_1. Hence we obtain a return map defined on $L_1/_\sim - \{12 \text{ points}\}$. This return map can be extended to the homeomorphism

$$f_g^{-1} : L_1/_\sim = \mathbb{R}/\mathbb{Z} = S^1 \to L_1/_\sim.$$

Then f_g is a PL homeomorphism whose left (right) differential coefficients are $(\lambda_g)^\sigma (\sigma = -1, 0, 1)$ as it is described below. (Here we parametrize L_1 by the x-coordinate.) f_g has four non-differentiable points and these points are caused by four of twelve points deleted from $L_1/_\sim$.

Now we prepare some notations. In order to describe a PL homeomorphism h of $S^1 = \mathbb{R}/\mathbb{Z}$, the lift homeomorphism of $h, \widetilde{h} : \mathbb{R} \to \mathbb{R}$, is described by using non-differentiable points of $\widetilde{h}$. If $a, b \in \mathbb{R} (a < b)$ are non-differentiable points of $\widetilde{h}$ and $\widetilde{h}|_{[a,b]}$ is

$$y = \lambda x + \nu (\lambda, \nu \in \mathbb{R}),$$

then $\widetilde{h}_{[a,b]}$ is denoted by

$$a \longmapsto c = \lambda a + \nu$$
$$[\lambda]$$
$$b \longmapsto d = \lambda b + \nu.$$

For example, $T(\theta)(\theta \in S^1 = \mathbb{R}/\mathbb{Z})$ is denoted by

$$T(\theta) : 0 \longmapsto \theta$$
$$[1]$$
$$1 \longmapsto \theta + 1.$$

Then f_g or its lift $\widetilde{f}_g$ is described as follows :

$$\frac{4g^2 - 2 + (1-4g)\sqrt{g^2-1}}{4(g+1)} \mapsto \frac{2+\sqrt{g^2-1}}{4(g+1)}$$
$$[1]$$
$$\frac{2g - \sqrt{g^2-1}}{4(g+1)} \mapsto \frac{(4g-1)\sqrt{g^2-1} - 4g^2 + 2g + 4}{4(g+1)}$$
$$[(\lambda_g)^{-1}]$$
$$\frac{4g + 2 + \sqrt{g^2-1}}{4(g+1)} \mapsto \frac{4g+2-3\sqrt{g^2-1}}{4(g+1)}$$
$$[1]$$
$$\frac{2g + 4 + 3\sqrt{g^2-1}}{4(g+1)} \mapsto \frac{2g+4-\sqrt{g^2-1}}{4(g+1)}$$
$$[\lambda_g]$$
$$\frac{4g^2 - 2 + (1-4g)\sqrt{g^2-1}}{4(g+1)} + 1 \mapsto \frac{2+\sqrt{g^2-1}}{4(g+1)} + 1.$$

Similarly, with respect to $\alpha_i (i = 2, 3, \ldots, 2g+2)$, we obtain a PL-homeomorphism

$$T\left(-\frac{i-1}{2(g+1)}\right) \circ f_g^{-1} \circ T\left(\frac{i-1}{2(g+1)}\right).$$

The next lemma is proved by the induction.

LEMMA 1. — *Let* $f_g^{(i)} (i = 1,2)$ *be* $f_g \circ T\left(-\frac{1}{i(g+1)}\right)$. $\mathcal{L}_g$ *and* $\mathcal{M}_g$ *denote* $2\lambda_g(g+1)$ *and* $\sqrt{\lambda_g}(g+1)$, *respectively.*

(1) *For* $m = 1,2,\ldots,g-1,g$, $\left\{f_g^{(1)}\right\}^m$:

$$\frac{4g^2+2+(1-4g)\sqrt{g^2-1}}{4(g+1)} - \frac{m-1}{\mathcal{M}_g} \longmapsto \frac{2+\sqrt{g^2-1}}{4(g+1)}$$

$$[1]$$

$$\frac{2g+4-\sqrt{g^2-1}}{4(g+1)} \longmapsto \frac{(4g-1)\sqrt{g^2-1}-4g^2+2g+4}{4(g+1)} + \frac{m-1}{\mathcal{M}_g}$$

$$[(\lambda_g)^{-1}]$$

$$\frac{4g+6+\sqrt{g^2-1}}{4(g+1)} \longmapsto \frac{4g+2-3\sqrt{g^2-1}}{4(g+1)} + \frac{m-1}{\mathcal{M}_g}$$

$$[1]$$

$$\frac{2g+8+3\sqrt{g^2-1}}{4(g+1)} - \frac{m-1}{\mathcal{M}_g} \longmapsto \frac{2g+4-\sqrt{g^2-1}}{4(g+1)}$$

$$[\lambda_g]$$

$$\frac{4g^2+4g+6+(1-4g)\sqrt{g^2-1}}{4(g+1)} - \frac{m-1}{\mathcal{M}_g} \longmapsto \frac{4g+6+\sqrt{g^2-1}}{4(g+1)}.$$

