ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 42, nº 1-2 (1992), p. 421-447
<http://www.numdam.org/item?id=AIF_1992__42_1-2_421_0>

Ann. Inst. Fourier, Grenoble
42, 1-2 (1992), 421-447

AREA FUNCTIONALS AND GODBILLON-VEY COCYLES

by Takashi TSUBOI

The Godbillon-Vey invariant is defined in [6] for the codimension one foliations of class C^2 as a 3-dimensional de Rham cohomology class of the manifold. When the foliation is given by a non-singular 1-form ω, there exists a 1-form η such that $d\omega = \eta \wedge \omega$ by the integrability condition, and the Godbillon-Vey invariant is the cohomology class represented by the closed 3-form $\eta \wedge d\eta$. By considering the foliations of circle bundles over surfaces transverse to the fibers, this gives rise to the Godbillon-Vey 2-cocycle of the group $\mathrm{Diff}^2_+(S^1)$ of orientation preserving C^2-diffeomorphisms of the circle.

The Godbillon-Vey cocycle is extended in larger or different groups of diffeomorphisms or homeomorphisms of the circle.

Duminy and Sergiescu observed that this 2-cocycle is defined in the group of diffeomorphisms of the circle of class P ([2]). Ghys and Sergiescu defined the discrete Godbillon-Vey 2-cocycle for the group of PL homeomorphism of the circle ([5]). Ghys observed that this discrete Godbillon-Vey cocycle is defined in the group of diffeomorphisms of class P ([3]).

On the other hand, in [9], Hurder and Katok defined the Godbillon-Vey cocycle for the group of diffeomorphisms of class $C^{1+\alpha}$, where $\alpha > 1/2$. They asked, in [9], whether one can define the Godbillon-Vey cocycle for the groups of diffeomorphisms of class $C^{1+\alpha}$ for $0 < \alpha \leq 1/2$. In [19],

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Key-words : Godbillon-Vey cocyle – Diffeomorphisms of the circle – Area – (total) variation.
A.M.S. Classification : 57R32 – 58H10 – 26A45 – 26B15.

we answered to this question. Namely, we showed that the Godbillon-Vey cocycle is not nicely defined for the group $\mathrm{Diff}_+^{1+\alpha}(S^1)$ for $0 < \alpha < 1/2$.

In [3] and [4], Ghys raised the question of determining the natural domain of definition of the Godbillon-Vey cocycle. This paper is a proposal of the group of homeomorphisms of the circle which could be more or less the natural domain of the definition of the Godbillon-Vey cocycle.

The Godbillon-Vey cocycle so far defined is in fact the area enclosed by the closed curve $(\log {f_1}' \circ f_2, \log {f_2}')$ in the Euclidean 2-plane for a pair of diffeomorphisms (f_1, f_2) of the circle ([15], [1]). This has been well known for the Godbillon-Vey cocycle for $\mathrm{Diff}^2(S^1)$. For the case of PL homeomorphisms, the image of the circle under the map $(\log {f_1}' \circ f_2, \log {f_2}')$ is a finite number of points in the Euclidean 2-plane. However, the discrete Godbillon-Vey cocycle defined by Ghys and Sergiescu ([5]) is also the area enclosed by the curve obtained from these points in the Euclidean 2-plane by interpolating the subsequent points in the cyclic order of the circle by line segments.

This leads us to consider the largest family of functions of S^1 such that, for a pair of these functions, one can define the area. It is intuitively clear that we cannot define the area enclosed by a continuous function from the circle to the Euclidean 2-plane (see the remark after the definition (3.1) of the area functional). For the area enclosed by a pair (φ, ψ) of smooth functions, we of course have the formula $\int_{S^1} \varphi d\psi$. Even if one of these two functions is not smooth, or not continuous, the formula has meaning when the other function is sufficiently smooth. However, what we need to do is to determine the family of functions such that for any pair of these functions, the area has meaning. We need the definition of the area, and that given in (3.1) is very simple. It is an antisymmetric bilinear form which is invariant under parameter changes. The invariance under parameter changes, which is very natural if we think of the usual area enclosed by a curve, implies that this area functional gives rise to a 2-cocycle of the group of homeomorphisms of the circle corresponding to the function space.

We will see that the family of functions where the area is defined is described by the quadratic variation introduced in §1. In fact, we define the β-variation for a real number β ($\beta \geq 1$) which corresponds to the usual variation when $\beta = 1$ and to our quadratic variation when $\beta = 2$.

Using the β-variation, we define various function spaces in §2. The interesting point of these spaces is that they are real vector spaces which are invariant under (topological) parameter changes.

For a pair of functions in a certain function space described by the quadratic variation, we define the area in §3. In particular, for a pair of (not necessarily continuous) functions with bounded β-variation ($\beta < 2$), the area is defined. We also see the existence of another area functional on the space of functions with bounded total gaps. We think that the functions with bounded β-variation ($\beta \geq 1$) would be of interest by themselves (see the diagram at the end of §2). In fact we can give a nice topology for the space of functions with bounded β-variation ($\beta \geq 1$) with respect to which the area is continuous ($1 \leq \beta < 2$).

In §4, we show that any area functional gives rise to a 2-cocycle of the group of the Lipschitz homeomorphisms of the circle which are written as the integrals of elements of the corresponding function spaces. Thus we exhibit a group of homeomorphisms of the circle which has the Godbillon-Vey cocycle or the discrete Godbillon-Vey cocycle, and which looks like very close to the natural domain of the definition of the Godbillon-Vey cocycle. We wonder that a reasonable theory on the Godbillon-Vey invariant could be achieved in such a group of Lipschitz homeomorphisms.

I would like to thank S. Hurder and S. Matsumoto for valuable comments during the preparation of this paper.

1. β-variations and the quadratic variation of functions on the circle.

Let ℓ be a positive real number, and $\mathbf{R}/\ell\mathbf{Z}$, the circle of length ℓ. Let A be a finite subset of $\mathbf{R}/\ell\mathbf{Z}$. Let mesh$(A)$ be the maximum in the lengths of the components of $(\mathbf{R}/\ell\mathbf{Z}) - A$, in other words,

$$\mathrm{mesh}(A) = \sup\{\mid x_i - x_{i-1} \mid ; 1 \leq i \leq k\} ,$$

where $A = \{x_1, ..., x_k\}$, $x_0 = x_k$ and every triple $(x_i,\ x_{i'},\ x_{i''})$ for $i < i' < i''$ is in the cyclic order. Hereafter, when we write a finite subset $A = \{x_1, ..., x_k\}$ of $\mathbf{R}/\ell\mathbf{Z}$, we understand the order of the elements of A in this way.

We are going to define the quadratic variation of a real valued (not necessarily continuous) function on $\mathbf{R}/\ell\mathbf{Z}$. We can, in fact, define the β-*variation* for a real number β ($\beta \geq 1$). Since the β-variation is interesting in itself, we give here the definition of the β-variation. When $\beta = 1$, it gives the usual total variation, and when $\beta = 2$, it gives the quadratic variation.

Let β be a positive real number. For a finite subset A of $\mathbf{R}/\ell\mathbf{Z}$ and a real valued (not necessarily continuous) function φ on $\mathbf{R}/\ell\mathbf{Z}$, put

$$v_\beta(\varphi, A) = \sum_{j=1}^{k} |\varphi(x_j) - \varphi(x_{j-1})|^\beta.$$

This quantity $v_\beta(\varphi, A)$ is called the *β-variation* of φ with respect to A.

As the usual total variation, the supremum of $v_\beta(\varphi, A)$ is an important quantity. Hence we define

$$V_\beta(\varphi) \;=\; \sup\{v_\beta(\varphi, A); A \subset \mathbf{R}/\ell\mathbf{Z}\}.$$

The β-variation ($\beta > 1$) is different from the usual total variation at the point that the global variation is not equal to the sum of the local variation, For a positive real number ε, we consider the supremum and the infimum of this β-variation for subsets A such that $\mathrm{mesh}(A) \leq \varepsilon$, and we put

$$v_\beta(\varphi, \varepsilon) = \sup\{v_\beta(\varphi, A); \mathrm{mesh}(A) \leq \varepsilon\}$$

and

$$u_\beta(\varphi, \varepsilon) = \inf\{v_\beta(\varphi, A); \mathrm{mesh}(A) \leq \varepsilon\}\ .$$

Since the functions $v_\beta(\varphi, \varepsilon)$ and $u_\beta(\varphi, \varepsilon)$ are monotonous with respect to ε by definition, the limits of them as ε tends to 0 exist. We put

$$v_\beta(\varphi) = \lim_{\varepsilon\to 0} v_\beta(\varphi, \varepsilon) \quad \text{and} \quad u_\beta(\varphi) = \lim_{\varepsilon\to 0} u_\beta(\varphi, \varepsilon)\ .$$

The quantity v_β is the more interesting. v_1 is of course the usual total variation and we call v_2 the quadratic variation. Note that it is easy to construct a continuous function φ such that $v_\beta(\varphi) > 0$ or $v_\beta(\varphi) = \infty$.

LEMMA 1.1. — *If $\beta > 1$ and φ is continuous, then $u_\beta(\varphi, \varepsilon) = 0$ for any positive real number ε. Hence $u_\beta(\varphi) = 0$ for a continuous function φ and $\beta > 1$.*

Proof. — Let $A = \{x_1, ..., x_k\}$ be a finite subset of $\mathbf{R}/\ell\mathbf{Z}$ such that $\mathrm{mesh}(A) \leq \varepsilon$. If φ is continuous, then there is a point y_i in each interval $[x_{i-1}, x_i]$, such that $\varphi(y_i) = (1/2)(\varphi(x_{i-1}) + \varphi(x_i))$. Take the union A' of A and $\{y_i; i = 1, ..., k\}$. Then $\mathrm{mesh}(A') \leq \varepsilon$ and $v_\beta(\varphi, A') = 2^{1-\beta} \cdot v_\beta(\varphi, A)$. The lemma follows from this equality.

LEMMA 1.2. — *If $\beta > 1$, then $v_\beta(\varphi, \varepsilon) \geq 2^{1-\beta} v_\beta(\varphi, 2\varepsilon)$.*

Proof. — For any $A = \{x_1, ..., x_k\}$ such that $\mathrm{mesh}(A) \leq 2\varepsilon$, let A' be the union of A and $\{(x_{i-1} + x_i)/2; i = 1, ..., k\}$. Then we have $v_\beta(\varphi, A') \geq 2^{1-\beta} v_\beta(\varphi, A)$. The lemma follows from this inequality.

