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BOUNDARY PROPERTIES OF FUNCTIONS WITH FINITE DIRICHLET INTEGRALS

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1. Introduction.

Let R be a Green space (Brelot-Choquet [4]) of dimension ≥ 2. Denote by $D(u, v)$ the Dirichlet bilinear functional of the pair (u, v) of functions on R, computed on the set $\check{R}$ of finite points of R. Denote by $D(u)$ $(= D(u, u))$ the Dirichlet integral of u on $\check{R}$. We shall as usual write « quasi-everywhere » for « except on a subset of R of outer capacity zero ». Here capacity is defined using the Green function of R. Note that if the dimensionality of R is at least 3 a set of outer capacity zero can contain no infinite point. Let u be a function defined quasi-everywhere on R, and let $\{u_n, n \geq 1\}$ be a sequence of infinitely differentiable functions on $\check{R}$, with finite Dirichlet integrals. Suppose that

$$(a) \lim_{m, n \to \infty} D(u_n - u_m) = 0$$

and that $(b) \lim_{n \to \infty} u_n = u$ quasi-everywhere on $\check{R}$. Then u is fine-continuous (continuous in the Cartan fine topology on R) quasi-everywhere on $\check{R}$. If (c) u is even fine-continuous quasi-everywhere on R we shall call u a « BLD function » (Beppo Levi-Deny) following Brelot. The class of BLD functions was introduced by Deny. (See also Aronszajn [2]). We refer to Deny [5], Deny-Lions [6], where such a function is called a « BL function made precise », and Brelot [3] for the fundamental properties of these functions.

If u is a BLD function, gradu is defined almost everywhere on $\check{R}$, and $D(u)$ is finite. If $\{u_n, n \geq 1\}$ is a sequence of BLD functions satisfying (a) above, and if there is a pointwise limit quasi-everywhere on $\check{R}$, then the limit u necessarily exists quasi-everywhere on R, is a BLD function, and

$$D(u - u_n) \to 0.$$

The sequence will be said to converge « in the BLD sense ». If only (a) is satisfied, there is a subsequence of the u_n sequence which, when centered by suitable additive constants, converges in the BLD sense. Thus the condition (b) is not so stringent as it appears to be at first sight. Finally, if u is a BLD function locally on $\check{R}$, and if $D(u)$ is finite, then u can be extended to be a BLD function on R.

The BLD harmonic functions on R are simply the harmonic functions with finite Dirichlet integrals over $\check{R}$. At the other extreme are the BLD functions which we shall call those « of potential type ». A BLD function will be said to be of potential type if it is the BLD limit of a sequence of infinitely differentiable (on $\check{R}$) functions with compact supports.

In this paper, **H** will denote the Hilbert space of BLD functions with inner product $D(u, v)$, two functions being identified if the restriction of their difference to the complement of some set of zero capacity is a constant function. The class of BLD harmonic functions corresponds to a closed linear manifold $\mathbf{H}_h$ of **H**, and the class of functionals of potential type corresponds to the orthogonal complement $\mathbf{H}_p$ of $\mathbf{H}_h$.

Brelot showed that every BLD function u has a limit in a certain L_1 sense along almost every Green line (orthogonal trajectory of level manifolds of the Green function with a preassigned pole) defining the « radial » of u, a function on the set of Green lines, and Godefroid [10] showed that u even has the radial as an ordinary limit along almost every Green line. Here the limit is to be taken along the Green line as the point on the line recedes to ∞, that is as the Green function decreases to 0, and the measure of a set of Green lines is the measure of the solid angle of their initial directions at the pole, normalized to have maximum value 1. In this paper the Brelot theorem just quoted will be strengthened to L_2 convergence, if the function is harmonic. Brelot also proved that BLD convergence of a sequence of functions implies L_1 convergence of the sequence of radials, and it will be shown that there is even L_2 convergence of the latter sequence.

Let R^M be the Martin boundary of R. In this paper it is shown that every BLD function has a fine topology limit at almost every (harmonic measure) point of R^M. Thus there is always a « fine boundary function ». The fine boundary function is in $L_2(R^M)$, and it is shown that BLD convergence of a sequence of functions implies L_2 convergence of the sequence of boundary functions. Moreover, if u is BLD harmonic, with fine boundary function u', $D(u)$ is evaluated in terms of u'. This evaluation reduces to that of Douglas [9] when R is a disc. It is known that when R is a domain in Euclidean N-space with a sufficiently smooth boundary, the boundary function of a sufficiently smooth BLD function is in L_2 relative to ordinary boundary « area » (Sobolev [14], Aronszajn [2]). The interest of the present version of this result lies in the absence of any smoothness hypothesis on either the space or the function.

These results are used in treating the first, second, mixed, and an unusual form of the third boundary value problem. Only in treating the latter problem (Sections 18 and 19) is any condition imposed on the Green space. The boundary of R is always taken as R^M, and it is accordingly necessary to define a generalized normal derivative, denoted by $\partial u/\partial g$, on R^M. There is always a Green function of R, by definition of a Green space (except in Doob [7] where the nomenclature was poorly chosen). It is shown that there is always a Green function of the second kind, with the usual properties, as well as a mixed Green function. Let u be a BLD harmonic function with fine boundary function u'. The characteristic value problem $\partial u/\partial g = \text{const.}\ \sigma u'$ is solved, for σ a positive (but not necessarily strictly positive) bounded function on R^M. The solution involves a complete orthogonal sequence in $\mathbf{H}_h$, corresponding to a sequence of BLD harmonic functions whose fine boundary functions form an orthonormal sequence on R^M (relative to a measure determined by σ and harmonic measure). Series expansions in terms of these orthogonal sequences are found for the kernels involved in the various Green functions studied.

2. Harmonic measure on R^M.

In the following we shall write $\mu(\xi, A)$ for the harmonic measure of a subset A of R^M relative to the point ξ of R. The class of sets on R^M of harmonic measure 0 is independent of the reference point ξ, and we write « almost everywhere on R^M » to mean « except for a set of $\mu(\xi, .)$ measure 0 ». The class of functions on R^M which are measurable and whose absolute values have integrable p-th powers with respect to $\mu(\xi, .)$ on a measurable subset A of R^M is independent of ξ and will be denoted by $\mathbf{L}_p(A)$. The property of mean convergence with specified index of a sequence of functions on R^M relative to $\mu(\xi, .)$ is also independent of ξ, so there is no need to mention the reference point in discussing mean convergence. We choose a point ξ_0 of R once and for all as a reference point, and any otherwise unspecified concepts involving a measure on R^M will always be relative to the measure $\mu(\xi_0, .)$.

Let $\{R_n, n \geq 1\}$ be an increasing sequence of open subsets of R, with union R, containing no infinite points on their boundaries, whose closures are compact subsets of R. In the following such a sequence of sets will be called a « standard nested sequence of subsets of R ». In fact, somewhat more generally, we shall even allow R_n not to have its full closure in R, as long as its relative boundary R_n' has harmonic measure 1 relative to points in it. For example, we can choose R_n as the set of points where the Green function of R, with specified pole, is greater than α_n where α_n is chosen so that R_n has no infinite point on its boundary $\alpha_n < \dfrac{1}{n}$ and α_n is supposed less than the value of the Green fonction at the pole, if the pole is an infinite point. Harmonic measure of subsets of R_n' relative to ξ will be denoted by $\mu_n(\xi, .)$. It is well known that $\mu_n(\xi, .) \to \mu(\xi, .)$ in the weak (vague) sense.

If a function on R has a limit at a point of $R \cap R^M$ on approach to the point in the fine topology of Cartan-Brelot-Naïm, it is said to have a fine limit at the point, denoted by « f lim ». If the function has a fine limit at almost every point of R^M, it will be said to have a fine boundary function on R^M. In general we shall use primes to denote fine boundary functions, so that u' will denote the fine boundary function of u.

It is known (Doob [7], [8]) that a superharmonic function on R, under suitable restrictions, for example if positive, has a finite fine boundary function, in $L_1(R^M)$. We shall use the following related fact. Let $\{R_n, n \geq 1\}$ be a standard nested sequence of subsets of R, and let ξ_0 be a point of R. Let u be a function harmonic on R. If the sequence of restrictions of u to $\{R_n', n \geq 1\}$ is uniformly integrable with respect to the sequence of measures $\{\mu_n(\xi_0, .), n \geq 1\}$, then u has a fine boundary function u' and

$$u(\xi) = \int_{R^M} u' \mu(\xi, d.) \tag{2. 1}$$

for all ξ in R. Conversely if u' is any function in $L_1(R^M)$ and if u is defined by (2. 1), u has the above uniform integrability property and has fine boundary function u'. Moreover u is then the solution to the Dirichlet problem corresponding to the assigned boundary function u' as derived using the Perron-Wiener-Brelot method. A function u defined by (2. 1) will be called a Dirichlet solution for the boundary function u'.

We shall also use the following fact, a slight extension of a well-known one, which we state as a lemma for ease in reference.

LEMMA 2.1. — *Let f be a bounded function on $R \cup R^M$ with the following properties.* (*a*) *The restriction of f to R is a Baire function.* (*b*) *The restriction of f to R^M is measurable.* (*c*) *f has the fine limit $f(\eta)$ at almost every point η of R^M, on approach from R. Then if $\{R_n, n \geq 1\}$ is a standard nested sequence of subsets of* R,

$$\lim_{n \to \infty} \int_{R_n'} f \mu_n(\xi, d\ .) = \int_{R^M} f \mu(\xi, d\ .). \tag{2. 2}$$

If fine limits are interpreted as limits along probability paths, this lemma is proved by an elementary transition to a limit under the sign of integration. In particular, if f is continuous on $R \cup R^M$ this lemma expresses the convergence $\mu_n \to \mu$ already noted. An easy corollary of the lemma, the only case we shall use, is that if u is a harmonic function which is a Dirichlet solution for the fine boundary function u', and if φ is a continuous bounded function from the reals to the reals, then

$$(2.3) \qquad \lim_{n\to\infty} \int_{R'_n} \varphi(u)\mu_n(\xi, d\,.) = \int_{R^M} \varphi(u')\mu(\xi, d\,.).$$

3. Decomposition of u^2.

All potentials, unless specifically described otherwise, will be defined by means of the Green function g of R as kernel. The potentials of positive measures are positive superharmonic functions and are characterized among these functions by the fact that they have fine boundary function 0 (almost everywhere) and enjoy the uniform integrability property described in the preceding section.

Throughout this paper we denote by q either 2π if R has dimension $N = 2$ or the product of $N - 2$ and the unit ball boundary « area » if $N > 2$.

THEOREM 3.1. — *Let u be a function harmonic on R.* (*i*) *Suppose that u is the Dirichlet solution corresponding to the fine boundary function u' and that $u' \in L_2(R^M)$. Then we can write u^2 in the form*

$$(3.1) \qquad u^2 = {}_hu - {}_pu$$

where ${}_hu$ is harmonic, the Dirichlet solution corresponding to the fine boundary function u'^2, and ${}_pu$ is the potential of a positive measure. Moreover the total value $M(\leq \infty)$ of this measure is $2D(u)/q$. (ii) *Conversely if u^2 is dominated by a harmonic function, the hypothesis of* (i) *is satisfied.*

Proof of (*i*) Under the hypotheses of (*i*), let ${}_hu$ be the Dirichlet solution corresponding to u'^2,

$$(3.2) \qquad {}_hu(\xi) = \int_{R^M} u'^2\mu(\xi, d\,.),$$

and define ${}_pu = {}_hu - u^2$. An application of Schwarz's inequality shows that $u^2 \leq {}_hu$. Then ${}_pu$ is a positive superharmonic function with fine boundary function 0. Since ${}_hu$ is a Dirichlet solution, it has the uniform integrability property described in the preceding section, so that the smaller function ${}_pu$ also has the property. Then ${}_pu$ is the potential of a positive measure. The evaluation of M is obvious from the fact that

$$-\Delta_p u = \Delta(u^2) = 2|\text{grad } u|^2.$$

Proof of (ii) Conversely if u^2 is dominated by some harmonic function, ${}_hu$, and if $\{R_n, n \geq 1\}$ is a standard nested sequence of subsets of R, with compact closures, omitting the members of the sequence not containing a preassigned point ξ, then

$$\sup_n \int_{R'_n} u^2 \mu_n(\xi, d.) \leq \sup_n \int_{R'_n} {}_hu\mu_n(\xi, d.) \leqslant {}_hu(\xi) < \infty.$$

Thus u has the uniform integrability property described in Section 2, so u is a Dirichlet solution corresponding to a fine boundary function u'. Since u'^2 is dominated by the fine boundary function of ${}_hu$, u' is in the class $L_2(R^M)$, as was to be proved.

As an application of the decomposition of u^2 in Theorem 3.1 we strengthen a theorem of Brelot. Let u be a BLD function, let ξ_0 be a point of R, and let $u_\alpha(l)$ be the value of u at the point of the Green line l from ξ_0 where the Green function with pole ξ_0 has value α. Then Brelot [3] proved that, if dl refers to the measure of Green lines (see Section 1), u_α has a mean limit $\hat{u}$ of index 1,

$$\lim_{\alpha \to 0} \int |u_\alpha(l) - \hat{u}(l)|\, dl = 0. \tag{3. 3}$$

The following theorem strengthens Brelot's result by increasing the index to 2, under the added hypothesis that u is harmonic.

