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MULTISUMMABILITY OF FORMAL POWER SERIES SOLUTIONS OF NONLINEAR MEROMORPHIC DIFFERENTIAL EQUATIONS

by Boele L. J. BRAAKSMA

0. Introduction.

In this paper we consider nonlinear ordinary differential equations

$$x^{\nu+1}\frac{dy}{dx} = F(x,y) \tag{0.1}$$

where $x \in \mathbb{C}$, $y \in \mathbb{C}^n$, $\nu \in \mathbb{N}$, $\nu > 0$ and F is an analytic function in a neighborhood of $(0,a) \in \mathbb{C} \times \mathbb{C}^n$. If (0.1) has a formal power series solution

$$\hat{y}(x) = \sum_{m=0}^{\infty} c_m x^m, \quad c_0 = a \tag{0.2}$$

then we will show that $\hat{y}$ can be summed by a new injective summation procedure, called multisummability, introduced by Ecalle (cf. [7], [8]). We use the description of this procedure given by Martinet and Ramis [13]. Equivalent forms of multisummability have been given by Balser [1], [2], Jurkat [10], Malgrange and Ramis [12], [15].

This multisummability property of formal power series solutions has been announced by Ecalle during the « Journées Résurgentes » in Paris 1989, with a rough idea of a proof. Later Ramis [14] announced several conjectures of which the statement concerning (0.1) given above

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constitutes the main conjecture. He gave a sketch of the principal ideas of a complete proof using essentially the techniques and ideas of his paper with Sibuya [16] extended to analytic infinitesimal neighborhoods of the origin. Here we give a proof in the spirit of Ecalle's work (cf. [6]) using convolution equations similar to our treatment of the linear case in [5] and to the nonlinear case with one level only in [4]. A different treatment of the linear case occurs in [3] and [13].

The organization of this paper is as follows. In section 1 we give a concise review of the definition of multisummability. In section 2 we reduce equation (0.1) to a normal form $D[y] = 0$ (cf. (2.2)) from which the different levels $k_1 > k_2 > \cdots k_r > 0$ of this equation can be read off. These levels are closely associated with the Newton polygon of (0.1). To each level correspond singular values and singular directions (cf. definition in section 2). The formal power series solutions $\hat{y}$ of $D[y] = 0$ is $(k_1, \ldots, k_r)$-summable in nonsingular directions to an analytic solution. In theorem 1 in section 2 this result is formulated for a somewhat more general case where F (cf. (0.1)) itself is the sum of a $(k_1, \ldots, k_r)$-summable power series $\hat{F}(x,y)$ in x with coefficients that are analytic in y.

In section 3 we derive convolution equations $Q_j\psi = 0$ which arise by application of some form of Borel transformation of order k_j to $D[y] = 0$, $j = 1, \ldots, r$. Here Q_j and Q_{j-1} are connected by an acceleration operator (cf. (7.2)). An analysis of these convolution equations leads to a proof of the main result which is given in section 3. This proof consists of four steps which are formulated as lemmas in that section : First it is shown that the convolution equation $Q_r\psi = 0$ corresponding to the lowest level has an analytic solution ψ_r in a neighborhood of the origin (lemma 2) and we show that ψ_r can be analytically continued on a sector (lemma 3) and has a certain exponential growth order μ_r (lemma 4). Since $\mu_r > k_r$ if $r > 1$ the Laplace transform of order k_r of ψ_r does not exist in general (it would have led to a Borel sum y of $\hat{y}$), but the growth order μ_r is such that we may apply the acceleration operator from level k_r to level k_{r-1} to ψ_r (lemma 5). Thus we obtain a solution ψ_{r-1} of $Q_{r-1}\psi = 0$. Repeating this procedure with lemmas 2-5 we obtain finally a solution ψ_1 of $Q_1\psi = 0$, the convolution equation of highest level k_1 which appears to have exponential growth of order $\leqslant k_1$. Therefore its Laplace transform y of order k_1 exists and this function y is analytic solution of $D[y] = 0$, it is the $(k_1, \ldots, k_r)$-sum of $\hat{y}$.

Lemma 2 is proven in section 4 by means of the contraction principle. Lemma 3 concerning the analytic continuation of solutions is derived in section 5 from a linearization of the convolution equations. In section 6 we prove the exponential estimates of the solutions ψ_j (Lemma 4) by means of a majorant equation. Section 7 contains the proof of lemma 5 concerning the acceleration of Q_j to Q_{j-1}. In section 8 we compare multisum-solutions on different sides of singular directions in view of the Stokes phenomenon.

1. Definition of multisummability.

In this section we give a concise review of the definitions of Laplace and Borel transforms, accelerations and multisummability as given by Martinet and Ramis [13] (cf. also Malgrange and Ramis [12], [15]).

If $\rho > 0$, $n \in \mathbb{N}$ then $\Delta_n(0,\rho) := \{\xi \in \mathbb{C}^n \mid |\xi| < \rho\}$. A direction d will be a half line $\{\xi \in \mathbb{C} \mid \xi \neq 0, \arg \xi = \theta\}$ where θ is some real number and $\arg d := \theta$. If $\alpha > 0$ then $S(d,\alpha) := \left\{\xi \in \mathbb{C} \,\middle|\, \xi \neq 0, |\arg \xi - \arg d| < \frac{1}{2}\alpha\right\}$. A neighborhood of 0 in $S(d,\alpha)$ will be a set $\{\xi \in S(d,\alpha) \mid 0 < |\xi| < \rho(\arg \xi)\}$ where ρ is some positive-valued continuous function on $\left(\arg d - \frac{1}{2}\alpha, \arg d + \frac{1}{2}\alpha\right)$.

Let $\mu > 0$. We say that a function f defined on a neighborhood of ∞ in $S(d,\alpha)$ is of exponential growth of order $\leqslant \mu$ if to every closed subsector S' of $S(d,\alpha)$ corresponds a positive constant c such that $f(\xi) = O(1) \exp |c\xi^\mu|$ as $\xi \to \infty$ on S'.

Let $k > 0$ and $f: S(d,\alpha) \to \mathbb{C}$ be analytic and of exponential growth of order $\leqslant k$ whereas $f(\xi) = O(\xi^{\varepsilon - k})$ as $\xi \to 0$ on $S(d,\alpha)$ for some $\varepsilon > 0$. Then the Laplace transform of order k in the direction d of f is defined by

$$(1.1) \qquad (\mathscr{L}_{k,d} f)(x) = \int_d f(\xi) \exp - (\xi/x)^k d(\xi^k),$$

where the path of integration d runs from 0 to ∞. Then $\mathscr{L}_{k,d} f$ is analytic in a neighborhood of 0 in $S(d, \alpha + \pi/k)$.

Let

$$(1.2) \qquad \hat{f}(\xi) = \sum_{m=1}^{\infty} c_m \xi^{m-k} \in \xi^{1-k}\mathbb{C}[[\xi]]$$

be a formal power series. Then we define

$$(1.3)\qquad (\hat{\mathscr{L}}_k\hat{f})(x) := \sum_{m=1}^{\infty} c_m\Gamma(m/k)x^m \in x\mathbb{C}[\![x]\!].$$

If $f(\xi) \sim \hat{f}(\xi)$ as $\xi \to 0$ in $S(d,\alpha)$ and the assumptions above on f are satisfied then $(\mathscr{L}_{k,d}f)(x) \sim (\hat{\mathscr{L}}_k\hat{f})(x)$ as $x \to 0$ on $S(d,\alpha+\pi/k)$.

Let U be a neighborhood of 0 in $S(d,\alpha+\pi/k)$ and $g : U \to \mathbb{C}$ be analytic and $g(\xi) = O(\xi^\delta)$ as $\xi \to 0$ where $\alpha > 0$, $\delta \in \mathbb{R}$, $d_\pm$ are directions in $S(d,\alpha+\pi/k)$, $\arg d_+ > \arg d + \pi/(2k)$, $\arg d_- < \arg d - \pi/(2k)$. Then the Borel transform of order k in the direction d of g is defined by

$$(1.4)\qquad (\mathscr{B}_{k,d}g)(\xi) = \frac{1}{2\pi i}\int_\gamma g(x)\exp(\xi/x)^k d(x^{-k}),$$

where γ is a loop from 0 to 0 in U with the first part in direction d_+ and the last part in direction d_-, $\xi \in S(d,\alpha)$. Then $\mathscr{B}_{k,d}g$ is analytic and of exponential growth of order $\leqslant k$ in $S(d,\alpha)$. If

$$(1.5)\qquad \hat{g}(x) := \sum_{m=1}^{\infty} a_m x^m \in \mathbb{C}[\![x]\!]$$

then

$$(1.6)\qquad (\hat{\mathscr{B}}_k\hat{g})(\xi) := \sum_{m=1}^{\infty} a_m\xi^{m-k}/\Gamma(m/k).$$

If $g(x) \sim \hat{g}(x)$ as $x \to 0$ on U and the assumptions above hold then $(\mathscr{B}_{k,d}g)(\xi) \sim (\hat{\mathscr{B}}_k\hat{g})(\xi)$ as $\xi \to 0$ on $S(d,\alpha)$. We have $\mathscr{B}_{k,d}\mathscr{L}_{k,d} = \mathrm{id}$, and $\mathscr{L}_{k,d}\mathscr{B}_{k,d} = \mathrm{id}$ on the spaces of functions f and g which satisfy the assumptions above.