(2) *For* $m = 3,4,\ldots,g,g+1$, $\left\{f_g^{(2)}\right\}^m$:

$$\frac{(8g^2-4g-1)\sqrt{g^2-1}-8g^3+4g^2+6g}{4(g+1)} + \frac{m-3}{\mathcal{L}_g} \mapsto \frac{2g+2-\sqrt{g^2-1}}{4(g+1)}$$

$$[\lambda_g]$$

$$\frac{(8g^2-8g-1)\sqrt{g^2-1}-8g^3+8g^2+6g-2}{4(g+1)} + \frac{m-3}{\mathcal{L}_g} \mapsto \frac{2g+4-\sqrt{g^2-1}}{4(g+1)}$$

$$[(\lambda_g)^2]$$

$$\frac{8g^2-2+(1-8g)\sqrt{g^2-1}}{4(g+1)} + \frac{m-3}{\mathcal{L}_g} \mapsto \frac{4g+4+\sqrt{g^2-1}}{4(g+1)}$$

$$[\lambda_g]$$

$$\frac{12g^2-4+(1-12g)\sqrt{g^2-1}}{4(g+1)}+\frac{m-3}{\mathcal{L}_g}\mapsto\frac{4g+6+\sqrt{g^2-1}}{4(g+1)}$$

$$[1]$$

$$\frac{2g+2-\sqrt{g^2-1}}{4(g+1)}\mapsto\frac{(12g-1)\sqrt{g^2-1}-12g^2+6g+12}{4(g+1)}-\frac{m-3}{\mathcal{L}_g}$$

$$[(\lambda_g)^{-1}]$$

$$\frac{2g+4-\sqrt{g^2-1}}{4(g+1)}\mapsto\frac{(8g-1)\sqrt{g^2-1}-8g^2+6g+10}{4(g+1)}-\frac{m-3}{\mathcal{L}_g}$$

$$[(\lambda_g)^{-2}]$$

$$\frac{4g+4+\sqrt{g^2-1}}{4(g+1)}\mapsto\frac{8g^3-8g^2+10-(8g^2-8g-1)\sqrt{g^2-1}}{4(g+1)}-\frac{m-3}{\mathcal{L}_g}$$

$$[(\lambda_g)^{-1}]$$

$$\frac{4g+6+\sqrt{g^2-1}}{5(g+1)}\mapsto\frac{8g^3-4g^2+8-(8g^2-4g-1)\sqrt{g^2-1}}{4(g+1)}-\frac{m-3}{\mathcal{L}_g}$$

$$[1]$$

$$\frac{(8g^2-4g-1)\sqrt{g^2-1}-8g^3+4g^2+10g+4}{4(g+1)}+\frac{m-3}{\mathcal{L}_g}\mapsto\frac{6g+6-\sqrt{g^2-1}}{4(g+1)}.$$

LEMMA 2. — *The map*

$$\Phi_g : \pi_1(\Sigma_g) \mapsto PL_+(S^1)$$

defined by

$$\Phi_g(a_i) = T\left(-\frac{i-1}{2(g+1)}\right)\circ f_g \circ T\left(\frac{i-1}{2(g+1)}\right)\ (i=1,2,\ldots,2(g+1))$$

is a group homomorphism.

Proof. — By Lemma 1,

$$\left\{f_g^{(1)}\right\}^{g+1}=1 \text{ and } \left\{f_g^{(2)}\right\}^{2(g+1)}=\left[\left\{f_g^{(2)}\right\}^{g+1}\right]^2=1.$$

Hence

$$\Phi_g(a_1a_3\dots a_{2g+1}) = \left\{f_g^{(1)}\right\}^g \circ f_g \circ T\left(\frac{2g}{2(g+1)}\right)$$
$$= \left\{f_g^{(1)}\right\}^g \circ f_g \circ T\left(-\frac{1}{g+1}\right) = \left\{f_g^{(1)}\right\}^{g+1} = 1,$$

$$\Phi_g(a_2a_4\dots a_{2(g+1)}) = T\left(-\frac{1}{g+1}\right) \circ \Phi_g(a_1a_3\dots a_{2g+1}) \circ T\left(\frac{1}{g+1}\right) = 1,$$

and

$$\Phi_g(a_1a_2\dots a_{2(g+1)}) = \left\{f_g^{(2)}\right\}^{2g+1} \circ f_g \circ T\left(\frac{2g+1}{2(g+1)}\right)$$
$$= \left\{f_g^{(2)}\right\}^{2g+1} \circ f_g \circ T\left(-\frac{1}{2(g+1)}\right) = \left\{f_g^{(2)}\right\}^{2(g+1)} = 1.$$

These verify that Φ_g is a group homomorphism. □

To sum up, we obtain the following theorem.

THEOREM. — *Let Σ_g be the orientable closed surface of genus $g(\geq 2)$. Consider a Riemannian metric on Σ_g of constant negative curvature -1, and let*

$$\Psi_g : \pi_1(\Sigma_g) \to PSL(2,\mathbb{R})$$

denote the total holonomy of the unstable foliation of the geodesic flow F_t on the unit tangent vector bundle. Then Ψ_g is topologically conjugate to the above homomorphism Φ_g. That is to say, there exists a homeomorphism $h: S^1 \to S^1$ such that

$$\Psi_g(\gamma)(\theta) = (h \circ \Phi_g(\gamma) \circ h^{-1})(\theta) (for\ all\ \gamma \in \pi_1(\Sigma_g)\ and\ \theta \in S^1).$$

Proof. — In [Gh] and [Ha1], it is shown that the unstable foliation of F_t is topologically equivalent to the unstable foliation of φ_t, that is, their total holonomies are topologically conjugate each other. On the other hand, Φ_g is the total holonomy of φ_t (cf. [CN], Chapter V). So Ψ_g is topologically conjugate to Φ_g. □

Remark. — Φ_g is independent of the choice of the metric of constant negative curvature -1 but dependent on the choice of the basis of the Birkhoff section.