Let φ be a (not necessarily continuous) function on $\mathbf{R}/\ell\mathbf{Z}$ such that $v_\beta(\varphi) < \infty$. Then it is easy to see that for any point x in $\mathbf{R}/\ell\mathbf{Z}$,

$$\varphi(x-0) = \lim_{\varepsilon \to -0} \varphi(x+\varepsilon) \quad \text{and} \quad \varphi(x+0) = \lim_{\varepsilon \to +0} \varphi(x+\varepsilon)$$

exist. When $\varphi(x-0)$ and $\varphi(x+0)$ exist, we put

$$\Delta\varphi(x) = \varphi(x+0) - \varphi(x-0) .$$

For a function φ such that $\varphi(x-0)$ and $\varphi(x+0)$ exist for any $x \in \mathbf{R}/\ell\mathbf{Z}$, we put

$$s_\beta(\varphi) = \sum_{x \in \mathbf{R}/\ell\mathbf{Z}} |\Delta\varphi(x)|^\beta .$$

Then we have the following lemma.

LEMMA 1.3. — *If $\beta > 1$, then $u_\beta(\varphi) \leq s_\beta(\varphi) \leq v_\beta(\varphi)$.*

Proof. — First we prove the latter inequality. As we mentioned, if $v_\beta(\varphi) < \infty$, then $\varphi(x-0)$ and $\varphi(x+0)$ exist for any point $x \in \mathbf{R}/\ell\mathbf{Z}$. Fix a positive real number ε. For any positive real number δ, let $\{a_i; i = 1, ..., N\}$ be a finite subset of $\mathbf{R}/\ell\mathbf{Z}$ such that

$$\sum_i |\Delta\varphi(a_i)|^\beta \geq s_\beta(\varphi) - \frac{\delta}{2} .$$

Let a_i^- and a_i^+ be the points sufficiently close to a_i such that

$$a_{i-1}^+ < a_i^- \leq a_i \leq a_i^+ , \quad a_i^+ - a_i^- \leq \varepsilon$$

and

$$\sum_i |\varphi(a_i^+) - \varphi(a_i^-)|^\beta \geq s_\beta(\varphi) - \delta .$$

By adding finitely many points between a_{i-1}^+ and a_i^- of $\mathbf{R}/\ell\mathbf{Z}$, we obtain a finite subset A of $\mathbf{R}/\ell\mathbf{Z}$ such that

$$\mathrm{mesh}(A) \leq \varepsilon \quad \text{and} \quad v_\beta(\varphi, A) \geq s_\beta(\varphi) - \delta .$$

Hence we have $v_\beta(\varphi, \varepsilon) \geq s_\beta(\varphi) - \delta$ for any ε. Thus $v_\beta(\varphi) \geq s_\beta(\varphi) - \delta$. Since δ is arbitrary, we have $v_\beta(\varphi) \geq s_\beta(\varphi)$. If $\varphi(x-0)$ and $\varphi(x+0)$ exist for any point $x \in \mathbf{R}/\ell\mathbf{Z}$ and $s_\beta(\varphi) = \infty$, we can show that $v_\beta(\varphi) = \infty$ by a similar argument.

Now we prove the former inequality. If $s_\beta(\varphi) = 0$, then this follows from Lemma 1.1. Suppose that $s_\beta(\varphi) > 0$ and the inequality does not hold. Then there exists a positive real number ε such that $u_\beta(\varphi, \varepsilon) > s_\beta(\varphi)$. Hence, for any positive real number δ there exists a subset $A = \{x_1, ..., x_k\}$ of $\mathbf{R}/\ell\mathbf{Z}$, such that

$$\operatorname{mesh}(A) \leq \varepsilon \quad \text{and} \quad v_\beta(\varphi, A) \leq u_\beta(\varphi, \varepsilon) + \delta \, .$$

For the sake of simplicity, suppose that φ is left continuous, *i.e.*, $\varphi(x-0) = \varphi(x)$. Put

$$s_\beta(\varphi, [x_{i-1}, x_i)) = \sum_{x \in [x_{i-1}, x_i)} |\Delta\varphi(x)|^\beta \, .$$

We compare $|\varphi(x_i) - \varphi(x_{i-1})|^\beta$ and $s_\beta(\varphi, [x_{i-1}, x_i))$ in each interval. Then there are intervals $[x_{i-1}, x_i)$ such that

$$|\varphi(x_i) - \varphi(x_{i-1})|^\beta \ \geq \ r^\beta \cdot s_\beta(\varphi, [x_{i-1}, x_i)) \ ,$$

where

$$r^\beta = \frac{u_\beta(\varphi, \varepsilon) + s_\beta(\varphi)}{2 s_\beta(\varphi)} \, .$$

For these intervals, we have

$$\sup_{x \in [x_{i-1}, x_i)} |\Delta\varphi(x)|^\beta \ \leq \ s_\beta(\varphi, [x_{i-1}, x_i)) \ \leq \ \frac{1}{r^\beta} |\varphi(x_i) - \varphi(x_{i-1})|^\beta \, .$$

Then we can find a point y_i in $[x_{i-1}, x_i)$ such that

$$\left|\varphi(y_i) - \frac{\varphi(x_{i-1}) + \varphi(x_i)}{2}\right| \ \leq \ \frac{1}{r} \left|\frac{\varphi(x_{i-1}) - \varphi(x_i)}{2}\right| \, ,$$

and we have

$$\begin{aligned} &|\varphi(y_i) - \varphi(x_{i-1})|^\beta + |\varphi(x_i) - \varphi(y_i)|^\beta \\ &\qquad \leq \left(\left\{\frac{r+1}{2r}\right\}^\beta + \left\{\frac{r-1}{2r}\right\}^\beta \right) |\varphi(x_i) - \varphi(x_{i-1})|^\beta \, . \end{aligned}$$

Note that, since $\beta > 1$, $\left(\{(r+1)/(2r)\}^\beta + \{(r-1)/(2r)\}^\beta\right) < 1$. On the other hand, we have

$$\begin{aligned} \sum |\varphi(x_i) - \varphi(x_{i-1})|^\beta \ &\geq v_\beta(\varphi, A) - \frac{u_\beta(\varphi, \varepsilon) + s_\beta(\varphi)}{2} \\ &\geq \frac{u_\beta(\varphi, \varepsilon) - s_\beta(\varphi)}{2} \, , \end{aligned}$$

where the sum is taken over the intervals $[x_{i-1}, x_i)$ such that

$$|\varphi(x_i) - \varphi(x_{i-1})|^\beta \ \geq \ r^\beta \cdot s_\beta(\varphi, [x_{i-1}, x_i)) \, .$$

Let A' be the union of A and the set of points y_i taken for the above intervals. Then we have $\mathrm{mesh}(A') \leq \varepsilon$ and

$$v_\beta(\varphi, A') \leq v_\beta(\varphi, A) - \frac{u_\beta(\varphi, \varepsilon) - s_\beta(\varphi)}{2} \cdot \left(1 - \{\frac{r+1}{2r}\}^\beta - \{\frac{r-1}{2r}\}^\beta\right) .$$

If we take δ smaller than the last term, this contradicts the definition of $u(\varphi, \varepsilon)$.

By this lemma there are only countably many points x such that $\Delta\varphi(x) \neq 0$ if $v_\beta(\varphi) < \infty$.

2. Function spaces on the circle.

In this section, we define several spaces of functions on the circle where the area functional is defined.

DEFINITION 2.1. — *Let β be a real number not smaller than 1. $\mathcal{CV}_\beta(\mathbf{R}/\ell\mathbf{Z})$ is the set of real valued continuous functions φ on $\mathbf{R}/\ell\mathbf{Z}$ such that $V_\beta(\varphi) < \infty$.*

It is easy to see that $\mathcal{CV}_\beta(\mathbf{R}/\ell\mathbf{Z})$ is a vector space. The following lemma is also easy.

LEMMA 2.2. — *For $\beta \geq 1$, $\mathcal{CV}_\beta(\mathbf{R}/\ell\mathbf{Z})$ contains the $1/\beta$ Hölder continuous functions.*

Proof. — If there is a real number C such that $|\varphi(x) - \varphi(y)| \leq C|x-y|^{1/\beta}$, then for $A = \{x_1, ..., x_k\} \subset \mathbf{R}/\ell\mathbf{Z}$, we have

$$v_\beta(\varphi, A) \leq \sum C^\beta |x_i - x_{i-1}| \ = \ C^\beta \ell \, .$$

For a $1/\beta$ Hölder continuous function on $\mathbf{R}/\ell\mathbf{Z}$ and a homeomorphism $h : \mathbf{R}/\ell\mathbf{Z} \longrightarrow \mathbf{R}/\ell\mathbf{Z}$, the composition $\varphi \circ h$ is not usually a $1/\beta$ Hölder continuous function, however it is easy to see that $\varphi \circ h$ is an element of $\mathcal{CV}_\beta(\mathbf{R}/\ell\mathbf{Z})$. On the other hand, we have the following proposition.

PROPOSITION 2.3. — *If $\varphi : \mathbf{R}/\ell\mathbf{Z} \longrightarrow \mathbf{R}$ is continuous and $V_\beta(\varphi) < \infty$ then there exist a positive real number ℓ' and a homeomorphism $h : \mathbf{R}/\ell'\mathbf{Z} \longrightarrow \mathbf{R}/\ell\mathbf{Z}$ such that $\varphi \circ h$ is a $1/\beta$ Hölder function.*

Proof. — Put

$$v_\beta(\varphi, [a,b]) \ = \ \sup\{v_\beta(\varphi, A); A \subset [a,b]\} \, .$$

First note that we have

$$v_\beta(\varphi,[0,x_0]) + v_\beta(\varphi,[x_0,x_1]) \le v_\beta(\varphi,[0,x_1])$$

for $0 \le x_0 \le x_1 \le \ell$, and $v_\beta(\varphi,[0,x])$ is a non-decreasing function. Put $h^{-1}(x) = v_\beta(\varphi,[0,x]) + x$. Then h is a homeomorphism from $\mathbf{R}/(V_\beta(\varphi)+\ell)\mathbf{Z}$ to $\mathbf{R}/\ell\mathbf{Z}$ and

$$\begin{aligned}
|(\varphi\circ h)(x_1) - (\varphi\circ h)(x_0)|^\beta &= |\varphi(h(x_1)) - \varphi(h(x_0))|^\beta \\
&\le v_\beta(\varphi,[h(x_0),h(x_1)]) \\
&\le v_\beta(\varphi,[0,h(x_1)]) - v_\beta(\varphi,[0,h(x_0)]) \\
&= \{h^{-1}(h(x_1)) - h(x_1)\} \\
&\quad - \{h^{-1}(h(x_0)) - h(x_0)\} \\
&= (x_1 - x_0) - (h(x_1) - h(x_0)) \\
&\le x_1 - x_0 \ .
\end{aligned}$$

We define another class of functions which is a little more complicated. We need this class to give an appropriate domain of definition for the area.

DEFINITION 2.4. — *Let β be a real number greater than 1. $\mathcal{RCV}_\beta(\mathbf{R}/\ell\mathbf{Z})$ is the set of real valued continuous functions φ on $\mathbf{R}/\ell\mathbf{Z}$ such that $v_\beta(\varphi) = 0$ and*

$$\int_0^\ell v_\beta(\varphi,\varepsilon)d\log\varepsilon < \infty \ .$$

By Lemma 1.2, the boundedness of the integral depends only on the behavior of $v_\beta(\varphi,\varepsilon)$ as ε tends to 0.