Let $(\rho_k f)(\xi) = f(\xi^{1/k})$. If f and φ are analytic in a neighborhood U of 0 in $S(d,\alpha)$ and $f(\xi), \varphi(\xi) = O(\xi^{\varepsilon-k})$ as $\xi \to 0$ in U for some $\varepsilon > 0$, then the k-convolution of f and φ in U is defined by

$$(1.7)\qquad (f \underset{k}{*} \varphi)(\xi) = \rho_k^{-1}((\rho_k f)*(\rho_k\varphi))(\xi),\quad \xi \in U.$$

If f and φ are also analytic and of exponential growth of order $\leqslant k$ in $S(d,\alpha)$ then $\mathscr{L}_k(f \underset{k}{*} \varphi) = \mathscr{L}_k f \cdot \mathscr{L}_k\varphi$ on a neighborhood of 0 in $S(d,\alpha+\pi/k)$. Similarly if g and ψ satisfy the assumptions in the definition of $\mathscr{B}_{k,d}$ on $S(d,\alpha+\pi/k)$ then $\mathscr{B}_k(g\psi) = (\mathscr{B}_k g)\underset{k}{*}(\mathscr{B}_k\psi)$ on $S(d,\alpha)$. Here $\mathscr{L}_k = \mathscr{L}_{k,d}$ and $\mathscr{B}_k = \mathscr{B}_{k,d}$.

Let $0 < k < k'$. Then the operator of (k',k)-acceleration in the direction d – $\mathbf{A}_{k',k;d}$ – is defined by $\mathbf{A}_{k',k;d} = \mathscr{B}_{k',d}\mathscr{L}_{k,d}$. This operator makes sense in the space of analytic functions $f\colon S(d,\alpha) \to \mathbb{C}$ which are of exponential growth in $S(d,\alpha)$ of order $\leqslant k$ and which satisfy $f(\xi) = O(\xi^{\varepsilon-k})$ as $\xi \to 0$ in $S(d,\alpha)$ for some $\varepsilon > 0$. However, Ecalle has shown that this operator may be extended to functions with the same conditions except that order $\leqslant k$ is replaced by order $\leqslant \mu$ where $\mu^{-1} := k^{-1} - (k')^{-1}$ (cf. [6], [7], [13]). Then $\mathbf{A}_{k',k;d}f$ is analytic in a neighborhood of 0 in $S(d,\alpha+\pi/\mu)$. Moreover if $f(\xi) \sim \hat{f}(\xi)$ as $\xi \to 0$ in $S(d,\alpha)$ (cf. (1.2)) then

$$(1.8)\quad (\mathbf{A}_{k',k;d}f)(\xi) \sim \sum_{1}^{\infty} c_m \frac{\Gamma(m/k)}{\Gamma(m/k')}\xi^{m-k'} \text{ as } \xi \to 0 \text{ in } S(d,\alpha+\pi/\mu).$$

If f and $g : S(d,\alpha) \to \mathbb{C}$ satisfy the assumptions of the definition of $\mathbf{A}_{k',k;d}f$ and $A_{k',k;d}g$ then we have

$$(1.9)\qquad \mathbf{A}_{k',k;d}(f \underset{k}{*} g) = (\mathbf{A}_{k',k;d}f) \underset{k'}{*} (\mathbf{A}_{k',k;d}g).$$

DEFINITION 1. – **k**-*summability of a formal power series in a multidirection* **d** *or multisector* **S**.

Let n and $r \in \mathbb{N}$, $\hat{f} \in x\mathbb{C}[\![x]\!]^n$, $\mathbf{k} = (k_1,\ldots,k_r)$, $0 < k_r < \cdots < k_1$, $\mathbf{d} = (d_1,\ldots,d_r)$, $\mathbf{S} = (S_1,\ldots,S_r)$ where d_j is a direction, $S_j = S(d_j,\varepsilon_j+\pi/k_j)$, $\varepsilon_j > 0$, $j = 1,\ldots,r$. Let

$$(1.10)\quad k_0 := +\infty,\quad \mu_j := (k_j^{-1}-k_{j-1}^{-1})^{-1},\quad j = 1,\ldots,r.$$

Then $\hat{f}$ is said to be **k**-*summable in the multidirection* **d** *or multisector* **S** *if*

a) $S_{j-1} \subset S_j$, $j = 2,\ldots,r$.

b) *$\hat{\mathscr{B}}_{k_r}\hat{f}$ is convergent in $\Delta_1(0;\rho)\backslash\{0\}$ for some $\rho > 0$. Let g_r be the sum of this series.*

c) *For $j = r, r-1,\ldots,1$ respectively the function g_j can be continued analytically on $S(d_j,\varepsilon_j)$ and is of exponential growth of order $\leqslant \mu_j$ on $S(d_j,\varepsilon_j)$, and if $j \neq 1$ we define $g_{j-1} := \mathbf{A}_{k_{j-1},k_j;d_j}g_j$ on a neighborhood of 0 in $S(d_j,\varepsilon_j+\pi/\mu_j)$.*

Then the **k**-*sum of $\hat{f}$ in multidirection* **d** *or multisector* **S** *is defined by $S_{\underline{k},\underline{d}}\hat{f} := \mathscr{L}_{k_1,d_1}g_1$ (cf. [13]).*

This sum is analytic in a neighborhood U of 0 in S_1 and satisfies $S_{\underline{k},\underline{d}}\hat{f}(x) \sim \hat{f}(x)$ as $x \to 0$ in U. The operator $S_{\underline{k},\underline{d}}$ is injective.

The definition above implies that $\hat{f}$ is **k**-summable on $(I_1, \dots, I_r)$ in the sense of Malgrange and Ramis [12], [15] if I_j is a closed subsector of S_j with length $\geqslant \pi/k_j$ and $I_1 \subset I_2 \subset \cdots \subset I_r$. This may be shown using a decomposition of multisums as for example in Lemma 4 in [5].

If $\hat{\varphi} = a + \hat{f}$ where a is a constant and $\hat{f}$ is as above we also say that $\hat{\varphi}$ is **k**-summable in direction **d** with $S_{\underline{k},\underline{d}}\hat{\varphi} := a + S_{\underline{k},\underline{d}}\hat{f}$. If $\hat{\varphi}$ is **k**-summable in all directions except finitely many then $\hat{\varphi}$ is said to be **k**-summable. If $\hat{\varphi}$ is **k**-summable in direction **d** with $d_1 = \cdots = d_r =: d$ then $\hat{\varphi}$ is said to be **k**-summable in direction d.

If no confusion arises we omit the directions d and **d** in the operators $\mathscr{L}_{k,d}$, $\mathscr{B}_{k,d}$, $\mathbf{A}_{k',k;d}$ and $S_{\underline{k},\underline{d}}$.

2. Normal form and statement of the result.

First we reduce (0.1) to a normal form (cf. [16] and [17]). Let $y(x) = P(x) + x^{\mu}\tilde{y}(x)$ where $P(x) = \sum_{m=0}^{M+\mu-1} c_m x^m$ (cf. (0.2)) and N, $\mu \in \mathbb{N}$ will be chosen later on. Substitution in (0. 1) gives

$$x^{\nu+1}\tilde{y}' = \tilde{F}(x,\tilde{y}) \tag{2.1}$$

where

$$\begin{aligned}
\tilde{F}(x,\tilde{y}) &:= x^M \tilde{F}_0(x) + \tilde{A}(x)\tilde{y} + x^{\mu}\tilde{F}_2(x,\tilde{y}),\\
\tilde{F}_0(x) &:= -x^{-M-\mu}\{F(x,P(x)) - x^{\nu+1}P'(x)\},\\
\tilde{A}(x) &:= -\mu x^{\nu} I + D_y F(x,P(x)),\\
\tilde{F}_2(x,\tilde{y}) &:= x^{-2\mu}\{F(x,P(x)+x^{\mu}\tilde{y}) - F(x,P(x)) - D_yF(x,P(x))x^{\mu}\tilde{y}\}.
\end{aligned}$$

Then $\tilde{F}_2$ is analytic near (0,0) in $\mathbb{C} \times \mathbb{C}^n$ and $O(|\tilde{y}|^2)$ as $\tilde{y} \to 0$. Since (2.1) has the formal solution $\hat{\tilde{y}}(x) = \sum_{m=M}^{\infty} c_{m+\mu} x^m$, we see that $\tilde{F}_0$ is analytic at 0.

Next we apply to the linear part of (2.1) the usual reduction procedure to a « normalized form » (cf. [9], [19], [11], [20]). Let $k_1 > k_2 > \cdots > k_r$ be the positive slopes of the Newton polygon associated with $x^{\nu+1}\dfrac{d}{dx} - \tilde{A}(x)$. If M is sufficiently large then $k_1, \ldots, k_r$ are independent of M. We may assume that $k_1, \ldots, k_r$ are integers since otherwise we first perform a ramification $x^* = x^{1/q}$ for some $q \in \mathbb{N}$. There exists an $n \times n$-matrix polynomial $S(x)$ with $\det S(x) \not\equiv 0$ such that

$$S^{-1}(x)\left\{x\frac{d}{dx} - \tilde{A}(x)\right\}S(x) = x\frac{d}{dx} - Q(x) - xA^*(x),$$

where $A^*(x)$ is an analytic matrix in a neighborhood of 0 and

$$Q(x) = \bigoplus_{h=1}^{r+1} x^{-k_h} A_h .$$

Here A_h is an $n_h \times n_h$-matrix, $n_h \in \mathbb{N}$, $n_1 + \cdots + n_{r+1} = n$, $k_{r+1} = 0$ and if $1 \leqslant h \leqslant r$ then A_h is invertible. Let $\det S(x)$ have a zero of order $\mu_0 \geqslant 0$ at $x = 0$. Then μ_0, $A_1, \ldots, A_r$ and $A_{r+1} + \mu I_r$ are independent of M and μ for sufficiently large M. Here I_h denote the $n_h \times n_h$-identity matrix.