THEOREM 3.2. — *Let u be a BLD harmonic function on R. Then*

$$\lim_{\alpha \to 0} \int |u_\alpha(l) - \hat{u}(l)|^2\, dl = 0. \tag{3. 4}$$

We use Theorem 3.1. Since ${}_hu$ is a Dirichlet solution, its restriction to the boundary S'_α of the set S_α where $g(\xi_0, \xi) > \alpha$

defines a family uniformly integrable with respect to harmonic measures relative to ξ_0 on the members of $\{S'_\alpha, \alpha > 0\}$ containing no infinite points. Since $u^2 \leq {}_h u$, u^2 has this same uniform integrability property. Now (Brelot-Choquet [4]) Green line measure dl determines a measure on S'_α a Green line corresponding to the point in which it meets S'_α, which is precisely harmonic measure relative to ξ_0. Hence the family of integrands in (3.4) is uniformly integrable. Since the integrand converges to 0 when $\alpha \to 0$, almost everywhere, (3.4) must be true.

4. The fine boundary functions of BLD harmonic functions.

THEOREM 4.1. — *Let u be a BLD harmonic function on R. Then the hypotheses of Theorem* 3.1 (ii) *are satisfied, so that u has a fine boundary function u' in* $L_2(R^M)$, *u is the Dirichlet solution corresponding to u', and* $D(u) = qM/2$.

To prove the theorem we first remark that $-u^2$ is superharmonic, with corresponding measure of total value $2D(u)/q$. The Riesz decomposition of $-u^2$ therefore yields (3.1), where now ${}_p u$ is the potential of a measure of the above total value and ${}_h u$ is a harmonic function. Since $u^2 \leq {}_h u$ the hypothèses of Theorem 3.1 (ii) are satisfied, as was to be proved.

Thus each BLD harmonic function u has both a radial $\hat{u}$ and a fine boundary function u'. (The radial is defined relative to a specified initial point of Green lines). The functions $\hat{u}$ and u' are each defined on a measure space of total value 1. We shall now prove that the distributions of these measurable functions are the same, that is, that if f is a continuous bounded function from the reals to the reals, and if the Green lines start at ξ_0,

$$(4.1) \qquad \int f[\hat{u}(l)]dl = \int_{R^M} f(u')\mu(\xi_0, d\,.).$$

To see this we use the notation of Theorem 3.2. Obviously

$$(4.2) \qquad \lim_{\alpha \to 0} \int f[u_\alpha(l)]dl = \int f[\hat{u}(l)]dl.$$

Moreover, since the measure on S'_α induced by dl measure by way of the transformation from l into the point in which l meets S'_α is harmonic measure relative to ξ_0,

$$(4.\ 3) \qquad \int f[u_\alpha(l)]dl = \int_{S'_\alpha} f(u)\mu_\alpha(d.),$$

where μ_α is harmonic measure on S'_α relative to ξ_0. Finally, as we remarked in discussing Lemma 2.1 the integral on the right in (4. 3) has as limit ($\alpha \to 0$) that on the right in (4. 1). Combining this fact with (4. 2) and (4. 3) we see that (4. 1) is true. Since this equation is true for bounded continuous f it is true for every Baire function f for which either integral exists, and we shall use this fact without further discussion in the next section.

We can obtain a stronger result in exactly the same way. If f is a continuous bounded function from n-space to the reals, and if $u_1, \ldots, u_n$ are BLD harmonic functions, then

$$(4.\ 1') \quad \int f[\hat{u}_1(l), \ldots, \hat{u}_n(l)]dl = \int_{R^M} f(u'_1, \ldots, u'_n)\mu(\xi_0, d.).$$

That is, the joint distribution of $\hat{u}_1, \ldots, \hat{u}_n$ is the same as that of $u'_1, \ldots, u'_n$.

THEOREM 4.2. — *Let u be a BLD harmonic function on R, with radial $\hat{u}$ and fine boundary function u'. Let ξ_0 be a point of R. Then there is a constant c, depending on R and ξ_0 but not on u such that*

$$(4.\ 4) \quad \int |\hat{u}(l) - u(\xi_0)|^2\, dl = \int_{R^M} |u' - u(\xi_0)|^2\, \mu(\xi_0, d.) \leq c\mathrm{D}(u).$$

We can and shall suppose in the following that $u(\xi_0) = 0$. Since $\hat{u}$ and u' have the same distribution, we need only prove the inequality involving u'. Using the notation of (3. 1) this inequality takes the form

$$(4.\ 5) \qquad {}_hu(\xi_0) = {}_pu(\xi_0) \leq c\mathrm{D}(u) = cq\mathrm{M}/2.$$

Unless the theorem is true, there is a sequence $\{u_n, n \geq 1\}$ for which in the obvious notation

$$(4.\ 6) \quad {}_hu_n(\xi_0) = {}_pu_n(\xi_0) = 1, \qquad \mathrm{D}(u_n) = q\mathrm{M}_n/2 \leq q2^{-n+1}.$$

But then $\sum_n {}_p u_n$ is the potential of a measure of total value 1 or less. Hence the series converges and ${}_pu_n \to 0$ except possibly on a set of zero capacity. Since $D(u_n) \to 0$ and $u_n(\xi_0) = 0$, $u_n \to 0$ uniformly on compact subsets of R. Combining these facts we obtain at once that ${}_h u_n \to 0$ except possibly on a set of zero capacity. Since ${}_h u_n$ is positive the Harnack inequality yields the fact that ${}_h u_n \to 0$ uniformly on compact subsets of R, contradicting (4.6). The theorem is therefore true.

The normalization by an additive constant in (4. 4) is not essential. In fact, for example, if A is a measurable subset of R^M, of strictly positive measure, and if we consider the class of BLD harmonic functions u with $u' = 0$ almost everywhere on A, then there is a constant γ for which

$$(4.\ 4') \qquad \int \hat{u}(l)^2\, dl = \int_{R^M} u'^2 \mu(\xi_0, d.) \leq \gamma D(u)$$

for every u in the class. The constant γ depends on A. If there were no such constant, there would be a sequence $\{u_n,\ n \geq 1\}$ of functions in the class such that

$$(4.\ 7) \qquad \int_{R^M} u_n'^2 \mu(\xi_0, d.) = 1, \qquad D(u_n) \to 0.$$

But then $u_n' - u_n(\xi_0) \to 0$ in the mean, according to Theorem 4.2, contradicting the fact that u_n' vanishes almost everywhere on A and that (4. 7) is true.

We observe that (Brelot [3]) if u is an arbitrary BLD function it is the sum of a uniquely determined harmonic BLD function and an orthogonal (in **H**) function of potential type with radial vanishing almost everywhere. It will be shown in Section 6 that the fine boundary function of a function of potential type exists and vanishes almost everywhere on R^M. It is therefore *obvious that* (4. 4) *remains true if u is merely supposed a BLD function, at the price of replacing $u(\xi_0)$ by the value at ξ_0 of the harmonic component of u.* Inequality (4. 4′) is true with no change for general BLD functions vanishing almost everywhere on A. Since the constant a_0 replacing $u(\xi_0)$ in (4. 4) is the average of the radial, as well as the $\mu(\xi_0, .)$ average of the fine boundary function of u, the constant a_0 is the constant minimizing

$$\int [\hat{u}(l) - a]^2\, dl = \int_{R^M} [u' - a]^2\, \mu(\xi_0, .)$$

for all a.

Suppose that u_n and u are BLD functions with radials $\hat{u}_n$, $\hat{u}$ and fine boundary functions u'_n, u' respectively, and that $u_n \to u$ in the BLD sense. Brelot proved that then $\hat{u}_n \to \hat{u}$ in the mean, index 1. We shall improve this result by showing that there is mean convergence of index, 2 for both $\{\hat{u}_n, n \geq 1\}$ and $\{u'_n, n \geq 1\}$. This result, together with our earlier results, illustrate the central role of L_2 spaces in the study of the Dirichlet integral. For analogous results in a somewhat more classical framework, involving domains in Euclidean space with smooth boundaries see Sobolev [1].

THEOREM 4.3. — *Let u_n, u be BLD functions on R, with radials $\hat{u}_n$, $\hat{u}$ and fine boundary functions u'_n, u'. Then if $u_n \to u$ in the BLD sense,*

$$(4.\ 8) \quad \lim_{n \to \infty} \int |\hat{u}_n(l) - \hat{u}(l)|^2 \, dl = \lim_{n \to \infty} \int_{R^M} |u'_n - u'|^2 \, \mu(\xi_0, d.) = 0.$$

Since the nth integrals involved are equal, we need discuss only the u'_n sequence. By our extension of (4. 4) there is a sequence of constants $\{a_n, n \geq 1\}$ such that $u'_n - u' - a_n \to 0$ in the mean of order 2. Since $u_n \to u$ quasi-everywhere on R, $a_n \to 0$, so that we can take $a_n = 0$. Then (4. 8) is true.

An alternative statement of this theorem is that if $D(u_n - u) \to 0$ then

$$(4.\ 8') \quad \lim_{n \to \infty} \int |\hat{u}_n(l) - \hat{u}(l) - u_n(\xi) + u(\xi)|^2 \, dl$$
$$= \lim_{n \to \infty} \int_{R^M} |u'_n - u' - u_n(\xi) + u(\xi)|^2 \, \mu(\xi_0, d.) = 0,$$

for quasi-all ξ, all ξ if the functions involved are harmonic. We omit the easy proof.

This theorem means that the transformation form H_h into $L_2(R^M)$ representing an element of H_h by a harmonic function vanishing at a preassigned point, the image being the fine boundary function of this harmonic function, is continuous.

5. Some results involving capacity.

THEOREM 5.1. — *Let u be a BLD function of potential type on* R. *Then if* $\alpha > 0$,

$$\text{cap}\,\{\xi : |u(\xi)| > \alpha\} \leq D(u)/(\alpha^2 q). \tag{5. 1}$$

Here the capacity is that relative to the whole space R, the Green function capacity. This theorem was proved by Deny and Lions [6] for R-Euclidean space of dimensionality > 2. Their proof is applicable to the more general case stated here, with the help of the decomposition technique used by Brelot [3, pp. 390-391].

We shall need the following rather trivial lemma.

LEMMA 5.2. — *Let* A *be a Borel subset of* R, *of finite capacity. Then* A *has almost no point of* R^M *as fine limit point.*

In fact if u is the equilibrium potential of A, u, like any potential, has fine limit 0 at almost every point of R^M. Since u has the value 1 quasi-everywhere on A, almost no point of R^M can be a fine limit point of A.

6. The fine boundary function of a BLD function.

In this section we shall prove that every BLD function u has a fine boundary function. Since u can be written as the sum of a BLD harmonic function and a BLD function of potential type, and since we have already proved the theorem for u harmonic, all that remains is to deal with functions of potential type.

If R is a bounded domain in N-dimensional Euclidean space, satisfying certain restrictive hypotheses, Deny [5] proved that a BLD function u can be extended to a BLD function v on the whole space. Since v is fine continuous quasi-everywhere, Deny concluded that u has a fine limit quasi-everywhere on the relative boundary of R. Here « fine limit » refers to the fine topology on the entire N-space. One can also conclude that at almost every (harmonic measure) point of R^M u has a fine limit in our sense, in which the fine topology is relative to $R \cup R^M$. [Since this conclusion is only incidental here, the argument will only be sketched. The continuity properties of BLD functions imply that such a function is continuous on almost all Brownian stochastic process paths in its domain, excluding the initial path point if the function is not fine-continuous there. Then φ is continuous on almost every Brownian stochastic process path from a fine-continuity point ξ of R. Hence u has a limit at the relative boundary points of R on almost every Brownian path from ξ to the first point at which the path hits the relative boundary, that is, u has a limit on almost every Brownian path from ξ to R^M (the paths involved are identical). This property is equivalent (Doob [7]) to the property that u have a fine limit at almost every point of R^M.]

Deny and Lions [6] proved that if R is any domain in N-dimensional Euclidean space (we take $N \geq 3$ to simplify the discussion) then a BLD function u on R is of potential type if and only if its extension by 0 is a BLD function on the entire space, also if and only if u has fine limit 0 (fine topology relative to the entire space) at quasi-every point of the relative boundary of R. Here ∞ is a point of the relative boundary if R is unbounded. This theorem implies that u is of potential type if and only if u has fine limit 0 in our sense at almost every point of R^M. [If u is of potential type the argument used in the preceding paragraph shows that u has limit 0 on almost every Brownian path from a point of R to R^M and so fine limit 0 almost everywhere on R^M. Conversely if u has fine limit 0 almost everywhere on R^M write u as the sum $u_1 + u_2$, where u_1 is a BLD harmonic function on R and u_2 is of potential type on R (see Deny-Lions [6] or Brelot [3]). We have already shown that u_2 has fine limit 0 almost everywhere on R^M so u_1 must have the same property. But u_1 is a Dirichlet solution, as such is the harmonic average of its fine boundary function, and hence must vanish identically. Thus $u = u_2$, as was to be proved.] The corresponding result for radials is due to Brelot [3] who proved that a BLD function is of potential type if and only if it has radial 0.

The Deny-Lions results are extended to Green spaces in the following theorem.

Theorem 6. 1. — *If u is a BLD function on R, it has an almost-everywhere finite fine boundary function. Moreover u is of potential type if and only if its fine boundary function vanishes almost everywhere on* R^M.