2. The discrete Godbillon-Vey invariant of Φ_g.

The discrete Godbillon-Vey invariant $\overline{GV}$ (see [Gr], [GS], [Gh] and [T]) is the $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}-$ valued 2-cocycle of $PL_+(S^1)$ defined by

$$\overline{GV}(h_1, h_2) = \frac{1}{2} \sum_{x \in S^1} C(h_2, h_1 \circ h_2)(x)(h_1, h_2 \in PL_+(S^1)),$$

where $C(k_1, k_2)(x) = \log k_1'(x+0) \otimes \Delta(\log k_2')(x) - \log k_2'(x+0) \otimes \Delta(\log k_1')(x)(k_1, k_2 \in PL_+(S^1), x \in S^1)$ and for a map $k : S^1 \to \mathbb{R}, \Delta k(x) = k(x+0) - k(x-0)$ if k has

$$k(x \pm 0) \lim_{\varepsilon \to 0} k(x \pm \varepsilon)(\text{for } x \in S^1).$$

From this definition, we have the next lemma.

LEMMA 3. — *For $\theta \in S^1$,*

(1) $\overline{GV}(T(\theta) \circ h_1, h_2) = \overline{GV}(h_1, h_2)$,
(2) $\overline{GV}(h_1 \circ T(\theta), h_2) = \overline{GV}(h_1, T(\theta) \circ h_2)$,
(3) $\overline{GV}(h_1, h_2 \circ T(\theta)) = \overline{GV}(h_1, h_2)$.

Let $\boldsymbol{\Sigma}_g \in H_2(\pi_1(\Sigma_g); \mathbb{Z}) = \mathbb{Z}$ be the fundamental class. According to [EM], $\boldsymbol{\Sigma}_g$ is represented by the 2-cycle

$$\begin{aligned}
\boldsymbol{\Sigma}_g &= (a_1, a_3) + (a_1 a_3, a_5) + \ldots + (a_1 a_3 a_5 \ldots a_{2g-3}, a_{2g-1}) \\
&+ (a_2, a_4) + (a_2 a_4, a_6) + \ldots + (a_2 a_4 a_6 \ldots a_{2g-2}, a_{2g}) \\
&- \{(a_1, a_2) + (a_1 a_2, a_3) + \ldots + (a_1 a_2 a_3 \ldots a_{2g-1}, a_{2g}) \\
&- (a_{2g+2}^{-1}, a_{2g+1}^{-1})\} \\
&= (a_1, a_3) + (a_1 a_3, a_5) + \ldots + (a_1 a_3 a_5 \ldots a_{2g-3}, a_{2g_{-1}}) \\
&+ (a_2, a_4) + (a_2 a_4, a_6) + \ldots + (a_2 a_4 a_6 \ldots a_{2g-2}, a_{2g}) \\
&- \{(a_1, a_2) + (a_1 a_2, a_3) + \ldots + (a_1 a_2 a_3 \ldots a_g, a_{g+1}) \\
&+ (a_{2g+2}^{-1} a_{2g+1}^{-1} a_{2g}^{-1} \ldots a_{g+2}^{-1}, a_{g+2}) + (a_{2g+2}^{-1} a_{2g+1}^{-1} a_{2g}^{-1} \ldots a_{g+3}^{-1}, a_{g+3}) \\
&+ \ldots + (a_{2g+2}^{-1} a_{2g+1}^{-1} a_{2g}^{-1} a_{2g-1}^{-1}, a_{2g-1}) + (a_{2g+2}^{-1} a_{2g+1}^{-1} a_{2g}^{-1}, a_{2g})\} \\
&+ (a_{2g+2}^{-1}, a_{2g+1}^{-1}).
\end{aligned}$$

The next lemma is proved by Lemma 3 and the fact $T\left(-\dfrac{2g+1}{2(g+2)}\right) = T\left(\dfrac{1}{2(g+1)}\right)$.

LEMMA 4.

(1) $\overline{GV}(\Phi_{g*}(a_1a_3\dots a_{2m-3}, a_{2m-1})) = \overline{GV}\left((f_g^{(1)})^{m-1}, f_g^{(1)}\right) (m=2,3,\dots,g)$.

(2) $\overline{GV}(\Phi_{g*}(a_2a_4\dots a_{2m-2}, a_{2m})) = \overline{GV}\left((f_g^{(1)})^{m-1}, f_g^{(1)}\right) (m=2,3,\dots,g)$.

(3) $\overline{GV}(\Phi_{g*}(a_1a_2\dots a_{m-1}, a_m)) = \overline{GV}\left((f_g^{(2)})^{m-1}, f_g^{(2)}\right) (m=2,3,\dots,g+1)$.

(4) $\overline{GV}(\Phi_{g*}(a_{2g+2}^{-1}a_{2g+1}^{-1}\cdots a_{2g-m}^{-1}, a_{2g-m})) = \overline{GV}\left((f_g^{(2)})^{-m-3}, f_g^{(2)}\right)$
$(m=0,1,\dots,g-2)$.

(5) $\overline{GV}(\Phi_{g*}(a_{2g+2}^{-1}, a_{2g+1}^{-1})) = \overline{GV}\left((f_g^{(2)})^{-1}, (f_g^{(2)})^{-1}\right)$.

If $h_1, h_2 \in PL_+(S^1)$, then it is the order of the non-differentiable points of h_2 and $h_1 \circ h_2$ that determines the value of $\overline{GV}(h_1, h_2)$. Let

$$d_g^{(i)}(m;\sigma,\tau) \in S^1 (i=1,2; m \in \mathbb{Z}; \sigma,\tau \in \{-2,-1,0,1,2\})$$

be the non-differentiable points of $\left(f_g^{(i)}\right)^m$ such that

$$\left\{\left(f_g^{(i)}\right)^m\right\}'(d_g^{(i)}(m;\sigma,\tau)-0) = (\lambda_g)^\sigma$$

and

$$\left\{\left(f_g^{(i)}\right)^m\right\}'(d_g^{(i)}(m;\sigma,\tau)+0) = (\lambda_g)^\tau.$$