Next we substitute $\tilde{y}(x) = S(x)y^*(x)$ in (2.1). After deleting stars we obtain

$$x\frac{dy}{dx} - Q(x)y = xA(x)y + x^{M-\nu}S^{-1}(x)\tilde{F}_0(x) + x^{\mu-\nu}S^{-1}(x)\tilde{F}_2(x,S(x)y).$$

Because $S^{-1}(x) = O(x^{-\mu_0})$ as $x \to 0$ we choose $\mu > \nu + \mu_0$ and $M > \nu + \mu_0$ sufficiently large. Thus we see that (2.1) is equivalent with $D[y] = 0$ where (cf. [16])

$$\text{(2.2)} \quad D[y](x) := \left(\bigoplus_{h=1}^{r+1} x^{k_h} I_h\right) x\frac{dy}{dx} - \left(\bigoplus_{h=1}^{r+1} A_h\right) y - G(x,y).$$

Here $G(x,y)$ is analytic in $\Delta_1(0;\rho) \times \Delta_n(0;\rho)$ for some $\rho > 0$,

$$\text{(2.3)} \quad G(x,0) = O(x^N) \text{ as } x \to 0, \quad G(x,y) = O(x) \text{ as } x \to 0,$$

uniformly in y on $\Delta_n(0;\rho)$ with some $N \in \mathbb{N}$. Since $D[y] = 0$ has a formal solution $S^{-1}(x)\sum_{m=M}^{\infty} c_{m+\mu}x^m$ we see that we may choose $N = M - \mu_0$ arbitrary large and that $D[y] = 0$ has a formal solution $\sum_{m=N}^{\infty} \tilde{c}_m x^m$. Note that G depends on the choice of N.

We will consider the case $r > 0$ only. If $r = 0$ then D has a regular singularity in 0 and so the formal power series solution $\hat{y}$ converges (theorem of Briot-Bouquet).

We now formulate the multisummability result for $D[y] = 0$ in a somewhat more general case. Instead of $G(x,y)$ analytic at $(0,0)$ in $\mathbb{C} \times \mathbb{C}^n$ we assume that $G(x,y)$ is the **k**-sum in a multidirection **d** of a formal power series $\hat{G}(x,y)$ in x:

$$(2.4) \qquad \hat{G}(x,y) := \sum_{m=1}^{\infty} G_m(y)x^m$$

with coefficients $G_m(y)$ analytic on $\Delta_n(0;\rho)$. Here we use the notation of section 1 with $\mathbf{k} = (k_1, \ldots, k_r)$, (1.10) and $U_j := S(d_j, \varepsilon_j + \pi/k_j)$, $U'_j := S(d_j, \varepsilon_j)$. More precisely we assume:

a) $U_{j-1} \subset U_j$, $j = 2, \ldots, r$.

b) $\hat{\mathscr{B}}_{k_r}\hat{G}(\cdot, y)(\xi)$ converges for $0 < |\xi| < \rho_r$, if $y \in \Delta_n(0;\rho)$, where $\rho_r > 0$; its sum will be denoted by $g_r(\xi, y)$.

c) If $j = r, \ldots, 1$ respectively then $g_j(\xi,y)$ can be analytically continued on $U'_j \times \Delta_n(0;\rho)$ and $g_j(\xi,y)$ is of exponential growth of order μ_j at most as $\xi \to \infty$ in U'_j uniformly for $y \in \Delta_n(0;\rho)$.

The latter means that to every closed subsector U' of U'_j correspond positive constants K and c_0 such that

$$(2.5) \quad |g_j(\xi,y)| \leqslant K \exp |c_0 \xi^{\mu_j}| \quad \text{on} \quad \{U' \backslash \Delta_1(0;1)\} \times \Delta_n(0;\rho).$$

If $j \neq 1$, then

$$(2.6) \qquad g_{j-1}(\xi,y) := \mathbf{A}_{k_{j-1},k_j;d_j} g_j(\cdot,y)(\xi).$$

Moreover

$$(2.7) \qquad g_j(\xi,y) \sim \hat{\mathscr{B}}_{k_j}\hat{G}(\cdot,y)(\xi) \text{ as } \xi \to 0 \text{ on } U'_j \text{ uniformly in } y \text{ on } \Delta_n(0;\rho).$$

d) $G(x,y) := \{\mathscr{L}_{k_1,d_1} g_1(\cdot,y)\}(x)$ on $U \times \Delta_n(0,\rho)$ where U is a neighborhood of 0 in U_1.

Then $G(x,y)$ is **k**-sum of $\hat{G}(x,y)$ in **d** and on $\mathbf{U} = (U_1, \ldots, U_r)$. The assumptions above are satisfied if $G(x,y)$ is analytic in $(0,0)$: now **d** and **U** may be chosen arbitrarily.

To formulate the result we also use

DEFINITION 2. — *Let* $j \in \{1, \dots, r\}$. *Then* ξ *is a singular value of level* k_j *for* D *and the direction of* ξ *is a singular direction of level* k_j *for* D *if* $k_j \xi^{kj}$ *is an eigenvalue of* A (cf. (2.2)). *The set of singular values and directions of level* k_j *for* D *will be denoted by* V_j *and* W_j *respectively. A direction* σ *is a Stokes direction for* D *if there exists* $d \in W_j$ *for some* j *such that* $\arg \sigma = \arg d \pm \pi/(2k_j)$.

Using the notation above we have

THEOREM 1. — *Let* D *be given by* (2.2) *where*

(i) $r \in \mathbb{N}$, $k_{r+1} = 0 < k_r < \cdots < k_1$, $k_h \in \mathbb{N}$, I_h *and* A_h *are* $(n_h \times n_h)$-*matrices,* $n_h \in \mathbb{N}$, $n_1 + \cdots + n_{r+1} = n$, I_h *is identity matrix,* $h = 1, \dots, r+1$, *and if* $1 \leqslant h \leqslant r$ *then* A_h *is invertible,*

(ii) $G(x,y)$ *is* **k**-*sum in multidirection* **d** *or multisector* **U** *of* $\hat{G}(x,y)$ (*cf.* 2.4)) *satisfying a) - d) with* $\mathbf{k} = (k_1, \dots, k_r)$, $\mathbf{d} = (d_1, \dots, d_r)$, $\mathbf{U} = (U_1, \dots, U_r)$, $U_j = S(d_j, \varepsilon_j + \pi/k_j)$, $\varepsilon_j > 0$, $j = 1, \dots, r$.

Let $\hat{y}(x) \in x\mathbb{C}[\![x]\!]^n$ *be a formal solution of* $D[y] = 0$. *Let* τ_j *be a direction and* $\delta_j > 0$ *such that* $S(\tau_j, \delta_j) \subset S(d_j, \varepsilon_j) \backslash W_j$ *and* $S_j \subset S_{j+1}$ *if* $j = 1, \dots, r$, *where* $S_j := S(\tau_j, \delta_j + \pi/k_j)$.

Then $\hat{y}(x)$ *is* **k**-*summable in multidirection* $\boldsymbol{\tau}$ *and on multisector* **S** *where* $\boldsymbol{\tau} = (\tau_1, \dots, \tau_r)$, $\mathbf{S} = (S_1, \dots, S_r)$.

Its multisum $y(x)$ *is analytic solution of* $D[y] = 0$ *in a neighborhood* V *of* 0 *in* S_1 *and* $y(x) \sim \hat{y}(x)$ *as* $x \to 0$ *in* V.

Remark 1. — If S_0 is a subsector of U_1 such that S_0 does not contain a pair of Stokes directions $\sigma \pm \pi/k_j$ with $\sigma \in W_j$ then we may choose $S_1, \dots, S_r$ in the theorem in such a way that the corresponding **k**-sum y exists in a neighborhood V_0 of 0 in S_0, $D[y] = 0$ on V_0 and $y(x) \sim \hat{y}(x)$ as $x \to 0$ in V_0.

COROLLARY. — *Combining theorem* 1 *with the reduction of* (0.1) *to* $D[y] = 0$ *described in the beginning of this section we see that the formal power series solution* $\hat{y}$ *in* (0.2) *of* (0.1) *is* $(k_1/q, \dots, k_r/q)$-*summable. Its multisums exist on all sectors which do not contain any pair of Stokes directions* $d_\pm$ *with* $\arg d_\pm = q(\arg \sigma \pm \pi/(2k_j))$, $\sigma \in W_j$. *The exceptional directions in which* $\hat{y}$ *is not* **k**-*summable are singular directions of* (0.1).

The proof of Theorem 1 and Remark 1 will be given in sect. 3.