It is easy to see that $\mathcal{RCV}_\beta(\mathbf{R}/\ell\mathbf{Z})$ is a vector space and $\mathcal{RCV}_\beta(\mathbf{R}/\ell\mathbf{Z}) \subset \mathcal{CV}_\beta(\mathbf{R}/\ell\mathbf{Z})$. The following lemma shows that $\mathcal{RCV}_\beta(\mathbf{R}/\ell\mathbf{Z})$ contains a reasonable class of continuous functions.

LEMMA 2.5. — *For $\beta > 1$, $\mathcal{RCV}_\beta(\mathbf{R}/\ell\mathbf{Z})$ contains the continuous functions with the modulus of continuity $|x|^{1/\beta}|\log|x||^{-1}$, hence $\mathcal{RCV}_\beta(\mathbf{R}/\ell\mathbf{Z})$ contains the α Hölder continuous functions for $\alpha > 1/\beta$.*

Proof. — If there is a real number C such that

$$|\varphi(x) - \varphi(y)| \le C|x-y|^{1/\beta}|\log|x-y||^{-1} \ ,$$

then for $A = \{x_1, ..., x_k\}$ with mesh$(A) \le \varepsilon$, we have

$$\begin{aligned}
v_\beta(\varphi,A) &\le \sum C^\beta |x_i - x_{i-1}||\log|x_i - x_{i-1}||^{-\beta} \\
&\le \sum C^\beta |x_i - x_{i-1}||\log\varepsilon|^{-\beta} \\
&\le C^\beta \ell|\log\varepsilon|^{-\beta} \ .
\end{aligned}$$

Hence $v_\beta(\varphi, \varepsilon) \le C^\beta \ell |\log \varepsilon|^{-\beta}$ and, for small δ,

$$\int_0^\delta v_\beta(\varphi,\varepsilon) d\log\varepsilon \le C^\beta \ell \int_0^\delta |\log\varepsilon|^{-\beta} d\log\varepsilon$$
$$= C^\beta \ell (\beta-1)^{-1} |\log\delta|^{1-\beta} < \infty.$$

For an element φ of $\mathcal{CV}_\beta(\mathbf{R}/\ell\mathbf{Z})$ and a homeomorphism $h : \mathbf{R}/\ell\mathbf{Z} \longrightarrow \mathbf{R}/\ell\mathbf{Z}$, the composition $\varphi \circ h$ is an element of $\mathcal{CV}_\beta(\mathbf{R}/\ell\mathbf{Z})$. However, for an element φ of $\mathcal{RCV}_\beta(\mathbf{R}/\ell\mathbf{Z})$ and a homeomorphism $h : \mathbf{R}/\ell\mathbf{Z} \longrightarrow \mathbf{R}/\ell\mathbf{Z}$, the composition $\varphi \circ h$ may not be an element of $\mathcal{RCV}_\beta(\mathbf{R}/\ell\mathbf{Z})$. We define a bigger class of functions as follows.

DEFINITION 2.6. — *$\mathcal{CQ}_\beta(\mathbf{R}/\ell\mathbf{Z})$ is the set of real valued continuous functions φ on $\mathbf{R}/\ell\mathbf{Z}$ such that there exists a homeomorphism $h : \mathbf{R}/\ell'\mathbf{Z} \longrightarrow \mathbf{R}/\ell\mathbf{Z}$ for some $\ell' > 0$ and $\varphi \circ h \in \mathcal{RCV}_\beta(\mathbf{R}/\ell\mathbf{Z})$.*

Remark. — The condition $\varphi \in \mathcal{CQ}_\beta(\mathbf{R}/\ell\mathbf{Z})$ does not depend on the metric structure of $\mathbf{R}/\ell\mathbf{Z}$, while the condition $\varphi \in \mathcal{RCV}_\beta(\mathbf{R}/\ell\mathbf{Z})$ does depend on the metric on $\mathbf{R}/\ell\mathbf{Z}$.

LEMMA 2.7. — *$\mathcal{CQ}_\beta(\mathbf{R}/\ell\mathbf{Z})$ is a real vector space.*

Proof. — Let φ_i $(i = 1, 2)$ be elements of $\mathcal{CQ}_\beta(\mathbf{R}/\ell\mathbf{Z})$. We show that $\varphi_1 + \varphi_2 \in \mathcal{CQ}_\beta(\mathbf{R}/\ell\mathbf{Z})$. Let $h_i : \mathbf{R}/\ell_i\mathbf{Z} \longrightarrow \mathbf{R}/\ell\mathbf{Z}$ be homeomorphisms such that $\varphi_i \circ h_i \in \mathcal{RCV}_\beta(\mathbf{R}/\ell_i\mathbf{Z})$. Since the homeomorphisms h_i^{-1} are with bounded variation, $d(h_i^{-1})$ are non-atomic positive measures on $\mathbf{R}/\ell\mathbf{Z}$. These measures of course satisfy the following equality :

$$\int_a^b d(h_i^{-1}) = h_i^{-1}(b) - h_i^{-1}(a) \; .$$

Now define $H : \mathbf{R}/(\ell_1 + \ell_2)\mathbf{Z} \longrightarrow \mathbf{R}/\ell\mathbf{Z}$ by

$$H^{-1}(x) = \int_0^x d(h_1^{-1}) + d(h_2^{-1}) \; .$$

We claim that if $\varphi_i \circ h \in \mathcal{RCV}_\beta(\mathbf{R}/\ell_i\mathbf{Z})$, then $\varphi_i \circ H \in \mathcal{RCV}_\beta(\mathbf{R}/(\ell_1 + \ell_2)\mathbf{Z})$. Let $A = \{x_1, ..., x_k\}$ be a finite subset of $\mathbf{R}/\ell\mathbf{Z}$. For $h_i^{-1}A = \{h_i^{-1}(x_1), ..., h_i^{-1}(x_k)\}$ $(\subset \mathbf{R}/\ell_i\mathbf{Z})$, we have

$$\mathrm{mesh}(h_i^{-1}A) = \max_i \int_{x_{i-1}}^{x_i} d(h_i^{-1}).$$

We have a similar formula for $\mathrm{mesh}(H^{-1}A)$. By the definition of H,

$$\mathrm{mesh}(H^{-1}A) \ge \mathrm{mesh}(h_i^{-1}A) \; .$$

Since $v_\beta(\varphi_i \circ h_i, h_i^{-1}A) = v_\beta(\varphi_i \circ H, H^{-1}A)$, we have

$$v_\beta(\varphi_i \circ h_i, \varepsilon) \geq v_\beta(\varphi_i \circ H, \varepsilon) \ .$$

Hence $v_\beta(\varphi_i \circ H) = 0$, $\varphi_i \circ H \in \mathcal{RCV}_\beta(\mathbf{R}/(\ell_1+\ell_2)\mathbf{Z})$ and

$$(\varphi_1 + \varphi_2) \circ H \in \mathcal{RCV}_\beta(\mathbf{R}/(\ell_1+\ell_2)\mathbf{Z}) \ .$$

We have a corollary which follows from Lemma 2.5 and Proposition 2.3.

COROLLARY 2.8. — *If $\beta' < \beta$, then $\mathcal{CV}_{\beta'}(\mathbf{R}/\ell\mathbf{Z}) \subset \mathcal{CQ}_\beta(\mathbf{R}/\ell\mathbf{Z})$. In particular, $\mathcal{CQ}_\beta(\mathbf{R}/\ell\mathbf{Z})$ contains $\mathcal{CV}_1(\mathbf{R}/\ell\mathbf{Z})$, the set of continuous functions with bounded variations.*

Now we look at not necessarily continuous functions. We define the function space $\mathcal{V}_\beta(\mathbf{R}/\ell\mathbf{Z})$ as follows.

DEFINITION 2.9. — *$\mathcal{V}_\beta(\mathbf{R}/\ell\mathbf{Z})$ is the set of real valued (not necessarily continuous) functions φ on $\mathbf{R}/\ell\mathbf{Z}$ such that $V_\beta(\varphi) < \infty$.*

PROPOSITION 2.10. — *$\mathcal{V}_\beta(\mathbf{R}/\ell\mathbf{Z})$ is a normed vector space with respect to the norm $\|\ \|_\beta$ defined by*

$$\|\varphi\|_\beta \ = \ (V_\beta(\varphi))^{1/\beta} \quad \textit{for} \quad \varphi \in \mathcal{V}_\beta(\mathbf{R}/\ell\mathbf{Z}) \ .$$

Proof. — As in the argument of defining L^p norm, we see that $\|\varphi + \psi\|_\beta \ = \ \|\varphi\|_\beta + \|\psi\|_\beta$. In fact, this follows from the following :

$$\begin{aligned}
(v_\beta(\varphi+\psi, A))^{1/\beta} \ &= \ \left(\sum_{j=1}^k |(\varphi+\psi)(x_j) - (\varphi+\psi)(x_{j-1})|^\beta\right)^{1/\beta} \\
&\leq \left(\sum_{j=1}^k |\varphi(x_j) - \varphi(x_{j-1})|^\beta\right)^{1/\beta} \\
&\qquad + \ \left(\sum_{j=1}^k |\psi(x_j) - \psi(x_{j-1})|^\beta\right)^{1/\beta} \\
&= \ (v_\beta(\varphi, A))^{1/\beta} + (v_\beta(\psi, A))^{1/\beta} \ .
\end{aligned}$$

Other verifications are easy.

The space $\mathcal{V}_\beta(\mathbf{R}/\ell\mathbf{Z})$ contains $\mathcal{CV}_\beta(\mathbf{R}/\ell\mathbf{Z})$ and, by Proposition 2.3, the space $\mathcal{V}_\beta(\mathbf{R}/\ell\mathbf{Z})$ contains the space $C^{1/\beta}(\mathbf{R}/\ell\mathbf{Z})$ of $1/\beta$ Hölder functions.

LEMMA 2.11. — *The norm* $\|\!|\ \|\!|_\beta$ *restricted to* $C^{1/\beta}(\mathbf{R}/\ell\mathbf{Z})$ *is weaker than the* $1/\beta$ *Hölder norm* $\|\ \|_{C^{1/\beta}}$ *given by*

$$\|\varphi\|_{C^{1/\beta}} = \sup \frac{|\varphi(x) - \varphi(y)|}{|x-y|^{1/\beta}}.$$

In fact, $\|\!|\varphi\|\!|_\beta \leq \ell^{1/\beta}\|\varphi\|_{C^{1/\beta}}$.

This lemma follows from the proof of Lemma 2.2.

We can define a map from $\mathcal{V}_\beta(\mathbf{R}/\ell\mathbf{Z})$ to $\mathcal{CV}_\beta(\mathbf{R}/\ell'\mathbf{Z})$ for some ℓ' which is a bounded map with respect to the above norm. To explain this, we need some notations.