Suppose first that u is of potential type. Then according to Theorem 5. 1 the set B_n where $u(\xi) \geq 1/n$ has finite capacity. Now a set of finite capacity has almost no fine limit points on R^M, according to Lemma 5. 2. Hence if A is the union of the set of non-minimal points of R^M and of the union over n of the set of fine limit points of B_n on R^M, A has harmonic measure 0 and $R - B_n$ is a deleted fine neighborhood of every point of $R^M - A$ for every value of n. Then u has fine limit 0 at every point of $R^M - A$, and therefore almost everywhere on R^M. If u is a BLD function on R, it is the sum of a uniquely determined BLD harmonic function u_1 and a BLD function u_2 of potential type (see Deny-Lions [6] or Brelot [3]). Theorem 4.1 and the result just obtained combine to show that u has a fine boundary function u'. Moreover u_1 is the Dirichlet solution for the boundary function u'. Then u is of potential type if and only if its fine boundary function vanishes almost everywhere. The proof of the theorem is complete.

7. The θ-potentials of Naïm.

Let R be a domain in N-dimensional Euclidean space, with relative boundary R′ so smooth that the following heuristic unrigorous reasoning looks plausible. We denote by $\mu(\xi, .)$ the harmonic measure relative to ξ in R of subsets of R′ and by g the Green function of R. The following directional derivatives are with respect to the second arguments, when there is any ambiguity in the notation, along inner normals. It is classical that

$$(7.\ 1) \qquad \frac{\partial g}{\partial n}(\xi, \eta)ds_\eta = q\mu(\xi, d\eta), \qquad \xi \in \mathrm{R}, \quad \eta \in \mathrm{R}'.$$

Thus if u is harmonic in a neighborhood of $\mathrm{R} \cup \mathrm{R}'$, with boundary function u',

$$(7.\ 2) \qquad u(\xi) = \frac{1}{q}\int_{\mathrm{R}'} u'(\eta)\frac{\partial g}{\partial n}\, ds_\eta$$

and

(7. 3)

$$D(u) = -\int_{R'} u'(\xi)\frac{\partial u}{\partial n}(\xi)\,ds_\xi = -\frac{1}{q}\int_{R'}\int_{R'} u'(\xi)u'(\eta)\frac{\partial^2 g}{\partial n_\xi\,\partial n_\eta}\,ds_\xi\,ds_\eta.$$

Since the integral of each side in (7. 1) is q, for all ξ, we can differentiate the integral to get 0,

$$\int_{R'}\frac{\partial^2 g}{\partial n_\xi\,\partial n_\eta}\,ds_\eta = 0. \tag{7. 4}$$

Combining (7. 3) with (7. 4) we « derive »

$$D(u) = \frac{1}{2q}\int_{R'}\int_{R'}[u'(\xi) - u'(\eta)]^2\frac{\partial^2 g}{\partial n_\xi\,\partial n_\eta}\,ds_\xi\,ds_\eta. \tag{7. 5}$$

See Osborn [13] for a justification of (7. 5) under suitable hypotheses on u, when $N = 2$ and R is sufficiently regular. The evaluation was given by Douglas [9] for $N = 2$ and R a disc.

Now let R be more generally a Green space, with Martin boundary R^M, and denote harmonic measure on R^M relative to ξ by $\mu(\xi, .)$. Let ξ_0 be a specified point of R, to serve as a reference point, and to be held fast throughout the discussion. Let $R_0 = R - \{\xi_0\}$. Naïm [11] has defined a kernel θ on $(R_0 \cup R^M)^2$ which on the direct product of the boundaries is a generalization of the one appearing in (7. 5)

$$\theta(\xi, \eta)\,\mu(\xi_0, d\xi)\,\mu(\xi_0, d\eta) \sim \frac{1}{q^2}\frac{\partial^2 g}{\partial n_\xi\,\partial n_\eta}\,ds_\xi\,ds_\eta. \tag{7. 6}$$

Let u' be a function on R^M measurable with respect to harmonic measure, and define $D'(u')$ by

(7. 7)

$$D'(u') = \frac{q}{2}\int_{R^M}\int_{R^M}[u'(\xi) - u'(\eta)]^2\,\theta(\xi, \eta)\,\mu(\xi_0, d\xi)\,\mu(\xi_0, d\eta).$$

n Section 9 it will be shown, as suggested by (7. 5), that if u is a BLD harmonic function on R, with fine boundary function u', then $D(u) = D'(u')$.

We now recall a few facts about θ and related matters (see Naïm [11]). Naïm defines θ on R_0^2 by

$$\theta(\xi, \eta) = \frac{g(\xi, \eta)}{g(\xi_0, \xi)\,g(\xi_0, \eta)}, \qquad \xi, \eta \in R_0. \tag{7. 8}$$

The domain of definition is then extended to $R_0 \times (R_0 \cup R^M)$ on which θ is continuous, and

$$(7.9)\quad \frac{\mu(\xi, d\eta)}{g(\xi_0, \xi)} = \theta(\xi, \eta)\, \mu(\xi_0, d\eta), \qquad \xi \in R_0, \qquad \eta \in R_0 \cup R^M.$$

The domain of θ is then enlarged to $(R_0 \cup R^M)^2$ in such a way that θ is lower semicontinuous as well as continuous in the fine topology in each variable. This function can be used to define « θ-potentials » of Radon measures on $R_0 \cup R^M$. A θ-potential is fine-continuous if the measure assigned to the class of nonminimal boundary points is 0. It is convenient to define the function K by

$$K(\eta, \xi) = \theta(\xi, \eta)\, g(\xi_0, \xi), \qquad \xi \in R_0, \qquad \eta \in R^M$$

and $K(\eta, \xi_0) = 1$. The function K is continuous.

8. Fine normal derivatives on R^M.

Let u be a function on $R \cup R^M$, and let η be a point of R^M. Suppose that the fine limit

$$f \lim_{\xi \to \eta} \frac{u(\xi) - u(\eta)}{g(\xi_0, \xi)}$$

exists. The point ξ_0 is a fixed reference point in R. Then u will be said to have this fine limit as « fine normal derivative » at η. We use the notation $\partial u/\partial g$ for this derivative, the analogue of the classical inner normal derivative. Its value, but not its existence or finiteness or vanishing, depends on the choice of reference point.

If u is any strictly positive superharmonic function on R, and if η is a minimal point of R^M, Naïm [11] has proved that $u/g(\xi_0, .)$ has a strictly positive not necessarily finite fine limit at η. Thus u has a fine normal derivative at that point, if u is defined as 0 at η.

Let u be the potential of a measure μ, with $\mu(\{\xi_0\}) = 0$. Then u has fine boundary function vanishing almost everywhere, and we define u on R^M as 0 in discussing the fine normal derivative. We can write u in the form

$$(8.1)\quad u(\xi) = g(\xi_0, \xi) \int_R \theta(\xi, \eta)\, \mu_1(d\eta), \quad \mu_1(d\eta) = g(\xi_0, \eta)\mu(d\eta).$$

The function $u/g(\xi_0, .)$ is a θ-potential, and Naïm's continuity theorem for these potentials implies that, if ξ is a minimal point of R^{M}, at which we define u to be 0 in computing the normal derivative at the boundary,

$$\frac{\partial u}{\partial g}(\xi) = \int_{\mathrm{R}} \theta(\xi, \eta)\mu_1(d\eta) \leq \infty. \tag{8. 2}$$

THEOREM 8. 1. — *Let v be a Dirichlet solution in* R, *with fine boundary function v', and let u be the potential of a positive measure μ. Then if $\partial u/\partial g$ is the fine normal derivative of u on* R^{M},

$$\int_{\mathrm{R}} v(\eta)\mu(d\eta) = \int_{\mathrm{R}^{\mathrm{M}}} \frac{\partial u}{\partial g} v'\mu(\xi_0, d.) \tag{8. 3}$$

whenever either integral is well-defined (in which case the other is also). In particular

$$\mu(\mathrm{R}) = \int_{\mathrm{R}^{\mathrm{M}}} \frac{\partial u}{\partial g}\mu(\xi_0, d.)\ (\leq \infty). \tag{8. 3'}$$

If one writes Green's first formula in the present context one obtains formally

$$\mathrm{D}(u, v) = -q\int_{\mathrm{R}^{\mathrm{M}}} \frac{\partial u}{\partial g} v'\mu(\xi_0, d.) + q\int_{\mathrm{R}} v\mu(d.). \tag{8. 4}$$

Since u is a potential and v is harmonic, $\mathrm{D}(u, v) = 0$ if u and v are both BLD functions. Thus in that case (8. 4) reduces to (8. 3). In other words this theorem extends Green's first formula to the present context. According to (8. 3'), $\partial u/\partial g$ must be finite almost everywhere on R^{M} if $\mu(\mathrm{R}) < \infty$.

Equation (8. 3) to be derived is trivially true if μ has support $\{\xi_0\}$. In view of the linearity here we can and shall suppose from now on that $\mu(\xi_0) = 0$. If $v \geq 0$ we find, using (8. 2) and reversing the order of an integration,

$$\begin{aligned}\int_{\mathrm{R}^{\mathrm{M}}} v'(\xi)\frac{\partial u}{\partial g}(\xi)\,\mu(\xi_0, d\xi) \\ = \int_{\mathrm{R}^{\mathrm{M}}} v'(\xi)\,\mu(\xi_0, d\xi)\int_{\mathrm{R}}\theta(\xi, \eta)\,\mu_1(d\eta) = \int_{\mathrm{R}} v(\eta)\,\mu(d\eta).\end{aligned} \tag{8. 5}$$

If v is not positive, we can write v' as the difference between two positive functions on R^M, $\varphi' = \varphi'_1 + \varphi'_2$, where φ'_i has Dirichlet solution φ_i, and the result already obtained for positive φ can be applied to φ_1 and φ_2.

9. The evaluation of $D(u)$ in terms of u'.

We shall now put (7. 5) into a precise form, using the following lemma.

LEMMA 9.1. — *Let u be a BLD harmonic function on* R, *with fine boundary function u'. Let ${}_pu$ be the potential component of u^2 ((see (3. 1)) and let ξ be a minimal point of* R^M *at which u' is defined. Then*

$$(9.\ 1)\qquad f\lim_{\zeta\to\xi}\frac{{}_pu(\zeta)}{g(\xi_0,\zeta)} = \int_{R^M}[u'(\eta)-u'(\xi)]^2\,\theta(\xi,\eta)\,\mu(\xi_0,d\eta).$$

If this limit is finite,

$$(9.\ 2)\qquad f\lim_{\zeta\to\xi}\frac{[u'(\xi)-u(\zeta)]^2}{g(\xi_0,\zeta)} = 0.$$

Note that if ξ is in the set of almost all boundary points at which ${}_pu$ has fine limit 0, the left side of (9. 1) can properly be interpreted as the fine normal derivative of ${}_pu$ at ξ. Define the function φ by

$$(9.\ 3)\qquad \varphi(\xi_1,\xi_2) = \int_{R^M}[u'(\eta)-u(\xi_1)]^2\,\theta(\xi_2,\eta)\,\mu(\xi_0,d\eta)$$
$$= \int_{R^M}[u'(\eta)-u(\xi_1)]^2\,\mu(\xi_2,d\eta)/g(\xi_2,\xi_0).$$

The right side of (9. 1) is then $\varphi(\xi,\xi)$. Applying elementary manipulations,

$$(9.\ 4)\qquad \frac{{}_pu(\zeta)}{g(\xi_0,\zeta)} = \int_{R^M}\frac{u'^2(\eta)-u^2(\zeta)}{g(\xi_0,\zeta)}\,\mu(\zeta,d\eta) = \varphi(\zeta,\zeta)$$
$$= \varphi(\xi,\zeta) - \frac{[u'(\xi)-u(\zeta)]^2}{g(\xi_0,\zeta)},\qquad \text{if}\quad \zeta\in R.$$

According to the fine limit theorem of Naïm quoted in Section 8, the fine limit of the left side of (9. 4) exists. Hence, applying Fatou's lemma,

$$(9.\ 5)\qquad f\lim_{\zeta\to\xi}\varphi(\zeta,\zeta) \geq \varphi(\xi,\xi).$$

For fixed ξ, $\varphi(\xi, .)$ is a θ-potential and as such is fine continuous, so that

$$(9.\ 6) \qquad f\lim_{\zeta \to \xi} \varphi(\xi, \zeta) = \varphi(\xi, \xi).$$

Applying (9. 5) and (9. 6) to (9. 4) we find that (9. 1) is true, and that (9. 2) is true if the left side of (9. 1) is finite.

We recall that the functional D' in the following theorem was defined in (7. 7).

THEOREM 9. 2. — *Let u be a harmonic function on the Green space R. Then if $D(u) < \infty$, u has a fine boundary function u' and $D(u) = D'(u')$. Conversely if u' is an arbitrary measurable function on R^M with $D'(u') < \infty$, then*

$$u' \in L_2(R^M)$$

and, if u is the Dirichlet solution for the boundary function u', $D(u) = D'(u')$.

If u is a BLD harmonic function on R and if we decompose u^2 as in (3. 1), $D(u) = qM/2$, according to Theorem 3. 1, where M is the total mass corresponding to the potential ${}_pu$. According to Theorem 8. 1 and Lemma 9. 1 this total mass is the integral of the right side of (9. 1) with respect to the measure $\mu(\xi_0, .)$, so that $D(u) = D'(u')$.