$\left(f_g^{(i)}\right)^m (d_g^{(i)}(m;\sigma,\tau))$ is denoted by $r_g^{(i)}(m;-\sigma,-\tau)$ which is the non-differentiable points of $\left(f_g^{(i)}\right)^{-m}$ such that

$$\left\{\left(f_g^{(i)}\right)^{-m}\right\}'(r_g^{(i)}(m;-\sigma,-\tau)-0) = (\lambda_g)^{-\sigma}$$

and

$$\left\{\left(f_g^{(i)}\right)^{-m}\right\}'(r_g^{(i)}(m;-\sigma,-\tau)+0) = (\lambda_g)^{-\tau}.$$

We know the values of $d_g^{(i)}(m;\sigma,\tau)$ and $r_g^{(i)}(m;\sigma,\tau)$ by Lemma 2 except $d_g^{(2)}(m;\sigma,\tau)$ and $r_g^{(2)}(m;\sigma,\tau)(m=1,2)$, but it is easy to calculate them.

$S^1 = \mathbb{R}/\mathbb{Z}$ has the cyclic order $\prec$ determined by the orientation of S^1. The orders of non-differentiable points which are used to calculate $\overline{GV}(\Phi_{g*}(\Sigma_g))$ are as follows.

LEMMA 5.

(1) $$d_g^{(1)}(m;1,0) \prec d_g^{(1)}(1;0,1) \prec d_g^{(1)}(1;1,0) \prec d_g^{(1)}(1;0,-1) = d_g^{(1)}(m;0,-1) \prec d_g^{(1)}(1;-1,0) = d_g^{(1)}(m;-1,0) \prec d_g^{(1)}(m;0,1) \prec d_g^{(1)}(m;1,0)$$
$$(m = 2,3,\dots,g).$$

(2) $$d_g^{(2)}(2;0,1) \prec d_g^{(2)}(2;1,2) \prec d_g^{(2)}(1;1,0) = d_g^{(2)}(2;2,1) \prec d_g^{(2)}(2;1,0) \prec d_g^{(2)}(1;0,-1) = d_g^{(2)}(2;0,-1) \prec d_g^{(2)}(2;-1,-2) \prec d_g^{(2)}(1;-1,0) = d_g^{(2)}(2;-2,-1) \prec d_g^{(2)}(1;0,1) = d_g^{(2)}(2;-1,0) \prec d_g^{(2)}(2;0,1).$$

(3) $$d_g^{(2)}(3;0,1) \prec d_g^{(2)}(1;1,0) \prec d_g^{(2)}(3;1,2) \prec d_g^{(2)}(3;2,1) \prec d_g^{(2)}(3;1,0) \prec d_g^{(2)}(1;0,-1) = d_g^{(2)}(3;0,-1) \prec d_g^{(2)}(3;-1,-2) \prec d_g^{(2)}(1;-1,0) = d_g^{(2)}(3;-2,-1) \prec d_g^{(2)}(1;0,1) \prec d_g^{(2)}(3;-1,0) \prec d_g^{(2)}(3;0,1).$$

(4) $$d_g^{(2)}(m;0,1) \prec d_g^{(2)}(m;1,2) \prec d_g^{(2)}(m;2,1) \prec d_g^{(2)}(m;1,0) \prec d_g^{(2)}(1;0,-1) = d_g^{(2)}(m;0,-1) \prec d_g^{(2)}(m;-1,-2) \prec d_g^{(2)}(1;-1,0) = d_g^{(2)}(m;-2,-1) \prec d_g^{(2)}(1;0,1) \prec d_g^{(2)}(m;-1,0) \prec d_g^{(2)}(1;1,0) \prec d_g^{(2)}(m;0,1)$$
$$(m = 4,5,\dots,g+1).$$

(5) $$r_g^{(2)}(2;0,-1) \prec r_g^{(2)}(2;-1,-2) \prec d_g^{(2)}(1;-1,0) = r_g^{(2)}(2;-2,-1) \prec d_g^{(2)}(1;0,1) \prec r_g^{(2)}(2;-1,0) \prec d_g^{(2)}(1;1,0) \prec r_g^{(2)}(2;0,1) \prec r_g^{(2)}(2;1,2) \prec r_g^{(2)}(2;2,1) \prec d_g^{(2)}(1;0,-1) = r_g^{(2)}(2;1,0) \prec r_g^{(2)}(2;0,-1).$$

(6) $$d_g^{(2)}(1;0,-1) = r_g^{(2)}(m;0,-1) \prec r_g^{(2)}(m;-1,-2) \prec d_g^{(2)}(1;-1,0) = r_g^{(2)}(m;-2,-1) \prec d_g^{(2)}(1;0,1) \prec r_g^{(2)}(m;-1,0)$$

$$\prec d_g^{(2)}(1;1,0) \prec r_g^{(2)}(m;0,1) \prec r_g^{(2)}(m;1,2) \prec r_g^{(2)}(m;2,1)$$
$$\prec r_g^{(2)}(m;1,0) \prec d_g^{(2)}(1;0,-1) = r_g^{(2)}(m;0,-1)$$
$$(m = 3,4,\dots,g).$$

(7) $r_g^{(2)}(1;1,0) = r_g^{(2)}(2;0,-1) \prec r_g^{(2)}(1;0,-1) = r_g^{(2)}(2;-1,-2)$

$$\prec r^{(2)}(2;-2,-1) \prec r_g^{(2)}(1;-1,0) = r_g^{(2)}(2;-1,0) \prec r_g^{(2)}(2;0,1)$$
$$\prec r_g^{(2)}(1;0,1) = r_g^{(2)}(2;1,2) \prec r_g^{(2)}(2;2,1) \prec r_g^{(2)}(2;1,0)$$
$$\prec r_g^{(2)}(1;1,0) = r_g^{(2)}(2;0,-1).$$