3. The convolution equations.

From the transformations in section 2 it follows that it is sufficient to consider the case that the first N terms in the formal solution vanish :

$$\hat{y}(x) = \sum_{m=N}^{\infty} c_m x^m \tag{3.1}$$

and that (2.3) holds with N sufficiently large.

First we consider $D[y]$ with $y = \mathscr{L}_{k_j}\psi$ for some $j \in \{1, \ldots, r\}$ where $\psi \in C_0^\infty(d, \mathbb{C}^n)$ and d is a direction in U_j'. Then we apply the Borel transformation $\mathscr{B}_{k_j}$ to a modification of $D[\mathscr{L}_{k_j}\psi]$. Here we utilize the relation

$$\left(\mathscr{B}_k x^{1+k}\frac{d}{dx}\mathscr{L}_k\psi\right)(\xi) = k\xi^k\psi(\xi). \tag{3.2}$$

Therefore we consider M_jD where

$$M_j = \bigoplus_{h=1}^{r+1} x^{m_{jh}} I_h \qquad \text{with} \qquad m_{jh} = \max(k_j - k_h, 0). \tag{3.3}$$

In this connection we use the following notation : if $w \in \mathbb{C}^n$ then $w^{(h)}$ denotes the projection of w onto the space spanned by the unit vectors of $\mathbb{C}^n$ with indices $n_1 + \cdots + n_{h-1} + 1, \cdots, n_1 + \cdots + n_h$ where $n_0 := 0$ and $h \in \{1, \ldots, r+1\}$. Hence

$$\{M_jD[y]\}^{(h)} = x^{k_j+1}\frac{d}{dx}y^{(h)} - x^{k_j-k_h}A_h y^{(h)} - x^{k_j-k_h}G^{(h)}(x,y) \tag{3.4}$$

if $h \geqslant j$ whereas if $h < j$ we replace k_j by k_h in the right hand side.

Next we define an operator Q_j on $C_0^\infty(d, \mathbb{C}^n)$ which formally is of the form $\mathscr{B}_{k_j}M_jD\mathscr{L}_{k_j}$. To give a meaning to the nonlinear part of this expression we use the function $g_j(\xi, y)$ of condition c) in section 2, which corresponds to $\hat{\mathscr{B}}_{k_j}\hat{G}(\cdot, y)$. We have the expansion

$$g_j(\xi, y) = \sum_{m \in \mathbb{N}^n} g_{mj}(\xi)y^m, \quad \text{if} \quad (\xi, y) \in U_j' \times \Delta_n(0;\rho). \tag{3.5}$$

Let $\psi_{j*m} = \mathscr{B}_{k_j}(\mathscr{L}_{k_j}\psi)^m$ if $m \in \mathbb{N}^n \backslash \{0\}$ and ψ as above. If $m = m_1 e_1 + \cdots + m_n e_n$ where e_h denotes the h^{th}-unit vector we have

$$(3.6) \qquad \psi_{j*m} = \psi_{j*m_1e_1} \underset{k_j}{*} \cdots \underset{k_j}{*} \psi_{j*m_ne_n}, \; \psi_{j*m_he_h} = \psi^{(h)} \underset{k_j}{*} \cdots \underset{k_j}{*} \psi^{(h)}$$

where in the last term $(m_h - 1)$ convolutions $\underset{k_j}{*}$ are performed.

If $\psi \in C_0^\infty(d,\mathbb{C}^n)$ we define

$$(3.7) \qquad g_{j*}(\xi,\psi) := \sum_{m \in \mathbb{N}^n} g_{mj}(\xi) \underset{k_j}{*} \psi_{j*m}.$$

In lemma 1 we will show that $g_{j*}(\xi,\psi)$ makes sense if $\psi \in C_0^\infty(d,\mathbb{C}^n), d \subset U_j'$ and also if ψ is analytic and bounded in a neighborhood of 0 in U_j'. In the latter case we define ψ_{j*m} by (3.6).

Let

$$(3.8) \qquad D_0 := \bigoplus_{h=1}^{r+1} \left(x^{k_h+1} I_h \frac{d}{dx} - A_h \right), \quad \text{so } D_0[y] = D[y] + G(x,y).$$

We define the operator Q_j on $C_0^\infty(d,\mathbb{C}^n)$ by:

$$(3.9) \qquad Q_j\psi = \mathscr{B}_{k_j} M_j D_0 \mathscr{L}_{k_j}\psi - (\mathscr{B}_{k_j} M_j) \underset{k_j}{*} g_{j*}(\xi,\psi), \quad \psi \in C_0^\infty(d,\mathbb{C}^n).$$

From (3.2) and (3.4) we now derive, if $k := k_j$:

$$(3.10) \quad \begin{cases} (Q_j\psi)^{(h)} = -A_h\psi^{(h)} + \dfrac{\xi^{k_h-2k}}{\Gamma(-1+k_h/k)} \underset{k}{*} (k\xi^k\psi)^{(h)} - g_{j_*}^{(h)}(\xi,\psi) \text{ if } h<j, \\ (Q_j\psi)^{(j)} = (k\xi^k I_j - A_j)\psi^{(j)} - g_{j_*}^{(j)}(\xi,\psi), \\ (Q_j\psi)^{(h)} = k\xi^k\psi^{(h)} - \dfrac{1}{\Gamma(1-k_h/k)} \xi^{-k_h} \underset{k}{*} \{A_h\psi^{(h)} + g_{j_*}^{(h)}(\xi,\psi)\} \\ \hfill \text{if } j<h\leqslant r+1. \end{cases}$$

Here the first terms in the righthand sides are the principal terms with the factors of $\psi^{(h)}$ invertible if $\xi \neq 0$ and ξ not a singular value of level k_j.

In view of lemma 1 we may extend the definition of $Q_j\psi$ to analytic functions ψ in a neighborhood of 0 in U_j' with

$$(3.11) \qquad \psi(\xi) \sim \sum_{l=N}^{\infty} \tilde{c}_l \xi^{l-k}, \qquad \xi \to 0 \quad \text{in} \quad U_j'.$$

We will solve $Q_j\psi = 0$ with $\psi(\xi) \sim \hat{\psi}_j(\xi)$ as $\xi \to 0$ in a subsector of U'_j. Here

$$(3.12) \qquad \hat{\psi}_j(\xi) := \hat{\mathscr{B}}_{k_j}\hat{y}(\xi) = \sum_{l=N}^{\infty} \frac{c_l}{\Gamma(l/k_j)} \xi^{l-k_j},$$

where $N \geqslant k_1$ will be suitably chosen later on.

The analysis of the equations $Q_j\psi = 0$ proceeds in 5 steps which we state as lemmas to be proved in later sections.

Let S'_j be a closed subsector of U'_j and $U := S'_j \cap \overline{\Delta_1(0;p)}$ for some $p > 0$. Consider the space $W(U)$ of continuous functions $\psi : U \to \mathbb{C}^n$ such that

$$(3.13) \qquad \|\psi\| := \sup_U |\xi^{k_j - N}\psi(\xi)| < \infty$$

and ψ is analytic in the interior of U. Then we have

LEMMA 1. — a) *If* $\psi \in W(U)$ *then*

$$(3.14) \qquad |\psi_{j*m}(\xi)| \leqslant \frac{\{\Gamma(N/k_j)\|\psi\|\,|\xi^N|\}^{|m|}}{\Gamma(|m|N/k_j)} |\xi^{-k_j}| \quad \text{if} \quad \xi \in U, \; m \in \mathbb{N}^n \backslash \{0\}.$$

b) *For* g_{mj} *defined by* (3.5) *we have :*

$$(3.15) \qquad |g_{mj}(\xi)| \leqslant K\rho^{-|m|}|\xi^{1-k_j}| \exp|c_0\xi^{\mu_j}|, \quad \xi \in S'_j, \quad m \in \mathbb{N}^n\backslash\{0\}$$

$$(3.16) \qquad |g_{oj}(\xi)| \leqslant K|\xi^{N-k_j}| \exp|c_0\xi^{\mu_j}|, \quad \xi \in S'_j$$

where K *and* c_0 *are certain positive constants.*

c) *The series for* $g_{j*}(\xi,\psi)$ *in* (3.7) *is uniformly and absolutely convergent on* U *if* $\psi \in W(U)$, *in particular if* $\psi \in C_0^\infty(d,\mathbb{C}^n)$ *and* $S'_j = d$.

This lemma implies that Q_j is well defined on U'_j. The solution of $Q_j\psi = 0$ will be deduced from the following lemmas :

LEMMA 2. — *The equation* $Q_r\psi = 0$ *has a unique solution* $\psi_r(\xi)$ *which is the sum of the formal series* $\hat{\psi}_r(\xi)$ *that converges on* $\Delta(0;r_1)$ *for some* $r_1 > 0$.

LEMMA 3. — *Let* $j \in \{1,\ldots,r\}$. *If* $j = r$ *we denote by* ψ_r *the solution* ψ_r *in lemma* 2. *If* $j < r$ *we assume that* ψ_j *is an analytic solution of* $Q_j\psi = 0$ *on a neighborhood* U *of* 0 *in* $S(\tau_{j+1},\delta_{j+1}+\pi/\mu_{j+1})$ *and*

$\psi_j(\xi) \sim \hat{\psi}_j(\xi)$ as $\xi \to 0$ in U. Here τ_{j+1} is some direction in U'_{j+1} and $\delta_{j+1} > 0$. Then ψ_j can be analytically continued on $\tilde{S}_j$ and $Q_j\psi_j = 0$ on $\tilde{S}_j$ where $\tilde{S}_r := U'_r \backslash V_r$ and $\tilde{S}_j := S(\tau_{j+1}, \delta_{j+1} + \pi/\mu_{j+1}) \cap U'_j \backslash V_j$ if $j < r$ with V_j given by definition 2.