Let φ be a function on $\mathbf{R}/\ell\mathbf{Z}$ such that $s_\beta(\varphi) < \infty$. Let

$$p : \mathbf{R}/\ell'\mathbf{Z} \longrightarrow \mathbf{R}/\ell\mathbf{Z} \quad (\ell' \geq \ell + s_\beta(\varphi))$$

be a weakly order preserving continuous map such that

$$\text{length}(p^{-1}([x,y])) \geq y - x + \sum_{z\in[x,y]} |\Delta\varphi(z)|^\beta.$$

Let $\varphi_p : \mathbf{R}/\ell'\mathbf{Z} \longrightarrow \mathbf{R}$ denote the function defined by

$$\varphi_p(x) = \varphi(p(x) - 0) + \frac{\Delta\varphi(p(x))}{\text{length}(p^{-1}p(x))} \cdot \{x - \min\{p^{-1}p(x)\}\}.$$

Then φ_p is obtained from φ by putting the intervals $p^{-1}p(x)$ at x where $\text{length}(p^{-1}p(x)) > 0$ and extending φ over these intervals linearly. Note that φ_p is a continuous function. Two functions φ_1 and φ_2 give rise to the same function $(\varphi_1)_p = (\varphi_2)_p$ if $\varphi_1(x \pm 0) = \varphi_2(x \pm 0)$ for any $x \in \mathbf{R}/\ell\mathbf{Z}$.

For a function φ such that $s_\beta(\varphi) < \infty$, there is a natural weakly order preserving function $p_\beta : \mathbf{R}/(\ell + s(\varphi))\mathbf{Z} \longrightarrow \mathbf{R}/\ell\mathbf{Z}$ such that

$$p_\beta(x + \sum_{0\leq y<x} |\ \Delta\varphi(y)\ |^\beta) = x.$$

Then $\varphi_{p_\beta} : \mathbf{R}/(\ell + s_\beta(\varphi))\mathbf{Z} \longrightarrow \mathbf{R}$ is defined by

$$\varphi_{p_\beta}(x) = \varphi(p_\beta(x) - 0) + \frac{\Delta\varphi(p_\beta(x))}{|\Delta\varphi(p_\beta(x))|^\beta}(x - \min\{p_\beta^{-1}p_\beta(x)\}).$$

PROPOSITION 2.12. — *If* $\varphi : \mathbf{R}/\ell\mathbf{Z} \longrightarrow \mathbf{R}$ *satisfies* $V_\beta(\varphi) < \infty$, *then*

$$\varphi_{p_\beta} \in \mathcal{CV}_\beta(\mathbf{R}/(\ell + s_\beta(\varphi))\mathbf{Z}).$$

Proof. — Let A be a finite subset of $\mathbf{R}/(\ell + s_\beta(\varphi))\mathbf{Z}$. For $x_i \in A$, put $I_i = {p_\beta}^{-1} p_\beta(x_i)$. I_i is either a point or an interval and $I_i = I_j$ if $p_\beta(x_i) = p_\beta(x_j)$. Let A_1 be the subset of $\mathbf{R}/(\ell + s_\beta(\varphi))\mathbf{Z}$ given by

$$A_1 = A \cup \bigcup_{i=1}^{k} \partial I_i.$$

Then each connected component of $(\mathbf{R}/(\ell + s(\varphi))\mathbf{Z}) - A$ contains at most 2 points of $A_1 - A$. Using the inequality

$$n^{\beta-1}(\sum_{i=1}^{n} |r_i|^\beta) \geq |\sum_{i=1}^{n} r_i|^\beta$$

for real numbers r_1, ..., r_n, we have

$$v_\beta(\varphi_{p_\beta}, A) \leq 3^{\beta-1} v_\beta(\varphi_{p_\beta}, A_1).$$

Let A_2 be the subset of $\mathbf{R}/(\ell + s(\varphi))\mathbf{Z}$ given by

$$A_2 = A_1 - \bigcup_{i=1}^{k} (A_1 \cap \mathrm{Int} I_i).$$

Then we have

$$v_\beta(\varphi_{p_\beta}, A_1) \leq v_\beta(\varphi_{p_\beta}, A_2).$$

Let A_3 be a subset of $\mathbf{R}/(\ell + s(\varphi))\mathbf{Z}$ obtained from A_2 by replacing the points ∂I_i by nearby points where p_β is injective. We can take A_3 so that

$$v_\beta(\varphi_{p_\beta}, A_2) \leq v_\beta(\varphi_{p_\beta}, A_3) + \varepsilon$$

for any positive real number ε. Since A_3 consists of the points where p_β is injective, we have $v_\beta(\varphi_{p_\beta}, A_3) = v_\beta(\varphi, p_\beta(A_3))$. Thus we have

$$v_\beta(\varphi_{p_\beta}, A) \leq 3^{\beta-1}\{v_\beta(\varphi, p_\beta(A_3)) + \varepsilon\} \leq 3^{\beta-1}\{v_\beta(\varphi, \ell) + \varepsilon\} < \infty.$$

We have proved the proposition.

Now we define the space $\mathcal{Q}_\beta(\mathbf{R}/\ell\mathbf{Z})$ corresponding to $\mathcal{CQ}_\beta(\mathbf{R}/\ell'\mathbf{Z})$.

DEFINITION 2.13. — *$\mathcal{Q}_\beta(\mathbf{R}/\ell\mathbf{Z})$ is the set of real valued (not necessarily continuous) functions φ on $\mathbf{R}/\ell\mathbf{Z}$ such that $v_\beta(\varphi) < \infty$ and $\varphi_{p_\beta} \in \mathcal{CQ}_\beta(\mathbf{R}/(\ell + s_\beta(\varphi))\mathbf{Z})$.*

If $\varphi \in \mathcal{Q}_\beta(\mathbf{R}/\ell\mathbf{Z})$, then φ_p belongs to $\mathcal{CQ}_\beta(\mathbf{R}/\ell'\mathbf{Z})$ for any p satisfying the above condition. Note that for a pair of elements $\varphi_1, \varphi_2 \in \mathcal{Q}_\beta(\mathbf{R}/\ell\mathbf{Z})$, there exists a weakly order preserving function p as above such that both $(\varphi_1)_p$ and $(\varphi_2)_p$ belong to $\mathcal{CQ}_\beta(\mathbf{R}/\ell'\mathbf{Z})$ or $\mathcal{RCV}_\beta(\mathbf{R}/\ell'\mathbf{Z})$. Hence we obtain the following lemma.

LEMMA 2.14. — *$\mathcal{Q}_\beta(\mathbf{R}/\ell\mathbf{Z})$ is a real vector space.*

For $\varphi \in \mathcal{V}_\beta(\mathbf{R}/\ell\mathbf{Z})$, we have $s_\beta(\varphi) < \infty$ by Lemma 1.3, and $\varphi_{p_\beta} \in \mathcal{CV}_\beta(\mathbf{R}/(\ell + s_\beta(\varphi))\mathbf{Z})$ by Proposition 2.12. The following corollary follows from Lemma 2.5 and Proposition 2.3.

COROLLARY 2.15. — *$\mathcal{V}_{\beta'}(\mathbf{R}/\ell\mathbf{Z}) \subset \mathcal{Q}_\beta(\mathbf{R}/\ell\mathbf{Z})$ if $\beta' < \beta$. In particular, $\mathcal{Q}_\beta(\mathbf{R}/\ell\mathbf{Z})$ ($\beta > 1$) contains $\mathcal{V}_1(\mathbf{R}/\ell\mathbf{Z})$, the set of functions with bounded variations.*

As a summary, we have the following inclusion maps between the function spaces we discussed

$$\begin{array}{ccccc}
C^{1/\beta'} & \xrightarrow{\subset} & AC^{1/\beta'} = \mathcal{CV}_{\beta'} & \xrightarrow{\subset} & \mathcal{V}_{\beta'} \\
\downarrow & & \downarrow\cap & & \downarrow\cap \\
\mathcal{RCV}_\beta & \xrightarrow{\subset} & \mathcal{CQ}_\beta & \xrightarrow{\subset} & \mathcal{Q}_\beta \\
& & \downarrow\cap & & \downarrow\cap \\
& & AC^{1/\beta} = \mathcal{CV}_\beta & \xrightarrow{\subset} & \mathcal{V}_\beta .
\end{array}$$

Here $1 \leq \beta' < \beta$ and $AC^{1/\beta}$ is the the set of real valued functions φ such that $\varphi \circ h$ are $C^{1/\beta}$ ($1/\beta$ Hölder) functions for some homeomorphisms h.

In later sections we have to consider the following function space on the circle.

DEFINITION 2.16. — *$\mathcal{B}_\ell$ is the set of bounded functions on $\mathbf{R}/\ell\mathbf{Z}$ such that $\varphi(x \pm 0)$ exist for any $x \in \mathbf{R}/\ell\mathbf{Z}$ and $\sum |\Delta\varphi(x)| < \infty$.*

This function space $\mathcal{B}(\mathbf{R}/\ell\mathbf{Z})$ has the following properties, which are easily verified.

LEMMA 2.17. — (1) *$\mathcal{B}(\mathbf{R}/\ell\mathbf{Z})$ is a real vector space.*

(2) *For any homeomorphism $\mathbf{R}/\ell'\mathbf{Z} \longrightarrow \mathbf{R}/\ell\mathbf{Z}$, $h^*\mathcal{B}(\mathbf{R}/\ell\mathbf{Z}) = \mathcal{B}(\mathbf{R}/\ell'\mathbf{Z})$.*

To end this section, we note that any function φ on the circle such that $\varphi(x-0)$ and $\varphi(x+0)$ exist is a measurable function and if it is bounded then it is integrable.

3. Area functionals.

First we define area functionals on a space $\mathcal{X}$ of functions on $S^1 = \mathbf{R}/\mathbf{Z}$.

DEFINITION 3.1. — *An area functional is an antisymmetric bilinear form*

$$\mathbf{A} : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbf{R}$$

satisfying the invariance under parameter changes, i.e., if (φ, ψ) *and* $(\varphi \circ h, \psi \circ h)$ *belong to* $\mathcal{X} \times \mathcal{X}$ *for a homeomorphism* $h : S^1 \longrightarrow S^1$, *then*

$$\mathbf{A}(\varphi, \psi) = \mathbf{A}(\varphi \circ h, \psi \circ h).$$

For the space of C^1 functions on S^1 there is the usual area functional defined by

$$\mathbf{A}(\varphi, \psi) = \int_{S^1} \varphi d\psi.$$

It is not difficult to see that this area is invariant under the C^1 parameter changes. It is not clear but it is true that this area is invariant under the parameter changes by homeomorphisms (see Lemma 3.3).