Suppose conversely that u' is a measurable function on R^M for which D' is finite. Then $\varphi(\xi, \xi)$ is finite for almost every ξ on R^M. Choose some ξ at which this function value is finite. Then $\varphi(\xi, .)$, a θ-potential which has $\varphi(\xi, \xi)$ as limit at ξ in the fine topology, is finite at some point $\xi_1 \neq \xi_0$. Hence $[u' - u(\xi_1)] \in L_2(R^M)$ so $u' \in L_2(R^M)$ also. Let u be the Dirichlet solution for u'. To prove the theorem we prove that $D(u) < \infty$. We decompose u^2 as in Theorem 3. 1. All we need show is that the total mass M for ${}_pu$ is finite. Just as above (9. 4) is true, so that

$$(9.\ 7) \qquad f\lim_{\zeta \to \xi} \frac{{}_pu(\zeta)}{g(\xi_0, \zeta)} \leq f\lim_{\zeta \to \xi} \varphi(\xi, \zeta) = \varphi(\xi, \xi).$$

Applying Theorem 8. 1 we find that

$$(9.\ 8) \quad D(u) = qM/2 \leq \frac{q}{2} \int_{R^M} \varphi(\xi, \xi)\mu(\xi_0, d\xi) = D'(u') < \infty,$$

as was to be proved.

From now on we shall call a function on R^M which is measurable and makes D' finite a « BLD boundary function ». Then a function is a BLD boundary function if and only if it coincides almost everywhere with the fine boundary function of a BLD harmonic function. If two BLD boundary functions are considered identical when their difference coincides almost everywhere with a constant function and if the bilinear functional $D'(u', v')$ (defined in the obvious way from our definition of D' in (7. 7)) is accepted as the inner product function, the space obtained is a Hilbert space $\mathbf{H}'_h$. The transformation from a BLD harmonic function to its fine boundary function defines a unitary operator from $\mathbf{H}_h$ onto $\mathbf{H}'_h$.

10. Linear functionals on $L_2(A)$.

Let ξ_0 be a fixed reference point of R, as usual, and let A be a measurable subset of R^M. Let $L^*(A)$ be the class of functions φ on A satisfying the following two conditions:

(i) $\varphi \in \mathbf{L}_1(A)$ and, if $\mu(\xi_0, A) = 1$, φ is orthogonal to 1, $\int_A \varphi\mu(\xi_0, d.) = 0$;

(ii) The integral in the following inequality is defined and the inequality is satisfied for some constant c_φ and every BLD harmonic function v with $v' = 0$ almost everywhere on $R^M - A$.

$$(10.\ 1) \qquad \left|\int_A \varphi v' \mu(\xi_0, d.)\right|^2 \leq c_\varphi D(v).$$

If $\mu(\xi_0, A) < 1$, and if $\varphi \in \mathbf{L}_2(A)$, φ can be extended to R^M in such a way that the extended function is in $\mathbf{L}_2(R^M)$ and is orthogonal to 1. Then, using Theorem 4. 2, if v' vanishes almost everywhere on $R^M - A$, the left side of (10. 1) can be written in the form

$$(10.\ 2) \quad \left|\int_{R^M} \varphi[v' - v(\xi_0)]\, \mu(\xi_0, d.)\right|^2 \leq c \int_{R^M} |\varphi|^2 \mu(\xi_0, d.) D(v),$$

so that $\mathbf{L}^*(A) \supset \mathbf{L}_2(A)$. If $\mu(\xi_0, A) = 1$ the obvious contraction of this reasoning shows that $\mathbf{L}^*(R^M)$ includes all functions in $\mathbf{L}_2(R^M)$ orthogonal to 1.

11. The generalized normal derivative on R^M.

The class of BLD harmonic functions, and hence the class of BLD boundary functions, may be small. For example, if R is finite N-dimensional Euclidean space, with $N > 2$, the only BLD harmonic functions are the constant functions. This is a degenerate case in which R^M contains only one point. Ahlfors and Royden [1] have given an example of a Riemann surface with a Green function but supporting no non-constant BLD harmonic functions even though harmonic measure on the boundary is not atomic. Because of such examples, we make the following definition. Let B be a measurable subset of R^M. A function φ in $L^*(B)$ which is orthogonal to every BLD boundary function φ' vanishing almost everywhere on $R^M - B$, in the sense that

$$(11.\ 1) \qquad \int_B \varphi\varphi'\mu(\xi_0, d.) = 0,$$

will be called « negligible on B ». The class of these functions is an invariant of the space. Thus the fact that this class consists of the almost everywhere on B vanishing functions when R is a plane disc implies that the class consists of the almost everywhere on B vanishing functions when R is any simply connected hyperbolic Riemann surface.

Let u_1 be a function on R, and let B be a measurable subset of R^M. Let u be a function on R with the following properties.

(*a*) u is the sum of a BLD harmonic function u_h and the potential u_p of a not necessarily positive measure μ.

(*b*) If φ is a BLD harmonic function and if φ' vanishes almost everywhere on $A = R^M - B$, then φ is integrable (finite-valued integral) on R relative to the absolute variation of μ.

(*c*) The restrictions of u and u_1 to R less some compact subset of R are identical.

We now make the following definition of an analogue of the classical inner normal derivative at the boundary, suggested by Green's first formula (8. 4). We shall say that « u_1 has a generalized normal derivative » φ on B if $\int_B \varphi\varphi'\mu(\xi_0, d.)$

is well-defined and finite for every BLD harmonic function ν whose fine boundary function vanishes almost everywhere on A, and if there is a function u with properties (a), (b), (c) above for which

$$(11.\ 2) \quad D(u_h, \nu) = -q\int_{R^M}\varphi\nu'\mu(\xi_0, d.) + q\int_R \nu\mu(d.),$$

for every ν as just described. Note that if μ has finite energy, so that u is a BLD function, the left side of (11. 2) is equal to $D(u, \nu)$.

We denote the generalized normal derivative on R^M, as well as the fine normal derivative there, by ∂—$/\partial g$ leaving it to subsidiary notation or explicit statement to distinguish between the two.

It is clear that if u_i has generalized normal derivatrice φ_i on B then $c_1u_1 + c_2u_2$ has generalized normal derivative $c_1\varphi_1 + c_2\varphi_2$ on B. If u_1 has generalized normal derivative φ on B and if B_0 is a measurable subset of B, then u_1 has a generalized normal derivative on B_0, the restriction of φ to B_0.

The function φ on B is generalized normal derivative of the function 0 if and only if φ is negligible on B. Then if u_1 has generalized normal derivative φ on B, the class of all generalized normal derivatives of u_1 on B is the class of all functions on B differing from φ by negligible functions on B. In the classical applications, the negligible functions on B are those vanishing almost everywhere on B, and functions then have essentially unique generalized normal derivatives.

Using Theorem 8.1, (11. 2) can be written in the form

$$(11.\ 2') \quad D(u_h, \nu) = -q\int_B\left[\varphi - \left(\frac{\partial u_p}{\partial g}\right)_f\right]\nu'\mu(\xi_0, d.),$$

where $(\partial u_p/\partial g)_f$ is the fine normal derivative on R^M of u_p. Note that if $\hat{u}$ is an arbitrary function satisfying the conditions (a), (b), (c) imposed on u, then, in the obvious notation, $u_h = \hat{u}_h$. Moreover the potentials u_p, $\hat{u}_p$ are identical outside some compact subset of R, and these potentials therefore have the same fine normal derivatives on R^M. Thus the condition (11. 2') is satisfied, if at all, independently of the choice of u. In particular, if u_1 itself has properties (a) and (b), (11. 2) and (11. 2') are satisfied with u replaced by u_1.

Applying Schwarz's inequality to the left side of (11. 2') we see that $\varphi - (\partial u_p/\partial g)_f$ (on B) is in the class $L^*(B)$.

If u is a BLD harmonic function, it has a generalized normal derivative φ on B if and only if $\varphi \in L^*(B)$ and

$$(11.\ 3) \qquad D(u, v) = -q\int_{R^M} \varphi v' \mu(\xi_0, d.)$$

for every v with the properties listed in (b) above.

If u is the potential of a measure μ satisfying the conditions listed in (b) above, u has the restriction of its fine normal derivative to B as generalized normal derivative on B. For example, every potential of a (not necessarily positive) measure with compact support has a generalized normal derivative on every set B, the restriction to B of its fine normal derivative. In particular, $g(\xi, .)$ has generalized normal derivative $K(., \xi)$ on R^M.

Finally, if u is the sum of a BLD harmonic function u_h and of the potential u_p of a measure μ with the properties listed in (b), then u has generalized normal derivative φ on B if and only if u_h has generalized normal derivative $\varphi - (\partial u_p/\partial g)_f$ on B.

THEOREM 11.1. — *Let* B *be a measurable subset of* R^M, $0 < \mu(\xi_0, B) \leq 1$. *Let f be the restriction to* $A = R^M - B$ *of a BLD boundary function, and let φ be a function in the class* $L^*(B)$. *Then there is a BLD harmonic function, unique if* $\mu(\xi_0, B) < 1$, *unique up to an additive constant if* $\mu(\xi_0, B) = 1$, *with fine boundary function coinciding almost everywhere on* A *with f and with generalized normal derivative φ on* B.

Let $\mathfrak{M}$ be the class of BLD harmonic functions whose fine boundary functions coincide with f almost everywhere on A. Then $\mathfrak{M}$ corresponds to an affine manifold $\mathfrak{M}^0$ in **H**, linear if $f = 0$. Moreover $\mathfrak{M}^0$ is closed, by Theorem 4.3. Let $\mathfrak{N}^0$ be the closed linear manifold of differences between pairs of elements of $\mathfrak{M}^0$, corresponding to the class $\mathfrak{N}$ of BLD harmonic functions whose fine boundary functions vanish almost everywhere on A. The functional of v defined by

$$-q\int_B \varphi v' \mu(\xi_0, d.) - D(u_0, v), \qquad v \in \mathfrak{N},$$

where u_0 is a fixed BLD harmonic function in $\mathfrak{M}$, defines a bounded linear functional on $\mathfrak{N}^0$. It follows that there is a unique element in $\mathfrak{N}^0$ corresponding to a function ν_0 in $\mathfrak{N}$ such that

$$(11.4)\quad D(\nu_0, \nu) = -q\int_B \varphi\nu'\mu(\xi_0, d.) - D(u_0, \nu), \qquad \text{if} \qquad \nu \in \mathfrak{N}.$$

That is, there is a function $u = u_0 + \nu_0$ in $\mathfrak{M}$, unique or unique up to an additive constant, as stated in the theorem, such that

$$(11.5)\quad D(u, \nu) = -q\int_B \varphi\nu'\mu(\xi_0, d.), \qquad \text{if} \qquad \nu \in \mathfrak{N}.$$

We shall indicate an alternative proof of this theorem in Section 13.

12. Weak convergence.

Let $\{u_n, n \geq 1\}$ be a sequence of BLD harmonic functions on R. Suppose that the fine boundary function sequence $\{u'_n, n \geq 1\}$ converges weakly in $L_2(R^M)$ to some function u'_∞. Such convergence is independent of the reference point for harmonic measure. We now show that the function u'_∞ is the fine boundary function of a harmonic function u_∞ and that $u_n \to u_\infty$ uniformly on compact subsets of R. Choosing some reference point ξ_0, our hypothesis is that

$$(12.1)\quad \lim_{n\to\infty}\int_{R^M} u'_n\nu'\mu(\xi_0, d.) = \int_{R^M} u'_\infty\nu'\mu(\xi_0, d.), \qquad \text{if} \qquad \nu' \in L_2(R^M).$$

Since $u_\infty \in L_2(R^M)$, u_∞ is the fine boundary function of its Dirichlet solution u_∞. If we choose $\nu' = K(., \xi)$, the limit equation (12.1) states that $u_n(\xi) \to u_\infty(\xi)$. Since the L_2 norm of u'_n is bounded independently of n, and since K is uniformly continuous on any direct product of R^M and of a compact subset of R, there is uniform convergence of the u_n sequence on compact subsets of R.

Now suppose that the sequence $\{u_n, n \geq 1\}$ of BLD harmonic functions converges weakly as a sequence of elements of $\mathbf{H}$ to some u_∞. Then the sequence of L_2 norms of the boundary functions is bounded, according to Theorem 4. 2 if the sequence is normalized by the proper additive constants. From now on we suppose that this has been done, supposing that every u_n and also u_∞ vanishes at our reference point ξ_0, and we show that $u'_n \to u'_\infty$ weakly in $\mathbf{L}_2(\mathrm{R}^{\mathrm{M}})$. Now if φ is a BLD harmonic function with generalized normal derivative on R^{M}, this derivative in $\mathbf{L}_2(\mathrm{R}^{\mathrm{M}})$, then $\mathrm{D}(u_n, \varphi) \to \mathrm{D}(u_\infty, \varphi)$, so that

$$(12.\ 2) \quad \lim_{n\to\infty} \int_{\mathrm{R}^{\mathrm{M}}} u'_n \frac{\partial \varphi}{\partial g} \mu(\xi_0, d.) = \int_{\mathrm{R}^{\mathrm{M}}} u'_\infty \frac{\partial \varphi}{\partial g} \mu(\xi_0, d.).$$

Since the generalized normal derivative can be any function in $\mathbf{L}_2(\mathrm{R}^{\mathrm{M}})$ orthogonal to 1, and since the limit equation (12. 2) is trivially true when the generalized normal derivative is replaced by 1, we have proved that $u'_n \to u'_\infty$ weakly.