LEMMA 4. — *Let j, ψ_j and $\tilde{S}_j$ be as in lemma 3, and $S(\tau_j, \delta_j) \subset \tilde{S}_j$ for some direction τ_j and $\delta_j > 0$. Then ψ_j is of exponential growth of order $\leqslant \mu_j$ in $S(\tau_j, \delta_j)$.*

LEMMA 5. — *Let $j \in \{2, \ldots, r\}$ and ψ_j be as in lemma 4. Then $\psi_{j-1} := \mathbf{A}_{k_{j-1}, k_j}\psi_j$ is an analytic solution of $Q_{j-1}\psi = 0$ in a neighborhood of 0 in $S(\tau_j, \delta_j + \pi/\mu_j)$ and $\psi_{j-1}(\xi) \sim \hat{\psi}_{j-1}(\xi)$ as $\xi \to 0$ on this sector.*

We postpone the proofs of these lemmas but now give the

Proof of theorem 1. — First apply lemmas 2, 3, 4 and 5 with $j = r$ consecutively. Then we obtain an analytic solution ψ_{r-1} of $Q_{r-1}\psi = 0$ in a neighborhood of 0 in $S(\tau_r, \delta_r + \pi/\mu_r)$ with $\psi_{r-1} \sim \hat{\psi}_{r-1}$. Then we apply lemmas 3, 4 and 5 consecutively first for $j = r - 1$, then $j = r - 2, \ldots,$ until $j = 2$. Finally lemmas 3 and 4 with $j = 1$ give an analytic solution ψ_1 of $Q_1\psi = 0$ on $S(\tau_1, \delta_1)$ which is of exponential growth of order $\leqslant \mu_1 = k_1$ (cf. (1.10)), and with $\psi_1 \sim \hat{\psi}_1$. Hence $\hat{y}(x)$ is $\mathbf{k}$-summable in multidirection $\boldsymbol{\tau}$ and on multisector $\mathbf{S}$ with sum $y(x) := (\mathscr{L}_{k_1, \tau_1}\psi_1)(x)$ on a neighborhood of 0 in S_1. Then it follows from (3.9) that

$$Q_1\psi_1 = \mathscr{B}_{k_1}\{M_1 D_0[y]\} - (\mathscr{B}_{k_1}M_1) \underset{k_1}{*} g_{1*}(\xi, \psi_1) = 0.$$

Hence using (3.7), (3.6), (3.15) and (3.5) we obtain

$$D_0[y] = \mathscr{L}_{k_1} g_{1*}(\cdot, \psi_1) = \sum_{m \in \mathbb{N}^n} \mathscr{L}_{k_1} g_{m1}(\cdot) y^m = \mathscr{L}_{k_1} g_1(\cdot, y)$$
$$= G(\cdot, y), \text{ i.e. } D[y] = 0 \text{ on } S_1. \quad \square$$

Proof of remark 1. — The construction of $S_1, \ldots, S_r$ is similar to that in [5], end of sect. 2. Let $S_0 = S(\tau_0, \delta_0 + \pi/k_1)$, $\delta_0 > 0$. Then there exists $h \in \mathbb{N}$ such that $\pi/k_{h-1} < \delta_0 + \pi/k_1 \leqslant \pi/k_h$. If τ_0 is a singular direction and/or $\delta_0 + \pi/k_1 = \pi/k_h$ we replace τ_0 by a nonsingular direction τ_1 near τ_0 and/or δ_0 by δ_1 near δ_0 such that $\pi/k_{h-1} < \delta_1 + \pi/k_1 < \pi/k_h$ and $S_1 := S(\tau_1, \delta_1 + \pi/k_1) \subset S_0$. Then we choose $S_j := S_1$ if $j = 2, \ldots, h - 1$. Now $S_j \subset U_1 \subset U_j$ and S_j does not contain a pair $\sigma_j \pm \pi/(2k_j)$, $\sigma_j \in W_j$. Therefore $S'_j := S(\tau_1, \delta_1 + \pi/k_1 - \pi/k_j)$ does not contain a singular direction of level k_j if $j \leqslant h - 1$.

Since the opening of S_1 is less than π/k_j and $S_1 \subset U_j \subset U_{j+1}$ if $j \geqslant h$, we may choose S_j for $j \geqslant h$ in such a way that for $j \geqslant h$ we have: $S_1 \subset S_j \subset S_{j+1}$, $S_j \subset U_j$, S_j does not contain a Stokes pair $\sigma_j \pm \pi/(2k_j)$ with $\sigma_j \in W_j$ and if $S_j = S(\tau_j, \delta_j + \pi/k_j)$ then $\delta_j > 0$. Hence $S(\tau_j, \delta_j) \cap W_j = \phi$.

Thus $\hat{y}$ is **k**-summable on **S** with multisum y on a neighborhood of 0 in S_1. If $\tau_0 \neq \tau_1$ and/or $\delta_0 \neq \delta_1$ we may vary τ_1 and/or δ_1 such that $\tau_1 \to \tau_0$ and $\delta_1 \to \delta_0$. Then we obtain an extension of the multisum y on a neighborhood of 0 in S_0 with $y(x) \sim \hat{y}(x)$ as $x \to 0$ in S_0. □

Finally we rewrite the equation $Q_j\psi = 0$ (cf. (3.10)) in a form which is more suitable for the proofs of lemmas 2, 3 and 4. Let U be a star-region with vertex 0 in U_j'/V_j. Let $W(U)$ be the space of analytic functions $\psi : U \to \mathbb{C}^n$ such that $\psi(\xi) = O(\xi^{1-k_j})$ as $\xi \to 0$ on U. Then we define the operator T_j on $W(U)$ as follows: let $k := k_j$ and $\psi \in W(U)$; then

$$\text{(3.17)} \quad \begin{cases} (T_j\psi)^{(h)} := A_h^{-1}\left\{\dfrac{\xi^{k_h-2k}}{\Gamma(-1+k_h/k)} \underset{k}{*} (k\xi^k\psi)^{(h)} - g_{j*}^{(h)}(\xi,\psi)\right\} \text{ if } h<j, \\ (T_j\psi)^{(j)} := (k\xi^k I_j - A_j)^{-1} g_{j*}^{(j)}(\xi,\psi), \\ (T_j\psi)^{(h)} := \{\Gamma(1-k_h/k)k\xi^k\}^{-1}\{\xi^{-k_h} \underset{k}{*} (A_h\psi^{(h)} + g_{j*}^{(h)}(\xi,\psi))\} \\ \hfill \text{if } j<h\leqslant r+1. \end{cases}$$

Then $Q_j\psi = 0$ is equivalent with $T_j\psi = \psi$.

4. Proof of lemmas 1 and 2.

Proof of lemma 1. — Let $k := k_j$, $\mu := \mu_j$. We have for $\alpha > 0$, $\beta > 0$:

$$\text{(4.1)} \qquad \xi^{\alpha-k} \underset{k}{*} \xi^{\beta-k} = B\left(\frac{\alpha}{k}, \frac{\beta}{k}\right) \xi^{\alpha+\beta-k},$$

where B denotes the beta-function. Hence

$$\text{(4.2)} \quad |(\xi^{\alpha-k} \underset{k}{*} \psi)(\xi)| \leqslant \|\psi\| \, |\xi^{\alpha-k} \underset{k}{*} \xi^{N-k}| = \|\psi\| \, B\left(\frac{\alpha}{k}, \frac{N}{k}\right) |\xi^{\alpha+N-k}|$$

if $\xi \in U$. Now a) follows by induction.

To prove b) we extend (2.5). From (2.3) and $g_j(\xi,y) \sim \hat{\mathscr{B}}_k G(\cdot,y)(\xi)$ as $\xi \to 0$ U' uniformly in $\Delta_n(0;\rho)$ it follows that we may choose K in (2.5) so large that

$$|g_j(\xi,y)| \leqslant K|\xi^{1-k}| \exp |c\xi^{\mu}| \text{ on } U' \times \Delta_n(0;\rho),$$

and that (3.16) holds. From (3.5) and Cauchy's inequality we deduce (3.15). Assertion c) follows from (3.7), (3.15) and (3.14) with (4.1).

Proof of lemma 2. — The proof is analogous to that in [4], sect. 4.3. It is sufficient to prove theorem 1 for N sufficiently large in (3.1). We will choose N so large that T_r is a contraction.