We show that there is an area functional on $\mathcal{Q}_2(\mathbf{R}/\ell\mathbf{Z})$ which is an extension of the above area functional. This is the widest space we could construct an area functional up to now.

Remark. — Such an extension cannot exist on a very big function space, for example, the space of all continuous functions on $\mathbf{R}/\ell\mathbf{Z}$, because such extension should give rise to a nontrivial second cohomology class (extending the Godbillon-Vey class, see §4) of the group of C^1-diffeomorphisms of the circle, however there is only the Euler class in the second cohomology of this group ([18], [11]).

DEFINITION 3.2. — *Let* $A = \{x_1, ..., x_k\}$ *be a finite subset of* $\mathbf{R}/\ell\mathbf{Z}$. *For a continuous map* $\gamma = (\varphi, \psi) : \mathbf{R}/\ell\mathbf{Z} \longrightarrow \mathbf{R}^2$, $\mathrm{Area}(\gamma, A) = \mathrm{Area}((\varphi, \psi), A)$ *is the algebraic area bounded by the polygon with vertices* $\gamma(x_i) = (\varphi(x_i), \psi(x_i))$;

$$\mathrm{Area}(\gamma, A) = \mathrm{Area}((\varphi, \psi), A) = \frac{1}{2} \sum_{i=1}^{k} \begin{vmatrix} \varphi(x_{i-1}) & \varphi(x_i) \\ \psi(x_{i-1}) & \psi(x_i) \end{vmatrix} .$$

We will show that $\mathrm{Area}((\varphi,\psi),A)$ converges as $\mathrm{mesh}(A) \longrightarrow 0$ under some condition on φ and ψ and define $\mathrm{Area}(\varphi,\psi)$ to be this limit.

LEMMA 3.3. — *Let A_0 and A_1 be finite subsets of $\mathbf{R}/\ell\mathbf{Z}$. If $A_0 \subset A_1$, $\mathrm{mesh}(A_0) \leq \varepsilon$, and φ, ψ are elements of $\mathcal{RCV}_2(\mathbf{R}/\ell\mathbf{Z})$, then*

$$|\,\mathrm{Area}((\varphi,\psi),A_1) - \mathrm{Area}((\varphi,\psi),A_0)\,| \leq 3\int_0^{2\varepsilon}\{v_2(\varphi,s)+v_2(\psi,s)\}d\log s.$$

Proof. — For A_0 and A_1, we define the sequence B_0, B_1, ..., B_k of subsets of $\mathbf{R}/\ell\mathbf{Z}$ such that $A_0 = B_0 \subset B_1 \subset B_2 \subset \ldots \subset B_k = A_1$ as follows. For each connected component (a,b) of $(\mathbf{R}/\ell\mathbf{Z}) - B_{j-1}$,

$$B_j \cap (a,b) = \left\{\frac{a+b}{2}\right\}$$

if $(a+b)/2 \in A_1$ and

$$B_j \cap (a,b) = \{x,y\} \cap (a,b)$$

if there is a connected component (x,y) of $(\mathbf{R}/\ell\mathbf{Z}) - A_1$ which contains $(a+b)/2$. In the latter case, $\{x,y\} \cap (a,b)$ can be two points, one point or empty and we have no points of A_1 in (x,y). Then the intervals of $(\mathbf{R}/\ell\mathbf{Z}) - B_{j-1}$ which contain a point of B_j are of length at most $2^{-j+1}\mathrm{mesh}(A_0)$.

For $\gamma = (\varphi,\psi)$, the area of the triangle with vertices $\gamma(a)$, $\gamma(x)$ and $\gamma(b)$ is not bigger than

$$\frac{1}{4}\{\|\gamma(x)-\gamma(a)\|^2 + \|\gamma(b)-\gamma(x)\|^2\}.$$

The area of the quadrilateral with vertices $\gamma(a)$, $\gamma(x)$, $\gamma(y)$ and $\gamma(b)$ is not bigger than

$$\frac{1}{2}\{\|\gamma(x)-\gamma(a)\|^2 + \|\gamma(y)-\gamma(x)\|^2 + \|\gamma(b)-\gamma(y)\|^2\}.$$

Hence we always have

$$|\,\mathrm{Area}(\gamma,B_j) - \mathrm{Area}(\gamma,B_{j-1})\,| \leq \frac{3}{2}\{v_2(\varphi,\mathrm{mesh}(B_{j-1})) + v_2(\psi,\mathrm{mesh}(B_{j-1}))\}.$$

We have a better estimate because the intervals of $(\mathbf{R}/\ell\mathbf{Z}) - B_{j-1}$ which contain a point of B_j are of length at most $2^{-j+1}\mathrm{mesh}(A_0)$

$$\begin{aligned}&|\,\mathrm{Area}((\varphi,\psi),B_j) - \mathrm{Area}((\varphi,\psi),B_{j-1})\,|\\ &\qquad\leq \frac{3}{2}\{v_2(\varphi,2^{-j+1}\mathrm{mesh}(A_0)) + v_2(\psi,2^{-j+1}\mathrm{mesh}(A_0))\}.\end{aligned}$$

Hence

$$
\begin{aligned}
&| \operatorname{Area}((\varphi,\psi),A_1) - \operatorname{Area}((\varphi,\psi),A_0) | \\
&\qquad \leq \sum_{j=1}^{k} \frac{3}{2}\{v_2(\varphi, 2^{-j+1}\mathrm{mesh}(A_0)) + v_2(\psi, 2^{-j+1}\mathrm{mesh}(A_0))\} \\
&\qquad \leq \frac{3}{2} \sum_{j=-\infty}^{0} \{v_2(\varphi, e^{\log\varepsilon+\lambda j}) + v_2(\psi, e^{\log\varepsilon+\lambda j})\} \\
&\qquad \leq \frac{3}{2\lambda} \int_{-\infty}^{\log\varepsilon+\lambda} \{v_2(\varphi, e^{s}) + v_2(\psi, e^{s})\} ds \\
&\qquad \leq \frac{3}{2\lambda} \int_{0}^{2\varepsilon} \{v_2(\varphi, s) + v_2(\psi, s)\} d\log s.
\end{aligned}
$$

Here $\lambda = -\log(1/2) = 0.693147....$

COROLLARY 3.4. — *If* $\mathrm{mesh}(A)$ *and* $\mathrm{mesh}(B)$ *are smaller than* ε, *then*

$$
| \operatorname{Area}((\varphi,\psi),A) - \operatorname{Area}((\varphi,\psi),B) | \leq 6 \int_0^{2\varepsilon} \{v_2(\varphi,s) + v_2(\psi,s)\} d\log s.
$$

By using this corollary, we can define $\operatorname{Area}(\varphi,\psi)$ for $\varphi,\psi \in \mathcal{RCV}_2(\mathbf{R}/\ell\mathbf{Z})$ by

$$
\operatorname{Area}(\varphi,\psi) = \lim_{\mathrm{mesh}(A)\to 0} \operatorname{Area}((\varphi,\psi),A).
$$

Note that for C^1 functions we have $\operatorname{Area}(\varphi,\psi) = \int \varphi d\psi$.

LEMMA 3.5. — *For* $\varphi,\psi \in \mathcal{CQ}_2(\mathbf{R}/\ell\mathbf{Z})$, $\lim_{\mathrm{mesh}(A)\to 0} \operatorname{Area}((\varphi,\psi),A)$ *exists.*

Proof. — For $\varphi,\psi \in \mathcal{CQ}_2(\mathbf{R}/\ell\mathbf{Z})$, we can find a homeomorphism $h : \mathbf{R}/\ell'\mathbf{Z} \longrightarrow \mathbf{R}/\ell\mathbf{Z}$ such that both $\varphi \circ h$ and $\psi \circ h$ belong to $\mathcal{RCV}_2(\mathbf{R}/\ell'\mathbf{Z})$. Then it is easy to see that if $\lim_{\mathrm{mesh}(A)\to 0} \operatorname{Area}((\varphi\circ h, \varphi\circ h),A)$ exists, then $\lim_{\mathrm{mesh}(A)\to 0} \operatorname{Area}((\varphi,\psi),A)$ exists and they are equal.

By this limit we define $\operatorname{Area}(\varphi,\psi)$ on $\mathcal{CQ}_2(\mathbf{R}/\ell\mathbf{Z})$. The proof of the following lemma is straightforward.

LEMMA 3.6. — $\operatorname{Area}(\varphi,\psi)$ *is an area functional on* $\mathcal{CQ}_2(\mathbf{R}/\ell\mathbf{Z})$.

For $\varphi, \psi \in \mathcal{Q}_2(\mathbf{R}/\ell\mathbf{Z})$, let $p : \mathbf{R}/\ell'\mathbf{Z} \longrightarrow \mathbf{R}/\ell\mathbf{Z}$ $(\ell' \geq \ell + s_2(\varphi) + s_2(\psi))$ be a weakly order preserving continuous map such that

$$
\mathrm{length}(p^{-1}([x,y])) \geq y - x + \sum_{z\in[x,y]} \{|\Delta\varphi(z)|^2 + |\Delta\psi(z)|^2\}.
$$

Then φ_p and ψ_p belong to $\mathcal{CQ}_2(\mathbf{R}/\ell'\mathbf{Z})$. We define $\mathrm{Area}(\varphi,\psi)$ by

$$\mathrm{Area}(\varphi,\psi)=\mathrm{Area}(\varphi_p,\psi_p).$$

This definition does not depend on the choice of p.

LEMMA 3.7. — *$\mathrm{Area}(\varphi,\psi)$ is an area functional on $\mathcal{Q}_2(\mathbf{R}/\ell\mathbf{Z})$.*

Proof. — For φ_1, φ_2 and $\psi\in\mathcal{Q}_2(\mathbf{R}/\ell\mathbf{Z})$, we can find $p:\mathbf{R}/\ell'\mathbf{Z}\longrightarrow\mathbf{R}/\ell\mathbf{Z}$ such that $(\varphi_1)_p$, $(\varphi_2)_p$ and ψ_p belong to $\mathcal{CQ}_2(\mathbf{R}/\ell'\mathbf{Z})$. Since $(\varphi_1+\varphi_2)_p=(\varphi_1)_p+(\varphi_2)_p$,

$$\mathrm{Area}(\varphi_1+\varphi_2,\psi)\;=\;\mathrm{Area}(\varphi_1,\psi)+\mathrm{Area}(\varphi_2,\psi)$$

follows from the bilinearity of Area in $\mathcal{CQ}_2(\mathbf{R}/\ell'\mathbf{Z})$ and the definition of Area in $\mathcal{Q}_2(\mathbf{R}/\ell\mathbf{Z})$. Other verifications are straightforward.

Since $\mathcal{V}_\beta(\mathbf{R}/\ell\mathbf{Z})$ $(1\le\beta<2)$ is contained in $\mathcal{Q}_2(\mathbf{R}/\ell\mathbf{Z})$, the functional Area is defined on on $\mathcal{V}_\beta(\mathbf{R}/\ell\mathbf{Z})$ $(1\le\beta<2)$. A remarkable point is that the area functional Area on $\mathcal{V}_\beta(\mathbf{R}/\ell\mathbf{Z})$ $(1\le\beta<2)$ is continuous with respect the norm $\|\ \|_\beta$. To see this, we give the following proposition.