In terms of fine boundary functions, $u_n \to u_\infty$ weakly in $\mathbf{H}$ if and only if $\mathrm{D}'(u'_n, \varphi') \to \mathrm{D}'(u'_\infty, \varphi')$ for every fine boundary function φ' of a BLD harmonic function, but this condition is not easy to use.

In many applications, that is, for many spaces R, whenever $u_n \to u_\infty$ weakly in $\mathbf{H}$, $u'_n \to u'_\infty$ strongly in $\mathbf{L}_2(\mathrm{R}^{\mathrm{M}})$, if suitable additive constants are adjoined. In other words in many applications the continuous transformation from $\mathbf{H}$ into $\mathbf{L}_2(\mathrm{R}^{\mathrm{M}})$ defined at the end of Section 4 takes the closed unit ball into a compact set. When this is true we shall say that « R satisfies the complete continuity condition ». This condition will be imposed only in Sections 18-19.

Theorem 12. 1. — *If $g(\xi_0, .)$ has limit 0 at every point of R^{M}, that is if $\{\xi : g(\xi_0, \xi) \geq \varepsilon\}$ is a compact subset of* R *for every $\varepsilon > 0$, then* R *satisfies the complete continuity condition.*

We observe that the hypothesis is independent of the reference point ξ_0, and that it is an intrinsic condition on a Green space. For example it is a conformally invariant property, so that every simply connected hyperbolic Riemann surface enjoys it. If R is a domain in Euclidean N-space, the hypothesis is satisfied if and only if every relative boundary point is regular for the Dirichlet problem.

To prove the theorem suppose that there is a sequence $\{u_n, n \geq 1\}$ of BLD harmonic functions vanishing at ξ_0, the sequence being weakly convergent in H to the function 0. Then we have proved that $u'_n \to 0$ weakly in $L_2(R^M)$ so that $u_n \to 0$ pointwise, uniformly on compact subsets of R. We write u_n as in (3. 1),

$$(12.\ 3) \qquad u_n^2 = {}_h u_n - {}_p u_n, \qquad D(u_n) = qM_n/2,$$

where we recall that M_n is the mass associated with the potential ${}_p u_n$. Since the sequence of L_1 norms of the ${}_h u'_n$ sequence is bounded, the ${}_h u_n$ sequence is locally uniformly bounded on R and therefore (going to a subsequence if necessary) we can suppose that this sequence of positive harmonic functions converges uniformly ou compact subsets of R to a function ${}_h u$. Then ${}_p u_n \to {}_h u$ also. There is strong convergence in $L_2(R^M)$ of the u'_n sequence if and only if ${}_h u = 0$. Suppose that ${}_p u_n$ is the potential of the measure μ_n. Since $\Delta_p u_n \to 0$ uniformly on compact subsets of R, the μ_n measure recedes to R^M. Our hypothesis implies that $g(\xi, .)$, extended by 0 to R^M, is continuous on $R \cup R^M$, aside from the infinity at ξ, if ξ is a finite point. Hence ${}_p u_n \to 0 = {}_h u$ as was to be proved.

13. A mixed boundary value problem.

If u is a BLD harmonic function with fine boundary function 0 almost everywhere on A and with generalized normal derivative $\partial u/\partial g$ on $B = R^M - A$, then the classical formula expressing $D(u)$ in terms of its boundary function and boundary normal derivative becomes in our context

$$(13.\ 1) \qquad D(u) = -q \int_B u' \frac{\partial u}{\partial g} \mu(\xi_0, d.).$$

This evaluation, simply an application of the definition of the generalized normal derivative, implies the uniqueness assertion of Theorem 11.1. We now take up again the problem treated in Theorem 11.1, using the notation of the proof of that theorem.

If u is a BLD harmonic function with generalized normal derivative φ on B, and *if* v is a BLD harmonic function whose fine boundary function, as well as that of u, coincides with f almost everywhere on A, then from (13. 1)

(13. 2)
$$D(v) + 2q\int_B \varphi v'\mu(\xi_0, d.) = D(u) + 2q\int_B \varphi u'\mu(\xi_0, d.) + D(v - u).$$

Hence u is the element of $\mathfrak{M}$, unique if $\mu(\xi_0, B) < 1$, unique up to an additive constant if $\mu(\xi_0, B) = 1$, minimizing the left side of (13. 2). The class $\mathfrak{M}^0$ is a closed affine manifold in **H** and the integral on the left in (13. 2) defines a bounded functional on $\mathfrak{M}^0$ so the existence of a minimum of the left side of (13. 2) also follows from the classical Beppo-Levi reasoning in accordance with which a minimizing sequence converges in **H** to the solution. We observe that the left side of (13. 2) is only increased if we add a function of potential type to v, so that u minimizes the left side for all BLD functions with fine boundary functions equal to f almost everywhere on A, determining in **H** a closed affine superspace of $\mathfrak{M}^0$.

If $f = 0$ (in particular if B has harmonic measure 1) the problem becomes homogeneous. In that case

$$(13.\ 3)\qquad D(v) + 2q\int_B v'\varphi\mu(\xi_0, d.) \geq -D(u) = q\int_B u'\varphi\mu(\xi_0, d.)$$

for every BLD harmonic function v in $\mathfrak{M} = \mathfrak{N}$ and, multiplying v by a suitable constant, this inequality becomes

$$(13.4)\qquad \frac{q^2\left[\int_B v'\varphi\mu(\xi_0, d.)\right]^2}{D(v)} \leq D(u).$$

Thus $D(u)$ is the maximum of the ratio on the left for all v in $\mathfrak{M}$. Then u is the unique up to a multiplicative constant BLD harmonic function, vanishing at ξ_0 if $\mu(\xi_0, B) = 1$, with fine boundary function equal to 0 almost everywhere on A, maximizing the left side of (13. 4).

As an application of the minimal property embodied in (13. 2) we prove the following intuitively obvious theorem. By « ess sup » we mean the supremum neglecting sets of measure 0. We always write « strictly » positive or negative if zero is to be excluded.

THEOREM 13. 1. — *Let u be a subharmonic function on* R, $u = u_h + u_p$, *where* u_h *is a BLD harmonic function and* u_p *is the potential of a negative measure. Let* B *be a measurable subset of* R^M, *and suppose that* $\mu(\xi_0, B) < 1$. *Then if* u *has a generalized normal derivative on* B, *which is positive, it follows that* $u \leq \operatorname{ess}\sup_{\xi \in A} u'(\xi)$, *where* $A = R^M - B$.

If B has harmonic measure 0, so that in effect $A = R^M$, u is the Dirichlet solution for its fine boundary function and as such is bounded from above by the essential supremum of u' on A. If B has strictly positive harmonic measure, let δ be the essential supremum in the theorem. It is sufficient to prove the theorem for u harmonic, $u = u_h$. In fact if the theorem is known to be true for u harmonic, the theorem is applicable to the component u_h, because δ is the essential supremum of u_h on A and u_h has a positive generalized normal derivative on B. Then $u \leq u_h \leq \delta$ as was to be proved. Thus in the following we can and shall assume that $u = u_h$, $u_p = 0$. Let $u'_1 = \min [u', \delta]$, and let u_1 be the Dirichlet solution for the boundary function u'_1. Then $u_1 \leq \delta$ and the restrictions of u' and u'_1 to A are identical. Hence according to the discussion centering around (13. 2),

(13. 5)
$$D(u_1) + 2q \int_B \frac{\partial u}{\partial g} u'_1 \mu(\xi_0, d.) \geq D(u) + 2q \int_B \frac{\partial u}{\partial g} u' \mu(\xi_0, d.)$$

and there is equality only when $u_1 = u$. Since (trivially) $D'(u'_1) \leq D'(u')$ and since the integral on the left must be at most equal to the one on the right, there must be equality in (13. 5) and we conclude that $u = u_1 \leq \delta$, as was to be proved.

We shall use the following remarks in later sections. Suppose that B (of strictly positive harmonic measure) and f are specified, and that u_i is the BLD harmonic function with fine boundary equal to f almost everywhere on A, with generalized normal derivative φ_i in $\mathbf{L}_2(B)$ on B. If A has harmonic measure 0 we take $u_i(\xi_0) = 0$ to ensure uniqueness. Then

$$(13.\ 6) \quad D(u_1 - u_2) = -q \int_B (\varphi_1 - \varphi_2)(u'_1 - u'_2)\mu(\xi_0, d.).$$

Hence if $|| \ ||_B$ denotes the $L_2(B)$ norm, and if either Theorem 4.2 or the discussion following the proof is applied, depending on the harmonic measure of B,

$$(13.7)\quad (1/\gamma^2)||u_1' - u_2'||_B^2 \leq D(u_1 - u_2) \leq q||\varphi_1 - \varphi_2||_B||u_1' - u_2'||_B \leq q\gamma||\varphi_1 - \varphi_2||_B D(u_1 - u_2)^{1/2},$$

where γ is a constant independent of the choice of u_1 and u_2. It follows that

$$(13.8)\quad ||u_1' - u_2'||_B^2 \leq \gamma^2 D(u_1 - u_2) \leq q^2\gamma^2||\varphi_1 - \varphi_2||_B^2.$$

These inequalities imply that the transformations from φ in $L_2(B)$ into u and u' are continuous.

14. Application to the geometry of various function classes.

A measurable subset A of R^M is supposed chosen, and various concepts will be defined relative to A. The point ξ_0 is a fixed reference point and we suppose that $0 \leq \mu(\xi_0, A) < 1$. Let f be a measurable function on A and let $\mathfrak{M}(f, A)$ be the class of BLD harmonic functions with fine boundary functions equal to f almost everywhere on A. Then $\mathfrak{M}(f, A)$ is an affine possibly empty manifold which determines a closed affine submanifold of **H** (Theorem 4.3); $\mathfrak{M}(f, A)$ is the class of all BLD harmonic functions if A has harmonic measure 0. Let $\mathfrak{N}(f, A)$ be the subclass of $\mathfrak{M}(f, A)$ whose functions have generalized normal derivatives on $B = R^M - A$, in $L_2(B)$. Then $\mathfrak{N}(f, A)$ is affine. Let $\mathfrak{M}'(f, A)$, $\mathfrak{N}'(f, A)$ be the classes of restrictions to B of the fine boundary functions of the members of $\mathfrak{M}(f, A)$, $\mathfrak{N}(f, A)$ respectively. The following theorem is phrased for $f = 0$. Corresponding results in the general case can be obtained by differencing.

THEOREM 14.1. — (*a*) (**H** *topology*) $\mathfrak{N}(0, A)$ *is a dense subset of* $\mathfrak{M}(0, A)$; (*b*) *The class of negligible functions on* B, *in* $L_2(B)$, *is the orthogonal complement of* $\mathfrak{N}'(0, A)$ *in* $L_2(B)$. *If* $\mathfrak{N}'(0, A)$ *is dense* ($L_2(B)$ *topology*) *in* $L_2(B)$, *a function on* B *is negligible on* B *if and only if it vanishes almost everywhere on* B.

To prove (*a*), let u be a BLD harmonic function in $\mathfrak{M}(0, A)$, orthogonal to $\mathfrak{N}(0, A)$. We shall prove that u must be a constant function, corresponding to the zero function in **H**. By definition of generalized normal derivative, if φ is in $\mathfrak{N}(0, A)$

$$(14.1) \qquad 0 = D(u, \varphi) = -q \int_B u' \frac{\partial \varphi}{\partial g} \mu(\xi_0, d.).$$

If A has strictly positive harmonic measure, φ can be chosen to make $\partial\varphi/\partial g$ equal to u' almost everywhere on B. Then u' must vanish almost everywhere on R^M, so $u = 0$. If A has zero harmonic measure, φ can be chosen to make $\partial\varphi/\partial g$ any function in $\mathbf{L}_2(R^M)$ orthogonal to 1. Hence u' must be a constant almost everywhere, so u is a constant function, as was to be proved. To prove (b), let ϕ be a function in $\mathbf{L}_2(B)$ orthogonal to $\mathfrak{N}'(0, A)$, that is

$$(14.2) \qquad \int_B \phi\varphi'\mu(\xi_0, d.) = 0 \qquad \text{if} \qquad \varphi \in \mathfrak{N}(0, A).$$

This is precisely the condition that ϕ be negligible. Conversely if ϕ is in $\mathbf{L}_2(B)$ and is negligible on B (14.2) is satisfied, so ϕ is orthogonal to $\mathfrak{N}'(0, A)$. Finally, if $\mathfrak{N}'(0, A)$ is dense in $\mathbf{L}_2(B)$, that is if there is no not almost everywhere 0 negligible function on B in $\mathbf{L}_2(B)$, let ϕ be any negligible function on B. Then (14.2) is true, that is this equation is true for functions φ' whose restrictions to B constitute a certain linear class dense in $\mathbf{L}_2(B)$. Since this class contains the functions $\max[0, \varphi']$, $\min[1, \varphi']$ when it contains φ', the class contains a sequence of functions, with values between 0 and 1, converging almost everywhere on B to the indicator function of any preassigned measurable subset C of B. But then $\int_C \phi\mu(\xi_0, d.) = 0$, so that ϕ vanishes almost everywhere on B, as was to be proved.

We now investigate the the infinities of Naïm's θ-kernel discussed in section 7. The measure used on $R^M \times R^M$ is the product measure $\mu(\xi_0, .) \times \mu(\xi_0, .)$.