Let $k := k_r$, $0 < p \leqslant p_0 < \operatorname{dist}(V_r,0)$ (cf. definition 2 for V_r in section 2). We define $W(p)$ as the space of continuous functions $\psi : \overline{\Delta_1(0,p)} \to \mathbb{C}^n$ which are analytic in $\Delta_1(0,p)$ and with

$$\|\psi\|_p := \frac{\max}{\overline{\Delta_1(0;p)}} |\xi^{k-N}\psi(\xi)| < \infty . \tag{4.3}$$

First we estimate the linear part T_{lin} of T_r in $W(p)$. Recall $k_{r+1} = 0$. If $\psi \in W(p)$ and $\xi \in \overline{\Delta_1(0;p)}$ we deduce from (4.1) in the same way as we obtained (4.2):

$$\begin{aligned} &|\xi^{k_h-2k} \underset{k}{*} (\xi^k\psi)| \leqslant \|\psi\|_p B(-1+k_h/k, 1+N/k)\, |\xi^{N+k_h-k}| \text{ if } h<r, \\ &|\xi^{-k}(1 \underset{k}{*} \psi)| \leqslant N^{-1}k\|\psi\|_p|\xi^{N-k}| . \end{aligned} \tag{4.4}$$

Moreover, if $|m| = 1$ we deduce from (3.15) and (4.1)

$$\begin{aligned} |g_{mr} \underset{k}{*} \psi_{r*m}| &\leqslant \rho^{-1}K_1\|\psi\|_p|\xi^{1-k} \underset{k}{*} \xi^{N-k}| \\ &= \rho^{-1}K_1\|\psi\|_p B(k^{-1},Nk^{-1})|\xi^{N+1-k}| \\ |\xi^{-1}(1 \underset{k}{*} g_{mr} \underset{k}{*} \psi_{r*m})| &\leqslant \rho^{-1}(N+1)^{-1}\, kK_1\|\psi\|_p B(k^{-1},Nk^{-1})|\xi^{N+1-k}| \end{aligned} \tag{4.5}$$

where $K_1 = K\exp(cp_0^{\mu_r})$. Because $B(a,b) \to \infty$ as $b \to \infty$ if $a > 0$ we may deduce from (3.17), (3.7), (4.4), (4.5) and the definition of T_{lin} that there exists $N_0 \in \mathbb{N}$, $N_0 > k_1$ such that for all $N \geqslant N_0$ and $p \in (0,p_0]$ we have $\|T_{lin}\|_p < \dfrac{1}{3}$ on $W(p)$. Therefore we choose $N \geqslant N_0$ in (3.1).

Because of (3.17), (3.7) and (3.16) we have $T_r(0) \in W(p_0) \subset W(p)$ if $0 < p \leqslant p_0$. Let $R_0 := \|T_r(0)\|_{p_0}$. Next we consider the higher order part of T_r : let $\tilde{T} := T_r - T_{lin} - T_r(0)$. We estimate $\tilde{T}$ on $B_p := \{\psi \in W(p) : \|\psi\|_p \leqslant 2R_0\}$ where $0 < p \leqslant p_0$.

The main ingredient in $\tilde{T}\psi$ is

$$(4.6) \qquad \tilde{g}(\xi,\psi) := \sum_{\substack{m \in \mathbb{N}^n \\ |m| \geq 2}} g_{mr} * \psi_{r*m}.$$

Utilizing (3.15), (3.14) and (4.1) we derive that

$$(4.7) \quad |\tilde{g}(\xi,\psi)| \leq K'|\xi^{2N+1-k}| \quad \text{if} \quad \psi \in B_p,\ \xi \in \overline{\Delta(0;p)},\ 0 < p \leq p_0.$$

Here K' is a positive constant independent of ψ, ξ and p. Hence $\tilde{g}(\xi,\psi) \in B_p$ if p is sufficiently small. Next we consider $\tilde{g}(\xi,\psi+\chi) - \tilde{g}(\xi,\psi)$ with ψ and $\psi + \chi \in B_p$. From (3.14) and (4.1) we deduce that on $\overline{\Delta_1(0;p)}$ we have

$$\begin{aligned} |(\psi+\chi)_{r*m} - \psi_{r*m}| &= \Bigg| \sum_{\substack{l \in \mathbb{N}^n \\ 0 \neq l \leq m}} \binom{m}{l} \psi_{r*(m-l)} \underset{k}{*} \chi_{r*l} \Bigg| \\ &\leq \{\Gamma(|m|N/k)\}^{-1}\{\Gamma(N/k)|\xi^N|\}^{|m|}|\xi^{-k}| \sum_{\substack{l \in \mathbb{N}^n \\ 0 \neq l \leq m}} \binom{m}{l} \|\psi\|_p^{|m-l|}\|\chi\|_p^{|l|} \\ &\leq \{2R_0\Gamma(|m|N/k)\}^{-1}\{4R_0\Gamma(N/k)|\xi^N|\}^{|m|}|\xi^{-k}|\,\|\chi\|_p \quad \text{if} \quad \xi \in \overline{\Delta_1(0;p)}. \end{aligned}$$

From this, (4.6), (3.15) and (4.1) we derive that

$$|\tilde{g}(\xi,\psi+\chi) - \tilde{g}(\xi,\psi)| \leq K''\|\chi\|_p|\xi^{2N+1-k}|,$$

if ψ, $\psi + \chi \in B_p$, $\xi \in \overline{\Delta_1(0;p)}$, where K'' is a positive constant independent of ψ, χ, ξ and p if $0 < p \leq p_0$.

From this, (4.7), (3.17), the definition of $\tilde{T}$ and (4.4) it follows by a similar reasoning as above for T_{lin} that there exists a positive constant K''' independent of ψ, χ, ξ and p such that

$$|(\tilde{T}\psi)(\xi)| \leq K'''|\xi^{2N+1-k}|, \qquad |\{\tilde{T}(\psi+\chi) - \tilde{T}\psi\}(\xi)| \leq K'''|\chi\|_p|\xi^{2N+1-k}|$$

with ψ, χ, ξ and p as above. Hence there exists $p \in (0,p_0]$ such that

$$\|\tilde{T}\psi\|_p \leq \frac{1}{3}R_0,\ \|\tilde{T}(\psi+\chi) - \tilde{T}\psi\| \leq \frac{1}{3}\|\chi\|_p \quad \text{if} \quad \psi, \psi + \chi \in B_p.$$

Combining this with the estimates in T_{lin} and $T_r(0)$ we see that T_r is a contraction on B_p. Therefore we get a unique solution ψ_r in B_p.

For every $M \geqslant N_0$ we obtain a unique solution $\psi_{r,M}$ depending on M on a disc around 0. From the construction of Q_r and T_r from D and (0.2) it follows that

$$\psi_{r,N}(\xi) - \psi_{r,M}(\xi) = B_{k_r}\left(\sum_{m=N}^{M-1} c_m x^m\right)(\xi) \text{ if } M > N \geqslant N_0 .$$

Hence if $\hat{y}(x)$ is given by (3.1) and ψ_r corresponds with $\psi_{r,N}$ then the Taylor coefficients of ψ_r and $\hat{\mathscr{B}}_{k_r}\hat{y}(\xi) = \hat{\psi}_r(\xi)$ are the same. So $\hat{\psi}_r$ converges and $\psi_r(\xi)$ is the sum of $\hat{\psi}_r(\xi)$ on $\Delta_1(0;p)$.

5. Analytic continuation of solutions.

Let $S'(\rho') := \{\xi \in \mathbb{C} \mid \alpha \leqslant \arg \xi \leqslant \beta, 0 < |\xi| \leqslant \rho'\}$ be a bounded closed subsector of $U_j' \backslash V_j$ on which a continuous solution ψ_j of $T_j\psi = \psi$ exists such that ψ_j is analytic in the interior of $S'(\rho')$ and $\psi_j(\xi) \sim \hat{\psi}_j(\xi)$ as $\xi \to 0$ on $S'(\rho')$. (V_j is defined as set of singular values of level k_j in section 2.) We prove lemma 3 in this section and show that ψ_j can be continued analytically in $U_j' \backslash V_j$ by linearizing $Q_j\psi = 0$ in a subsector (cf. [6], [4], section 4.3). We fix j and denote $k := k_j$.

Choose $\xi_0 \in S'(\rho')$ with $|\xi_0| =: \rho_1$, $0 < \rho_1 < \rho'$. Let $p \in (0, \rho_1]$, S'' a subsector of S' with vertex 0, $S''(p) = \{\xi \in S'' \mid 0 < |\xi| \leqslant p\}$ such that $\tilde{S} := \{\xi \in \mathbb{C} \mid (\xi^k - \xi_0^k)^{1/k} \in S''(p) \text{ or } \xi = \xi_0\}$ satisfies $\tilde{S} \subset U_j' \backslash V_j$ and $\tilde{S} \cap S'(\rho_1) = \{\xi_0\}$.

For the analytic continuation of ψ_j on $\tilde{S}$ we utilize the space W of continuous functions $\varphi : \tilde{S} \to \mathbb{C}^n$ which are analytic in the interior of $\tilde{S}$. Let $S_0 := S'(\rho_1) \cup \tilde{S}$ and W_0 be the space of functions $\varphi : S_0 \to \mathbb{C}^n$ which are continuous on $S_0 \backslash \{\xi_0\}$ and analytic in its interior whereas $\lim \varphi(\xi)$ exists as $\xi \to \xi_0$ on $|\xi| < \rho_1$ and $|\xi| > \rho_1$ respectively.

If $\varphi \in W$ we define $\varphi_0 = \varphi$ on $\tilde{S}$ and $\varphi_0 = 0$ on $S'(\rho_1) \backslash \{\xi_0\}$. Then $\varphi_0 \in W_0$. Moreover we define $\tilde{\psi} \in W_0$ by $\tilde{\psi} = \psi_j$ on $S'(\rho_1)$ and $\tilde{\psi} = \psi_j(\xi_0)$ on $\tilde{S}$.