PROPOSITION 3.8. — *Let φ and ψ be elements of $\mathcal{V}_\beta(\mathbf{R}/\ell\mathbf{Z})$ $(1\le\beta<2)$. Then*

$$\mathrm{Area}(\varphi,\psi)\le\frac{1}{2}\frac{(\ell+8)^{2/\beta}}{2^{2/\beta-1}-1}\,\|\varphi\|_\beta\cdot\|\psi\|_\beta.$$

Proof. — Since Area is bilinear with respect to φ and ψ, it is sufficient to prove that, if $\|\varphi\|_\beta=\|\psi\|_\beta=1$, then $\mathrm{Area}(\varphi,\psi)$ is bounded by the coefficient in the inequality.

For elements φ and ψ of $\mathcal{V}_\beta(\mathbf{R}/\ell\mathbf{Z})$ $(\beta<2)$, we have a nondecreasing function

$$p:\mathbf{R}/(\ell+s_\beta(\varphi)+s_\beta(\psi))\mathbf{Z}\longrightarrow\mathbf{R}/\ell\mathbf{Z}$$

such that φ_p and ψ_p are elements of $\mathcal{CV}_\beta(\mathbf{R}/(\ell+s_\beta(\varphi)+s_\beta(\psi))\mathbf{Z})$ and

$$\|\varphi_p\|_\beta=3^{1-1/\beta}\|\varphi\|_\beta\quad\text{and}\quad\|\psi_p\|_\beta=3^{1-1/\beta}\|\psi\|_\beta$$

(Proposition 2.12). Then by Proposition 2.3, for

$$\ell'=\ell+s_\beta(\varphi)+s_\beta(\psi)+3^{\beta-1}V_\beta(\varphi)+3^{\beta-1}V_\beta(\psi),$$

there is a homeomorphism

$$h:\mathbf{R}/\ell'\mathbf{Z}\longrightarrow\mathbf{R}/(\ell+s_\beta(\varphi)+s_\beta(\psi))\mathbf{Z}$$

such that $\varphi_p \circ h$ and $\psi_p \circ h$ are $C^{1/\beta}$ functions on $\mathbf{R}/\ell'\mathbf{Z}$ and their norms are estimated as follows :

$$\|\varphi_p \circ h\|_{C^{1/\beta}} \leq 1 \quad \text{and} \quad \|\psi_p \circ h\|_{C^{1/\beta}} \leq 1.$$

We already know (Lemma 3.5) that $\mathrm{Area}(\varphi, \psi)$ is defined by

$$\mathrm{Area}(\varphi, \psi) = \lim_{\mathrm{mesh}(A)\to 0} \mathrm{Area}((\varphi_p \circ h, \psi_p \circ h), A).$$

We take the finite set A_n consisting of 2^n points as in [9], that is,

$$A_n = \{k2^{-n}\ell' \ ; \ k = 1, ..., 2^n\}.$$

Then for $\gamma = (\varphi_p \circ h, \psi_p \circ h)$, the area of the triangle with vertices

$$\gamma(k2^{-n}\ell'), \ \gamma((2k+1)2^{-(n+1)}\ell') \quad \text{and} \quad \gamma((k+1)2^{-n}\ell')$$

is not bigger than

$$\frac{1}{2}\{(\max - \min)(\varphi_p \circ h|[2^{-n}\ell', (k+1)2^{-n}\ell'])\}$$
$$\{(\max - \min)(\psi_p \circ h|[2^{-n}\ell', (k+1)2^{-n}\ell'])\}.$$

Here $(\max - \min)$ is the difference of the maximal value and minimal value of the continuous function. Since $\|\varphi_p \circ h\|_{C^{1/\beta}} \leq 1$ and $\|\psi_p \circ h\|_{C^{1/\beta}} \leq 1$, this quantity is not bigger than

$$\frac{1}{2}((2^{-n}\ell')^{1/\beta})^2.$$

Hence $\mathrm{Area}(\varphi, \psi)$ is estimated by

$$\sum_{n=1}^{\infty} 2^n \frac{1}{2}((2^{-n}\ell')^{1/\beta})^2 = \frac{1}{2}\frac{\ell'^{2/\beta}}{2^{2/\beta - 1} - 1}.$$

Now if $|\!|\!|\varphi|\!|\!|_\beta = |\!|\!|\psi|\!|\!|_\beta = 1$, then by Lemma 1.3,

$$\ell' \leq \ell + (1 + 3^{\beta-1})V_\beta(\varphi) + (1 + 3^{\beta-1})V_\beta(\psi) \leq \ell + 8.$$

Hence we obtain $\mathrm{Area}(\varphi, \psi) \leq 2^{-1}(\ell + 8)^{2/\beta}/(1 - 2^{1-2/\beta})$.

There is an area functional on $\mathcal{B}(\mathbf{R}/\ell\mathbf{Z})$ which have somewhat different from that on $\mathcal{Q}_2(\mathbf{R}/\ell\mathbf{Z})$. Let $\mathbf{A}_d : \mathcal{B}(\mathbf{R}/\ell\mathbf{Z}) \times \mathcal{B}(\mathbf{R}/\ell\mathbf{Z}) \longrightarrow \mathbf{R}$ be the functional defined by

$$\mathbf{A}_d(\varphi, \psi) = \frac{1}{2}\sum \begin{vmatrix} \varphi(x-0) & \Delta\varphi(x) \\ \psi(x-0) & \Delta\psi(x) \end{vmatrix}$$

for φ, $\psi \in \mathcal{B}(\mathbf{R}/\ell\mathbf{Z})$. It is easy to show the following lemma.

LEMMA 3.9. — *$\mathbf{A}_d$ is an area functional on $\mathcal{B}(\mathbf{R}/\ell\mathbf{Z})$.*

Remark. — This area functional corresponds to the discrete Godbillon-Vey class given by Ghys-Sergiescu ([5]) and Ghys ([3]).

As for the uniqueness of the area functional, the two area functionals Area and $\mathbf{A}_d$ on $\mathcal{Q}_2(\mathbf{R}/\ell\mathbf{Z}) \cap \mathcal{B}(\mathbf{R}/\ell\mathbf{Z})$ are nontrivial and independent (see Ghys [3]). It is possible that the area functional is unique on $\mathcal{CQ}_2(\mathbf{R}/\ell\mathbf{Z})$, $\mathcal{Q}_2(\mathbf{R}/\ell\mathbf{Z})$ or $\mathcal{B}(\mathbf{R}/\ell\mathbf{Z})$ though we have no proof up to now. There are spaces where the area functional is unique.

PROPOSITION 3.10. — *Let $\mathcal{PL}(\mathbf{R}/\ell\mathbf{Z})$ be the vector space of continuous functions on $\mathbf{R}/\ell\mathbf{Z}$ which are piecewise linear. Then the nontrivial ($\mathbf{R}$-bilinear) area functional which is invariant under the parameter change by the piecewise linear homeomorphisms of $\mathbf{R}/\ell\mathbf{Z}$ is unique up to a real multiple.*

Proof. — For three distinct points x_0, x_1, x_2 of $\mathbf{R}/\ell\mathbf{Z}$ in the cyclic order, let α_{x_0,x_1,x_2} denote the function on $\mathbf{R}/\ell\mathbf{Z}$ such that

$$\alpha_{x_0,x_1,x_2}(y) = \begin{cases} 0 & \text{if} \quad y \in [x_2, x_0] \\ (y - x_0)/(x_1 - x_0) & \text{if} \quad y \in [x_0, x_1] \\ (y - x_2)/(x_1 - x_2) & \text{if} \quad y \in [x_1, x_2]. \end{cases}$$

Let $\mathbf{A}$ be an area functional. By the $\mathbf{R}$-bilinearity, it is necessary to know the values of

$$\mathbf{A}(\alpha_{x_0,x_1,x_2}, \alpha_{x_3,x_4,x_5}), \mathbf{A}(\alpha_{x_0,x_1,x_2}, \alpha_{x_2,x_3,x_4}), \\ \mathbf{A}(\alpha_{x_0,x_1,x_2}, \alpha_{x_1,x_2,x_3}), \mathbf{A}(\alpha_{x_0,x_1,x_2}, \alpha_{x_0,x_1,x_2})$$

for distinct points x_0, x_1, x_2, x_3, x_4, x_5 in the cyclic order. By the antisymmetricity, the last one is 0. By the bilinearity,

$$\begin{aligned} \mathbf{A}(\alpha_{x_0,x_1,x_2}, \alpha_{x_3,x_4,x_5}) &= \mathbf{A}(\alpha_{x_0,x_1,x_2}, \frac{1}{2}\alpha_{x_3,(x_3+x_4)/2,x_4}) \\ &\quad + \mathbf{A}(\alpha_{x_0,x_1,x_2}, \alpha_{(x_3+x_4)/2,x_4,(x_4+x_5)/2}) \\ &\quad + \mathbf{A}(\alpha_{x_0,x_1,x_2}, \frac{1}{2}\alpha_{x_4,(x_4+x_5)/2,x_5}). \end{aligned}$$

By the invariance under the parameter change and the bilinearity, the right-hand-side is equal to $2\mathbf{A}(\alpha_{x_0,x_1,x_2}, \alpha_{x_3,x_4,x_5})$. Hence the first one is 0. For the second one, by replacing the indices in the above formula, we see that

$$\mathbf{A}(\alpha_{x_0,x_1,x_2}, \alpha_{x_2,x_3,x_4}) = \mathbf{A}(\alpha_{x_0,x_1,x_2}, \frac{1}{2}\alpha_{x_2,(x_2+x_3)/2,x_4}).$$

By the invariance under the parameter change and the bilinearity, the right-hand-side is equal to $(1/2)\mathbf{A}(\alpha_{x_0,x_1,x_2}, \alpha_{x_2,x_3,x_4})$. Hence the second one is also 0.

Now by the invariance under the parameter change, the value $\mathbf{A}(\alpha_{x_0,x_1,x_2}, \alpha_{x_1,x_2,x_3})$ does not depend on the points x_0, x_1, x_2, x_3 if they are in the cyclic order. Suppose that $\mathbf{A}(\alpha_{x_0,x_1,x_2}, \alpha_{x_1,x_2,x_3}) = 1/2$ which is the area of the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$. Then $\mathbf{A}(\varphi,\psi)$, $(\varphi, \psi \in \mathcal{PL}(\mathbf{R}/\ell\mathbf{Z}))$ is equal to the usual area of the polygon.

In a way similar to the proof of the above proposition, we can show the following proposition.