THEOREM 14.2. — *Let $\xi_1, \xi_2, \ldots,$ be the points on R^M of strictly positive harmonic measure, if any. Then $\theta(\xi_n, \xi_n) = \infty$, $n \geq 1$ and, if every negligible function on R^M vanishes almost everywhere, θ is finite almost everywhere on $R^M \times R^M - \cup_n \{(\xi_n, \xi_n)\}$.*

The minimal harmonic function corresponding to ξ_n is

$K(\xi_n, .) = \mu(., \{\xi_n\})/\mu(\xi_0, \{\xi_n\})$. This function has fine limit $1/\mu(\xi_0, \{\xi_n\})$ at ξ_n, and the Green function with an arbitrary pole has ordinary limit 0 at ξ_n because it has ordinary limit 0 almost everywhere on R^M, by a theorem of Naïm [11].

It follows that

$$(14.\ 3) \quad \theta(\xi_n, \xi_n) = f\lim_{\xi \to \xi_n} \theta(\xi, \xi_n) = f\lim_{\xi \to \xi_n} \frac{K(\xi_n, \xi)}{g(\xi_0, \xi)} = \infty.$$

If every negligible function on R^M vanishes almost everywhere, suppose, contrary to the assertion of the theorem, that θ is infinite on a set $A_2 \subset R^M \times R^M$ of strictly positive harmonic measure, containing no point either of whose coordinates is a ξ_n. Let u be any BLD harmonic function. Then $D'(u') < \infty$, so that $u'(\xi) = u'(\eta)$ almost everywhere on A_2, say on the subset $A_2(u)$ of the same measure as A_2. Let $\{u_n, n \geq 1\}$ be a sequence of BLD harmonic functions chosen, as is possible according to Theorem 14. 1, in such a way that the corresponding sequence of boundary functions is dense in $L_2(R^M)$ and let $A_2' = \cap_1^\infty A_2(u_n)$. Then there is a point η_0 of R^M, not of strictly positive harmonic measure, such that the set A_1 of points η with (η_0, η) in A_2' has strictly positive measure. Clearly every u_n' is constant on A_1, and hence every function in $L_2(R^M)$ is constant almost everywhere on A_1. This situation is impossible, however, because A_1 contains no point of strictly positive measure. Hence the theorem is true.

15. Green functions of the second kind.

Let ξ_0 be a fixed reference point for the kernels θ and K defined in Section 7 and let ξ be any point of R. The function $\varphi = 1 - K(., \xi)$ is continuous on R^M and, if u is a BLD harmonic function,

$$(15.\ 1) \qquad \int_{R^M} \varphi u' \mu(\xi, d.) = u(\xi_0) - u(\xi).$$

Then φ is in the class $L^*(R^M)$ defined in Section 10. Let $\alpha(\xi, .)$ be the BLD harmonic function with generalized normal derivative φ on R^M, vanishing at ξ_0. The function $g_2 = \alpha + g$ is our version of the Green function of the second kind. It has and is uniquely determined by the following two properties, once ξ_0 is specified.

(i) $g_2(\xi, .) - g(\xi, .)$ can be defined at ξ to be continuous there, and is then a BLD harmonic function, vanishing at ξ_0.

(ii) $g_2(\xi, .)$ has generalized normal derivative 1 on R^M. We shall write $\alpha'(\xi, .)$ and $g_2'(\xi, .)$ respectively for the (almost everywhere equal) fine boundary functions of $\alpha(\xi, .)$ and $g_2(\xi, .)$. Using the defining property of generalized normal derivatives, if u is any BLD harmonic function,

$$(15.\ 2)\qquad D(\alpha(\xi, .)) = q\alpha(\xi, \xi), \quad D(\alpha(\xi, .), u) = q(u(\xi) - u(\xi_0)).$$

From (13. 4),

$$(15.\ 3)\qquad \frac{|u(\xi) - u(\xi_0)|^2}{D(u)} \leq \alpha(\xi, \xi)/q,$$

with equality if and only if $u - u(\xi_0)$ is a multiple of $\alpha(\xi, .)$. From (15. 2,)

$$(15.\ 4)\qquad D(\alpha(\xi, .),\ \alpha(\eta, .)) = q\alpha(\eta, \xi).$$

Hence α and g_2 are symmetric functions. According to (13. 8), $\alpha(\xi, .)$ as an element of **H** depends continuously on its normal derivative viewed as an element of $\mathbf{L}_2(R^M)$. We conclude that α is a continuous function on $R \times R$.

The function $\alpha(\xi, .)$ is bounded, and in fact is bounded uniformly for ξ restricted to a compact subset of R. We only sketch the proof. Fix ξ and consider the function

$$u = g_2(\xi, .) - g(\xi_0, .).$$

This function is harmonic on R except at ξ and ξ_0 and has generalized normal derivative 0 on R^M. Let R_0 be R less a compact neighborhood of $\{\xi\} \cup \{\xi_0\}$. Then it can be shown that the restriction of u to R_0 is a BLD harmonic function, with generalized normal derivative 0 on R^M considered as a subset of R_0^M, with a bounded boundary function at the set B (the rest of R_0^M) of boundary points of R_0 in R. This presupposes that B is sufficiently smooth. It follows from Theorem 13. 1 that u is bounded in R_0 by the bounds of

u on B. Hence $\alpha(\xi, .)$ is bounded as stated above. It follows readily that for almost all points η of R^M $f\lim_{\xi\to\eta} \alpha(\xi, \zeta)$ exists, uniformly on compact ξ-sets. The limit harmonic function will be denoted by $\alpha(., \eta)$; $\alpha(\eta, .)$ is defined similarly.

Theorems 15. 1 and 15. 2 are our versions of the representations of a harmonic function using the second Green function. Note that in these theorems only the fine boundary function $g_2'(\xi, .)$ is involved, which can be replaced by the almost everywhere equal $\alpha'(\xi, .)$.

THEOREM 15. 1. — *Let u be a BLD harmonic function. Then*

$$u(\xi) - u(\xi_0) = D'(g_2'(\xi, .), u')/q. \tag{15. 5}$$

This theorem is a trivial consequence of (15. 2).

THEOREM 15. 2. — *Let u be a BLD harmonic function with generalized normal derivative on* R^M. *Then*

$$u(\xi) - u(\xi_0) = -\int_{R^M} g_2'(\xi, .)\frac{\partial u}{\partial g}\mu(\xi_0, d.). \tag{15. 6}$$

The representation (15. 6) is derived from (15. 2) with the help of the defining property of generalized normal derivatives.

Let μ be a (positive) measure on R with compact support and define the α-potential u by

$$u(\xi) = \int_R \alpha(\xi, \eta)\mu(d\eta). \tag{15. 7}$$

Since $\alpha(., \eta)$ is harmonic, the function is, in small open sets, the harmonic average of its boundary values. Hence u has the same property, so u is harmonic. Obviously $u(\xi_0) = 0$. A similar argument shows that differentiation under the integral sign is admissible and we obtain, if v is any BLD harmonic function,

(15. 8)

$$D(u, v) = \int_R D_\xi(\alpha(\xi, \eta), v)\mu(d\eta) = q\int_R [v(\eta) - v(\xi_0)]\mu(d\eta).$$

In particular u is itself BLD harmonic and has a generalized normal derivative on R^M,

$$\text{(15. 9)} \qquad D(u) = q\int_R\int_R \alpha(\xi, \eta)\,\mu(d\xi)\,\mu(d\eta),$$
$$\frac{\partial u}{\partial g}(\xi) = \int_{R^M}[1 - K(\xi, \eta)]\mu(d\eta).$$

If φ is the α-potential of the measure ν with compact support (15. 8) becomes

$$\text{(15. 10)} \qquad D(u, \varphi) = q\int_R\int_R \alpha(\xi, \eta)\mu(d\xi)\nu(d\eta).$$

Now let μ be any measure for which the double integral in (15. 9) exists as an absolutely convergent integral. Then if $R = \cup_n A_n$, where A_n is a set whose closure is a compact subset of R, with $A_1 \subset A_2 \subset \ldots$, if $\mu_n(B) = \mu(B \cap A_n)$, and if u_n is the α-potential of the measure μ_n, $D(u_n - u_m) \to 0$ so that $\lim_{n\to\infty} u_n$ exists, pointwise locally uniformly and also in the BLD sense. Hence the α-potential u of μ is defined and BLD harmonic. If μ is a signed measure whose absolute variation makes the double integral in (15. 9) converge absotely, we define the potential of μ in the obvious way. In all cases the potential u is BLD harmonic, has a generalized normal derivative on R^M given by (15. 9), and (15. 10) is valid if ν satisfies the same conditions as μ.

Now let μ be a measure of finite energy for whose absolute variation the double integral in (15. 9) converges absolutely, and let u be the g_2-potential of μ,

$$\text{(15. 11)} \qquad u(\xi) = \int_R g_2(\xi, \eta)\mu(d\eta).$$

Then u is the sum of the g-potential (ordinary Green potential) of u and of the α-potential. The function is therefore a BLD function. If $\mu(R)$ is finite, the generalized normal derivative of u on R^M exists and is $\mu(R)$. This potential is the unique (up to an additive constant) BLD function which has constant generalized normal derivative on R^M and whose component of potential type is a Green potential with an assigned measure, corresponding to the classical problem of finding a function u with constant normal derivative at the boundary and assigned Laplacian Δu.

16. Mixed Green functions.

We have used two Green functions, g and g_2. The function $g(\xi, .)$ has fine boundary function (and even ordinary boundary function) 0 almost everywhere; the function $g_2(\xi, .)$ has generalized normal derivative 1. The two functions have the same singularity at ξ. The classical idea of a mixed Green function is the following. Let A be a boundary set. Then $g_A(\xi, .)$ is to be a function which is harmonic except at ξ, where it has the same singularity as $g(\xi, .)$, and is to have the boundary function 0 on A, boundary normal derivative 0 on $R^M - A = B$. When A is the whole boundary, g_A is then g. When is the empty set, a change must be made because the integral of the normal derivative is the mass due to the singularity, so that the boundary normal derivative is required to be 1 instead of 0. This leads to g_2. We accordingly can and shall restrict A by the inequality $0 < \mu(\xi_0, A) < 1$ below.

Since the argument developing the properties of g_A is similar to that for g_2, it will only be sketched. Let ξ_0 be a fixed reference point and suppose that both A and its complement B on the Martin boundary have strictly positive harmonic measure. Let $\alpha_A(\xi, .)$ be the BLD harmonic function whose fine boundary function vanishes almost everywhere on A, and which has generalized normal derivative $- K(., \xi)$ on B. Define $g_A = \alpha_A + g$. Then g_A has and is uniquely determined by the following two properties, when ξ_0 and A are specified.

(i) $g_A(\xi, .) - g(\xi, .)$ can be defined at ξ to be continuous there, and is then a BLD harmonic function.

(ii) The fine boundary function of $g_A(\xi, .)$ vanishes almost everywhere on A; $g_A(\xi, .)$ has the generalized normal derivative 0 on B.

The same kind of reasoning as that used to prove the boundedness of $\alpha(\xi, .)$ can be applied here to prove the positivity of $g_A(\xi, .)$ and the fact that $\alpha_A(\xi, .)$ is uniformly bounded for ξ in a compact subset of R. Since $g(\xi, .)$ is dominated by every positive superharmonic function with the same singularity at ξ, $\alpha_A(\xi, .)$ must be positive.

If u is any BLD harmonic function whose fine boundary function vanishes almost everywhere on A,

$$D(\alpha_A(\xi, .), u) = qu(\xi). \tag{16. 1}$$

Then

$$D(\alpha_A(\xi, .), \alpha_A(\eta, .)) = q\alpha_A(\eta, \xi). \tag{16. 2}$$

We conclude as in Section 15 that α_A is symmetric and is continuous on $R \times R$.

Following the reasoning in Section 15, for almost every η on R^M there is a harmonic function which we denote by $\alpha_A(., \eta)$ and $\alpha_A(\eta, .)$ such that $\alpha_A(., \xi) \to \alpha_A(., \eta)$ uniformly on compact subsets of R when $\xi \to \eta$ in the fine topology. We denote by $\alpha'_A(\xi, .)$ and $g'_A(\xi, .)$ the fine boundary functions of the functions denoted without primes.

The difference $\alpha_{A_2}(\xi, .) - \alpha_{A_1}(\xi, .)$ is BLD harmonic, and, if $A_2 \supset A_1$, this function has a fine boundary function ≤ 0 on A_2, vanishing generalized normal derivative on $R^M - A_2$. Applying Theorem 13. 1 we deduce that the difference is negative: α_A decreases when A increases. We now prove that $g_{A_n} \downarrow g$ is $A_n \uparrow R^M$. In fact if ξ is a point of R and if $\alpha_{A_n}(\xi) \downarrow u$ say, u is a positive harmonic function, which we show vanishes identically. In the first place it is a BLD harmonic function, because

$$D(u) \leq \lim_{n\to\infty} D(\alpha_{A_n}(\xi, .)) = q \lim_{n\to\infty} \alpha_{A_n}(\xi, \xi) = qu(\xi).$$

In the second place, since $u \leq \alpha_{A_n}(\xi, .)$, the fine boundary function of u must vanish almost everywhere. Hence $u = 0$, and we have proved more than we stated, since we have proved that $\alpha_{A_n}(\xi, .) \to 0$ in the BLD sense. A slight extension of the reasoning shows that $\alpha_A(\xi, .) \to 0$ when $\mu(\xi_0, A) \to 1$.