Consequently,

PROPOSITION 1.

$$(\Phi_g^*(\overline{GV}))(\boldsymbol{\Sigma}_g) = \overline{GV}(\Phi_{g*}(\boldsymbol{\Sigma}_g)) = -4(g+1)\log\lambda_g \otimes \log\lambda_g.$$

Proof. — For $m = 2,3,\dots,g$, Lemma 4 (1), (2) and Lemma 5 (1) imply that

$$\overline{GV}(\Phi_{g*}(a_1a_3\dots a_{2m-3},a_{2m-1})) = \overline{GV}(\Phi_{g*}(a_2a_4\dots a_{2m-2},a_{2m}))$$
$$= \overline{GV}\left((f_g^{(1)})^{m-1}, f_g^{(1)}\right) = \frac{1}{2}\sum_{x\in S^1} C\left(f_g^{(1)},(f_g^{(1)})^m\right)(x)$$

$$= \frac{1}{2}\Big\{C\left(f_g^{(1)},(f_g^{(1)})^m\right)(d_g^{(1)}(m;1,0)) + C\left(f_g^{(1)},(f_g^{(1)})^m\right)(d_g^{(1)}(1;0,1))$$
$$+ C\left(f_g^{(1)},(f_g^{(1)})^m\right)(d_g^{(1)}(1;1,0)) + C\left(f_g^{(1)},(f_g^{(1)})^m\right)(d_g^{(1)}(1;0,-1))$$
$$+C\left(f_g^{(1)},(f_g^{(1)})^m\right)(d_g^{(1)}(1;-1,0)) + C\left(f_g^{(1)},(f_g^{(1)})^m\right)(d_g^{(1)}(m;0,1))\Big\}$$

(where $d_g^{(1)}(1;0,-1) = d_g^{(1)}(m;0,-1)$
and $d_g^{(1)}(1;-1,0) = d_g^{(1)}(m;-1,0)$)

$$= \frac{1}{2}\{(0\otimes(-\log\lambda_g) - 0\otimes 0) + (\log\lambda_g \otimes 0 - 0\otimes \log\lambda_g)$$
$$+ (0\otimes 0 - 0\otimes(-\log\lambda_g))$$
$$+ (\log(\lambda_g)^{-1}\otimes\log(\lambda_g)^{-1} - \log(\lambda_g)^{-1}\otimes\log(\lambda_g)^{-1})$$
$$+ (0\otimes(-\log(\lambda_g)^{-1} - 0\otimes(-\log(\lambda_g)^{-1}))$$
$$+(0\otimes\log\lambda_g - \log\lambda_g\otimes 0)\} = 0.$$

Similarly,

$$\overline{GV}(\Phi_{g*}(a_1, a_2)) = \overline{GV}\left(f_g^{(2)}, f_g^{(2)}\right) = \frac{1}{2}\sum_{x\in S^1} C\left(f_g^{(2)}, (f_g^{(2)})^2\right)(x) = 3\log\lambda_g \otimes \log\lambda_g,$$

$$\overline{GV}(\Phi_{g*}(a_1a_2, a_3)) = \overline{GV}\left((f_g^{(2)})^2, f_g^{(2)}\right) = \frac{1}{2}\sum_{x\in S^1} C\left(f_g^{(2)}, (f_g^{(2)})^3\right)(x) = 3\log\lambda_g \otimes \log\lambda_g,$$

$$\overline{GV}(\Phi_{g*}(a_1a_2\dots a_{m-1}, a_m)) = \overline{GV}\left((f_g^{(2)})^{m-1}, f_g^{(2)}\right) = \frac{1}{2}\sum_{x\in S^1} C\left(f_g^{(2)}, (f_g^{(2)})^m\right)(x) = 2\log\lambda_g \otimes \log\lambda_g \quad (m = 4, 5, \dots, g+1),$$

$$\overline{GV}(\Phi_{g*}(a_{2g+2}^{-1}a_{2g+1}^{-1}\dots a_{2g-m}^{-1}, a_{2g-m})) = \overline{GV}\left((f_g^{(2)})^{-m-3}, f_g^{(2)}\right) = \frac{1}{2}\sum_{x\in S^1} C\left(f_g^{(2)}, (f_g^{(2)})^{-m-2}\right)(x) = 2\log\lambda_g \otimes \log\lambda_g \quad (m = 1, 2, \dots, g-2),$$

$$\overline{GV}(\Phi_{g*}(a_{2g+2}^{-1}a_{2g+1}^{-1}a_{2g}^{-1}, a_{2g})) = \overline{GV}\left((f_g^{(2)})^{-3}, f_g^{(2)}\right) = \frac{1}{2}\sum_{x\in S^1} C\left(f_g^{(2)}, (f_g^{(2)})^{-2}\right)(x) = 3\log\lambda_g \otimes \log\lambda_g,$$

$$\overline{GV}(\Phi_{g*}(a_{2g+2}^{-1}, a_{2g+1}^{-1})) = \overline{GV}\left((f_g^{(2)})^{-1}, (f_g^{(2)})^{-1}\right) = \frac{1}{2}\sum_{x\in S^1} C\left((f_g^{(2)})^{-1}, (f_g^{(2)})^{-2}\right)(x) = -3\log\lambda_g \otimes \log\lambda_g.$$

Therefore,

$$\begin{aligned}&\overline{GV}(\Phi_{g*}(\Sigma_g))\\ &= \{0\times 2(g-1) - (3+3+2\times 2(g-2)+3) - 3\}\log\lambda_g \otimes \log\lambda_g\\ &= -4(g+1)\log\lambda_g \otimes \log\lambda_g.\end{aligned}$$

□

Let $\mathrm{Homeo}_{+}^{\sim}(S^1)$ be the group of orientation preserving homeomorphism f of $\mathbb{R}$ satisfying that

$$f(x+1) = f(x) + 1, \text{ for all } x \in \mathbb{R}.$$

Definition ([He]). — *For $f \in \mathrm{Homeo}_{+}^{\sim}(S^1)$, we say that f is of class P if f is differentiable except at most countably many points of $\mathbb{R}$ and there exists a function $h: \mathbb{R} \to \mathbb{R}$ satisfying that*

(i) $h(x+1) = h(x)$ for all $x \in \mathbb{R}$,

(ii) $h(x) > a > 0$ for all $x \in \mathbb{R}$,

(iii) $h|_{[0,1]}$ is of bounded variation,

(iv) f' coincides with h except at most countably many points of $\mathbb{R}$.