PROPOSITION 3.11. — *Let $\mathcal{S}(\mathbf{R}/\ell\mathbf{Z})$ be the vector space of left-continuous functions on $\mathbf{R}/\ell\mathbf{Z}$ which are constant out of finitely many points. Then the nontrivial ($\mathbf{R}$-bilinear) area functional which is invariant under the parameter change by the piecewise linear homeomorphisms of $\mathbf{R}/\ell\mathbf{Z}$ is unique up to a real multiple.*

Remark. — These propositions are closely related to the scissors congruence of polygons on the Euclidean plane, where the area is the only invariant. We also note that in both cases, the space of $\mathbf{Q}$-bilinear area functionals is isomorphic to $\mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R}$ (cf. [5], [7]).

These propositions may optimistically imply that the area functional is unique on $\mathcal{CQ}_2(\mathbf{R}/\ell\mathbf{Z})$ under a certain continuity assumption. See the remark in §4 for the group of Lipschitz homeomorphisms which are the integrals of functions of bounded variations.

4. The Godbillon-Vey cocycle.

Let $\mathcal{X}$ be a space of integrable functions on $S^1 = \mathbf{R}/\mathbf{Z}$. Let $\mathbf{G}^{L,\mathcal{X}}$ denote the set of Lipschitz homeomorphisms of S^1 which are written of the form

$$f(x) = \int_0^x e^{\varphi(y)}dy + f(0),$$

where $\varphi \in \mathcal{X}$. Let $\mathcal{RCV}_\beta$, $\mathcal{CV}_\beta$, $\mathcal{CQ}_\beta$, $\mathcal{V}_\beta$, $\mathcal{Q}_\beta$ and $\mathcal{B}$ denote the spaces of corresponding functions on $\mathbf{R}/\mathbf{Z}$.

LEMMA 4.1. — *For $\mathcal{X} = \mathcal{RCV}_\beta, \mathcal{CV}_\beta, \mathcal{CQ}_\beta, \mathcal{V}_\beta, \mathcal{Q}_\beta$ and $\mathcal{B}$, the set $\mathbf{G}^{L,\mathcal{X}}$ is a group.*

Proof. — First note that the derivative $f'(x)$ exists almost everywhere and the limits $f'(x \pm 0)$ exist everywhere. Then we have

$$\varphi(x \pm 0) = \log f'(x \pm 0)$$

for any point $x \in S^1$. Since

$$\begin{aligned}\log(f_1 \circ f_2)'(x \pm 0) &= \log f_1'(f_2(x \pm 0)) + \log f_2'(x \pm 0) \\ &= \varphi_1(f_2(x \pm 0)) + \varphi_2(x \pm 0),\end{aligned}$$

$f_1 \circ f_2$ belongs to $\mathcal{X}$, if f_1 and f_2 belong to $\mathcal{X}$. For the inversion, we have

$$f^{-1}(x) = \int_0^x e^{-\varphi(f(y))} dy + f^{-1}(0),$$

and this shows that if f belongs to $\mathbf{G}^{L,\mathcal{X}}$, then f^{-1} belongs to $\mathbf{G}^{L,\mathcal{X}}$.

If there is an area functional $\mathbf{A}$ on $\mathcal{X}$, then $\mathbf{A}$ gives rise to a 2-dimensional cocycle of the group $\mathbf{G}^{L,\mathcal{X}}$. A 2-dimensional cocycle of the group $\mathbf{G}^{L,\mathcal{X}}$ is a function $C : \mathbf{G}^{L,\mathcal{X}} \times \mathbf{G}^{L,\mathcal{X}} \longrightarrow \mathbf{R}$ such that

$$C(f_1, f_2) - C(f_0 \circ f_1, f_2) + C(f_0, f_1 \circ f_2) - C(f_0, f_1) = 0.$$

We define the 2-dimensional cocycle $C_{\mathbf{A}}$ by

$$C_{\mathbf{A}}(f_1, f_2) = \mathbf{A}(\varphi_1 \circ f_2, \varphi_2)$$

for

$$f_i(x) = \int_0^x e^{\varphi_i(y)} dy + f_i(0) \quad (i = 1, 2).$$

Then the cocycle condition is verified as follows. Since

$$\log(f_i \circ f_j)'(x \pm 0) = \varphi_i(f_j(x \pm 0)) + \varphi_j(x \pm 0),$$

by using the bilinearity and the invariance under parameter changes, we have

$$\begin{aligned}&\mathbf{A}(\varphi_1 \circ f_2, \varphi_2) - \mathbf{A}(\varphi_0 \circ f_1 \circ f_2 + \varphi_1 \circ f_2, \varphi_2) \\ &\qquad + \mathbf{A}(\varphi_0 \circ f_1 \circ f_2, \varphi_1 \circ f_2 + \varphi_2) - \mathbf{A}(\varphi_0 \circ f_1, \varphi_1) \\ &= \mathbf{A}(\varphi_0 \circ f_1 \circ f_2, \varphi_1 \circ f_2) - \mathbf{A}(\varphi_0 \circ f_1, \varphi_1) \\ &= 0.\end{aligned}$$

If $\mathbf{A}$ is antisymmetric then we have

$$\mathbf{A}(\log(f_1 f_2)', \log(f_2)') = \mathbf{A}(\log(f_1)' \circ f_2, \log(f_2)').$$

Note that the above verification of the cocycle condition does not use the antisymmetricity, however, it is sufficient to consider the antisymmetric area functionals because a symmetric area functional gives rise to a coboundary. Let $\mathbf{A}_S$ be a symmetric bilinear form on $\mathcal{X}$ invariant under the parameter change. Then, by putting $U(f) = \mathbf{A}_S(\log f', \log f')$ we have

$$\begin{aligned}(\delta U)(f_1, f_2) &= \mathbf{A}_S(\log f_2', \log f_2') - \mathbf{A}_S(\log(f_1 \circ f_2)', \log(f_1 \circ f_2)') \\ &\qquad + \mathbf{A}_S(\log f_1', \log f_1') \\ &= -2\mathbf{A}_S(\log f_1' \circ f_2, \log f_2') .\end{aligned}$$

By the result of §2, $\mathbf{G}^{L,\mathcal{Q}_2}$ contains the group of $C^{1+\alpha}$ diffeomorphisms ($\alpha > 1/2$) of the circle and the group of diffeomorphisms of class P ([8]), *i.e.*, $\mathbf{G}^{L,\mathcal{V}_1}$. By using the area functional Area defined in §3, we have the following theorem.

THEOREM 4.2. — *There is a well-defined 2-cocycle for the group $\mathbf{G}^{L,\mathcal{Q}_2}$ which is an extension of the Godbillon-Vey cocycle.*

Ghys and Sergiescu ([5]) and Ghys ([3]) showed that the group of diffeomorphisms of class P has the discrete Godbillon-Vey cocycle. This corresponds to the area functional $\mathbf{A}_d$ and the discrete Godbillon-Vey cocycle is defined in $\mathbf{G}^{L,\mathcal{B}}$ which is a group. Hence we have the following theorems.

THEOREM 4.3. — *There is a well-defined 2-cocycle for the group $\mathbf{G}^{L,\mathcal{B}}$ which is an extension of the discrete Godbillon-Vey cocycle.*

THEOREM 4.4. — *There are two well-defined 2-cocycles for the group $\mathbf{G}^{L,\mathcal{Q}_2} \cap \mathbf{G}^{L,\mathcal{B}}$ which are extensions of the Godbillon-Vey cocycle and the discrete Godbillon-Vey cocycle.*

These cocycles are not invariant under the conjugation by homeomorphisms of the circle. Hurder and Katok showed that GV varies continuously without changing the topological isomorphism class of foliations ([9]). These foliations are the stable foliations of the geodesic flows of a surface (see also Mitsumatsu [14]). It is easy to see that GV_d also varies continuously without changing the topological isomorphism class of foliations ([3]). For real numbers a and b, let f_a and g_b be PL homeomorphisms of $\mathbf{R}$ with support in $[-1,0]$ and in $[0,1]$ such that $\log f_a'(-0) = a$ and $\log g_b'(+0) = b$, respectively. By a straightforward computation, it is easy to see that GV_d for the 2-cycle $(f_a, g_b) - (g_b, f_a)$ is equal to ab.

Among the possible extensions of the Godbillon-Vey invariant, that to the group $\mathbf{G}^{L,\mathcal{V}_\beta}$ $(1 \leq \beta < 2)$ given by Theorem 4.2 might be most interesting. The reason is the group $\mathbf{G}^{L,\mathcal{V}_\beta}$ has a metric coming from $\mathcal{V}_\beta$-norm and the invariant is continuous with respect to this topology. More precisely, we have the following. (See [2].)

PROPOSITION 4.5. — *$\mathbf{G}^{L,\mathcal{V}_\beta}$ $(1 \leq \beta)$ has the following right invariant metric. For f_1 and f_2 of $\mathbf{G}^{L,\mathcal{V}_\beta}$ $(1 \leq \beta)$,*

$$\mathrm{dist}(f_1, f_2) = \|f_1 \circ f_2^{-1} - \mathrm{id}\|_{C^0} + \|\!|\log(f_1 \circ f_2^{-1})'(x-0)|\!\|_\beta.$$

There is a 2-cocycle GV for the group $\mathbf{G}^{L,\mathcal{V}_\beta}$ $(1 \leq \beta < 2)$ which is an extension of the Godbillon-Vey cocycle, and

$$(f_1 \circ f_2, f_2) \longmapsto GV(f_1, f_2) = \mathrm{Area}(\log(f_1 \circ f_2)', \log f_2')$$

is continuous with respect to the above metric.

Proof. — In the definition of the distance, we are considering the derivatives of elements of $\mathbf{G}^{L,\mathcal{V}_\beta}$ which are left continuous. Note that

$$\begin{aligned} &\| \log(f_1 \circ f_2^{-1})'(x-0) \|_\beta \\ &= \| \log f_1' \circ f_2^{-1}(x-0) - \log f_2' \circ f_2^{-1}(x-0) \|_\beta \\ &= \| \log f_1'(x-0) - \log f_2'(x-0) \|_\beta . \end{aligned}$$

The fact that dist is a metric follows easily from the definition.

If $(f_3, f_2) = (f_1 \circ f_2, f_2)$ is close to $(g_3, g_2) = (g_1 \circ g_2, g_2)$, then

$$\begin{aligned} &\mathrm{Area}(\log f_3', \log f_2') - \mathrm{Area}(\log g_3', \log g_2') \\ &= \mathrm{Area}(\log(f_3 \circ g_3^{-1} \circ g_3)', \log(f_2 \circ g_2^{-1} \circ g_2)') - \mathrm{Area}(\log g_3', \log g_2') \\ &= \mathrm{Area}(\log(f_3 \circ g_3^{-1})' \circ g_3 + \log g_3', \log(f_2 \circ g_2^{-1})' \circ g_2 + \log g_2') \\ &\qquad - \mathrm{Area}(\log g_3', \log g_2') \\ &= \mathrm{Area}(\log(f_3 \circ g_3^{-1})' \circ g_3, \log(f_2 \circ g_2^{-1})' \circ g_2) \\ &\qquad + \mathrm{Area}(\log(f_3 \circ g_3^{-1})' \circ g_3, \log g_2') \\ &\qquad + \mathrm{Area}(\log g_3', \log(f_2 \circ g_2^{-1})' \circ g_2). \end{aligned}$$

Each term of the last expression is small by Proposition 3.8.