Going in the other direction we can only show that if $A_n \downarrow 0$ then there is a symmetric continuous function φ on $R \times R$ such that

$$\alpha_{A_n}(\xi, .) - \alpha_{A_n}(\xi_0, \xi_0) \to \varphi(\xi, .), \tag{16. 3}$$

uniformly on compact subsets of R, for each point ξ in R and that

$$\alpha_{A_n}(\xi, .) - \alpha_{A_n}(\xi_0, .) \to \varphi(\xi, .) \tag{16. 4}$$

in the BLD sense. Moreover $\alpha(\xi, .) - \varphi(\xi, .)$ has generalized normal derivative 0 on A_n for every n, and

$$(16.5) \qquad \varphi(\xi_0, .) = 0, \qquad D(\varphi(\xi, .)) = q\varphi(\xi, \xi).$$

We would of course like to identify φ with α, but this seems impossible without further hypotheses. These results can be deduced using standard Hilbert space reasoning based on the following facts. Let $\varphi_A(\xi, .) = \alpha_A(\xi, .) - \alpha_A(\xi_0, .)$. Then ($\mathbf{H}$ — geometry) $\alpha(\xi, .) - \varphi_A(\xi, .)$ is orthogonal to every $\varphi_C(\xi, .)$ with $C \supset A$ and $\varphi_A(\xi, .) - \varphi_C(\xi, .)$ is orthogonal to every $\varphi_E(\xi, .)$ with $E \supset C \supset A$. We omit the details.

If u is a BLD harmonic function whose fine boundary function vanishes almost everywhere on A, and if neither A nor its complement B with respect to R^M has harmonic measure 0, u can be expressed in terms of its fine boundary function, using (16. 1),

$$(16.6) \quad u(\xi) = D'(\alpha'_A(\xi, .), u)/q = D'(g'_A(\xi, .), u)/q.$$

If in addition u has a generalized normal derivative on B we can manipulate (16. 1) to obtain

(16. 7)
$$u(\xi) = -\int_B \frac{\partial u}{\partial g}\alpha'_A(\xi, .)\mu(\xi_0, d.) = -\int_B \frac{\partial u}{\partial g} g'_A(\xi, .)\mu(\xi_0, d.).$$

Geometrically the integral (16. 7) represents a projection. The classes of BLD harmonic functions with fine boundary functions vanishing on A and with generalized normal derivatives existing and 0 on B are images of closed linear manifolds in $\mathbf{H}$ which are orthogonal and span $\mathbf{H}_h$. If u is *a* BLD harmonic function with a generalized normal derivative on B, the integral in (16. 7) represents the projection of u on the first class. It thus solves the problem of minimizing $D(u)$ for the generalized normal derivative assigned on B, the dual of a minimum problem considered in Section 13.

We shall now find the projection of u on the second of the above classes and then combine the two results to obtain an expression for u. We make an extra, undesirable, hypothesis. Suppose that $\alpha_A(\xi, .)$ has a generalized normal derivative on

R^M. Then if u is a BLD harmonic function with a generalized derivative on R^M which vanishes on B,

$$(16.\ 8)\quad D(\alpha(\xi,.)u) = -q\int_{R^M}\alpha(\xi,.)\frac{\partial u}{\partial g}\mu(\xi_0,.) = 0$$
$$= -q\int_{R^M}\frac{\partial\alpha_A(\xi,.)}{\partial g}u'\mu(\xi_0,d.)$$
$$= -q\int_{R^M}\frac{\partial g_A(\xi,.)}{\partial g}u'\mu(\xi_0,d.) + qu(\xi).$$

Hence

$$(16.\ 9)\qquad u(\xi) = \int_{R^M}\frac{\partial g_A(\xi,.)}{\partial g}u'\mu(\xi_0,d.).$$

Combining our results, and making use of the decomposition of u into the sum of its projections on the two classes described above we find that *if u is a BLD harmonic function with a generalized normal derivative on* B, *then (under our extra hypothesis that $\alpha_A(\xi,.)$ has a generalized normal derivative on* R^M),

(16. 10)
$$u(\xi) = \int_{R^M}\frac{\partial g_A(\xi,.)}{\partial g}u'\mu(\xi_0,d.) - \int_{R^M}g'_A(\xi_0,.)\frac{\partial u}{\partial g}\mu(\xi_0,d.).$$

This is the desired representation of u in terms of the mixed Green function.

17. The third boundary value problem.

Let ξ_0 be a reference point of R and let σ be a positive bounded measurable function on R^M, not vanishing almost everywhere. Let A be the set of zeros of σ, and define the measure ν of subsets of $B = R^M - A$ by setting $\nu(d\xi) = \mu(\xi_0, d\xi)/\sigma(\xi)$. Let $\mathbf{L}_2(\nu)$ be the $\mathbf{L}_2$ — space of functions on B relative to the measure ν with inner product denoted by (,). Then $\mathbf{L}_2(\nu) \subset \mathbf{L}_2(B)$. Let $\mathbf{L}$ be the subclass of $\mathbf{L}_2(\nu)$ satisfying the condition $(\varphi, \sigma) = 0$. If $\varphi \in \mathbf{L}$ define $S\varphi$ as the restriction to B of $\sigma u'$, where u is the unique BLD harmonic function with generalized normal derivative on R^M, and satisfying

$$\frac{\partial u}{\partial g} = 0 \text{ on A}, \qquad \sigma u' \in \mathbf{L}.$$
$$= \varphi \text{ on B}$$

The last condition is always possible, with the help of an additive constant in u. Then $S\varphi = 0$ if and only if the extension by 0 of φ to R^M is negligible on R^M. (See Section 11 for the definition of negligibility.) If u_i is determined by φ_i as u is by φ above,

$$(17.\ 1) \qquad (S\varphi_1, \varphi_2) = -D(u_1, u_2)/q.$$

Thus S is a negative definite symmetric linear transformation from **L** into itself. It is therefore bounded and selfadjoint. We shall use below the fact that 1 is in the resolvent set of S.

We now consider the following boundary value problem. Let f be a function in the class $L^*(R^M)$ defined in Section 10 or at least a function differing by a constant multiple of σ from such a function. For example f may be any function in $\mathbf{L}_2(R^M)$. The problem is to find a BLD harmonic function u, with generalized normal derivative on R^M, one of whose generalized normal derivatives satisfies the boundary condition

$$(17.\ 2) \qquad \frac{\partial u}{\partial g} = \sigma u' + f.$$

Note that the solution to (17. 2), if there is one, is unique, since 1 is not a characteristic value of S. An application of Theorem 13. 1 shows that if u is *a* solution of our boundary value problem

$$(17.\ 3) \qquad \inf(-f/\sigma) \leq u \leq u \sup(-f/\sigma)$$

where the extremes are on the set where σf does not vanish. Thus u is a bounded function if these extremes are finite.

If a is any constant, it is sufficient to solve the problem with the boundary condition

$$(17.\ 3') \qquad \frac{\partial u}{\partial g} = \sigma u' + (f - a\sigma)$$

instead of (17. 2), because if u is a solution of (17. 2′) $u - a$ is a solution of (17. 2) and conversely. Hence it is no restriction to solve (17. 2) under the restriction that $f \in L^*(R^M)$ and we shall do so. Let f_1 be the restriction to B of $\sigma u'_f$, whre u_f is the BLD harmonic function with generalized normal derivative f on R^M, and whith $\sigma u'_f$ in **L**. Since 1 is in the resolvent set of S, there is a unique function φ in **L** satisfying the equation $S\varphi - \varphi = -f_1$. By definition of S, $(\varphi - f_1)/\sigma$ coincides on B with a certain BLD boundary function u'. The function $u + u_f$ is the desired solution of (17. 2).

The functions u(**H**-topology) and u' ($\mathbf{L}_2(R^M)$-topology) depend continuously on $f(\mathbf{L}_2(R^M)$-topology) if f is supposed in $\mathbf{L}_2(R^M)$.

We now define a Green function for our boundary value problem. For each point ξ of R let $\beta(\xi, .)$ be the BLD harmonic function with generalized normal derivative $\sigma\beta(\xi, .) - K(., \xi)$ on R^M. Let $g_3 = \beta + g$. Then g_3 has and is uniquely characterized by the following properties.

(i) $g_3(\xi, .) - g(\xi, .)$ can be defined at ξ to be continuous there and if so defined is a BLD harmonic function.

(ii) $g_3(\xi, .)$ has generalized normal derivative $\sigma g_3(\xi, .)$ on R^M.

Let $\beta'(\xi, .)$ and $g'_3(\xi, .)$ be the almost everywhere equal fine boundary functions of the functions denoted without the primes. By definition of the generalized normal derivative, if u is a BLD harmonic function,

$$(17.\ 4)\quad D(\beta(\xi, .), u) = qu(\xi) - q\int_{R^M} \sigma\beta'(\xi, .)u'\mu(\xi_0, d.)$$

Hence

(17. 5)

$$D(\beta(\xi, .), \beta(\eta, .)) = q\beta(\eta, \xi) - q\int_{R^M} \sigma\beta'(\xi, .)\beta'(\eta, .)\mu(\xi_0, d.).$$

From which it follows, as in the analogous discussions in previous sections, that β and g_3 are symmetric continuous functions.

Theorem 17. 1. — *If u is a BLD harmonic function satisfying* (17. 2), *with f and σ as described, u can be represented in the form*

$$(17.\ 6)\quad u(\xi) = -\int_{R^M} f\beta'(\xi, .)\mu(\xi_0, d.) = -\int_{R^M} fg'_3(\xi, .)\mu(\xi_0, d.).$$

This representation follows from an elementary manipulation of (17. 4).

If f is a negligible function on R^M, the corresponding solution u vanishes identically. More generally, different choices of f will lead to the same solution if and only if the difference between the choices is a negligible function on R^M.

The g_3-potential of a signed measure on R,

$$u(\xi) = \int_R g_3(\xi, \eta)\mu(d\eta), \tag{17. 7}$$

is, at least formally, a function which is the sum of a harmonic function and the Green potential of μ, and satisfies the boundary condition $\partial u/\partial g = \sigma u'$. It is easy to verify that u is in fact a BLD function for which these assertions are true if μ is a signed measure of finite energy for which the integrals

$$q\int_R\int_R \beta(\xi, \eta)\mu(d\xi)\mu(d\eta),$$

$$q\int_{R^M}\int_R\int_R \beta(\xi, \zeta)\beta(\eta, \zeta)\mu(d\xi)\mu(d\eta)\mu(\xi_0, d\zeta)$$

converge absolutely when μ is replaced by its absolute variation. Under this condition $D(u)$ is equal to the first of these integrals less the second.

18. A characteristic value problem.

The characteristic value problem treated in this section goes back to Pleijel (1951). See Odhnoff [12] for references and a more recent treatment.

In this and the next section we suppose that R satisfies the complete continuity condition discussed in Section 12. The transformation S of the preceding section is accordingly completely continuous. There is therefore a sequence $\{\varphi_n, n \geq 1\}$ of characteristic functions of S, corresponding to the non-vanishing characteristic values of S, an orthonormal sequence in L, complete in the orthogonal complement of the characteristic manifold for the characteristic value 0. If U_n is the BLD harmonic function in terms of which $S\varphi_n = \sigma U_n'$ is defined (see the beginning of Section 17),

$$\begin{aligned} \frac{\partial U_n}{\partial g} &= -\,\delta_n \sigma U_n' \\ \delta_{mn} &= D(U_m, U_n)/q\delta_n = (\sigma U_m', \sigma U_n'), \end{aligned} \tag{18. 1}$$

where $\delta_1, \delta_2, \ldots,$ are the negative reciprocals of the non-vanishing characteristic values of S. We can suppose that $0 < \delta_1 \leq \delta_2 \leq \ldots$. Thus the U_n and $\sigma U'_n$ sequences are both orthogonal in their respective spaces **H** and **L**. The sequence $\{\sigma U'_n, n \leq 1\}$ together with the class of functions φ in $\mathbf{L}_2(\nu)$ whose extensions by 0 to R^M are negligible on R^M, and with the constant multiples of σ, span $\mathbf{L}_2(\nu)$.

Instead of relying on the general theory of completely continuous selfadjoint transformations on a Hilbert space we could also have proceeded using extremal procedures. In fact there is a BLD harmonic function u_1 minimizing $D(u)$ for u a BLD harmonic function with $\int_B \sigma u'^2 \mu(\xi_0, d.) = 1$ and the usual argument yields the fact that $\partial u_1/\partial g = -\sigma u'_1/qD(u_1)$. Thus u_1 is a solution of our characteristic value problem, and $\delta_1 = 1/qD(u_1)$. The function u_1 is a linear combination of the members of the U_j sequence with $\delta_j = \delta_1$. At the next step we would minimize $D(u)$ for u a BLD harmonic function with $\int_B \sigma u'^2 \mu(\xi, d.) = 1$ and $D(u, u_1) = 0$, and so on.