$\widetilde{\mathcal{D}}_+(S^1)$ denotes the homeomorphisms of class P of $\mathbb{R}$. In [He], some remarks about $\widetilde{\mathcal{D}}_+(S^1)$ are stated.

Remark.

(1) If $f \in \widetilde{\mathcal{D}}_+(S^1)$ then both of f, f^{-1} are absolutely continuous and Lipschitz continuous.

(2) If $f \in \widetilde{\mathcal{D}}_+(S^1)$ then so is f^{-1} and if $f, g \in \widetilde{\mathcal{D}}_+(S^1)$ then so is $f \circ g$. Hence $\widetilde{\mathcal{D}}_+(S^1)$ is a group.

(3) In the above definition, $\log h$ is of bounded variation.

(4) f is of class P if and only if for all $x \in \mathbb{R}, f$ has the right derivative $f'(x+0)$ and $\log f'(\cdot + 0)$ has bounded variation on $[0,1]$.

$\mathcal{D}_+(S^1)$ denotes the orientation preserving homeomorphisms of S^1 whose lifts belong to $\widetilde{\mathcal{D}}_+(S^1)$. So $\mathcal{D}_+(S^1)$ is a group and $PL_+(S^1) \cup PSL(2,\mathbb{R}) \subset \mathcal{D}_+(S^1) \subset \mathrm{Homeo}_+(S^1)$. Let $\rho : \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}$ be the homomorphism defined by $\rho(a \otimes b) = a \times b (a, b \in \mathbb{R})$. $\overline{gv}$ denotes the composition $\rho \circ \overline{GV}$ which can define the $\mathbb{R}$-valued 2-cocycle of $\mathcal{D}_+(S^1)$. On the other hand, for $h_1, h_2 \in \mathrm{Diff}_+^2(S^1)$, the Godbillon-Vey cocycle is defined by

$$gv(h_1, h_2) = \frac{1}{2} \int_{S^1} \begin{vmatrix} \log h_2'(x) & \log(h_1 \circ h_2)'(x) \\ (\log h_2')'(x) & (\log(h_1 \circ h_2)')'(x) \end{vmatrix} dx.$$

From Remark (3) and (4), this integral has a finite value for $h_1, h_2 \in \mathcal{D}_+(S^1)$. So the Godbillon-Vey cocycle can be defined for $\mathcal{D}_+(S^1)$ by the same formula.

Now we prove that each of non-trivial linear combinations gv and $\overline{gv}$ is not a topological invariant in $\mathcal{D}_+(S^1)$.

COROLLARY ([Gh], THÉRORÈME 1). — *Each*

$$\alpha gv + \beta\overline{gv}(\alpha, \beta \in \mathbb{R}, \alpha^2 + \beta^2 \neq 0)$$

is not a topological invariant in $\mathcal{D}_+(S^1)$.

Proof. — The above proposition implies that

$$\left(\Phi_g^*(\overline{gv})\right)(\Sigma_g) = -4(g+1)(\log\lambda_g)^2.$$

It is well known that

$$\begin{aligned}\left(\Psi_g^*(gv)\right)(\Sigma_g) &= -2\pi \cdot \text{volume of}\{PSL(2,\mathbb{R})/\Psi_g(\pi_1(\Sigma_g))\}\\ &= (2\pi)^2 2(1-g) = -8(g-1)\pi^2.\end{aligned}$$

By definitions,

$$\left(\Psi_g^*(\overline{gv})\right)(\Sigma_g) = \left(\Phi_g^*(gv)\right)(\Sigma_g) = 0.$$

Hence,

$$\begin{aligned}\left(\Psi_g^*(\alpha gv + \beta\overline{gv})\right)(\Sigma_g) &= -8(g-1)\pi^2\alpha,\\ \left(\Phi_g^*(\alpha gv + \beta\overline{gv})\right)(\Sigma_g) &= -4(g+1)(\log\lambda_g)^2\beta.\end{aligned}$$

From Theorem in §1, Ψ_g is topologically conjugate to Φ_g. So, if $\alpha gv + \beta\overline{gv}$ is a topological invariant, then

$$-8(g-1)\pi^2\alpha = -4(g+1)(\log\lambda_g)^2\beta (\text{for all } g(\geq 2)).$$

Therefore,

$$\beta = \frac{2(g-1)\pi^2}{(g+1)(\log\lambda_g)^2}\alpha.$$

On the other hand,

$$\lim_{g\to\infty} \frac{g-1}{g+1} \cdot \frac{1}{(\log\lambda_g)^2} = 0.$$

This implies that $\beta = 0$, therefore, $\alpha = 0$. This contradicts the assumption, $\alpha^2 + \beta^2 \neq 0$. □