$\mathbf{G}^{L,\mathcal{V}_\beta}$ is not a topological group with respect to the topology given by the above metric. However, it is easy to see that the composition and the inversion is continuous at the identity. Moreover, we have the following proposition.

PROPOSITION 4.6. — *The composition*

$$\mathbf{G}^{L,\mathcal{V}_\beta} \times \mathbf{G}^{L,\mathcal{V}_\beta} \longrightarrow \mathbf{G}^{L,\mathcal{V}_\beta}$$
$$(f_1, f_2) \longmapsto f_1 \circ f_2$$

is continuous at (f_1, f_2) with f_1 being a C^1 diffeomorphism such that $\log f_1'$ has the modulus of continuity $|x|^{1/\beta} |\log |x||^{-1}$. The inversion

$$\mathbf{G}^{L,\mathcal{V}_\beta} \longrightarrow \mathbf{G}^{L,\mathcal{V}_\beta}$$
$$f \longmapsto f^{-1}$$

is continuous at a C^1 diffeomorphism f such that $\log f'$ has the modulus of continuity $|x|^{1/\beta} |\log |x||^{-1}$.

Proof. — For $(g_1, g_2) \in \mathbf{G}^{L,\mathcal{V}_\beta} \times \mathbf{G}^{L,\mathcal{V}_\beta}$,

$$\begin{aligned}
\log(g_1 \circ g_2)'(x-0) - \log(f_1 \circ f_2)'(x-0) &= (\log g_1'(g_2(x-0)) - \log f_1'(f_2(x))) \\
&\qquad + (\log g_2'(x-0) - \log f_2'(x-0)) \\
&= (\log g_1'(g_2(x-0)) - \log f_1'(g_2(x))) \\
&\qquad + (\log f_1'(g_2(x)) - \log f_1'(f_2(x))) \\
&\qquad + (\log g_2'(x-0) - \log f_2'(x-0)).
\end{aligned}$$

The first term and the last term are small if (f_1, f_2) and (g_1, g_2) are near. Hence the propostion for the composition follows from the following lemma.

For $g \in \mathbf{G}^{L,\mathcal{V}_\beta}$,

$$\begin{aligned}
\log(g^{-1})'(x-0) - \log(f^{-1})'(x) &= -\log g' \circ g^{-1}(x-0) + \log f' \circ f^{-1}(x) \\
&= (-\log g' \circ g^{-1}(x-0) + \log f' \circ g^{-1}(x)) \\
&\qquad + (-\log f' \circ g^{-1}(x) + \log f' \circ f^{-1}(x)).
\end{aligned}$$

Since the first term is small if g is near to f, the propostion for the inversion also follows from the following lemma.

LEMMA 4.7. — *Let φ be a continuous function on $\mathbf{R}/\ell\mathbf{Z}$ with the modulus of continuity $|x|^{1/\beta}|\log|x||^{-1}$. Let f be a Lipschitz homeomorphism of $\mathbf{R}/\ell\mathbf{Z}$ such that $\|f - \mathrm{id}\|_{C^0} < \varepsilon$ and $|f(x) - f(y)| \leq 2|x-y|$. Then there is a positive real number c such that*

$$V_\beta(\varphi \circ f - \varphi) \leq c\,|\log \varepsilon|^{-\beta}.$$

Proof. — Let C be a real number such that

$$|\varphi(x) - \varphi(y)| \leq C|x-y|^{1/\beta}|\log|x-y||^{-1}.$$

Let $A = \{x_1, \ldots., x_k\}$ be a finite subset of $\mathbf{R}/\ell\mathbf{Z}$. Since $\|f - \mathrm{id}\|_{C^0} < \varepsilon$,

$$|(\varphi \circ f(x_i) - \varphi(x_i)) - (\varphi \circ f(x_{i-1}) - \varphi(x_{i-1}))|$$

is always smaller than $2C\varepsilon^{1/\beta}|\log\varepsilon|^{-1}$. For those x_{i-1} and x_i such that $|x_i - x_{i-1}| > \varepsilon$, we use this estimate and we obtain an estimate for the sum over such x_{i-1} and x_i

$$\begin{aligned}
\sum |(\varphi \circ f(x_i) - \varphi(x_i)) &- (\varphi \circ f(x_{i-1}) - \varphi(x_{i-1}))|^\beta \\
&< 2^\beta C^\beta \varepsilon |\log\varepsilon|^{-\beta} \frac{\ell}{\varepsilon} \\
&= 2^\beta C^\beta \ell |\log\varepsilon|^{-\beta}.
\end{aligned}$$

For those x_{i-1} and x_i such that $|x_i - x_{i-1}| \leq \varepsilon$, by the assumption on f, we obtain $|f(x_i) - f(x_{i-1})| \leq 2|x_i - x_{i-1}|$, and

$$\begin{aligned}
&|(\varphi \circ f(x_i) - \varphi(x_i)) - (\varphi \circ f(x_{i-1}) - \varphi(x_{i-1}))| \\
&\qquad \leq |\varphi \circ f(x_i) - \varphi \circ f(x_{i-1})| + |\varphi(x_i) - \varphi(x_{i-1})| \\
&\qquad \leq 3C|x_i - x_{i-1}|^{1/\beta}|\log|x_i - x_{i-1}||^{-1}.
\end{aligned}$$

The sum over such x_{i-1} and x_i is estimated by

$$\begin{aligned}
&\sum |(\varphi \circ f(x_i) - \varphi(x_i)) - (\varphi \circ f(x_{i-1}) - \varphi(x_{i-1}))|^\beta \\
&\qquad \leq \sum 3^\beta C^\beta |x_i - x_{i-1}| \, |\log|x_i - x_{i-1}| \, |^{-\beta} \\
&\qquad \leq \sum 3^\beta C^\beta |x_i - x_{i-1}| \, |\log \varepsilon|^{-\beta} \\
&\qquad \leq 3^\beta C^\beta \ell |\log \varepsilon|^{-\beta}.
\end{aligned}$$

Thus we obtain $V_\beta(\varphi \circ f - \varphi) \leq (2^\beta + 3^\beta) C^\beta \ell \, |\log \varepsilon|^{-\beta}$.

Remark. — We can define similar groups of Lipschitz homeomorphisms of $\mathbf{R}$ with compact support and Theorems 4.2 - 4.4 and Propositions 4.5 - 4.6 are also true for the corresponding groups of Lipschitz homeomorphisms of $\mathbf{R}$ with compact support.

Remark. — For the group $\mathbf{G}^{L,\mathcal{V}_1}$ of Lipschitz homeomorphisms which are the integrals of functions of bounded variations, we have a singular cocycle. Let f be such a diffeomorphism. Then $d(\log f')$ is a measure on $\mathbf{R}/\mathbf{Z}$. We have the decomposition of $d(\log f')$ into its regular, atomic and singular parts.

$$d(\log f') = d(\log f')_{\text{reg}} + d(\log f')_{\text{atom}} + d(\log f')_{\text{sing}}.$$

This is used by Mather ([12], III) to show that 1-dimensional homology of this group $\mathbf{G}^{L,\mathcal{V}_1}$ is nontrivial. Then we have the following cocycles :

$$C_{\text{reg}}(f_1, f_2) = \frac{1}{2} \int \begin{vmatrix} \log f_1' \circ f_2 & d(\log f_1' \circ f_2)_{\text{reg}} \\ \log f_1' & d(\log f_1')_{\text{reg}} \end{vmatrix},$$

$$C_{\text{atom}}(f_1, f_2) = \frac{1}{2} \int \begin{vmatrix} \log f_1' \circ f_2 & d(\log f_1' \circ f_2)_{\text{atom}} \\ \log f_1' & d(\log f_1')_{\text{atom}} \end{vmatrix} \quad \text{and}$$

$$C_{\text{sing}}(f_1, f_2) = \frac{1}{2} \int \begin{vmatrix} \log f_1' \circ f_2 & d(\log f_1' \circ f_2)_{\text{sing}} \\ \log f_1' & d(\log f_1')_{\text{sing}} \end{vmatrix}.$$

The corresponding area functionals are invariant under absolutely continuous parameter changes. Ghys showed no nontrivial linear combinations of C_{reg} and C_{atom} are invariant under conjugation by homeomorphisms ([3]). Up to now, we know no examples to show that C_{sing} is nontrivial.

According to Mather ([12], III), there are a huge number of ways of decompositions of $d(\log f')$ invariant under C^1 parameter changes. Hence there are in fact many other cocycles constructed as above. Note also that any pair of surjective homomorphisms $a_1, a_2 : H_1(\mathbf{G}^{L,\mathcal{V}_1}; \mathbf{Z}) \longrightarrow \mathbf{R}$ induces a surjective homomorphism $a_1 \wedge_{\mathbf{Q}} a_2 : H_2(\mathbf{G}^{L,\mathcal{V}_1}; \mathbf{Z}) \longrightarrow \mathbf{R} \wedge_{\mathbf{Q}} \mathbf{R}$, but the above 2-cocycles are not of this type.

It should be interesting to know dynamical and homological properties of the groups $\mathbf{G}^{L,\mathcal{X}}$ for $\mathcal{X} = \mathcal{RCV}_\beta, \mathcal{CV}_\beta, \mathcal{CQ}_\beta, \mathcal{V}_\beta, \mathcal{Q}_\beta, \mathcal{B}$, or $\mathbf{G}^{L,\mathcal{Q}_\beta} \cap \mathbf{G}^{L,\mathcal{B}}$, $\mathbf{G}^{L,\mathcal{V}_\beta} \cap \mathbf{G}^{L,\mathcal{B}}$. Note that there are inclusions between these groups which correspond to those in the diagram at the end of §2. In [20] we will study these properties, *e.g.*, the perfectness of these groups, the property of 2-cycles GV and GV_d.

Remark. — For $\mathcal{S}$ of Proposition 3.11, $\mathbf{G}^{L,\mathcal{S}}$ is the group of PL homeomorphisms of the circle. The homological property of $\mathbf{G}^{L,\mathcal{S}}$ is rather well understood. See [5] and [7].

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Cet article contient la première partie de mon exposé oral "Une caractérisation de l'invariant de Godbillon-Vey" au Colloque de Géometrie "Feuilletages et Singularités". Les autres parties se trouveront dans mes articles "Small commutators in piecewise linear homeomorphisms of the real line", "Homological and dynamical study on certain groups of Lipschitz homeomorphisms of the circle" and "The Godbillon-Vey invariant and the foliated cobordism group".

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