The formulation of the results becomes more elegant after a change in the reference measure. Define the measure ν_1 of subsets of R^M by $\nu_1(d\xi) = \sigma(\xi)\mu(\xi_0, d\xi)$, and denote by $\mathbf{L}_2(\nu_1)$ the $\mathbf{L}_2$ space for this measure. Define $U_0 = 1\Big/\left[\int_{R^M} \sigma\mu(\xi_0, d.)\right]^{1/2}$. Then $\{U'_n, n \geq 0\}$ is an orthogonal sequence in $\mathbf{L}_2(\nu_1)$ and the orthogonal complement is the class of functions φ in $\mathbf{L}_2(\nu_1)$ for which $\sigma\varphi$ is negligible on R^M. Then any function in $\mathbf{L}_2(\nu_1)$ can be expressed as the sum of its Fourier series (convergent in the mean with the weighting ν_1) and a function φ in the class just described. In particular, let u' be a BLD boundary function. Then u' is orthogonal in $\mathbf{L}_2(\nu_1)$ to the class of functions φ and we obtain the following Fourier expansion

$$(18.2) \qquad u' = \sum_{n \geq 0} a_n U'_n, \qquad a_n = \int_{R^M} u' U'_n \sigma\mu(\xi_0, d.)$$

convergent in the mean with weighting ν_1.

The sequence $\{U_n, n \geq 1\}$ corresponds to an orthogonal sequence in $\mathbf{H}_h$, and the orthogonal complement of the latter sequence corresponds to the family of BLD harmonic functions

φ for which $\sigma\varphi'$ is negligible on R^M. For such a function φ' must vanish almost everywhere on B because $\int_{R^M} \varphi'^2\sigma\mu(\xi_0, d.) = 0$, and conversely this vanishing of φ' characterizes the family. We thus obtain the representation

$$u = \sum_{n \geq 0} a_n U_n + U, \tag{18. 3}$$

where U is a BLD harmonic function whose fine boundary function vanishes almost everywhere on B. The coefficient a_n is the same as the nth coefficient in (18. 2). The series converges in $\mathbf{H}_h$, and the term a_0U_0 is of course equivalent to 0 in considering the representation as one in $\mathbf{H}_h$. If the representation is to be made into a functional representation, additive constants can be adjoined, say to make each summand vanish at ξ_0, which will make the series converge pointwise, that is, in the BLD sense. But then the corresponding sequence of fine boundary functions converges in the mean relative to harmonic measure, and so in the mean relative to ν_1-measure. On comparing this conclusion with the representation (18. 2) we see that the series in (18. 3) itself converges in the BLD sense and that the representation of u can be taken as an ordinary representation of the function.

Applying Bessel's inequality to (18. 3) we find

$$\sum_{n \geq 0} a_n^2\delta_n < \infty, \tag{18. 4}$$

and an application of the Riesz-Fischer theorem then completes the proof of the following theorem.

THEOREM 18. 1. — *A function in the class* $\mathbf{L}_2(\nu_1)$ *coincides almost everywhere (harmonic measure) on B with a BLD boundary function if and only if* (*a*) *it is orthogonal in* $\mathbf{L}_2(\nu_1)$ *to every* φ *for which* $\sigma\varphi$ *is negligible on* R^M *and if* (*b*) *its Fourier coefficients relative to the* U'_n *sequence satisfy* (18. 4).

The condition (*a*) is equivalent to the following: (*a'*) *it is orthogonal in* $\mathbf{L}_2(B)$ *to every function in* $\mathbf{L}_2(B)$ *whose extension by* 0 *to* R^M *is negligible on* R^M.

The complete continuity hypothesis we have imposed is necessary to obtain our results. In fact if $\sigma = 1$ the complete continuity condition on R follows from (18. 3) and (18. 4) knowing that $\delta_n \to \infty$ if there are infinitely many characteristic values.

For R a disc of radius 1, central angle λ, with center ξ_0, if $\sigma = 1$,

$$U_{2k-1}(\lambda) = \sqrt{2}\cos k\lambda, \qquad U'_{2k}(\lambda) = \sqrt{2}\sin k\lambda,$$
$$\delta_{2k-1} = \delta_{2k} = k, \qquad k \geqslant 1.$$

Every negligible function on R^M vanishes almost everywhere.

THEOREM 18. 2. — *If u is a BLD harmonic function, given by* (18. 3), *and if u has a generalized normal derivative on* R^M *which vanishes on* A *and whose restriction to B is in the class* $L_2(\nu)$, *then* $U = 0$ *and*

$$\sum_n a_n^2\delta_n^2 < \infty. \tag{18. 5}$$

Conversely, if (18. 5) *is true* $\sum_{n\geq 0} a_n U_n$ *has generalized normal derivative* $-\sigma\sum_{n\geq 1} a_n\delta_n U'_n$ *which has the two stated properties.*

If the generalized normal derivative of u vanishes on A, u is orthogonal (H) to every BLD harmonic function whose fine boundary function vanishes almost everywhere on B. Hence $U = 0$ in (18. 3). Moreover, using the other stated property of $\partial u/\partial g$, there is a constant γ for which

$$D(u, \varphi)^2 \leq \gamma \int_{R^M} \varphi'^2 \nu_1(d.) \tag{18. 6}$$

for every BLD boundary function φ'. If (18. 6) is true, choose $\varphi = \sum_1^k a_n\delta_n U_n$. Then (18. 6) becomes

$$\left[q\sum_1^k a_n^2\delta_n^2\right]^2 \leq \gamma\sum_1^k a_n^2\delta_n^2$$

and we conclude that (18. 5) is true. Conversely if (18. 5) is true $\sum_{n\geq 1} a_n\delta_n U_n$ converges in $L_2(\nu_1)$, that is in the mean with weighting ν_1. The sum multiplied by $-\sigma$ can be verified directly to be a generalized normal derivative of $\sum_{n\geq 0} a_n U_n$ and to have the two properties stated in the theorem.

It is convenient for some purposes to modify the characteristic value problem we have treated in this section when, as we shall suppose in the remainder of this section, A has strictly positive harmonic measure. Define the transformation T on $L_2(\nu)$ by $T\varphi = \sigma u'$ where u is the BLD harmonic function defined by

$$\frac{\partial u}{\partial g} = \varphi \quad \text{on B}$$
$$u' = 0 \quad \text{almost everywhere on A.}$$

Then (17. 1) is true for T as well as S; T is a bounded self-adjoint negative definite transformation from $L_2(\nu)$ into itself, and $T\varphi = 0$ if and only if φ is negligible on B.

Just as in the discussion at the beginning of this section we find that there is a sequence $\{V_n, n \geq 1\}$ of BLD harmonic functions (a sequence which may be empty or finite) satisfying

$$\text{(18. 1')} \qquad \begin{aligned} &\frac{\partial V_n}{\partial g} = -\delta_n \sigma V_n' \quad \text{on B} \\ &V_n' = 0 \quad \text{almost everywhere on A,} \\ &\delta_{mn} = D(V_m, V_n)/q\delta_n = (\sigma V_m', \sigma V_n'), \end{aligned}$$

where $0 < \delta_1 \leq \delta_2 \leq \ldots$ and these numbers are the negative reciprocals of the nonvanishing characteristic values of T, repeated according to their multiplicity. The δ_n-sequence here is not the same as that in (18. 1). The sequence $\{V_n', n \geq 1\}$ is orthonormal in $L_2(\nu_1)$ and, together with the functions φ with $\sigma\varphi$ negligible on B, span $L_2(\nu_1)$. If u is a BLD harmonic function for which u' vanishes almost everywhere on A, u' has the Fourier expansion

$$\text{(18. 2')} \quad u' = \sum_{n \geq 1} a_n V_n', \qquad a_n = \int_{R^M} u' V_n' \sigma \mu(\xi_0, d.)$$

and

$$\text{(18. 3')} \qquad u = \sum_{n \geq 1} a_n V_n.$$

The first series converges in the $L_2(\nu_1)$ topology, the second in the BLD sense. The function U in (18. 3) has no counterpart in this development. Theorem 18. 1 becomes the following.

Theorem 18. 1′. — *A function in the class* $L_2(\nu_1)$ *which vanishes almost everywhere (harmonic measure) on A coincides almost everywhere on* R^M *with a BLD harmonic function if and only if* (*a*) *it is orthogonal in* $L_2(\nu_1)$ *to every* φ *for which* $\sigma\varphi$ *is negligible on B and if* (*b*) *its Fourier coefficients relative to the* V'_n *sequence satisfy* (18. 4).

The condition (*a*) is equivalent to the following: (*a*′) *it is orthogonal in* $L_2(B)$ *to every function in* $L_2(B)$ *which is negligible on B.*

Theorem 18. 2 becomes Theorem 18. 2′ which we do not state explicitly but which differs from the former theorem only in that U'_n is replaced by V'_n that u' is to vanish almost everywhere on A, and that the generalized normal derivative is on B rather than on R^M.

19. Series expansions.

Under the hypotheses of Section 18, namely the complete continuity hypothesis for R and the positivity and boundedness of σ, we can express our fundamental kernels in Fourier series.

If σ is strictly positive,

(19. 1)

$$\begin{aligned}\alpha(\xi, \eta) &= \int_B \alpha'(\xi, .)\sigma\mu(\xi_0, d.)/\sigma_0 + \sum_{n\geq 1} [U_n(\xi) - U_n(\xi_0)]U_n(\eta)/\delta_n,\\ &= \sum_{n\geq 1} [U_n(\xi) - U_n(\xi_0)][U_n(\eta) - U_n(\xi_0)]/\delta_n,\\ &\qquad \xi, \eta \in R, \qquad \sigma_0 = \int_B \sigma\mu(\xi_0, d.).\end{aligned}$$

The series converge in the BLD sense for fixed ξ, and uniformly on compact subsets of $R \times R$. The representation is also valid for η in R^M, if we use the relevant boundary functions. For fixed ξ in R the series obtained in this way converge in the mean, weighting ν_1 on R^M.

If σ is strictly positive,

(19. 2)

$$\begin{aligned}K(\eta, \xi) &\simeq \sigma(\eta)/\sigma_0 + \sigma(\eta)\sum_{n\geq 1} U_n(\xi)U'_n(\eta)\\ &\simeq 1 + \sigma(\eta)\sum_{n\geq 1} [U_n(\xi) - U_n(\xi_0)]U'_n(\eta), \quad \xi \in R, \eta \in R^M\end{aligned}$$

where « $\simeq$ » means that, for fixed ξ, there is equality up to a function negligible on R^M.

Still supposing that σ is strictly positive,

$$(19.\ 3)\quad \beta(\xi, \eta) = 1/\sigma_0 + \sum_{n\geq 1} U_n(\xi)U_n(\eta)/(1 + \delta_n),\ \xi,\ \eta \in R,$$

where the series converges in the BLD sense for fixed ξ, and uniformly on compact subsets of $R \times R$. The representation is also valid for η in R^M, if we use the relevant boundary functions, and is then convergent in the mean, weighting ν_1, for fixed ξ.

If the set A of zeros of σ has strictly positive harmonic measure,

$$(19.\ 1')\quad \alpha_A(\xi, \eta) = \sum_{n\geq 1} V_n(\xi)V_n(\eta)/\delta_n, \qquad \xi, \eta \in R.$$

We recall that the characteristic values here are not the same as those in the preceding series. The series converges in the BLD sense for each ξ and converges uniformly on compact subsets of $R \times R$. With the same conventions as above, the representation is also valid for one argument on the boundary.

APPENDIX

Since the theory of uniform integrability of function families is less well known than it should be, the following outline of that part of the theory most useful in studying boundary value problems is appended. Let f_t be a function from a measure space X_t to the reals, for t in an index set I. The measure on X_t will be denoted by μ_t. It is supposed that this measure is positive and that $\sup_t \mu_t (X_t) < \infty$. The function family is called uniformly intégrable if

$$\lim_{\alpha\to\infty} \int_{A_{\alpha,t}} |f_t|\, d\mu_t = 0$$

uniformly on the index set I, where $A_{\alpha,t}$ is the set where the integrand exceeds α. For proofs of the following facts see the fundamental papers by C. de la Vallée Poussin [*Trans. Amer. Math. Soc.* 16 (1915), 435-501] and S. Saks [*Trans. Amer. Math. Soc.*, 35 (1933), 549-556, 965-970].

(1) If the set I is finite, the family is uniformly integrable if and only if each member has a finite integral.

(2) The family is uniformly integrable if and only if the following two conditions are satisfied:

$$(a) \qquad \sup_t \int_{X_t} |f_t| \, d\mu_t < \infty ;$$

$$(b) \qquad \lim_{\delta \to 0} \sup \left| \int_{A_t} f_t \, d\mu_t \right| = 0.$$

where the supremum is taken for all t in I and measurable subsets A_t of X_t with $\mu_t(A_t) \leq \delta$.

(3) The family is uniformly integrable if and only if there is a Baire function Φ from the positive reals to the positive reals such that,

$$(a) \qquad \lim_{r \to \infty} \frac{\Phi(r)}{r} = \infty,$$

$$(b) \qquad \sup_t \int_{X_t} \Phi(|f_t|) \, d\mu_t < \infty.$$

Moreover in this case there exists a Φ which is even monotone increasing and convex.

In (4) and (5) it is supposed that the index set I is the set of positive integers that there is only a single measure space, $X_t = X$, $\mu_t = \mu$, and that $f_n \to f$ in measure.

(4) There is L_1-convergence if and only if the sequence is uniformly integrable.

(5) The sequence is uniformly integrable if and only if f is integrable and $\int_A f_n \, d\mu \to \int_A f \, d\mu$ for every measurable A (or for A = X if the functions are all positive).

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