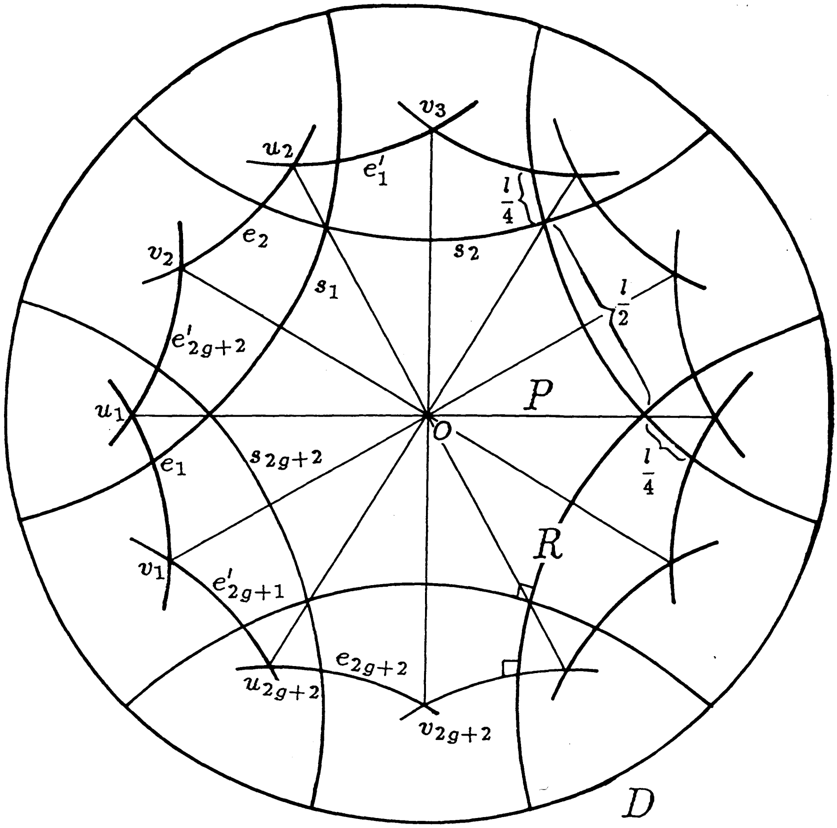


Figure 3

3. Some remarks on Ψ_g and Φ_g.

In previous sections, we use the presentation of $\pi_1(\Sigma_g)$ whose generators are $2g+2$ loops $a_1, a_2, a_3, \ldots, a_{2g+2}$. The fundamental domain of Σ_g corresponding to this presentation is a $4g+4$ gon in the Poincaré disk D, where $D = \{z \in \mathbb{C}; |z| < 1\}$ with the Poincaré metric

$$ds = \frac{2|dz|}{1-|z|^2}.$$

Now we will construct a symmetric fundamental domain R as is shown in Figure 3.

Let P be the regular orthogonal $2g+2$ gon whose center coincides with $o \in D \subset \mathbb{C}$ and whose edges are a part of geodesics called $s_1, s_2, s_3, \ldots, s_{2g+2}$ in the clockwise order. $\frac{l}{2}$ denotes the length of edges of P. Let e_i and $e'_i (i = 1, 2, 3, \ldots, 2g+2)$ be the geodesics satisfying the following conditions (see Figure 3) :

(1) they are outside P and orthogonal to s_i,

(2) the distance between $e_i(e'_i)$ and P is $\frac{l}{4}$.

Then, the $4g+4$ gon surrounded by $e_1, e'_1, e_2, e'_2, \ldots, e_{2g+2}, e'_{2g+2}$ is the desired fundamental domain R. In order to obtain Σ_g from R, we identify e_i with e'_i by the translation by the length l along s_i, for $i = 1, 2, 3, \ldots, 2g+2$. Hence, we have the presentation of $\pi_1(\Sigma_g)$ (see [Ma]).

PROPOSITION 2.

$$\pi_1(\Sigma_g) = \langle a_1, a_2, a_3, \ldots, a_{2g+2}; \quad a_1 a_2 a_3 \ldots a_{2g+2} = a_1 a_3 \ldots a_{2g+1} = a_2 a_4 \ldots a_{2g+2} = 1 \rangle.$$

Let $h_g \in PSL(2, \mathbb{R})$ be the hyperbolic element corresponding to the translation by the length l along s_1 such that $h_g(e'_1 \cap s_1) = e_1 \cap s_1$. Then we obtain the next proposition.

PROPOSITION 3. — *The total holonomy of the unstable foliation of the geodesic flow* F_t

$$\Psi_g : \pi_1(\Sigma_g) \to PSL(2, \mathbb{R})$$

is defined as follows :

$$\Psi_g(a_i) = T\left(-\frac{i-1}{2(g+1)}\right) \circ h_g \circ T\left(\frac{i-1}{2(g+1)}\right) \; (i = 1, 2, 3, \ldots, 2g+2).$$

Moreover, $h_g = h \circ f_g \circ h^{-1}$*, where* f_g *and* h *are homeomorphism of* S^1 *obtained in* §*1.*

In the rest of this section, we will show that Φ_g factors through

$$\phi_g : \Gamma_g \to PL_+(S^1),$$

where Γ_g is the triangle group (see [Mi])

$$\Gamma(g+1, 2g+2, 2g+2) = \langle \tau_1, \tau_2, \tau_3; \\ (\tau_1)^{g+1} = (\tau_2)^{2g+2} = (\tau_3)^{2g+2} = \tau_1\tau_2\tau_3 = 1\rangle.$$

PROPOSITION 4.