If $\varphi, \chi \in W_0$ we define $\varphi \mathbin{\tilde{*}} \chi$ as follows: $\varphi \mathbin{\tilde{*}} \chi = \varphi \underset{k}{*} \chi$ on $S'(\rho_1)$ whereas if $\xi \in \tilde{S} \backslash \{\xi_0\}$ we define

$$(\varphi \mathbin{\tilde{*}} \chi)(\xi) = \int_{C(\xi)} \varphi((\xi^k - t^k)^{1/k}) \chi(t) d(t^k)$$

where $C(\xi)$ is a path from 0 to ξ in S_0 (so $\xi_0 \in C(\xi)$) such that $t \in C(\xi)$ implies $(\xi^k - t^k)^{1/k} \in S_0$. In particular if $t \in C(\xi) \cap \tilde{S}$ then $(\xi^k - t^k)^{1/k} \in S'(\rho_1)$. Such paths $C(\xi)$ exist because of the definition of $\tilde{S}$ and S_0. Now $\varphi \tilde{*} \chi \in W_0$.

If the m-fold convolution of $\chi \in W_0$ with respect to $\tilde{*}$ is denoted by $\chi_{\tilde{*}m} (m \in \mathbb{N}^n)$, then we have $\tilde{\psi}_{\tilde{*}m} = (\psi_j)_{j*m}$ on $S'(\rho_1)$ and if $\varphi \in W$ and $|m| \geqslant 2$ then $(\varphi_0)_{\tilde{*}m} = 0$ on S_0.

From this we may deduce

$$(\tilde{\psi} + \varphi_0)_{\tilde{*}m} = \tilde{\psi}_{\tilde{*}m} + \sum_{l=1}^{n} m_l \tilde{\psi}_{\tilde{*}(m-e_l)} \tilde{*} \varphi_0^{(l)},$$

where $m = (m_1, \ldots, m_n)$ and e_l denotes the $l-th$ unit vector. Let

$$\tilde{g}(\xi,\chi) := g_{0j}(\xi) + g_{1j}(\xi) \tilde{*} \chi(\xi) + \sum_{\substack{m \in \mathbb{N}^n \\ |m| \geqslant 2}} g_{mj}(\xi) \tilde{*} \chi_{\tilde{*}m}(\xi) \text{ if } \chi \in W_0.$$

Then it follows that for $\varphi \in W$ we have

$$\tilde{g}(\xi, \tilde{\psi} + \varphi_0) = \tilde{g}(\xi, \tilde{\psi}) + B(\xi) \tilde{*} \varphi_0(\xi)$$

where $B(\xi)$ is an $(n \times n)$-matrix valued function with (h,l)-element $B^{(h,l)}(\xi)$ given by

$$B^{(h,l)}(\xi) = g_{1j}^{(h,l)}(\xi) + \sum_{\substack{m \in \mathbb{N}^n \\ |m| \geqslant 2}} g_{mj}^{(h)}(\xi) \tilde{*} m_l \tilde{\psi}_{\tilde{*}(m-e_l)}(\xi).$$

To get a solution of $\psi = T_j \psi$ on $\tilde{S}$ we substitute $\psi = \tilde{\psi} + \varphi_0$ with $\varphi \in W$ in this equation, and replace the convolutions $\underset{k}{*}$ in the definition of T_j by $\tilde{*}$. Then the equation for φ becomes $\varphi = L\varphi + \chi$ where $\chi = \{(T_j \tilde{\psi}) - \tilde{\psi}\}|_{\tilde{S}}$ and L is the linear operator given by

$$\begin{gathered}(L\varphi)^{(h)} := A_h^{-1}[\{\Gamma(-1 + k_h/k)\}^{-1} \xi^{k_h - 2k} \tilde{*} (k\xi^k \varphi_0) - B(\xi) \tilde{*} \varphi_0]^{(h)} \text{ if } h<j, \\ (L\varphi)^{(j)} = (k\xi^k I_j - A_j)^{-1} B(\xi) \tilde{*} \varphi_0)^{(j)}, \\ (L\varphi)^{(h)} := \{\Gamma(1 - k_h/k) k\xi^k\}^{-1} [\xi^{-k_h} \tilde{*} \{A_h \varphi_0 + B(\xi) \tilde{*} \varphi_0\}^{(h)}] \quad \text{if} \quad h > j,\end{gathered}$$

where in the right hand sides we take the restrictions to $\tilde{S}$.

Let $\|\varphi\|$ be the supremum norm of $\varphi \in W$. Then for $\xi \in \tilde{S}$ we have

$$|B(\xi)\tilde{*}\varphi_0(\xi)| = \left|\int_{\xi_0}^{\xi} B((\xi^k - t^k)^{1/k})\varphi(t)d(t^k)\right|$$

$$\leqslant \|\varphi\| \int_0^{\eta} |B(\tau)d(\tau^k)| \text{ where } \eta = (\xi^k - \xi_0^k)^{1/k} \in S''(p) \subset S'(\rho').$$

Similarly for B replaced by $\xi^{-k_h} \tilde{*} B$. Hence L only depends on the values of B in $S''(p)$, so on ψ_j in $S''(p)$.

Therefore there exists $p_0 > 0$ independent of ρ' and ρ_1 such that for $0 < p \leqslant p_0$ we have $\|L\| < 1$. Hence we get a unique solution $\varphi \in W$ of $\varphi = L\varphi + \chi$ if $p \leqslant p_0$. Thus $\psi = \varphi + \psi_j(\xi_0)$ is unique solution of $T_j\psi = \psi$ on $\tilde{S}$. If $p < \rho' - \rho_1$ then we have already the analytic solution ψ_j on $\tilde{S} \subset S'(\rho')$. Hence $\varphi + \psi_j(\xi_0)$ is analytic continuation of ψ_j on $\tilde{S}$ if $\rho' - \rho_1 = \rho' - |\xi_0| < p < p_0$. Repeated application of this procedure gives the analytic continuation of ψ_j on the star region with vertex 0 in $U_j' \backslash V_j$. Moreover, we obtain analytic continuation on the singular directions in U_j' outside the singular points of level k_j. This implies lemma 3. We remark that the analytic continuation may be obtained arbitrary on Riemann surfaces above $U_j' \backslash V_j$ by means of more complicated symmetric paths of integration for the convolution integrals as in Ecalle [6].

6. Exponential estimates.

In this section we prove lemma 4 using a majorant equation for $\psi = T_j\psi$ and the following lemma.

LEMMA 6. – *Let $a > 0$, $b \geqslant 0$, $\mu > 1$ and $a + b \leqslant \mu a$. Then there exists a constant $K > 0$ such that for all $c \geqslant 1$ we have for all positive p*

$$p^{a-1} * \{p^b \exp(cp^{\mu})\} \leqslant Kc^{-(a+b)/\mu} \exp(cp^{\mu}) \tag{6.1}$$

and if $a \leqslant \mu$

$$\{p^{a-1} \exp(cp^{\mu})\} * \exp(cp^{\mu}) \leqslant Kc^{-a/\mu} \exp(cp^{\mu}). \tag{6.2}$$

We postpone the proof of lemma 6 till the end of this section. Let S' be a closed subsector of $\tilde{S}_j$ (cf. lemma 3) with vertex 0 and let

$k := k_j$, $\mu := \mu_j/k_j$, $p_1 > 0$ and ψ_j be the analytic solution of $Q_j\psi = 0$ as in lemma 3. We define $\Psi(p) := \sup\{|\psi_j(\xi)| : \xi \in S', |\xi| = p^k\}$ if $p > 0$. We first derive an integral inequality for Ψ. From (3.15) we deduce that if $|\xi| = p^k$ and $j < h \leqslant r+1$ then

$$|\xi^{-k_h} \underset{k}{*} g_{mj}(\xi)| \leqslant K\rho^{-|m|}\{p^{-k_h/k} * (p^{-1+1/k}\exp(c_0 p^\mu))\}$$
$$\leqslant K\rho^{-|m|}\{\exp(c_0 p^\mu)\} B(1-k_h/k, 1/k) p^{(1-k_h)/k} \quad \text{if} \quad |m| \geqslant 1 .$$

A similar estimate holds for $m = 0$ (cf. (3.16)).

From this, (3.17), $T_j\psi_j = \psi_j$ and (3.7) it follows that there exist positive constants l and q such that for $p \geqslant 1$ we have $\Psi(p) < (M\Psi)(p)$ where

$$(M\Psi)(p) = l\left[\exp(2c_0 p^\mu) + \sum_{h=1}^{j-1} p^{-2+k_h/k} * (p\Psi) + \sum_{h=j+1}^{r+1} p^{-k_h/k} * \Psi \right.$$
$$\left. + \left\{\sum_{h=j}^{r+1} p^{(1-k_h)/k} \exp(c_0 p^\mu)\right\} * \sum_{m=1}^{\infty} q^m \Psi_{*m}\right](p),$$

with Ψ_{*m} the ordinary m-fold convolution of Ψ. Since $\Psi(p)$ is bounded on $(0,1)$ we may choose l so large that $\Psi(p) < (M\Psi)(p)$ also holds for $0 < p < 1$.

We now consider separately the cases $1 < j \leqslant r$ and $j = 1$.