(1) *A quadrangle* $ov_1u_1v_2$ *is the fundamental domain for some action of the group* Γ_g *on* D. *The quotient space* D/Γ_g *is the 2-dimensional sphere* $\Sigma(g+1, 2g+2, 2g+2)$ *with three elliptic points of order* $g+1, 2g+2, 2g+2$. Γ_g *also acts on the unit tangent bundle* T_1D *of* D *and the quotient space* T_1D/Γ_g *is the Brieskorn manifold* $M(g+1, 2g+2, 2g+2) = \{(z_1, z_2, z_3) \in \mathbb{C}^3; z_1^{g+1} + z_2^{2g+2} + z_3^{2g+2} = 0, |z_1|^2 + |z_2|^2 + |s_3|^2 = 1\}$.

(2) $\overline{\pi} : M(g+1, 2g+2, 2g+2) \to \Sigma(g+1, 2g+2, 2g+2)$ *is a Seifert fibration with a transverse foliation* $\mathcal{F}'$. $\mathcal{F}'$ *is induced from the bundle foliation of* $e : T_1D \to \partial D$ *where* $e(v) \in \partial D(v \in T_1D)$ *is the end point of the geodesic starting at the base point of* v *in the direction of* $-v$.

(3) *The commutative diagram*

$$\begin{array}{ccc} T_1\Sigma_g & \xrightarrow{p_0} & M(g+1, 2g+2, 2g+2) \\ \pi\downarrow & & \overline{\pi}\downarrow \\ \Sigma_g & \xrightarrow{p_1} & \Sigma(g+1, 2g+2, 2g+2) \end{array}$$

is held where p_0 *is a* $(2g+2)$*-fold covering and* p_1 *is a* $(2g+2)$*- fold ramified covering. Moreover,* $\mathcal{F} = p_0^*(\mathcal{F}')$ *is the unstable foliation of the geodesic flow* F_t.

The fundamental group $\pi_1(M(g+1, 2g+2, 2g+2))$ (resp.$\pi_1(T_1\Sigma_g)$) is the central extension of Γ_g(resp.$\pi_1(\Sigma_g)$) by the infinite cyclic group, *i.e.*,

$$\pi_1(M(g+1, 2g+2, 2g+2)) = \overline{\Gamma}_g = \langle \tau_1, \tau_2, \tau_3, z; \tau_i z = z\tau_i (i = 1, 2, 3), (\tau_1)^{g+1} \\ = (\tau_2)^{2g+2} = (\tau_3)^{2g+2} = \tau_1\tau_2\tau_3 = z\rangle,$$

$$(\text{resp.}\pi_1(T_1\Sigma_g) = \langle a_1, a_2, a_3, \ldots, a_{2g+2}, z; a_i z = z a_i (i = 1, 2, \ldots, 2g+2), \\ a_1a_2a_3\ldots a_{2g+2} = z^{2g+2}, a_1a_3\ldots a_{2g+1} = z^2, a_2a_4\ldots a_{2g+2} = z^2\rangle)$$

where z is the class of a general fibre.

Let

$$p_{0*} : \pi_1(T_1\Sigma_g) \to \overline{\Gamma}_g$$

denote the homomorphism induced by p_0.

LEMMA 6.

$$p_{0*}(a_i) = (\tau_2)^{1-i}\tau_1(\tau_2)^{i+1} (i = 1, 2, 3, \dots, 2g+2) \text{ and } p_{0*}(z) = z.$$

By Theorem 3.5 in [EHN], there exist homomorphisms

$$\widetilde{\Phi}_g : \pi_1(T_1\Sigma_g) \to PL_+^\sim(S^1)$$

and

$$\widetilde{\phi}_g : \overline{\Gamma}_g \to PL_+^\sim(S^1)$$

corresponding to transverse foliations $\mathcal{F}$ and $\mathcal{F}'$, respectively, where $PL_+^\sim(S^1) \subset \text{Homeo}_+^\sim(S^1)$ is the universal covering group of $PL_+(S^1)$. In Proposition 5, we see that $\mathcal{F}$ is induced from $\mathcal{F}'$. In fact, there exists a lift $\tilde{f}_g \in PL_+^\sim(S^1)$ of $f_g \in PL_+(S^1)$ satisfying that

$$\widetilde{\Phi}_g(a_i) = T\left(-\frac{i-1}{2(g+1)}\right) \circ \tilde{f}_g \circ T\left(\frac{i-1}{2(g+1)}\right)$$
$$(i = 1, 2, 3, \dots, 2g+2), \widetilde{\Phi}_g(z) = T(1)$$

and

$$\tilde{\phi}_g(\tau_1) = \tilde{f}_g \circ T\left(-\frac{1}{g+1}\right), \tilde{\phi}_g(\tau_2) = T\left(\frac{1}{2(g+1)}\right), \tilde{\phi}_g(z) = T(1).$$

Consequently,

LEMMA 7.

$$\widetilde{\Phi}_g = \widetilde{\phi}_g \circ p_{0*}.$$

We can consider a homomorphism ϕ_g satisfying the next commutative diagram,

$$\begin{array}{ccc} \overline{\Gamma}_g & \xrightarrow{\widetilde{\phi}_g} & PL_+^\sim(S^1) \\ \downarrow & & \downarrow \\ \Gamma_g & \xrightarrow{\phi_g} & PL_+(S^1), \end{array}$$

i.e., defined by

$$\phi_g(\tau_1) = f_g \circ T\left(-\frac{1}{g+1}\right), \phi_g(\tau_2) = T\left(\frac{1}{2(g+1)}\right).$$

p_{0*} induces the homomorphism

$$\underline{p_{0*}} : \pi_1(\Sigma_g) = \pi_1(T_1\Sigma_g)/\langle z\rangle \to \Gamma_g = \overline{\Gamma}_g/\langle z\rangle.$$

From Lemma 7, we have the next proposition which says that Φ_g factors through ϕ_g.

PROPOSITION 7.

$$\Phi_g = \phi_g \circ \underline{p_{0*}}.$$

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