I. Case $1 < j \leqslant r$. Then $\mu > 1$. Let $v(p) := l\exp(cp^\mu)$ where $c > 2c_0$, $c > 1$. We show that it is possible to choose c so large that $v(p) \geqslant (Mv)(p)$. From lemma 6 it follows that $v_{*m}(p) \leqslant (lKc^{-1/\mu})^{m-1} v(p)$. Next we apply lemma 6 to the other terms in Mv. Here we use also that if $\alpha > \mu$ and $c \geqslant 2c_0$ then there exists a positive K_0 independent of c such that $p^{\alpha-1}\exp(c_0 p^\mu) \leqslant K_0 p^{\mu-1}\exp(cp^\mu)$.

Thus we may derive

$$(Mv)(p) \leqslant l[\exp(2c_0 p^\mu) + K_2 c^{-1/\mu} v(p)]$$

if $qlKc^{-1/\mu} < \frac{1}{2}\cdot$ Here K_2 is a constant independent of c. Thus we see that we may choose c so large that $Mv \leqslant v$. Now $\Psi < M\Psi$, $\Psi(0) < v(0)$, $Mv \leqslant v$ and M is a monotone operator. So if $\Psi(t) < v(t)$ on $0 \leqslant t < p$ then $\Psi(p) < M\Psi(p) < Mv(p) \leqslant v(p)$. Hence $\Psi(p) < v(p)$, $\forall p > 0$ and so ψ_j is of exponential growth of order $\leqslant \mu_j$ if $j < r$.

II. Case $j = 1$. We now have $\mu = 1$ and lemma 6 does not apply. We proceed similarly as in [4], section 4.5 and solve $v = Mv$ using Laplace transforms. Let $u := L_1 v$. Then $v = Mv$ becomes

$$u(x) = l\left[(x^{-1}-2c_0)^{-1} + \sum_{h=2}^{r+1} \Gamma(1-k_k/k)x^{1-k_h/k}u(x) \right.$$
$$\left. + \sum_{h=1}^{r+1} \Gamma(1+(1-k_h)/k)(x^{-1}-c_0)^{-1-(1-k_h)/k}(1-qu(x))^{-1}qu(x)\right].$$

This equation has a unique solution u which is analytic in $x^{1/k}$ in a neighborhood of 0, real valued for $x > 0$ and $u(x) = lx(1+o(1))$ as $x \to 0$. Hence we obtain a solution $v = \mathscr{B}_1 u$ of $v = Mv$. Then $v(p)$ is real-valued for $p > 0$, $v(p) = O(\exp(cp))$ as $p \to \infty$ we choose $c > 2c_0$, $v(p) = l(1+o(1))$ as $p \to 0$. Again monotonicity of M now implies $\Psi(p) < v(p)$ and so ψ_1 is of exponential growth of order $\leqslant k_1$.

Proof of lemma 6. — A proof of (6.1) has been given in [5], (I) in section 3. To prove (6.2) we use

$$\{p^{a-1}\exp(cp^{\mu})\} * \exp(cp^{\mu}) = p^a \int_0^1 g(t)\,dt \tag{6.3}$$

where

$$g(t) := t^{a-1}\exp[cp^{\mu}\{t^{\mu}+(1-t)^{\mu}\}].$$

We now use that $f(t) := t^{\mu} + (1-t)^{\mu}$ is convex on $0 \leqslant t \leqslant 1$ since $\mu > 1$. Hence if $0 \leqslant t \leqslant \frac{1}{2}$ then $f(t) \leqslant (1-2t)f(0) + 2tf\left(\frac{1}{2}\right) = 1 - \mu_1 t$ where $\mu_1 = 2(1-2^{1-\mu}) > 0$. Therefore

$$\int_0^{\frac{1}{2}} g(t)\,dt \leqslant \int_0^{\frac{1}{2}} t^{a-1}\exp\{cp^{\mu}(1-\mu_1 t)\}\,dt \tag{6.4}$$
$$\leqslant \Gamma(a)\{c\mu_1 p^{\mu}\}^{-a}\exp(cp^{\mu}\}.$$

Similarly

$$\int_{\frac{1}{2}}^1 g(t)\,dt \leqslant (1+2^{1-a})\int_0^{\frac{1}{2}} \exp\{cp^{\mu}(1-\mu_1 t)\}\,dt \tag{6.5}$$
$$\leqslant (1+2^{1-a})(c\mu_1 p^{\mu})^{-1}\exp(cp^{\mu}).$$

If $cp^{\mu} \geqslant 1$ then

$$p^a(cp^{\mu})^{-a} = c^{-a/\mu}(cp^{\mu})^{-a+a/\mu} \leqslant c^{-a/\mu}$$

and

$$p^a(cp^\mu)^{-1} = c^{-a/\mu}(cp^\mu)^{-1+a/\mu} \leqslant c^{-a/\mu}.$$

If $cp^\mu \geqslant 1$ we now obtain (6.2) by combining these estimates with (6.3), (6.4) and (6.5). If $cp^\mu \leqslant 1$ then $p^a \leqslant c^{-a/\mu}$ and $\int_0^1 g(t)\,dt \leqslant \int_0^1 t^{a-1}e\,dt = e/a$ and now also (6.2) follows.

7. Proof of lemma 5.

The proof of lemma 5 is analogous to that of lemma 3 in [5]. Let $\psi \in C_0^\infty(d,\mathbb{C}^n)$ where d is a direction in $S(\tau_j,\delta_j)$ and

$$\tilde{\psi} = \mathbf{A}_{k_{j-1},k_j}\psi. \tag{7.1}$$

From the definition of Q_j in (3.9) we deduce

$$Q_{j-1}\tilde{\psi} = \mathscr{B}_{k_{j-1}}(M_{j-1}D_0\mathscr{L}_{k_j}\psi) - (\mathscr{B}_{k_{j-1}}M_{j-1}) \underset{k_{j-1}}{*} g_{j-1*}(\xi,\tilde{\psi})$$
$$= \mathscr{B}_{k_{j-1}}(M_{j-1}M_j^{-1}) \underset{k_{j-1}}{*} \chi$$

where

$$\chi = \mathscr{B}_{k_{j-1}}(M_jD_0\mathscr{L}_{k_j}\psi) - (\mathscr{B}_{k_{j-1}}M_j) \underset{k_{j-1}}{*} g_{j-1*}(\xi,\tilde{\psi}).$$

From (3.7), (1.9) and lemma 1 we may derive

$$g_{j-1*}(\xi,\tilde{\psi}) = \mathbf{A}_{k_{j-1},k_j}g_{j*}(\xi,\psi).$$

Hence

$$\chi = \mathbf{A}_{k_{j-1},k_j}\{\mathscr{B}_{k_j}(M_jD_0\mathscr{L}_{k_j}\psi) - (\mathscr{B}_{k_j}M_j) \underset{k_j}{*} g_{j*}(\xi,\psi)\} = \mathbf{A}_{k_{j-1},k_j}(Q_j\psi),$$

and so

$$Q_{j-1}\tilde{\psi} = \mathscr{B}_{k_{j-1}}(M_{j-1}M_j^{-1}) \underset{k_{j-1}}{*} \mathbf{A}_{k_{j-1},k_j}(Q_j\psi). \tag{7.2}$$

By a density argument we may extend (7.2) with (7.1) to the case that $\Psi \in C^\infty(d,\mathbb{C}^n)$, ψ continuous at 0 on d and of exponential growth of order $\leqslant \mu_j$ at ∞ on d. So (7.2) holds for ψ_j as in lemma 4. Hence $Q_{j-1}\psi_{j-1} = 0$ on the set where $\mathbf{A}_{k_{j-1},k_j}\psi_j$ exists, so on a neighborhood of 0 in $S(\tau_j,\delta_j+\pi/\mu_j)$. Furthermore $\psi_{j-1} \sim \hat{\psi}_{j-1}$ at 0 in this sector because of (1.8) and (3.11).

8. Stokes phenomenon.

Suppose τ_0 is a singular direction for $D[y]$ and k is the highest level for which τ_0 is singular (cf. section 2). Assume moreover that $G(\cdot, y)$ is multisum of $\hat{G}(\cdot, y)$ in all directions τ in a neighborhood of τ_0. Then theorem 1 gives two solutions y_+ and y_- of $D[y] = 0$ which are multisums of $\hat{y}(x)$ in all directions τ with $\arg\tau_0 < \arg\tau < \arg\tau_0 + \varepsilon$ and $\arg\tau_0 - \varepsilon < \arg\tau < \arg\tau_0$ respectively. Hence $y_\pm(x) \sim \hat{y}(x)$ and $y_+(x) - y_-(x) \sim 0$ as $x \to 0$ on $S(\tau_0, \pi/k_1)$. Therefore y_+ and y_- exhibit a Stokes phenomenon.

From the construction of the multisums y_+ and y_- given in section 3 we may extract more precise information on $y_+ - y_-$ in an analogous way as in [5]. In particular we may show that

$$y_+(x) - y_-(x) = O(1)\exp(-(c/x)^k) \text{ as } x \to 0 \quad \text{on} \quad S(\tau_0, \pi/k)$$

where $c \in (0, c_0)$ arbitrary close to c_0, if c_0 is the singular value of level k which is closest to the origin. It is possible to give more accurate estimates for $y_+ - y_-$ using the analysis of [5]. Compare also Sibuya [17], [18] and Ramis and Sibuya [16].

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